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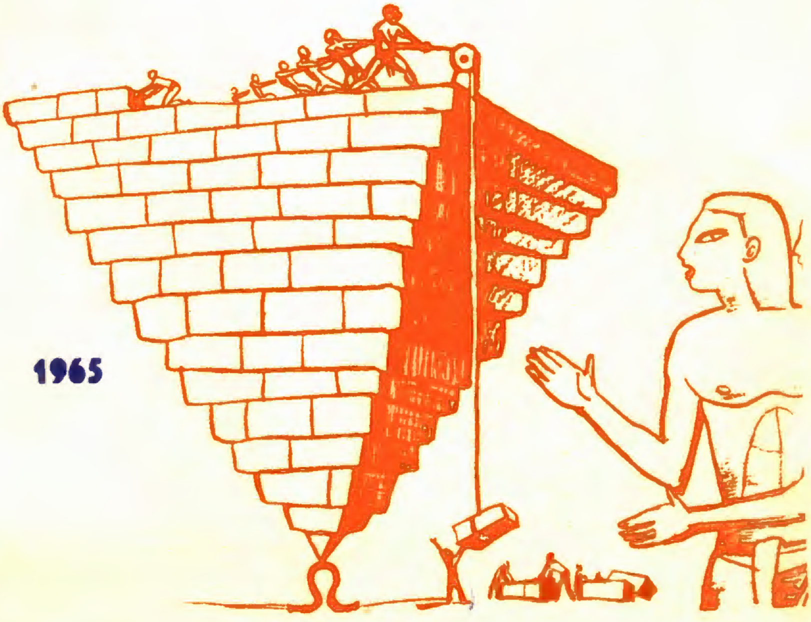
ПОЛУПРОСТЫЕ ГРУППЫ И СИСТЕМАТИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ

(Сборник статей)

Том I

ЛАБОРАТОРИЯ ТЕОРЕТИЧЕСКОЙ ФИЗИКИ



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ПОЛУПРОСТЫЕ ГРУППЫ И СИСТЕМАТИКА ЭЛЕМЕНТАРНЫХ ЧАСТИЦ

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Издание настоящего сборника вызвано возрастающим интересом к групповым подходам в теории элементарных частиц. В сборник включен ряд новейших статей, относящихся как к общей теории компактных групп, так и к применению некоторых из них, таких как $SU(3)$ и $SU(6)$, в теории элементарных частиц.

ОГЛАВЛЕНИЕ

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Simple Groups and Strong Interaction Symmetries*

R. E. BEHRENDS,† J. DREITLEIN,‡ C. FRONSDAL,§ AND W. LEE||

University of Pennsylvania, Philadelphia, Pennsylvania

CONTENTS

INTRODUCTION

ONE of the most natural questions when one looks at the mass of uncorrelated data on elementary particle interactions[1] is whether a systematic pattern is emerging from this complexity. The penetration of controlled laboratory experiments into the multi-Bev energy region can only make such a question more acute. Several attempts[2] have already been made to unfurl the underlying symmetry of strong interactions, such as might exist above and beyond those symmetries, e.g., isotopic symmetry,[3] which have already survived experimental tests.

In this article, we sharpen some tools which prove useful in formulating the consequences of proposed symmetries of a rather special type, namely, those symmetries which are characteristic of the simple Lie groups. Since it is as yet too early to establish a definite

* Supported in part by the National Science Foundation and by the Atomic Energy Commission.

† Now at the Graduate School of Science, Yeshiva University, New York.

‡ National Science Foundation Postdoctoral Fellow 1960–1961. Now at Stanford University, Stanford, California.

§ On leave of absence from the University of Bergen, Norway. Now at University of California, Los Angeles, California.

|| Now on leave of absence at the Institute for Advanced Study, Princeton, New Jersey.

[1] See, for example, the *Proceedings of the Tenth Annual Conference on High Energy Nuclear Physics, Rochester, 1960, University of Rochester* (Interscience Publishers, Inc., New York, 1960).

[2] See, for example, B. d'Espagnat and J. Prentki, Nuclear Phys. 1, 33 (1956); J. Schwinger, Ann. Phys. 2, 407 (1957); M. Gell-Mann, Phys. Rev. **106**, 1296 (1957); A. Pais, *ibid.* **110**, 574 (1958); J. Tiomno, Nuovo cimento **6**, 69 (1957); R. E. Behrends, *ibid.* 11, 424 (1959); D. C. Peaslee, Phys. Rev. 117, 873 (1960); J. J. Sakurai, *ibid.* **115**, 1304 (1959).

[3] See, for example, W. Heisenberg, Z. Physik **77**, 1 (1932); B. Cassen and E. U. Condon, Phys. Rev. **50**, 846 (1936); G. Breit, E. U. Condon, and R. D. Present, *ibid.* **50**, 825 (1936); G. Breit and E. Feenberg, *ibid.* **50**, 850 (1936).

symmetry of the strong interactions, both because of the lack of experimental data and the theoretical uncertainties about the way in which the symmetries will manifest themselves, the formalism developed is left quite flexible in order to accommodate a wide range of conceivable symmetries.

Much of the material is an exposition of the theory of Lie groups and, although most of the results have been known for many years, several new features appear. Thus the material on the composition and decomposition of Lie algebras by point set theory, the explicit construction of the Lie algebras, the tensor analysis of the groups B_2 and G_2, and the possible physics associated with the group B_2 is believed to be novel. A large portion of the remaining material is possibly unfamiliar to many physicists (as it was to us), and so is pedagogical in nature. Although the discussions are directed primarily to applications in elementary particle physics, many of the techniques have been used before in group theoretical treatments of atomic and nuclear spectroscopy.[4]

An admirable summary of the elementary properties of semi-simple Lie algebras is contained in the lecture notes of Racah,[5] which treat both the classification of semi-simple groups, following Cartan,[6] and their linear representations. A complete and rigorous derivation of the properties of semi-simple Lie algebras can be found in the work of Dynkin,[7] while Weyl's original work[8] remains the standard reference on the representation theory of semi-simple groups. For the tensor analysis associated with particular groups and with the Young tableaux, Weyl's *Classical Groups*[9] and *Group Theory and Quantum Mechanics*[10] is recommended. We assume that the reader is mildly conversant with the group theoretical treatment of angular momentum as given by Wigner,[11] for example. Finally, we give various references[12] to the basic mathematical literature.

As far as the physical application of the group theoretical methods is concerned, we are immediately faced with the problem of justifying the specific course which we pursue in attributing symmetries to strong particle interactions. The hope that symmetries exist, other than those associated with space-time structure, is kindled by the observation that some such "internal" symmetries are already apparent. First of all, charge independence has so far run the gauntlet of experimental tests[13] and has become a commonly accepted symmetry. In addition, a second kind of symmetry, slightly more mysterious than the former, is afforded by the electrodynamic[14] and weak-dynamic equivalence[15] of the muon and electron. Both of these symmetries call for a closer discussion.

It is well known that particles belonging to the same isotopic multiplet exhibit a remarkable similarity in their strong-interaction dynamics. Differences in behavior and in mass of isotopic spin multiplet members are quite naturally attributed to the charge-dependent electromagnetic interaction, which acts as a weak perturbation on the strong-interaction dynamics. Indeed, the breakdown of isotopic symmetry is evidenced in the high Z nuclear species where the coherent Coulomb field no longer can be treated as a perturbation. By analogy, we may conjecture that a basic symmetry exists among, say, baryon-baryon interactions, but that the full force of this symmetry is diluted by a relatively weak symmetry-breaking interaction. The answer to the question "under what circumstances will the symmetry-masking interaction be minimized?" is not yet clear, since the answer undoubtedly depends on the specific nature of the symmetry-breaking interaction. Of course, the latter interaction would most likely, produce the baryon mass differences besides its other effects.

In the case of the dynamic symmetry of muon and electron, no interaction is known which can serve to break the symmetry and account for the mass difference. Most physicists seem to feel that a specific difference in muon and electron interactions will ultimately emerge even if present experimental circumstances have not revealed it. If the proposed strong interaction symmetry resembles that of the muon and electron, it could conceivably be discernible even in the presence

[4] See, for example, G. Racah, Phys. Rev. **61**, 186 (1942); **62**, 438 (1942); **63**, 367 (1943); **76**, 1352 (1949). T. H. R. Skyrme, "Lectures in Nuclear Structure (I), General Theory and Shell Model," Department of Physics, University of Pennsylvania, Philadelphia, Pennsylvania, 1958.

[5] G. Racah, "Group Theory and Spectroscopy," Institute for Advanced Study, Lecture notes, Princeton, New Jersey, 1951.

[6] E. Cartan, *Thèse Paris* (1894) reprinted in E. Cartan, *Oeuvres Complètes* (Gauthiers-Villars, Paris, France, 1952). E. Cartan, Bull. Soc. Math. de France 41, 53 (1913).

[7] E. B. Dynkin, Am. Math. Soc. Translations, No. 17 (1950).

[8] H. Weyl, Z. Math. 24, 328, 377 (1925), reprinted in H. Weyl, *Selecta* (Birkhauser Verlag, Basel und Stuttgart, Germany, 1956), p. 262.

[9] H. Weyl, *Classical Groups* (Princeton University Press, Princeton, New Jersey, 1946), 2nd ed.

[10] H. Weyl, *Group Theory and Quantum Mechanics* (reprint, Dover Publications, New York).

[11] E. P. Wigner, *Group Theory and Its Applications to Atomic Structure* (Academic Press, Inc., New York, 1960).

[12] S. Lie and F. Engels, Theorie der Transformationsgruppen (B. G. Teubner, Leipzig, Germany, 1888–1893). V. Killing, Math. Ann. **31**, 252 (1888); **33**, 1 (1889); **34**, 57 (1889); **36**, 161 (1890). L. P. Eisenhart, *Groups of Continuous Transformations* (reprint, Dover Publications, New York, 1961). L. S. Pontrjagin, Topological Groups (Princeton University Press, Princeton, New Jersey, 1958). H. Freudenthal, "Lie Groups," Lecture notes, Department of Mathematics, Berkeley, California, 1960). D. Montgomery, "Topological Groups," Lecture notes, Haverford College, Haverford, Pennsylvania, 1956.

[13] See, for example, J. M. Blatt and V. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952).

[14] J. Garwin, L. Lederman, and M. Weinrich, Phys. Rev. **105**, 1415 (1957). G. Charpak, F. Farley, R. Garwin, T. Muller, J. Sens, V. Telegdi, and A. Zichichi, Phys. Rev. Letters **6**, 128 (1961).

[15] M. Ruderman and R. J. Finkelstein, Phys. Rev. **76**, 1458 (1949); J. A. Wheeler and J. Tiomno, Revs. Modern Phys. **21**, 144 (1949); O. Klein, Nature **161**, 897 (1948); E. Clementel and G. Puppi, Nuovo cimento **5**, 505 (1948); T. D. Lee, M. Rosenbluth, and C. N. Yang, Phys. Rev. **75**, 905 (1949).

of baryon mass differences, just as is the case with muon and electron interactions.

In summary, we are unable to give any *a priori* justification for the existence of strong interaction symmetries, but share the widespread feeling that such symmetries are plausible and not entirely unprecedented.

In Sec. I, the embryonic elements of the application of symmetry considerations to elementary particle interactions are presented to motivate physically the following sections. Section II is devoted to a necessarily abbreviated form of the theory of Lie groups, in which an attempt is made to appeal as much as possible to a physicist's intuition. There then follows (Sec. III) the properties and the construction of linear representations of Lie groups, of which it is hoped that elementary particles provide an instance. The next two sections (Secs. IV and V) solve the problem of finding certain properties of the Lie algebra representations, in particular, the "weights" of the representations and the decomposition of direct products of representations (generalized Clebsch-Gordan series). Two approaches are employed; one predominantly geometric (Sec. IV), the second predominantly algebraic (Sec. V). Section V is essentially the tensor analysis associated with simple groups. All roads lead to Sec. VI which is concerned with physical applications of the mathematical complex of the previous sections. From this summit, we briefly view the expanding vistas of possible strong interaction symmetries.

I. SYMMETRIES OF THE LAGRANGIAN

The basic idea behind Heisenberg's introduction of the concept of isotopic spin[3] was the realization that the neutron and the proton are, after all, quite similar. The differences in mass and in electromagnetic interactions are small in the context of the strong interactions. The fact that only two baryons were known led unambiguously to the assignment of a doublet structure to the "nucleon." Later, when strange particles were discovered, these were found, as is reflected in the name, to have properties so widely different from the nucleons that the assignment of the proton and the neutron to a doublet was retained without question. When one attempts to introduce symmetries which treat particles of widely different masses as states of the same field, however, it is not wise to be so categorical about the number of particles to be included in the scheme. Specific conjectures are made in the last section; for the present let n be the number of baryons treated as states of the same field, i.e., as belonging to the same supermultiplet. A favorite choice for n is 8, if all the observed baryons are included.[16–18] It could be less than eight if the baryons separate into two or more supermultiplets,[19] or it could be larger than 8 if some hypothetical baryons not yet discovered are included.

Let ψ_a, $a=1, 2, \cdots, n$, denote the n-component baryon field, where each component is a Dirac four-spinor, and let $\bar{\psi}^a=\psi_a^\dagger\gamma_4$. The free Lagrangian is

$$\mathcal{L}_0=\left(\frac{1}{2i}\right)\sum_{a=1}^{n}\bar{\psi}^a(\not{p}-im_a)\psi_a.$$

In the introduction we mentioned several different points of view, according to which the mass differences may be argued to be nonessential in the first analysis. When the n masses are put equal, $m_1=m_2=\cdots=m_n$, $\mathcal{L}_0$ is invariant under a set of linear transformations, acting on the set $\psi_a=\{\psi_1\cdots\psi_n\}$. In fact, let $U_a{}^b$ be a square $n\times n$ matrix, and consider the transformation

$$\psi_a\rightarrow\psi_a'=U_a{}^b\psi_b,$$

$$\bar{\psi}^a\rightarrow\bar{\psi}'^a=(U\psi)^{\dagger a}\gamma_4=\bar{\psi}^b(U^\dagger)_b{}^a.$$

Clearly, $\mathcal{L}_0$ is invariant if and only if U is unitary, i.e.,

$$(U^\dagger)_a{}^c(U)_c{}^b=\delta_a{}^b.$$

Hence, in matrix notation

$$\psi\rightarrow U\psi,\quad \bar{\psi}\rightarrow\bar{\psi}U^{-1},\quad UU^\dagger=U^\dagger U=1. \qquad \text{(I.1)}$$

The set of all $n\times n$ unitary matrices forms a *group*.[20] That is, if U, V are unitary, so are UV and U^{-1}. Hence $\mathcal{L}_0$ is invariant under the group of unitary transformations (I.1). This group contains an invariant subgroup, which is usually called the baryon gauge group. Any unitary matrix U may be written

$$U=e^{i\varphi}\mathfrak{U},$$

where φ is real and $\mathfrak{U}$ is unitary and unimodular:

$$\mathfrak{U}^\dagger\mathfrak{U}=\mathfrak{U}\mathfrak{U}^\dagger=1,\quad \det\mathfrak{U}=1. \qquad \text{(I.2)}$$

Invariance under the gauge transformation, represented by the factor $e^{i\varphi}$, corresponds to the conservation of baryons. This conservation law is taken for granted, and it is therefore unnecessary to include the gauge transformations in our analysis. From now on we deal with transformation matrices that are unimodular as well as unitary. The set of all such matrices forms a group[21] which is denoted SU_n.

In general, the interaction between the fields will break part of the symmetry of the free Lagrangian. Invariance under SU_n represents the maximum symmetry between the n baryons, and any group of transformations admitted by the fields in interaction is a subgroup of SU_n. In order to explore, in a systematic manner, the various groups of interest, it is helpful to review some topics from the theory of Lie groups. The basic concepts of the theory of Lie groups and of their

[16] R. E. Behrends and D. C. Peaslee, reference 2.

[17] T. D. Lee and C. N. Yang, Phys. Rev. **122**, 1954 (1961).

[18] M. Gell-Mann, Phys. Rev. (to be published).

[19] R. E. Behrends and A. Sirlin, Phys. Rev. **121**, 324 (1961).

[20] This group is called the *unitary group* U_n.

[21] It is the *factor group* of U_n with respect to the gauge group.

representations are reviewed in the next two sections. Before that, however, we say a few words about the problem of writing down interactions. It is convenient to deal with a simple specific example only, without any implication that the problems and their solution are peculiar to this case, or to this point of view. By way of an example, let us treat the case of a Yukawa-type interaction, invariant under SU_8, between the 8 baryons and a number m of bosons. The interaction Lagrangian is of the form

$$\mathcal{L}'=\bar{\psi}^a(\Gamma_\sigma)_a{}^b\psi_b\varphi^\sigma,$$

where the sum over σ runs from 1 to m. The φ^σ may or may not transform under SU_8, but once the transformation character of the φ^σ is fixed, generally it is not possible, to find matrices $(\Gamma_\sigma)_a{}^b$ such that $\mathcal{L}'$ is invariant. In order to answer questions of this kind, it is necessary to know the theory of direct products and reduction of representations. This is taken up in Sec. IV by one method, and in Sec. V by another. The answer, in the special case mentioned, is that there exist matrices $(\Gamma_\sigma)_a{}^b$ that make $\mathcal{L}'$ invariant in two cases only. Either all the φ^σ are invariant under SU_8, or there are at least 63 of them.[22]

II. LIE ALGEBRAS OF SIMPLE GROUPS

An important tool in the study of groups is the concept of an infinitesimal transformation. Since $\mathfrak{U}$ is unitary, it can be written $\exp(i\epsilon^A L_A)$ with L_A Hermitian, where the ϵ^A are a set of real continuous parameters.[23] For an infinitesimal transformation the exponential may be approximated by[24]

$$\mathfrak{U}=1+i\epsilon^A L_A,$$
or
$$\mathfrak{U}_a{}^b=\delta_a{}^b+i\epsilon^A(L_A)_a{}^b. \qquad \text{(II.1)}$$

The set of linear combinations, with arbitrary complex coefficients, of the Hermitian matrices L_A, associated with the transformations $\mathfrak{U}$, form the *Lie algebra* of the group. The $\mathfrak{U}_a{}^b$ determine the $(L_A)_a{}^b$ uniquely, and the converse is almost true. In fact, the L_A determine the $\mathfrak{U}$ up to a discrete set of transformations which commute with all the $\mathfrak{U}$.[25] We have taken $\mathfrak{U}$ to be unimodular, and this requires L_A to be traceless:

$$(L_A)_a{}^a=0. \qquad \text{(II.2)}$$

According to the fundamental theorem proved by Lie and Engels,[12] the structure of the group is completely specified by the commutation relations among the generators L_A of infinitesimal transformations,

$$[L_A,L_B]=C_{AB}{}^D L_D \qquad \text{(II.3)}$$

where the $C_{AB}{}^D$ are called the *structure constants* and satisfy the conditions

$$C_{AB}{}^D=-C_{BA}{}^D \quad \text{(Antisymmetry)}$$
$$C_{AB}{}^E C_{EF}{}^G+C_{BF}{}^E C_{EA}{}^G+C_{FA}{}^E C_{EB}{}^G=0 \quad \text{(Jacobi identity).} \qquad \text{(II.4)}$$

Many different sets of matrices may be found that satisfy the same commutation relations (II.3), with the same structure constants. Such matrix sets may be regarded as different realizations (or representations, see next section) of the same set of *abstract operators*. The latter, whose only properties are the commutation relations, is designated by a caret, as $\hat{H}_i$, $\hat{E}_\alpha$, etc., in order to emphasize that we are not dealing with any particular realization.

A group is *simple* if it has no invariant subgroups[26] except the unit element. A group is *semi-simple* if it has no Abelian (commutative) invariant subgroups. We have disposed of an Abelian invariant subgroup which is the baryon gauge group at the beginning. The distinction between groups which have Abelian invariant subgroups, and those which do not, rests upon the fact that the Abelian subgroups are most troublesome to handle from the viewpoint of representations.[27] We therefore restrict ourselves to the study of simple groups.[28] There are certain cases of simple or semi-simple groups with discrete transformations added, such as that discussed by Lee and Yang,[17] which have equal claim for attention, but these are not discussed in this paper.

It is worthwhile to draw an analogy between the possible symmetries of elementary particles and the three dimensional rotation group in ordinary quantum mechanics.[11,29] In quantum mechanics, one observes that when the potential is spherically symmetric, the angular-momentum operators, which are the generators of infinitesimal rotations, commute with the Hamiltonian. Since the three angular momentum operators do not commute among themselves one can diagonalize only one of them at a time, call it H_1. This is a linear operator, and so the eigenvalue of H_1 for a compound state is the sum of the eigenvalues associated with the component states (to be contrasted with the properties of L^2, say).

The conservation of the additive quantum numbers charge and strangeness[30] (or, equivalently, the third component of the isotopic spin and the hypercharge) in strong interactions is so well established that any group of practical interest must contain at least two commuting linear operators whose eigenvalues are the isotopic spin and the hypercharge. Let us denote these two operators by H_1 and H_2. Since the group is assumed to be the group of the Hamiltonian, i.e., every element of the group commutes with the Hamiltonian, one can diagonalize H_1 and H_2 simultaneously with the Hamiltonian, so that the eigenstates of the Hamiltonian have definite eigenvalues of H_1 and H_2, proportional to the I_3 and hypercharge quantum numbers.

The number of mutually-commuting linear operators[31] is called the *rank* of the group. Hence the rank of the three-dimensional rotation group is one. If the rank of the group is larger than two, there exists at least one more operator, H_3 say, which commutes with H_1 and H_2. But such an operator can mix states which are degenerate with respect to H_1 and H_2 only. Among the eight baryons, only the Λ and Σ^0 have equal charge and strangeness. Among the seven mesons no such degeneracy occurs. Thus, if H_3 is independent of H_1 and H_2, one or more of the following four possibilities can be considered:

(1) H_3 is the same for all eight baryons[32] and has a different value on a set of other baryons or physical states;

(2) H_3 mixes observed baryons with other physical states;

(3) H_3 splits the Λ, Σ^0 degeneracy, but is of the form aH_1+bH_2 for the other six baryons[33];

(4) The eight baryons are eigenstates of H_3 with eigenvalues which cannot be written in the form aH_1+bH_2+c.

Sometimes (3) and (4) leads to the forbidding of certain observed processes.[22] Although we can offer no arguments against the first two possibilities, we note that for these cases any group of rank three which accommodates the baryons will have a subgroup of rank two whose predictions will be less restrictive. If any of these should be acceptable, the "parent" groups of higher ranks should be investigated.

In the case of the angular momentum, the commutation relations, or the Lie algebra of the angular momentum operators, are sufficient to specify the physical content of the spherical symmetry of the system as in the classification of states and deduction of selection rules, etc. We now present a way of constructing the algebra of all simple groups, specializing later to those of rank two.

We call the number of independent elements of the algebra the *order* (r) of the group, or the *dimension* of the algebra. A particular choice of r linearly independent operators forms a *basis* of the Lie algebra. As an illustration, let us take the three dimensional rotation group R_3. The order of the group is three and the usualy choice of the basis is $\hat{T}_x$, $\hat{T}_y$, and $\hat{T}_z$. Instead, we may choose a basis as follows. Take an operator $\hat{H}_1=\hat{T}_z$ and consider an "eigenvalue" problem:

$$[\hat{T}_z,\hat{E}_\alpha]=r(\alpha)\hat{E}_\alpha.$$

The "eigenvectors" are $\hat{E}_{\pm1}=\hat{T}_\pm=\hat{T}_x\pm i\hat{T}_y$ with "eigenvalues," $r(\pm1)=\pm1$ ($\hat{T}_+$ and $\hat{T}_-$ are the "raising and lowering" operators). Here $\hat{T}_+$, $\hat{T}_-$ and $\hat{T}_z$ form an alternative basis of the algebra. Note that, while $\hat{T}_x$ and $\hat{T}_y$ are Hermitian in the usual representation, $\hat{T}_+$ and $\hat{T}_-$ are not; instead they are related by Hermitian conjugation.[23]

For simple groups of rank l, the basis of the algebra may be so chosen that $\hat{H}_1$, $\cdots$, $\hat{H}_l$ are l elements of the basis and

$$[\hat{H}_i,\hat{H}_j]=0. \quad i,j=1,2,\cdots l. \qquad \text{(II.5)}$$

The rest of the basis may be chosen to be the $r-l$ elements E_α of the algebra satisfying

$$[\hat{H}_i,\hat{E}_\alpha]=r_i(\alpha)\hat{E}_\alpha, \qquad \text{(II.6)}$$

where $r_i(\alpha)$ is the ith component of the *root* $\mathbf{r}(\alpha)$, that is, the $r_i(\alpha)$ form a "vector" in an l-dimensional *root space*. If $\mathbf{r}(\alpha)$ is a root, then $-\mathbf{r}(\alpha)\equiv\mathbf{r}(-\alpha)$ is also a root, and we denote the corresponding operator by $\hat{E}_{-\alpha}$. Then it can be shown that

$$[\hat{E}_\alpha,\hat{E}_{-\alpha}]=C_{\alpha,-\alpha}{}^i\hat{H}_i, \quad \alpha=\pm1,\pm2,\cdots\pm\tfrac{1}{2}(r-l), \qquad \text{(II.7)}$$

and that

$$[\hat{E}_\alpha,\hat{E}_\beta]=C_{\alpha,\beta}{}^\gamma\hat{E}_\gamma, \quad \text{(not summed)}, \qquad \text{(II.8)}$$

if $\mathbf{r}(\gamma)\equiv\mathbf{r}(\alpha)+\mathbf{r}(\beta)$ is a nonvanishing root and $[\hat{E}_\alpha,\hat{E}_\beta]=0$, otherwise. These statements can be easily verified for R_3. It is possible to normalize the $\hat{H}_i$, such that

$$\textstyle\sum_\alpha r_i(\alpha)r_j(\alpha)=\delta_{ij}. \qquad \text{(II.9)}$$

Then it can be shown that

$$C_{\alpha,-\alpha}{}^i\equiv r^i(\alpha)=r_i(\alpha), \qquad \text{(II.10)}$$

so that

$$[\hat{E}_\alpha,\hat{E}_{-\alpha}]=r^i(\alpha)\hat{H}_i. \qquad \text{(II.11)}$$

Collecting these results, we have the standard form

[22] D. Speiser and J. Tarski (to be published).

[23] Unitarity of $\mathfrak{U}$ requires the $(\epsilon^A L_A)$ be Hermitian; sometimes we shall use non-Hermitian L_A, in which case it is implied that the ϵ^A have appropriate reality properties.

[24] For a more complete discussion see reference 12, or reference 5, Chap. I.

[25] L. S. Pontrjagin, reference 12, Chap. IX, Sec. 54. In the case of SU_2, for example, we find that it has the same Lie algebra as the three dimensional rotation group R_3, although the two groups differ in that a rotation by 2π is the identity transformation in R, while it is -1 in SU_2.

[26] A *subgroup* is a subset of the elements of the group that has the group property. A subgroup S of a group G is an *invariant subgroup* if gSg^{-1} is in S for every g in G and s in S. In keeping with convention, we shall call a group simple if the only invariant subgroup is discrete. The reason for this is that the Lie algebra of such groups are often simple. (An algebra is *simple* if it has no invariant subalgebra.)

[27] See reference 5, p. 55.

[28] The study of semi-simple groups can be reduced in a trivial manner to that of simple groups.

[29] E. U. Condon and G. H. Shortley, *Theory of Atomic Spectra* (Cambridge University Press, New York, 1935); A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957); M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).

[30] T. Nakano and K. Nishijima, Progr. Theoret. Phys. (Kyoto) 10, 581 (1953); M. Gell-Mann, Phys. Rev. 92, 833 (1953).

[31] To be contrasted with commuting operators of the group, e.g., Casimir operators which are non-linear in the L_A.

[32] Or, what is equivalent, of the form $H_3=aH_1+bH_2+c$.

[33] The eigenstates of H_3 would then be linear combinations of Λ and Σ, as in the doublet symmetry of Pais, reference 2. A model based on the seven-dimensional rotation group comes under this category, see R. E. Behrends, and D. C. Peaslee, reference 2. Pais has shown that the doublet symmetry scheme leads to difficulties, (which are shared by the R_7 model), A. Pais, Phys. Rev. 110, 574 (1958).

of the commutation relations:

$$[\hat{H}_i,\hat{H}_j]=0,$$
$$[\hat{H}_i,\hat{E}_\alpha]=r_i(\alpha)\hat{E}_\alpha,$$
$$[\hat{E}_\alpha,\hat{E}_{-\alpha}]=r^i(\alpha)\hat{H}_i,\qquad \text{(II.12)}$$
$$[\hat{E}_\alpha,\hat{E}_\beta]=N_{\alpha\beta}\hat{E}_\gamma,$$

if $\mathbf{r}(\gamma)=\mathbf{r}(\alpha)+\mathbf{r}(\beta)$ is a nonvanishing root; $N_{\alpha\beta}=C_{\alpha,\beta}{}^{\gamma}$. The explicit form of $N_{\alpha\beta}$ is given in Eq. (II.14).

The graphical representation of the root vectors is called a *root diagram*. All simple groups can be classified by root diagrams.[34] Since roots and structure constants $N_{\alpha\beta}$ can be deduced simply from the vector diagram for all simple groups, we describe the vector diagrams for simple groups of rank two in some detail. The following theorem plays a central role in the construction of the vector diagram:

Theorem[35]: If $\mathbf{r}(\alpha)$ and $\mathbf{r}(\beta)$ are two roots, then $2[\mathbf{r}(\alpha)\cdot\mathbf{r}(\beta)]/[\mathbf{r}(\alpha)\cdot\mathbf{r}(\alpha)]$ is an integer and $\mathbf{r}(\beta)-2\mathbf{r}(\alpha)\times[\mathbf{r}(\alpha)\cdot\mathbf{r}(\beta)]/|\mathbf{r}(\alpha)|^2$ is also a root.

Graphically, this means that a new root $\mathbf{r}(\beta)-2\mathbf{r}(\alpha)\times[\mathbf{r}(\alpha)\cdot\mathbf{r}(\beta)]/|\mathbf{r}(\alpha)|^2$ can be obtained from $\mathbf{r}(\beta)$ by reflection with respect to a hyperplane perpendicular to $\mathbf{r}(\alpha)$.

Suppose we have two roots, $\mathbf{r}(\alpha)$ and $\mathbf{r}(\beta)$, and let φ be the angle between them. Then it follows from the theorem that

$$\mathbf{r}(\alpha)\cdot\mathbf{r}(\beta)=\tfrac{1}{2}m|\mathbf{r}(\alpha)|^2=\tfrac{1}{2}n|\mathbf{r}(\beta)|^2,\qquad \text{(II.13a)}$$

where m and n are integers. From this we further obtain

$$\cos^2\varphi=\tfrac{1}{4}mn.\qquad \text{(II.13b)}$$

We see that φ can have only the values 0°, 30°, 45°, 60°, and 90°. From Eq. (II.13a) one deduces that the ratios of the lengths of the two vectors are $\sqrt{3}$ for 30°, $\sqrt{2}$ for 45°, 1 for 60°, and undetermined for 90°.

It is easy to see that the only possible two dimensional diagrams corresponding to simple groups of rank two, compatible with Eqs. (13a) and (13b), are those drawn in Fig. 1. The first one corresponds to the three-dimensional special unitary group $SU_3(A_2)$; the second to the five-dimensional orthogonal group $O_5(B_2)$, which is also isomorphic to the two-dimensional symplectic group $Sp_2(C_2)$; the last to the exceptional group G_2. The notations in parenthesis are those used by Cartan. The number of parameters of a group (order) is equal to the sum of the number of root vectors and the rank of the group: SU_3 is a $8(=6+2)$ parameter group; O_5 a 10 parameter group; G_2 a 14 parameter group.

Once the vector diagram of a simple group is known, it is a trivial matter to construct the standard form of the commutation relations (12). This is due to the theorem:

Theorem[36]: Form $[\hat{E}_\beta,\hat{E}_\alpha]$, $[[\hat{E}_\beta,\hat{E}_\alpha],\hat{E}_\alpha]$, $\cdots$ and

[34] B. L. van der Waerden, Math. Z. 37, 446 (1933).
[35] See, for example, G. Racah, reference 5, p. 21.
[36] See, for example, G. Racah, reference 5, p. 24.

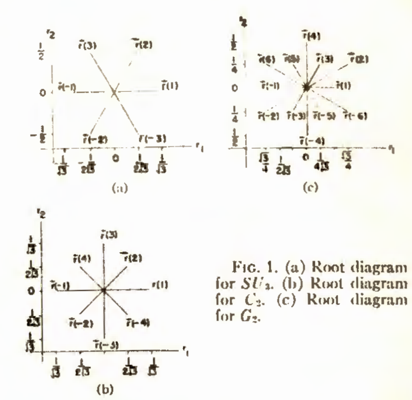


FIG. 1. (a) Root diagram for SU_3. (b) Root diagram for C_2. (c) Root diagram for G_2.

$[\hat{E}_\beta,\hat{E}_{-\alpha}]$, $[[\hat{E}_\beta,\hat{E}_{-\alpha}],\hat{E}_{-\alpha}]$, $\cdots$, where $\mathbf{r}(\beta)\neq\pm\mathbf{r}(\alpha)$. These series must terminate. A series of $\hat{E}_\lambda$'s are generated in this manner. Let

$$\mathbf{r}(\lambda)=\mathbf{r}(\beta)-m\mathbf{r}(\alpha),$$
$$\mathbf{r}(\beta)-(m-1)\mathbf{r}(\alpha),\ \cdots\mathbf{r}(\beta),\ \cdots,\ \mathbf{r}(\beta)+n\mathbf{r}(\alpha)$$

be the corresponding nonvanishing roots. Then

$$N_{\alpha\beta}=\pm[\tfrac{1}{2}(m+1)n|\mathbf{r}(\alpha)|^2]^{\frac{1}{2}}.\qquad \text{(II.14)}$$

Here the signatures of $N_{\alpha\beta}$ must be chosen so that

$$N_{\alpha\beta}=-N_{\beta\alpha}=-N_{-\alpha,-\beta},\qquad \text{(II.15)}$$

$$N_{\alpha\beta}=N_{\beta,-\alpha-\beta}=N_{-\alpha-\beta,\alpha}.\qquad \text{(II.16)}$$

As an example, let us construct the standard commutation relations for SU_3. Label the root vectors as in Fig. 1(a), where the lengths of the $\mathbf{r}(\alpha)$ are normalized according to Eq. (10):

$$\textstyle\sum_\alpha r_i(\alpha)r_j(\alpha)=\delta_{ij}.$$

Let us consider $[\hat{E}_1,\hat{E}_3]$. Since $\mathbf{r}(3)+\mathbf{r}(1)$ is a root while $\mathbf{r}(3)+2\mathbf{r}(1)$ is not, we have $n=1$; since $\mathbf{r}(3)-\mathbf{r}(1)$ is not a root, $m=0$. We choose the sign such that[37]

$$N_{13}=-N_{31}=\sqrt{\tfrac{1}{6}}.$$

Equations (15) and (16) give 5 other constants:

$$N_{-3,-1}=N_{3,-2}=N_{-2,1}=N_{2,-3}=N_{-1,2}=\sqrt{\tfrac{1}{6}}.\qquad \text{(II.17)}$$

The roots are

$$\mathbf{r}(1)=(1/\sqrt{3})(1,0);$$
$$\mathbf{r}(2)=(1/2\sqrt{3})(1,\sqrt{3});\qquad \text{(II.18)}$$
$$\mathbf{r}(3)=(1/2\sqrt{3})(-1,\sqrt{3}).$$

The $N_{\alpha\beta}$ and the roots listed above give a complete set of commutation relations when inserted in Eq. (14).

We summarize a choice of the $N_{\alpha\beta}$ for C_2, and for G_2.

[37] The number of signs that can be chosen independently is the number of different pairs of roots with positive α's whose sums are roots.

[The roots can be read off immediately from Figs. 1(b) and (c).]

C_2:

$$N_{24}=N_{4,-2}=N_{-4,-2}=N_{2,-4}=N_{1,4}=N_{-2,1}$$
$$=N_{-2,3}=N_{3,-4}=N_{-1,2}=N_{-4,-1}$$
$$=N_{4,-3}=N_{-3,2}=\sqrt{\tfrac{1}{6}}.\qquad \text{(II.19)}$$

G_2:

$$N_{26}=N_{4,-6}=N_{-2,4}=N_{2,-1}=N_{3,1}=N_{-2,3}$$
$$=N_{5,-6}=N_{1,6}=N_{-1,5}=N_{3,-4}=N_{-5,4}$$
$$=N_{-3,-5}=1/2\sqrt{2};$$
$$N_{1,5}=N_{3,-5}=N_{-1,3}=\sqrt{\tfrac{1}{6}}.\qquad \text{(II.20)}$$

III. REPRESENTATIONS OF LIE ALGEBRAS

A. General Properties of Representations

In a previous section we discussed the r infinitesimal operators of a Lie group and their commutation relations from an abstract point of view, without using an explicit form of the operators. In order to make connection with physical situations, it is necessary to introduce specific realizations of these operators. If we associate a matrix with each operator $\hat{H}_i$ and $\hat{E}_\alpha$ such that these r matrices satisfy the commutation relations of the r operators, then the matrices are said to constitute a *representation* of the group.[38] In what follows, the symbols H_i and E_α denote a matrix representation. The dimension of these matrices N is called the dimension (or degree) of the representation. If the r matrices of a particular representation can be simultaneously brought into block diagonal form, by a similarity transformation, the representation is said to be *decomposable* (or fully reducible) into lower dimensional representations. When this is not possible, the representation is called *irreducible*.[39]

From the commutation relations, we see that the H_i commute among themselves, so that it is possible to diagonalize simultaneously these l matrices. We choose a representation in which the H_i are diagonal, and write ψ for an N-component basis vector. The eigenfunctions and eigenvalues of $\hat{H}_i$ are defined by

$$H_i\psi=m_i\psi.$$

The l-component vector $\mathbf{m}=(m_1,m_2,\cdots,m_l)$ is called the *weight*,[6] and the l-dimensional vector space spanned by the set of weights is called the *weight space*.

In order to develop some physical intuition for what we are doing, consider the isotopic-spin rotation group. The commutation relations are the usual angular-momentum set. We know that only one of the three matrices can be diagonalized at a time (it then corresponds to H_1, $l=1$ for this group), and the eigenvalues of this matrix are the components of isotopic spin. The E_1 and E_{-1} in this case are proportional to the usual isotopic spin raising and lowering operators. This algebra is the only simple or semi-simple Lie algebra of rank one. The three groups of rank two $(l=2)$ were given in a previous section, i.e., B_2, G_2 and SU_3. For these groups we might identify the eigenvalues of $\hat{H}_1$ and $\hat{H}_2$ with the third component of isotopic spin and with the hypercharge,[40] $Y=N+S$, two good quantum numbers for the strong interactions as well as the electromagnetic interactions. The ψ's which would represent the various particles or states, would then be labeled by their eigenvalues of $\hat{H}_i$, i.e., weights $\mathbf{m}$. The ψ's having different weights are obviously linearly independent, so that there are at most N different weights. If a weight belongs to only one eigenvector, it is called *simple* (for groups of rank greater than one, not all weights are simple).

Let us consider the weights more closely. The following powerful theorem is very useful.

Theorem[41]: For any weight $\mathbf{m}$ and root $\mathbf{r}(\alpha)$, the quantity $2\mathbf{m}\cdot\mathbf{r}(\alpha)/\mathbf{r}(\alpha)\cdot\mathbf{r}(\alpha)$ is an integer and $\mathbf{m}'=\mathbf{m}-\mathbf{r}(\alpha)2\mathbf{m}\cdot\mathbf{r}(\alpha)/\mathbf{r}(\alpha)\cdot\mathbf{r}(\alpha)$ is also a weight, and has the same multiplicity as $\mathbf{m}$. It can be easily verified that this prescription for obtaining $\mathbf{m}'$ from $\mathbf{m}$ corresponds geometrically, in the weight space, to reflecting $\mathbf{m}$ through a hyperplane perpendicular to the root $\mathbf{r}(\alpha)$. Weights that are related by a reflection or a product of reflections are said to be *equivalent*. Reflections and the product of reflections give the set of all equivalent weights. We denote by S the group generated by these reflections.[42]

A weight $\mathbf{m}$ is said to be *higher* than a weight $\mathbf{m}'$ if $\mathbf{m}-\mathbf{m}'$ has a positive number for its first non-vanishing component, e.g., if $m_1-m_1'=0$ and $m_2-m_2'>0$, then $\mathbf{m}$ is higher than $\mathbf{m}'$. A *dominant* weight is the highest member of a set of equivalent weights, and the *highest* weight is the dominant weight which is higher than any other dominant weight in a representation. For an irreducible representation, the highest weight is simple.[43] This concept of a highest weight is useful because two irreducible representations which are related by a similarity transformation (the representations are called *equivalent*) have the same highest weight, and vice versa.

With regard to dominant weights, Cartan[6] has proved that for every simple group of rank l there are l *fundamental* dominant weights $\mathbf{M}^{(1)}\cdots\mathbf{M}^{(l)}$ such that

[38] A representation is faithful if the correspondence between $\hat{L}_A$ and L_A is one-to-one. For simple algebras, all except the identity representation $(L_A=0)$ are faithful.
[39] A noncompact group has no finite dimensional unitary representations (see Pontrjagin, reference 12, Chap. III). Therefore all admissible groups are compact. Representations of compact groups are either irreducible or fully reducible.

[40] Here S is the strangeness quantum number and N is the baryon number. This is the usual definition of hypercharge, although some authors define it as $\frac{1}{2}(N+S)$.
[41] See, for example, G. Racah, reference 5, p. 35.
[42] This group was first introduced by H. Weyl, reference 8, (*Selecta*), p. 338.
[43] See, G. Racah, reference 5, p. 37.

any other dominant weight **M** is a linear combination

$$\mathbf{M}=\sum_{i=1}^{l}\lambda_i\mathbf{M}^{(i)}\equiv\mathbf{M}(\lambda_1\cdots\lambda_l), \qquad \text{(III.1)}$$

with λ_i as non-negative integral coefficients, and that there exist l *fundamental* irreducible representations which have the fundamental weights as their highest weights.[44]

Let us return to the isotopic spin rotation group. The weights m are $\pm I_3$ (weight space is one-dimensional, in this case, since the group is of rank $l=1$). The weight $-I_3$ is obtained from I_3 by reflection through the "plane" perpendicular to the root $\mathbf{r}(1)$ and I_3 is the dominant weight. For each I_3 which appears in an irreducible representation, there will be a $-I_3$, which is equivalent and has the same multiplicity. The fundamental dominant weight is $\frac{1}{2}$ in order that $2\mathbf{m}\cdot\mathbf{r}(\alpha)/\mathbf{r}(\alpha)\cdot\mathbf{r}(\alpha)$ be an integer for all weights. The highest weight is $I=\lambda\frac{1}{2}$, where λ is a non-negative integer, and is simple in an irreducible representation. The corresponding statements for the groups of rank two are postponed until later.

In order to distinguish the different irreducible representations of a group. Weyl has utilized extensively a quantity called the *character*. This is a function of l real variables $\varphi^1,\cdots,\varphi^l$ defined by

$$\chi(\varphi^1,\cdots,\varphi^l)\equiv\text{trace}\exp(iH_i\varphi^i)$$
$$=\sum_a\exp[i(H_i\varphi^i)_a{}^a],$$

where, in the last expression, $(H_i)_a{}^b$ has been assumed to be in diagonal form. Since the trace of a matrix is invariant under a similarity transformation, the characters of two representations are equal if and only if the two representations are equivalent. In particular, a representation and its complex conjugate are equivalent, if and only if the trace is real.

Weyl[8] has given an explicit formula for calculating the character of any representation of any simple group, namely,

$$\chi(\lambda_i,\varphi)=\frac{\xi(\lambda_i)}{\xi(0)},\quad \xi(\lambda_i)\equiv\sum_S\delta_S\exp[i(S\mathbf{K})\cdot\varphi], \qquad \text{(III.2)}$$

where the sum is over the reflection operations S defined above and $\delta_S=+1$ for an even number of reflections and -1 for an odd number. If **R** is defined by

$$\mathbf{R}=\tfrac{1}{2}\sum_{\alpha,+}\mathbf{r}(\alpha) \qquad \text{(III.3)}$$

where the sum is over the positive roots, i.e., those roots which have a positive first nonvanishing component, then **K** is **R** plus the highest weight of the representation, **M**

$$\mathbf{K}=\mathbf{R}+\mathbf{M}(\lambda_1,\cdots,\lambda_l). \qquad \text{(III.4)}$$

[44] In fact, every **M** determines uniquely an irreducible representation with **M** as the highest weight.

It is obvious from the above definition of the character as a trace that it may also be written as

$$\chi(\lambda_i,\varphi)=\textstyle\sum_{\mathbf{m}}\gamma_{\mathbf{m}}\exp(i\mathbf{m}\cdot\varphi), \qquad \text{(III.5)}$$

where the sum is over all the weights and $\gamma_{\mathbf{m}}$ is the number of times a weight **m** occurs, i.e., the multiplicity of the weight. For $\varphi=0$, the character is just the dimensionality of the representation, i.e.,

$$N(\lambda_i)=\textstyle\sum_{\mathbf{m}}\gamma_{\mathbf{m}}=\chi(\lambda_i,0). \qquad \text{(III.6)}$$

The above can be exemplified by referring once again to the isotopic-spin-rotation group. There is one positive root, $\mathbf{r}(1)=1$, therefore, $\mathbf{R}=\frac{1}{2}$; $\mathbf{M}=I=\lambda\frac{1}{2}$. Thus,

$$\mathbf{K}=\tfrac{1}{2}(\lambda+1)=I+\tfrac{1}{2}.$$

Since there is only one reflection,

$$\xi(\lambda)=e^{i(I+\frac{1}{2})\varphi}-e^{-i(I+\frac{1}{2})\varphi},$$

and

$$\chi(\lambda,\varphi)=(e^{i(I+\frac{1}{2})\varphi}-e^{-i(I+\frac{1}{2})\varphi})/(e^{\frac{1}{2}i\varphi}-e^{-\frac{1}{2}i\varphi}),\quad I=\tfrac{1}{2}\lambda.$$

This may easily be shown to be

$$\chi(\lambda,\varphi)=\sum_{I_3=-I}^{+I}e^{iI_3\varphi},$$

so that the multiplicity of each weight is one, $\gamma_{\mathbf{m}}=1$. The dimensions of the irreducible representations are $N=\chi(\lambda,0)=2I+1$.

So far, in order to distinguish the various eigenvectors, or bases, we have the l integers $(\lambda_1,\cdots\lambda_l)$ which are necessary to form the highest weight **M**. These numbers distinguish between representations of different dimensionalities as well as inequivalent representations of the same dimensionality. However, within an irreducible representation, in addition to the weights we still need $\frac{1}{2}(r-3l)$ more numbers, $\mu=(\mu_1,\mu_2,\cdots,\mu_{\frac{1}{2}(r-3l)})$ in order to distinguish the various eigenvectors of the same weight.[5] Given these numbers, it would then be possible to determine the explicit form of the matrix element

$$\psi^\dagger(\mathbf{M},\mathbf{m},\mu)\hat{E}_\alpha\psi(\mathbf{M},\mathbf{m}',\mu')=f(\mathbf{M},\mathbf{m},\mathbf{m}',\mu,\mu').$$

For example, in the isotopic spin rotation group $\frac{1}{2}(r-3l)=0$, so that we need no additional numbers. This matrix element is then the well known[29]

$$\psi^\dagger(I,I_3)I_+\psi(I,I_3')=[\tfrac{1}{2}(I-I_3')(I+I_3)]^{\frac{1}{2}}\delta_{I_3,I_3'+1}.$$

We shall show how to circumvent the task of finding the operators whose eigenvalues are the μ's for groups of higher rank.

Thus far we have used the isotopic spin rotation group as an example. Let us now demonstrate the method with the rank two groups SU_3, G_2, and C_2.

B. Characters of Representations of SU_3

In order to satisfy the condition that $2\mathbf{m}\cdot\mathbf{r}(\alpha)/\mathbf{r}(\alpha)\cdot\mathbf{r}(\alpha)$ be an integer for an arbitrary weight $\mathbf{m}=(m_1,m_2)$ and any root, $\mathbf{r}(\alpha)$, it is necessary that $m_1=(1/2\sqrt{3})(a+b)$ and $m_2=\frac{1}{6}(a-b)$, where a and b are integers. Thus $\mathbf{m}=\frac{1}{6}a(\sqrt{3},1)+\frac{1}{6}b(\sqrt{3},-1)$. By noting that $\frac{1}{6}(\sqrt{3},1)$ and $\frac{1}{6}(\sqrt{3},-1)$ each lie in a plane perpendicular to a root, we see that each belongs to a set of 3 equivalent weights and that each is a dominant weight of its set, in fact, a fundamental dominant weight. Thus

$$\mathbf{M}(\lambda_1\lambda_2)=\tfrac{1}{6}\lambda_1(\sqrt{3},1)+\tfrac{1}{6}\lambda_2(\sqrt{3},-1).$$

The quantity **R** for SU_3 is

$$\mathbf{R}=\tfrac{1}{2}\sum_{\alpha,+}\mathbf{r}(\alpha)=(1/\sqrt{3})(1,0),$$

so that **K** is

$$\mathbf{K}=\mathbf{M}+\mathbf{R}=\tfrac{1}{6}(\sqrt{3}\lambda_1+\sqrt{3}\lambda_2+2\sqrt{3},\lambda_1-\lambda_2).$$

Thus, $\xi(\lambda_1,\lambda_2)$ may be written

$$\begin{aligned}\xi(\lambda_1,\lambda_2)=&\exp\tfrac{1}{6}i[(\lambda_1+\lambda_2+2)\sqrt{3}\varphi_1+(\lambda_1-\lambda_2)\varphi_2]\\&-\exp\tfrac{1}{6}i[-(\lambda_1+\lambda_2+2)\sqrt{3}\varphi_1+(\lambda_1-\lambda_2)\varphi_2]\\&-\exp\tfrac{1}{6}i[(\lambda_2+1)\sqrt{3}\varphi_1-(2\lambda_1+\lambda_2+3)\varphi_2]\\&+\exp\tfrac{1}{6}i[-(\lambda_2+1)\sqrt{3}\varphi_1-(2\lambda_1+\lambda_2+3)\varphi_2]\\&-\exp\tfrac{1}{6}i[(\lambda_1+1)\sqrt{3}\varphi_1+(\lambda_1+2\lambda_2+3)\varphi_2]\\&+\exp\tfrac{1}{6}i[-(\lambda_1+1)\sqrt{3}\varphi_1+(\lambda_1+2\lambda_2+3)\varphi_2].\end{aligned}$$

It should be apparent that dividing $\xi(\lambda_1,\lambda_2)$ by $\xi(0,0)$ in order to obtain the character in the form $\sum\gamma_{\mathbf{m}}\exp(i\mathbf{m}\cdot\varphi)$ is no trivial matter for this group. In the next section, we develop a technique for handling this problem. First let us find the dimensions N of the irreducible representations. In terms of the character, $N=\chi(\lambda_1\lambda_2,\varphi_1=\varphi_2=0)$. Since $\xi(\lambda_1,\lambda_2)$ is zero for $\varphi_1=\varphi_2=0$, we use L'Hôpital's rule[45] to find

$$N=[1+\tfrac{1}{2}(\lambda_1+\lambda_2)](1+\lambda_1)(1+\lambda_2).$$

The numbers $\lambda_1,\cdots,\lambda_l$ are sufficient to identify a representation. For this reason we label the representations by $D^{(N)}(\lambda_1\lambda_2)$ [or occasionally by just $D(\lambda_1\lambda_2)$ or $D^{(N)}$]. Thus $D^{(3)}(1,0)$ denotes one of the 3-dimensional representations, while $D^{(3)}(0,1)$ denotes the complex conjugate (χ^*) inequivalent 3-dimensional representation.

We note that $\chi^*=\chi$ only for values of $\lambda_1=\lambda_2$.[46] Thus, only in this case are the complex conjugate representations equivalent. In Fig. 2 we have drawn the weight diagrams for a few of the lower dimensional representations of SU_3. The solid lines with arrows represent the weight vectors while the dotted lines which are perpendicular to the roots represent the planes of reflection that leave the weight diagram unchanged (the set of operations S defined above). The 3-dimensional representations $D^{(3)}(1,0)$ and $D^{(3)}(0,1)$ are the fundamental irreducible representations, $D^{(8)}(1,1)$ is the regular representation.[47]

[45] Marquis G. F. A. de l'Hôpital, *Analyse des Infiniment Petits* (Paris, 1730).

[46] This follows from the identity $\chi^*(\lambda_1,\lambda_2)=\chi(\lambda_2,\lambda_1)$ satisfied by the SU_3 characters.

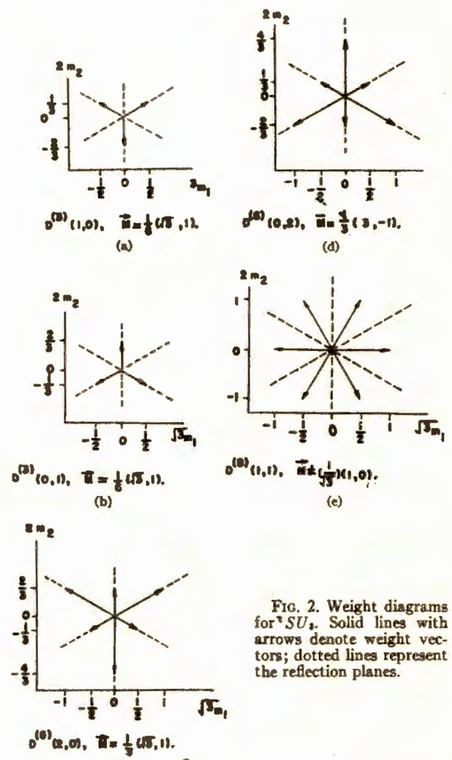


FIG. 2. Weight diagrams for SU_3. Solid lines with arrows denote weight vectors; dotted lines represent the reflection planes.

C. Characters of Representations of G_2

In order that $2\mathbf{m}\cdot\mathbf{r}(\alpha)/\mathbf{r}(\alpha)\cdot\mathbf{r}(\alpha)$ be an integer for an arbitrary weight $\mathbf{m}=(m_1,m_2)$ and any root, $\mathbf{r}(\alpha)$, it is necessary that $m_1=(1/4\sqrt{3})(2a+3b)$ and $m_2=\frac{1}{4}b$, where a and b are integers. Thus $\mathbf{m}=(a/2\sqrt{3})(1,0)+(b/2\sqrt{3})\times(3/2,\sqrt{3}/2)$. By noting that $(1/2\sqrt{3})(1,0)$ and $(1/2\sqrt{3})(3/2,\sqrt{3}/2)$ each lie in a plane perpendicular to a root, we see that each belongs to a set of 6 equivalent weights and that each is a dominant weight of its set,

[47] The regular representation is very important and plays a prominent role in later sections. It is defined by $L_A\to -C_A$, where the components of the matrix C_A are the structure constants $C_{AB}{}^D$. That this is a representation can be seen by rewriting the Jacobi identity (II.4) in the form $C_{AE}{}^FC_{BF}{}^G-C_{BE}{}^FC_{AF}{}^G=-C_{AB}{}^FC_{EF}{}^G$. It can easily be proved that the regular representation is irreducible if and only if the group is simple.

in fact, a fundamental dominant weight. Thus

$$\mathbf{M}(\lambda_1,\lambda_2)=(\lambda_1/2\sqrt{3})(1,0)+(\lambda_2/4\sqrt{3})(3,\sqrt{3}).$$

The quantity $\mathbf{R}$ for G_2 is

$$\mathbf{R}=(1/4\sqrt{3})(5,\sqrt{3}),$$

so that

$$\mathbf{K}=(1/4\sqrt{3})(2\lambda_1+3\lambda_2+5,\ \sqrt{3}\lambda_2+\sqrt{3}).$$

Then, $\xi(\lambda_1,\lambda_2)$ may be written

$$\begin{aligned}\xi(\lambda_1,\lambda_2) &=\{\exp[i(2\lambda_1+3\lambda_2+5)\varphi_1/4\sqrt{3}]\\ &\quad-\exp[-i(2\lambda_1+3\lambda_2+5)\varphi_1/4\sqrt{3}]\}\\ &\quad\times\{\exp[i(\lambda_2+1)\varphi_2/4]-\exp[-i(\lambda_2+1)\varphi_2/4]\}\\ &\quad-\{\exp[i(\lambda_1+3\lambda_2+4)\varphi_1/4\sqrt{3}]\\ &\quad-\exp[-i(\lambda_1+3\lambda_2+4)\varphi_1/4\sqrt{3}]\}\\ &\quad\times\{\exp[i(\lambda_1+\lambda_2+2)\varphi_2/4]\\ &\quad-\exp[-i(\lambda_1+\lambda_2+2)\varphi_2/4]\}\\ &\quad+\{\exp[i(\lambda_1+1)\varphi_1/4\sqrt{3}]-\exp[-i(\lambda_1+1)\varphi_1/4\sqrt{3}]\}\\ &\quad\times\{\exp[i(\lambda_1+2\lambda_2+3)\varphi_2/4]\\ &\qquad-\exp[-i(\lambda_1+2\lambda_2+3)\varphi_2/4]\}.\end{aligned}$$

The dimensions N of the irreducible representations are $N=\chi(\lambda_1,\lambda_2,\ \varphi_1=\varphi_2=0)$. The result is

$$N=(1+\lambda_1)(1+\lambda_2)[1+\tfrac{1}{2}(\lambda_1+\lambda_2)][1+\tfrac{1}{3}(\lambda_1+2\lambda_2)]\times[1+\tfrac{1}{4}(\lambda_1+3\lambda_2)][1+\tfrac{1}{5}(2\lambda_1+3\lambda_2)].$$

We note that $\chi^*=\chi$, so that representations related by complex conjunction are always equivalent.

In Fig. 3 we have drawn the weight diagram for the 7- and 14-dimensional representations of G_2. The solid lines with arrows denote the weight vectors while the dotted lines, which are perpendicular to the roots, represent the planes of reflection that leave the weight diagram unchanged (the set of reflections S defined above). These two representations are the fundamental irreducible representations of G_2, and $D^{(14)}(0,1)$ is the regular representation.[47]

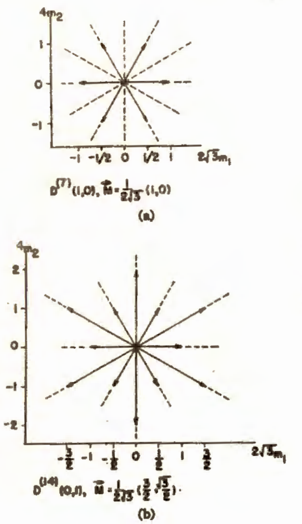


FIG. 3. Weight diagrams for G_2. Solid lines with arrows denote weight vectors; dotted lines represent the reflection plane.

D. Characters of Representations of C_2

In order that $2\mathbf{m}\cdot\mathbf{r}(\alpha)/\mathbf{r}(\alpha)\cdot\mathbf{r}(\alpha)$ be an integer for an arbitrary weight $\mathbf{m}=(m_1,m_2)$ and any root $\mathbf{r}(\alpha)$, it is necessary that $m_1=(2\sqrt{3})^{-1}(a+b)$ and $m_2=b/2\sqrt{3}$, where a and b are integers. Thus, $\mathbf{m}=(a/2\sqrt{3})(1,0)+(b/2\sqrt{3})(1,1)$. By noting that $(1/2\sqrt{3})(1,0)$ and $(1/2\sqrt{3})(1,1)$ each lie in a plane perpendicular to a root, we see that each belongs to a set of 4 equivalent weights and that each is a dominant weight of its set, in fact, a fundamental dominant weight. Thus

$$\mathbf{M}(\lambda_1,\lambda_2)=(\lambda_1/2\sqrt{3})(1,0)+(\lambda_2/2\sqrt{3})(1,1).$$

The quantity $\mathbf{R}$ for C_2 is

$$\mathbf{R}=(1/2\sqrt{3})(2,1),$$

so that

$$\mathbf{K}=\mathbf{R}+\mathbf{M}=(1/2\sqrt{3})(\lambda_1+\lambda_2+2,\ \lambda_2+1).$$

Then, $\xi(\lambda_1,\lambda_2)$ may be written

$$\begin{aligned}\xi(\lambda_1,\lambda_2) &=\{\exp[i(\lambda_1+\lambda_2+2)\varphi_1/2\sqrt{3}]\\ &\quad-\exp[-i(\lambda_1+\lambda_2+2)\varphi_1/2\sqrt{3}]\}\\ &\quad\times\{\exp[i(\lambda_2+1)\varphi_2/2\sqrt{3}]-\exp[-i(\lambda_2+1)\varphi_2/2\sqrt{3}]\}\\ &\quad-\{\exp[i(\lambda_2+1)\varphi_1/2\sqrt{3}]-\exp[-i(\lambda_2+1)\varphi_1/2\sqrt{3}]\}\\ &\quad\times\{\exp[i(\lambda_1+\lambda_2+2)\varphi_2/2\sqrt{3}]\\ &\qquad-\exp[-i(\lambda_1+\lambda_2+2)\varphi_2/2\sqrt{3}]\}.\end{aligned}$$

The dimensions of the irreducible representations, $N=\chi(\lambda_1,\lambda_2,\ \varphi_1=\varphi_2=0)$, are

$$N=(1+\lambda_1)(1+\lambda_2)[1+\tfrac{1}{2}(\lambda_1+\lambda_2)][1+\tfrac{1}{3}(\lambda_1+2\lambda_2)].$$

We note that $\chi^*=\chi$, so that representations related by complex conjugation are always equivalent.

In Fig. 4 we have drawn the weight diagrams for the 4, 5, and 10 dimensional representations of C_2. The solid lines with arrows denote the weight vectors while the dotted lines, which are perpendicular to the roots, represent the planes of reflection that leave the weight diagram unchanged (the set of reflections S defined above). The 4 and 5 dimensional representations, $D^{(4)}(1,0)$ and $D^{(5)}(0,1)$, are the fundamental irreducible representations of B_2, while $D^{(10)}(2,0)$ is the regular representation.[47]

E. Synthesis of Representations of Lie Algebras

For physical application, it is imperative to have explicit matrix representations of the low dimensional Lie algebras. As has been implied in the preceding

paragraph, the straightforward generalization of the favorite method of constructing the matrix representation of a rank one group is somewhat awkward for higher rank groups. Of the several alternative methods which offer promise, we choose one which has useful by-products. In particular, the generalized Clebsch-Gordan coefficients[48] will materialize as part of the fallout of results.

As a warming-up exercise, we recall certain facts about the group SU_2. Let the basis for an irreducible representation $D(J)$, uniquely characterized by the total angular momentum $J(J+1)$, be labeled as $\psi_M{}^J$ where J is an integer or a half-integer and M runs from J to $-J$ in integral steps. In particular, select the spin $\frac{1}{2}$ representation $D(\frac{1}{2})$ whose highest weight is the fundamental dominant weight of SU_2. Then the representation is given in terms of Pauli matrices[49]:

$$\hat{H}_1=\tfrac{1}{2}\sigma_3;\qquad \hat{T}_+=(1/2\sqrt{2})(\sigma_1+i\sigma_2);\qquad \hat{T}_-=(1/2\sqrt{2})(\sigma_1-i\sigma_2)$$
$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix};\quad \sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix};\quad \sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}\qquad\text{(III.7)}$$

and the basis is $\psi_m{}^{\frac{1}{2}}$, $m=\frac{1}{2},-\frac{1}{2}$. It is possible to arrive at a new representation inequivalent to $D(\frac{1}{2})$ by forming the direct product representation in the space spanned by the $\psi_m{}^{\frac{1}{2}}\psi_{m'}{}^{\frac{1}{2}}$. The action of $\hat{H}_1$ and $\hat{T}_\pm$ on the product basis is, of course,

$$\begin{aligned}\hat{T}_A\psi_m{}^{\frac{1}{2}}\psi_{m'}{}^{\frac{1}{2}}&=(\hat{T}_A\psi_m{}^{\frac{1}{2}})\psi_{m'}{}^{\frac{1}{2}}+\psi_m{}^{\frac{1}{2}}(\hat{T}_A\psi_{m'}{}^{\frac{1}{2}})\\ \hat{T}_A\psi_m{}^{\frac{1}{2}}&=\textstyle\sum_{m'}(\hat{T}_A)_m{}^{m'}\psi_{m'}{}^{\frac{1}{2}},\end{aligned}\qquad\text{(III.8)}$$

where $\hat{T}_A$ is $\hat{T}_z=H_1$, $\hat{T}_+$, or $\hat{T}_-$. The product representation is, in general, reducible; for example,

$$\psi_m{}^{\frac{1}{2}}\psi_{m'}{}^{\frac{1}{2}}=\sum_{M,J}(JM|\tfrac{1}{2}m,\tfrac{1}{2}m')\psi_M{}^J,\qquad\text{(III.9)}$$

where $(JM|\frac{1}{2}m,\frac{1}{2}m')$ are the Clebsch-Gordan coefficients which reduce the representation. To accomplish the reduction, we note that $\hat{T}_+$ and $\hat{T}_-$ commute with $\hat{T}^2$, and, since the eigenvalue of $\hat{T}^2$ uniquely characterizes an irreducible representation, they cannot lead out of an irreducible representation when applied in any order and any number of times to a single basis vector. The highest weight M in the product representation $\psi_m{}^{\frac{1}{2}}\psi_{m'}{}^{\frac{1}{2}}$, namely $M=\frac{1}{2}+\frac{1}{2}$, belongs to an irreducible representation and hence the space spanned by the vectors generated by application of $\hat{T}_+$ and $\hat{T}_-$ to $\psi_{\frac{1}{2}}{}^{\frac{1}{2}}\psi_{\frac{1}{2}}{}^{\frac{1}{2}}$ is irreducible under SU_2. Thus the orthonormal

[48] We refer here to the coefficients prescribing the linear combinations of direct product states relative to which the representation reduces.

[49] The operators $\hat{T}_\pm$ are usually defined without the factor $1/\sqrt{2}$. Throughout this paper, we shall adopt the sign convention of Condon and Shortley (reference 29) for isotopic spin. This implies that all the signs of $L_\pm$ matrix elements are positive although the physical particles are sometimes identified as the negative of the bases defining this representation.

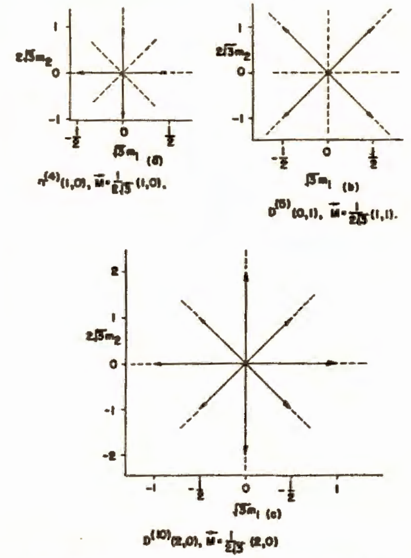


FIG. 4. Weight diagrams for C_2. Solid lines denote weight vectors; dotted lines represent the reflection planes.

vectors

$$\begin{aligned}\psi_1{}^1&\equiv\psi_{\frac{1}{2}}{}^{\frac{1}{2}}\psi_{\frac{1}{2}}{}^{\frac{1}{2}}\\ \psi_0{}^1&\equiv\hat{T}_-(\psi_{\frac{1}{2}}{}^{\frac{1}{2}}\psi_{\frac{1}{2}}{}^{\frac{1}{2}})=(1/\sqrt{2})(\psi_{-\frac{1}{2}}{}^{\frac{1}{2}}\psi_{\frac{1}{2}}{}^{\frac{1}{2}}+\psi_{\frac{1}{2}}{}^{\frac{1}{2}}\psi_{-\frac{1}{2}}{}^{\frac{1}{2}})\qquad\text{(III.10a)}\\ \psi_{-1}{}^1&\equiv2(\hat{T}_-)^2(\psi_{\frac{1}{2}}{}^{\frac{1}{2}}\psi_{\frac{1}{2}}{}^{\frac{1}{2}})=\psi_{-\frac{1}{2}}{}^{\frac{1}{2}}\psi_{-\frac{1}{2}}{}^{\frac{1}{2}}\end{aligned}$$

are a basis for an irreducible representation $D(1)$ of SU_2 and the remaining linear independent vector in the direct product space $\psi_m{}^{\frac{1}{2}}\psi_{m'}{}^{\frac{1}{2}}$ is

$$\psi_0{}^0\equiv(1/\sqrt{2})(\psi_{\frac{1}{2}}{}^{\frac{1}{2}}\psi_{-\frac{1}{2}}{}^{\frac{1}{2}}-\psi_{-\frac{1}{2}}{}^{\frac{1}{2}}\psi_{\frac{1}{2}}{}^{\frac{1}{2}}).\qquad\text{(III.10b)}$$

This $\psi_0{}^0$ generates $D(0)$ for SU_2. The Clebsch-Gordan coefficients are read off from Eqs. (III.10a) and (III.10b) while the irreducible Lie algebra follows by computing the $\hat{T}_A$ matrix elements by using Eq. (III.8).

To find an arbitrary irreducible representation $D(J)$, it is only necessary to split off the highest irreducible representation of the direct product space

$$\psi_{m(1)}{}^{\frac{1}{2}}\psi_{m(2)}{}^{\frac{1}{2}}\cdots\psi_{m(2J)}{}^{\frac{1}{2}}\equiv(\psi_m{}^{\frac{1}{2}})^{2J}.\qquad\text{(III.11)}$$

The orthonormal basis which results is[50]:

$$\psi_M{}^J=N(J,M)(\hat{T}_-)^{J-M}(\psi_{\frac{1}{2}}{}^{\frac{1}{2}})^{2J},\qquad M=J,\cdots,-J,$$
$$N(J,M)=\left[\frac{(J+M)!\,2^{J-M}}{(J-M)!\,(2J)!}\right]^{\frac{1}{2}}\qquad\text{(III.12)}$$

[50] To derive $N(J,M)$, use the identity
$$\hat{T}_+\hat{T}_-=\tfrac{1}{2}(\hat{T}_+\hat{T}_-+\hat{T}_-\hat{T}_++T_z)=\tfrac{1}{2}(\hat{T}^2-\hat{T}_z^2-\hat{T}_z)$$
to obtain a recursion relation.

and the operators $\hat{T}_s$, $\hat{T}_+$, $\hat{T}_-$ enjoy the properties:

$$\hat{T}_-\psi_M{}^J=[N(J,M)/N(J,M-1)]\psi_{M-1}{}^J$$
$$=[\sqrt{\tfrac{1}{2}}(J+M)(J-M+1)]^{\frac{1}{2}}\psi_{M-1}{}^J \quad \text{(III.13)}$$
$$\hat{T}_+\psi_M{}^J=[\tfrac{1}{2}(J-M)(J+M+1)]^{\frac{1}{2}}\psi_{M+1}{}^J$$
$$\hat{T}_s\psi_M{}^J=M\psi_M{}^J$$

which then gives the constitution of the $D(J)$ representation. We now develop the generalization of the foregoing conclusions to simple groups of higher rank.

To construct the irreducible representations of Lie algebras of rank two and higher, we show that all that is required is:

(a) The l fundamental irreducible representations whose highest weights are characterized by one of the l fundamental dominant weights.

(b) a reduction procedure for direct product representations.

Before deriving the theorems needed to synthesize representations, a few words on the characterization of the representation space are in order. In order to specify the representation of the algebra, it is sufficient to give the representations of the basis elements $\hat{H}_i$ and $\hat{E}_\alpha$. We define the representation by prescribing the action of $\hat{H}_i$ and $\hat{E}_\alpha$ on an orthonormal complete set of ket vectors $|\{\lambda_1\cdots\lambda_l\},\nu\rangle$ spanning the N dimensional representation space ($\nu=1,\cdots,N$). When no ambiguities arise, the ket $|\{\lambda_1\cdots\lambda_l\},\nu\rangle$ will often be abbreviated as $|\{N\},\nu\rangle$ and even $|\nu\rangle$. Since the H_i intercommute, they can be simultaneously diagonalized, and, since they are taken to be Hermitian, their eigenvalues are real. We choose a representation in which the H_i are diagonal. Thus the label ν in $|\{N\},\nu\rangle$ stands for a fixed eigenvalue of each of the H_i (the weight $\mathbf{m}$) in addition to other discriminating labels (g) which are needed in the case of multiple weights. Furthermore, the matrices E_α satisfy the relation: $(E_\alpha)^\dagger=E_{-\alpha}$.

If $|\{N\},\nu\rangle$ is the basis for one representation of a Lie algebra and $|\{N'\},\nu'\rangle$ a basis for a second representation, the direct product space spanned by the basis $|\{N\},\nu;\{N'\},\nu'\rangle$ is again a representation of the Lie algebra whose elements L_A act upon the kets $|\{N\},\nu;\{N'\},\nu'\rangle$ in the following manner:

$$L_A|\{N\},\nu;\{N'\},\nu'\rangle$$
$$=L_A{}^{(N)}\otimes 1^{(N')}|\{N\},\nu;\{N'\},\nu'\rangle$$
$$+1^{(N)}\otimes L_A{}^{(N')}|\{N\},\nu;\{N'\},\nu'\rangle. \quad \text{(III.14)}$$

Here $L_A{}^{(N)}$, $1^{(N)}$ and $L_A{}^{(N')}$, $1^{(N')}$ act only on the N and N' dimensional representations, respectively. The direct product representation defined by Eq. (III.14) is, in general, reducible in a way which is shown below.

Given the abstract Lie algebra as presented in Sec. II, we now seek to construct in a systematic way the matrix sets representing the algebra. The method is essentially predicated upon four theorems:

Theorem I. If $H_i|\mathbf{m},g\rangle=m_i|\mathbf{m},g\rangle$, then $H_iE_{-\alpha}|\mathbf{m},g\rangle=[m_i-r_i(\alpha)]E_{-\alpha}|\mathbf{m},g\rangle$.

Proof: $[H_i,E_{-\alpha}]=-r_i(\alpha)E_{-\alpha}$ by Eq. (II.6).

Therefore

$$H_iE_{-\alpha}|\mathbf{m},g\rangle=E_{-\alpha}H_i|\mathbf{m},g\rangle-r_i(\alpha)E_{-\alpha}|\mathbf{m},g\rangle$$
$$=[m_i-r_i(\alpha)]E_{-\alpha}|\mathbf{m},g\rangle.$$

We seek the value of a such that the ket $aE_{-\alpha}|\mathbf{m},g\rangle$ is of unit length. Note, incidentally, that $aE_{-\alpha}|\mathbf{m},g\rangle$ is orthogonal to $|\mathbf{m},g\rangle$ since the H_i eigenvalues of these two states differ.

Theorem II. If $E_\alpha|\mathbf{m},g\rangle=0$, then the normalization constant a is $a=[\mathbf{r}(\alpha)\cdot\mathbf{m}]^{\frac{1}{2}}$.

Proof: $[E_\alpha,E_{-\alpha}]=\mathbf{r}(\alpha)\cdot\mathbf{H}$ by Eq. (II.7).

Therefore

$$\langle\mathbf{m},g|[E_\alpha,E_{-\alpha}]|\mathbf{m},g\rangle$$
$$=\langle\mathbf{m},g|E_\alpha E_{-\alpha}|\mathbf{m},g\rangle=|a|^{-2}$$
$$=\langle\mathbf{m},g|\mathbf{r}(\alpha)\cdot\mathbf{H}|\mathbf{m},g\rangle=\mathbf{r}(\alpha)\cdot\mathbf{m} \text{ Q.E.D.}$$

In a direct product representation, the greatest dominant weight $\mathbf{M}$ is the sum of the greatest dominant weights $\mathbf{M}^{(N)}$ and $\mathbf{M}^{(N')}$ of the constituent N and N' dimensional representations.

Theorem III. The space spanned by the basis vectors generated by application of $\hat{H}_i$ and $\hat{E}_\alpha$, in any order and any number of times, to $|\mathbf{M}\rangle$ is irreducible under the Lie algebra.

Proof: $|\mathbf{M}\rangle$ is a basis vector of an irreducible representation. Hence, the space spanned by application of $\hat{H}_i$ and $\hat{E}_\alpha$ to $|\mathbf{M}\rangle$ provides an irreducible representation for the algebra by the very definition of irreducibility.

The number of orthonormal vectors which span the reduced direct product space generated in the above manner is the dimension of the resulting representation.

To construct the irreducible representations contained in a direct product representation, we proceed as follows:

(a) Select the ket in the direct product space with the highest weight $|\mathbf{M}\rangle$.

(b) Apply the operators E_α, $E_\alpha E_\beta$, $\cdots$, to $|\mathbf{M}\rangle$. Orthonormalize by the Schmidt process all resulting kets. The orthonormalization is carried out by using the orthonormal properties of the constituent representations, i.e.,

$$\langle\{N\},\nu;\{N'\},\nu'|\{N\},\nu'';\{N'\},\nu'''\rangle$$
$$=\delta_{\nu\nu''}\delta_{\nu'\nu'''}. \quad \text{(III.15)}$$

Kets having different weights will automatically come out orthogonal to each other. The dimensions of the irreducible representations of the algebra have been evaluated in a previous

section from the character of the associated group so that this information can be used to predict the number of linearly independent vectors.

(c) Next, in the subspace orthogonal to that generated from $|\mathbf{M}\rangle$, select the ket $|\mathbf{M}'\rangle$ with the highest weight. Generate from $|\mathbf{M}'\rangle$ another space irreducible under the Lie algebra in the same way as an irreducible space was generated from $|\mathbf{M}\rangle$.

(d) The action of the elements of the Lie algebra on the orthonormal vector basis thus generated is readily ascertained by noting the action of $\hat{H}_i$ and $\hat{E}_\alpha$ on the spaces from which the direct product was constructed [Eq. (III.14)].

Given the l explicit representations characterized by each of the l fundamental dominant weights, every irreducible representation of the algebra can be generated by reducing a suitably chosen direct product. Let $\mathfrak{a}^{(s)}$ be the matrix algebra $D(0,0,\cdots,1,\cdots 0)$, where the 1 is in the sth position, whose highest weight is the fundamental dominant weight:

$$\mathbf{M}^{(s)}=(M_1{}^{(s)},M_2{}^{(s)},\cdots,M_l{}^{(s)}).$$

The highest weight of an arbitrary irreducible representation is $\mathbf{M}=\sum_s\lambda_s\mathbf{M}^{(s)}$.

Theorem IV. The irreducible representation of the Lie algebra characterized by the highest weight $\mathbf{M}=\sum_s\lambda_s\mathbf{M}^{(s)}$ is the first irreducible representation obtained by reduction of the product algebra

$$\underbrace{\mathfrak{a}^{(1)}\times\cdots\times\mathfrak{a}^{(1)}}_{\lambda_1\text{ times}}\times\underbrace{\mathfrak{a}^{(2)}\times\cdots\times\mathfrak{a}^{(2)}}_{\lambda_2\text{ times}}\times\cdots\times\underbrace{\mathfrak{a}^{(l)}\times\cdots\times\mathfrak{a}^{(l)}}_{\lambda_l\text{ times}}.$$

Proof: The highest weight in the product algebra is $\mathbf{M}=\sum_s\lambda_s\mathbf{M}^{(s)}$. By generating a space irreducible under the Lie algebra from the ket $|\mathbf{M}\rangle$, by a generalization of the procedure illustrated above for the direct product of two spaces, an irreducible representation results.

We now go on to use the above method to construct some irreducible representations of SU_3, C_3, and G_2. In particular, all the fundamental representations which go into making the direct product representations will be generated.

F. Matrix Representations of SU_3

The fundamental representations are $D^{(3)}(1,0)$ and $D^{(3)}(0,1)$. Besides constructing these representations we also reduce the regular representation $D^{(8)}(1,1)$ out of the product $D^{(3)}(1,0)\otimes D^{(3)}(0,1)$.

$D^{(3)}(1,0)$. The weight diagram was given in Fig. 2(a); the highest weight is the fundamental dominant weight

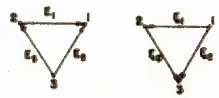


FIG. 5. Action of E_α on $D^{(3)}(1,0)$ of SU_3.

$$\mathbf{M}^{(1)}=\tfrac{1}{6}(\sqrt{3},1) \quad \text{(III.16)}$$

We may write $|\{3\},a\rangle$, $a=1$, 2, 3 or simply $|a\rangle$ for the three states, and use the labeling of Fig. 5. Then the H_i are the diagonal matrices whose eigenvalues are the respective components m_i of the weights. That is[21]

$$H_1=\sum_a m_1(a)|a\rangle\langle a|=\frac{1}{2\sqrt{3}}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix},$$
$$H_2=\sum_a m_2(a)|a\rangle\langle a|=\frac{1}{6}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}. \quad \text{(III.17)}$$

According to theorem I (Sec. III E), when $\hat{E}_{-\alpha}$ operates on a state with weight $\mathbf{m}$, it creates a state with weight $\mathbf{m}-\mathbf{r}(\alpha)$. This is symbolized in Fig. 5. Clearly, if $\mathbf{m}-\mathbf{r}(\alpha)$ is not a weight, then $\hat{E}_{-\alpha}|\mathbf{m}\rangle=0$. Therefore, in this simple case, all the constants of proportionality are given by theorem II (Sec. III E) to be $\pm[\mathbf{r}(\alpha)\cdot\mathbf{m}]^{\frac{1}{2}}$. Hence

$$E_{-1}|\{3\},1\rangle=[\mathbf{r}(1)\cdot\mathbf{m}(1)]^{\frac{1}{2}}|\{3\},2\rangle$$
$$=6^{-\frac{1}{2}}|\{3\},2\rangle,$$
$$E_{-2}|\{3\},1\rangle=[\mathbf{r}(2)\cdot\mathbf{m}(1)]^{\frac{1}{2}}|\{3\},3\rangle$$
$$=6^{-\frac{1}{2}}|\{3\},3\rangle, \quad \text{(III.18)}$$
$$E_{-3}|\{3\},2\rangle=[\mathbf{r}(3)\cdot\mathbf{m}(2)]^{\frac{1}{2}}|\{3\},3\rangle$$
$$=6^{-\frac{1}{2}}|\{3\},3\rangle.$$

The phases of E_{-1} and E_{-2} are arbitrary, but once they have been selected, the phase of E_{-3} is determined by the convention (II.17), since

$$[E_{-1},E_2]=N_{-1,2}E_3=6^{-\frac{1}{2}}E_3. \quad \text{(III.19)}$$

In the form of matrices, (III.18) becomes

$$E_{-1}=6^{-\frac{1}{2}}\begin{pmatrix}0&0&0\\1&0&0\\0&0&0\end{pmatrix}=6^{-\frac{1}{2}}|2\rangle\langle 1|,$$
$$E_{-2}=6^{-\frac{1}{2}}\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}=6^{-\frac{1}{2}}|3\rangle\langle 1|, \quad \text{(III.20)}$$
$$E_{-3}=6^{-\frac{1}{2}}\begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}=6^{-\frac{1}{2}}|3\rangle\langle 2|, \quad E_\alpha=E_{-\alpha}{}^\dagger.$$

$D^{(3)}(0,1)$. If $\mathfrak{U}$ is a unitary matrix representation of the group SU_3, then $\mathfrak{U}^*$, the complex conjugate matrices, are also a representation. Let $\mathfrak{U}$ be of the

[21] Here $m_i(a)$ is the ith component of the weight of the ath state.

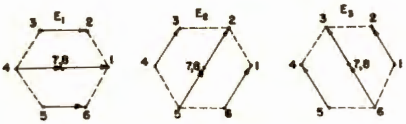


FIG. 6. Action of E_α on $D^{(8)}(1,1)$ of SU_3.

form $\mathcal{U}=\exp(i\epsilon^A L_A)$, then $\mathcal{U}^*=\exp(-i\epsilon^A\tilde{L}_A)$ since $\mathcal{U}^+=\mathcal{U}^{-1}$. Hence the "contragredient" representation of the Lie algebra is $L_A'=-\tilde{L}_A$ where L_A are given in Eq. (20). In view of the reality of these L_A, we find

$$H_i'=-\tilde{H}_i=-H_i,\qquad E_\alpha'=-\tilde{E}_\alpha=-E_\alpha^\dagger=-E_{-\alpha}. \tag{III.21}$$

The first of Eq. (21) shows that the weight diagrams for contragrediently related representations are transformed into each other by reflection through the origin. Thus we get the weight diagram of Fig. 2(b). Equation (21) would not hold with a different labeling. From (17), (20), and (21):

$$H_1'=\frac{1}{2\sqrt{3}}\begin{pmatrix}-1&&\\&1&\\&&0\end{pmatrix},\quad H_2'=\frac{1}{6}\begin{pmatrix}-1&&\\&-1&\\&&2\end{pmatrix},$$
$$E_1'=-6^{-\frac{1}{2}}|2\rangle\langle 1|,\qquad E_2'=-6^{-\frac{1}{2}}|3\rangle\langle 1|,$$
$$E_3'=-6^{-\frac{1}{2}}|3\rangle\langle 2|,\qquad E_{-\alpha}'=E_\alpha'^\dagger. \tag{III.22}$$

Thus

$$|\{3^*\},2\rangle=-6^{\frac{1}{2}}E_1'|\{3^*\},1\rangle,$$
$$|\{3^*\},3\rangle=-6^{\frac{1}{2}}E_2'|\{3^*\},1\rangle,$$
$$|\{3^*\},3\rangle=-6^{\frac{1}{2}}E_3'|\{3^*\},2\rangle.$$

This representation is inequivalent to $D^{(3)}(1,0)$ because the set of eigenvalues of H_i' is different from that of H_i. (See also Secs. III B and V.)

$D^{(8)}(1,1)$. The highest weight $\mathbf{M}$ of this representation is $\mathbf{M}^{(1)}+\mathbf{M}^{(2)}$, where $\mathbf{M}^{(1)}=\frac{1}{6}(\sqrt{3},1)$ and $\mathbf{M}^{(2)}=\frac{1}{6}(\sqrt{3},-1)$ are the fundamental dominant weights of $D(1,0)$ and $D(0,1)$, respectively. Hence $D(1,1)$ is contained in $D(1,0)\otimes D(0,1)$. The weight diagram is given in Fig. 2(e); we shall label the states as in Fig. 6, writing $|\{8\},A\rangle$ for the Ath state.

Each product state $|\{3\},a;\{3^*\},b\rangle$ has a unique weight equal to the sum of the weights of $|\{3\},a\rangle$ and $|\{3^*\},b\rangle$. Conversely, for $A=1,\cdots,6$ there is only one product state with the weight of $|\{8\},A\rangle$. Hence, with a choice of phases that turns out to be convenient later:

$$|\{8\},1\rangle=|\{3\},1;\{3^*\},2\rangle,$$
$$|\{8\},2\rangle=|\{3\},1;\{3^*\},3\rangle,$$
$$|\{8\},3\rangle=|\{3\},2;\{3^*\},3\rangle,$$
$$|\{8\},4\rangle=-|\{3\},2;\{3^*\},1\rangle,$$
$$|\{8\},5\rangle=-|\{3\},3;\{3^*\},1\rangle,$$
$$|\{8\},6\rangle=|\{3\},3;\{3^*\},2\rangle. \tag{III.23}$$

The states with weight zero are of the form

$$\sum_{a=1}^{3}\rho_a|\{3\},a;\{3^*\},a\rangle, \tag{III.24}$$

with real coefficients ρ_a. The transformations of the states by the $\hat{E}_\alpha$ is given by the action of $\hat{E}_\alpha$ on $|\{3\},a\rangle$ and on $|\{3^*\},a\rangle$, and is symbolized in Fig. 6. When Eqs. (23) are operated on by the $\hat{E}_\alpha$ or by products of the $\hat{E}_\alpha$, it is easy to see from (21) that the states (III.24) always occur in such linear combinations that

$$\sum\rho_a=0.$$

Hence, only two linearly independent combinations (III.24) occur in $D^{(8)}(1,1)$, as is in fact obvious from the fact that $D^{(8)}(1,1)$ is eight-dimensional. A possible choice of two orthonormal states is

$$|\{8\},7\rangle=(1/\sqrt{2})[|\{3\},2;\{3^*\},2\rangle-|\{3\},1;\{3^*\},1\rangle],$$
$$|\{8\},8\rangle=6^{-\frac{1}{2}}[-|\{3\},2;\{3^*\},2\rangle-|\{3\},1;\{3^*\},1\rangle+2|\{3\},3;\{3^*\},3\rangle]. \tag{III.25}$$

Here the state $|\{8\},7\rangle$ has been chosen, in anticipation of future convenience, as the state obtained by applying $\hat{E}_{-1}$ to $|\{8\},1\rangle$. Once $|\{8\},7\rangle$ has been chosen, $|\{8\},8\rangle$ is unique. The E_α are given by their effect on each of the product states, for example, using (20) and (22):

$$E_2|\{8\},6\rangle=E_2|\{3\},3;\{3^*\},2\rangle=6^{-\frac{1}{2}}|\{3\},1;\{3^*\},2\rangle=6^{-\frac{1}{2}}|\{8\},1\rangle.$$

In this way we get

$$6^{\frac{1}{2}}E_1=\sqrt{2}|1\rangle\langle 7|+\sqrt{2}|7\rangle\langle 4|+|2\rangle\langle 3|+|6\rangle\langle 5|,$$
$$6^{\frac{1}{2}}E_2=\tfrac{1}{2}\sqrt{2}|2\rangle(\langle 7|+\sqrt{3}\langle 8|)+\tfrac{1}{2}\sqrt{2}(|7\rangle+\sqrt{3}|8\rangle)\langle 5|+|3\rangle\langle 4|+|1\rangle\langle 6|,$$
$$6^{\frac{1}{2}}E_3=-\tfrac{1}{2}\sqrt{2}|3\rangle(\langle 7|-\sqrt{3}\langle 8|)+\tfrac{1}{2}\sqrt{2}(|7\rangle-\sqrt{3}|8\rangle)\langle 6|+|4\rangle\langle 5|-|2\rangle\langle 1|,$$
$$E_{-\alpha}=E_\alpha^\dagger. \tag{III.26}$$

The H_i, are the diagonal matrices

$$H_i=\sum_{A=1}^{8}m_i(A)|A\rangle\langle A|, \tag{III.27}$$

where $\mathbf{m}(A)$ is the weight of the Ath state. The phases here are consequences of the phases in (23).

We found that $D^{(8)}(1,1)$ contains only the two linear combination (25) of the three states (24). The third linear combination, orthogonal to (25) and normalized, is

$$|\{1\}1\rangle=(1/\sqrt{3})\sum_a|\{3\}a;\{3^*\}a\rangle. \tag{III.28}$$

This is an invariant, $E_\alpha=H_i=0$. Thus the decomposition of $D(1,0)\otimes D(0,1)$ is

$$D^{(3)}(1,0)\otimes D^{(3)}(0,1)=D^{(8)}(1,1)\oplus D^{(1)}(0,0). \tag{III.29}$$

An easier way of finding $D^{(8)}(1,1)$ uses the fact that this is the regular representation, as we shall show. The regular representation[47] is that in which L_A is represented by the matrices $-(C_A)_B{}^D$ whose components are the structure constants $-C_{AB}{}^D$. When the commutation relations are in the standard form (II.12), the capital latin index $A=1,\cdots,8$ is replaced by $i=1,2$ and $\alpha=\pm1,\pm2,\pm3$. Thus, referring to (II.12), the $\hat{H}_i$ are represented by $-(C_i)_A{}^B$, whose nonvanishing matrix elements are $-C_{i\alpha}{}^\alpha=-r_i(\alpha)$; the $\hat{E}_\alpha$ are represented by $-(C_\alpha)_A{}^B$, whose nonvanishing matrix elements are $-C_{\alpha i}{}^\alpha=+r_i(\alpha)$, $-C_{\alpha-\alpha}{}^i=-r_i(\alpha)$, and $-C_{\alpha\beta}{}^\gamma=-N_{\alpha\beta}$. Summarizing

$$H_i=-C_i=-\sum_\alpha r_i(\alpha)|\alpha\rangle\langle\alpha|, \tag{III.30}$$

$$E_\alpha=-C_\alpha=+\sum_i r_i(\alpha)|i\rangle\langle\alpha| -\sum_i r_i(\alpha)|-\alpha\rangle\langle i|-\sum_\beta N_{\alpha\beta}|\beta\rangle\langle\gamma|. \tag{III.31}$$

Here, as in (II.12), $|\gamma\rangle$ is the state whose root is $\mathbf{r}(\alpha)+\mathbf{r}(\beta)$.

Comparing (30) and (31) with (27) and (26), and taking $r_i(\alpha)$ from (II.18), we find complete agreement with the following identifications

$$\begin{aligned}
|\alpha\rangle\to|A\rangle:\ & |-1\rangle\to|1\rangle, && |-2\rangle\to|2\rangle,\\
& && |-3\rangle\to|3\rangle,\\
& |+1\rangle\to-|4\rangle, && |+2\rangle\to-|5\rangle,\\
& && |+3\rangle\to|6\rangle,\\
|i\rangle\to|A\rangle:\ & |1\rangle\to-|7\rangle,\\
& |2\rangle\to-|8\rangle.
\end{aligned} \tag{III.32}$$

The complex conjugate of $D^{(8)}(1,1)$ is related to it by reflection through the origin of the weight diagram. This gives the same diagram with a different labeling. The operator reflecting through the origin is[48]

$$C=-|1\rangle\langle4|\pm|2\rangle\langle5|\pm|3\rangle\langle6|\pm|4\rangle\langle1| \pm|5\rangle\langle2|\pm|6\rangle\langle3|\pm|7\rangle\langle7|\pm|8\rangle\langle8|.$$

The signs are determined by

$$CL_AC^{-1}=L_A'=-\tilde{L}_A, \tag{III.33}$$

where L_A are the matrices (26), (27). The solution is

$$\begin{aligned}C&=-|1\rangle\langle4|-|4\rangle\langle1|+|3\rangle\langle6|+|6\rangle\langle3|\\ &\quad-|5\rangle\langle2|-|2\rangle\langle5|+|7\rangle\langle7|+|8\rangle\langle8|\\ &=\tilde{C}=C^{-1}.\end{aligned} \tag{III.34}$$

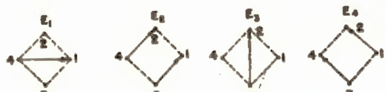


FIG. 7. Action of E_α on $D^{(4)}(1,0)$ of C_2.

[48] This operator is the same as will be introduced later as a "metric tensor." It could also have been defined by the property that

$$\sum_{A,B} C_{AB}|\{8\},A;\{8\},B\rangle=|\{1\}\rangle$$

is an invariant.

The existence of C means that $D^{(8)}(1,1)$ is equivalent to its contragredient representation. This is both displayed and proved in (33).

G. Matrix Representations of C_2

The fundamental representations are $D^{(4)}(1,0)$ and $D^{(5)}(0,1)$ and the regular representation is $D^{(10)}(2,0)$.

$D^{(4)}(1,0)$: The weight diagram was given in Fig. 4(a), we label the states as in Fig. 7, $|\{4\},a\rangle$, $a=1,2,3,4$. The actions of E_α, $\alpha=1,\cdots,4$ are summarized in Fig. 7; the action of $E_{-\alpha}$ are the same with the arrows reversed. As in the case of SU_3, Theorem II is sufficient to allow one to write down the explicit forms of E_α almost immediately. Thus, the analogs of (17) and (18) are

$$H_i=\sum_{a=1}^{4} m_i(a)|a\rangle\langle a| \tag{III.35}$$

$$\begin{aligned}
E_1&=6^{-\frac12}|1\rangle\langle4|,\\
E_2&=(1/2\sqrt{3})(|1\rangle\langle3|-|2\rangle\langle4|),\\
E_3&=-6^{-\frac12}|2\rangle\langle3|,\\
E_4&=(1/2\sqrt{3})(|2\rangle\langle1|+|4\rangle\langle3|),\\
E_{-\alpha}&=(E_\alpha)^\dagger.
\end{aligned} \tag{III.36}$$

The weight diagram for the contragredient representation is obtained by reflection through the origin. In this case we get the same diagram with a different labeling. Hence, the operator reflecting through the origin is

$$C=-|1\rangle\langle4|\pm|4\rangle\langle1|\pm|2\rangle\langle3|\pm|3\rangle\langle2|.$$

The phases must be chosen to agree with (21), that is

$$CL_AC^{-1}=L_A'=-\tilde{L}_A. \tag{III.37}$$

The solution is

$$\begin{aligned}C&=-|1\rangle\langle4|+|4\rangle\langle1|+|2\rangle\langle3|-|3\rangle\langle2|\\ &=-\tilde{C}=-C^{-1}=-C^\dagger.\end{aligned} \tag{III.38}$$

The existence of C means that $D^{(4)}(1,0)$ is equivalent to its contragredient representation. This is both displayed and proved in (35).

$D^{(5)}(0,1)$. The weight diagram is that of Fig. 4(b); we use the labeling of Fig. 8. The action of E_α is also symbolized in Fig. 8. The matrices are obtained

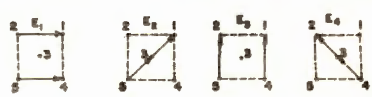


FIG. 8. Action of E_α on $D^{(5)}(0,1)$ of C_2.

exactly as before, namely:

$$H_i=\sum_{k=1}^{5} m_i(k)|k\rangle\langle k|, \tag{III.39}$$

$$\begin{aligned}
E_1&=6^{-\frac{1}{2}}(|1\rangle\langle 2|+|4\rangle\langle 5|),\\
E_2&=6^{-\frac{1}{2}}(|1\rangle\langle 3|-|3\rangle\langle 5|),\\
E_3&=6^{-\frac{1}{2}}(|1\rangle\langle 4|+|2\rangle\langle 5|),\\
E_4&=6^{-\frac{1}{2}}(|2\rangle\langle 3|+|3\rangle\langle 4|),\\
E_{-\alpha}&=(E_\alpha)^\dagger.
\end{aligned} \tag{III.40}$$

Again the contragredient representation is equivalent. The matrix C in this case is

$$C=|5\rangle\langle 1|+|1\rangle\langle 5|-|2\rangle\langle 4|-|4\rangle\langle 2|+|3\rangle\langle 3| =\tilde{C}. \tag{III.41}$$

$D^{(10)}(2,0)$. The weight diagram was given in Fig. 4(c). The highest weight is exactly twice the highest weight of $D^{(4)}(1,0)$, and $D^{(10)}(2,0)$ is contained in $D^{(4)}(1,0)\otimes D^{(4)}(1,0)$. We begin by calling $|\{10\},1\rangle$ the state $|\{4\},1;\{4\},1\rangle$. Since the E_α operate in the same way on the two factors, it is evident that $E_\alpha|\{10\},1\rangle$, $E_\alpha E_\beta|\{10\},1\rangle$, etc., are all symmetric in the two factors. Hence we have, with a convenient set of phases,

$$\begin{aligned}
|\{10\},1\rangle&=|\{4\},1;\{4\},1\rangle &&=-|-1\rangle,\\
|\{10\},2\rangle&=\tfrac{1}{2}\sqrt{2}|\{4\},1;\{4\},2\rangle\\
&\quad+\tfrac{1}{2}\sqrt{2}|\{4\},2;\{4\},1\rangle&&=+|-2\rangle,\\
|\{10\},3\rangle&=|\{4\},2;\{4\},2\rangle &&=-|-3\rangle,\\
|\{10\},4\rangle&=\tfrac{1}{2}\sqrt{2}|\{4\},2;\{4\},4\rangle\\
&\quad+\tfrac{1}{2}\sqrt{2}|\{4\},4;\{4\},2\rangle&&=+|-4\rangle,\\
|\{10\},5\rangle&=|\{4\},4;\{4\},4\rangle &&=+|+1\rangle,\\
|\{10\},6\rangle&=\tfrac{1}{2}\sqrt{2}|\{4\},4;\{4\},3\rangle\\
&\quad+\tfrac{1}{2}\sqrt{2}|\{4\},3;\{4\},4\rangle&&=+|+2\rangle,\\
|\{10\},7\rangle&=|\{4\},3;\{4\},3\rangle &&=+|+3\rangle,\\
|\{10\},8\rangle&=\tfrac{1}{2}\sqrt{2}|\{4\},1;\{4\},3\rangle\\
&\quad+\tfrac{1}{2}\sqrt{2}|\{4\},3;\{4\},1\rangle&&=-|+4\rangle,\\
|\{10\},9\rangle&=\tfrac{1}{2}\sqrt{2}|\{4\},1;\{4\},4\rangle\\
&\quad+\tfrac{1}{2}\sqrt{2}|\{4\},4;\{4\},1\rangle&&=+|1\rangle,\\
|\{10\},10\rangle&=\tfrac{1}{2}\sqrt{2}|\{4\},2;\{4\},3\rangle\\
&\quad+\tfrac{1}{2}\sqrt{2}|\{4\},3;\{4\},2\rangle&&=-|2\rangle.
\end{aligned} \tag{III.42}$$

The labeling on the right-hand side is the one that allows us to use Eqs. (30) and (31) directly. The simplest derivation is by means of [cf. (V.5)]

$$6^{\frac{1}{2}}\sum_{a,b,d} C_{ab}(E_{\pm\alpha})_{bd}|ad\rangle=|\pm\alpha\rangle;$$

$$6^{\frac{1}{2}}\sum_{a,b,d} C_{ab}(H_i)_{bd}|ad\rangle=|i\rangle.$$

H. Matrix Representations of G_2

The fundamental representations are $D^{(7)}(1,0)$ and $D^{(14)}(0,1)$, the latter being the regular representation.

$D^{(7)}(1,0)$. The weight diagram is that of Fig. 3(a), and we use the labeling indicated there, thus $|\{7\},k\rangle$, $k=1, \cdots, 7$. As in the other examples, Theorem II (Sec. III E) suffices to determine the matrix elements of E_α. The result is

$$\begin{aligned}
H_i&=\sum_{k=1}^{7} m_i(k)|k\rangle\langle k|,\\
E_1&=(1/2\sqrt{3})(\tfrac{1}{2}\sqrt{2}|1\rangle\langle 2|+\tfrac{1}{2}\sqrt{2}|3\rangle\langle 4|+|5\rangle\langle 6|+|6\rangle\langle 7|),\\
E_2&=(1/2\sqrt{2})(|5\rangle\langle 4|+|1\rangle\langle 7|),\\
E_3&=(1/2\sqrt{3})(\tfrac{1}{2}\sqrt{2}|5\rangle\langle 3|-|6\rangle\langle 4|+|1\rangle\langle 6|-\tfrac{1}{2}\sqrt{2}|2\rangle\langle 7|),\\
E_4&=(1/2\sqrt{2})(|1\rangle\langle 3|+|2\rangle\langle 4|),\\
E_5&=(1/2\sqrt{3})(|6\rangle\langle 3|-\tfrac{1}{2}\sqrt{2}|7\rangle\langle 4|-\tfrac{1}{2}\sqrt{2}|1\rangle\langle 5|+|2\rangle\langle 6|),\\
E_6&=(1/2\sqrt{2})(-|2\rangle\langle 5|+|7\rangle\langle 3|);\ E_{-\alpha}=(E_\alpha)^\dagger.
\end{aligned} \tag{III.43}$$

The C operator which changes $D^{(7)}$ into its complex conjugate and reflects the weight-diagram through the origin is defined by

$$CL_AC^{-1}=-\tilde{L}_A.$$

The solution is

$$C=|5\rangle\langle 7|+|7\rangle\langle 5|-|1\rangle\langle 4|-|4\rangle\langle 1| +|2\rangle\langle 3|+|3\rangle\langle 2|-|6\rangle\langle 6|. \tag{III.44}$$

$D^{(14)}(0,1)$. Since this is the regular representation, the matrices H_i and E_α are given by (30) and (31). The weight diagram is that of Fig. 3(b).

IV. COMPOSITION AND DECOMPOSITION OF LIE ALGEBRA REPRESENTATIONS

The basis of the vector space affording a representation of a simple group may be characterized by the simultaneous eigenvalues of the maximum number of mutually commuting Lie algebra operators, designated by the symbols $H_1, H_2, \cdots, H_l$ where l is the rank of the group. However, the characterization of the representation space basis is not complete if only the H_i eigenvalues are assigned to the basis vectors because the same set of eigenvalues of the H_i, the weight $\{m_1\cdots m_l\}\equiv\mathbf{m}$, can occur more than once in a specific representation, i.e., weights other than the dominant weight $\{M_1\cdots M_l\}\equiv\mathbf{M}$ are, in general, not simple. The goals of this section are (a) to find the set of weights and their multiplicities in every representation, and (b) to reduce the direct product of irreducible representations into a direct sum of irreducible representations. The course which we pursue is a purely geometric one, and represents an extension of the classical method.

A. Geometric Characterization of a Representation

Let us restrict the considerations to the groups of rank two. The seven-dimensional representation of G_2 can be characterized by plotting the array of points (m_1,m_2) whose coordinates are the weights of the representation.[53] Figure 9(a) shows the resulting array of points. In this specific example, the multiplicity of each of the weights is one and so each weight is associated with one and only one point. When the multiplicity of a weight is greater than one, this will be indicated. Such is the case in the eight-dimensional representation of SU_3, for which the associated point set is given in Fig. 9(b).

Before proceeding with the task of composing and reducing representations, we introduce the formal operations on sets of points which are utilized in the subsequent sections.

B. Algebra of Sets of Points

To illustrate the algebraic manipulations to which sets of points can be subjected, consider first sets of collinear points. A set of points on a line with a center * and with signed multiplicities attached to each point will be associated with a function which is a sum of powers of a single variable x, as follows:

(a) Each point is associated with a term in the function; the latter has as many terms as there are points,
(b) The coordinate of each point relative to the set center * represents the power of x in the relevant term,
(c) The numerical coefficient of the term is the attached signed multiplicity.

Thus the set of points in Fig. 10(a) represents the algebraic expression $0.3x^{-4}-x^{-1}+2x^2$.

In what follows, only integral multiplicities come into consideration and if a single point without indicated multiplicity but with an attached sign occurs, the associated term in the algebraic expression is assigned a coefficient ±1 depending on the indicated sign. A final liberty with the above conventions is to assume that, in the absence of an indicated center of a point set, this coincides with the geometric center of the point set.

FIG. 9. (a) Representation set for G_2, (b) representation set for SU_3.

[53] These arrays are nothing but the weight diagrams of Sec. III, with the multiplicities added.

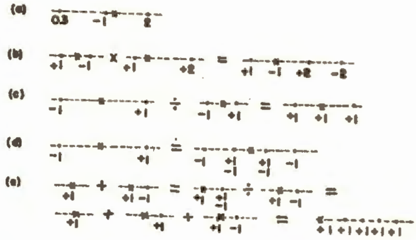


FIG. 10. Algebraic processes on linear point sets.

Addition of two sets of points (ζ and ζ') with a common center is defined to be the union of the two sets; $\zeta+\zeta'\equiv\zeta\cup\zeta'$, the multiplicities adding algebraically. *Subtraction* of two sets of points ζ and ζ' is defined to be the addition of ζ to the set $-\zeta'$ obtained from ζ by changing the signs of all multiplicities.

To *multiply* one set of points ζ by another set ζ', the center of the set ζ' is placed on each of the points of the set ζ and each term of ζ' is multiplied by the multiplicity of the point of ζ upon which its center sits. The new set of points obtained in such a manner is defined to be the product set $\zeta\times\zeta'$. For example, Fig. 10(b) is the geometric equivalent of $(x^{-1}-x)\times(x^{-1}+2x^3)=(x^{-2}-1+2x^2-2x^4)$.

Division is defined to be the inverse of multiplication. The most trivial case of division is the case in which the two sets of points ζ and ζ' are identical. The result of the division $\zeta\div\zeta'$ is simply a single point at the common center of ζ and ζ'. In general, one set of points ζ' exactly divides a congruent set ζ if the multiplicities of every point of ζ is a fixed multiple Z of its image point in ζ'. The result of this division operation is a point of multiplicity Z which sits where the center of ζ' falls when superimposed on ζ. If the set ζ' is not congruent to the set ζ, it is possible to create a subset of ζ, denoted by ζ'' and exactly divisible by ζ', by adding and subtracting points, of the same multiplicity at appropriate positions in the set ζ. After dividing such a subset ζ'' away, we are left with the problem of dividing the residual set $\zeta-\zeta''$ by ζ'. By continuing this process, we may ultimately arrive at a residual set itself exactly divisible by ζ' without modification. As an example, consider the problem illustrated in Fig. 10(c) whose algebraic analog is $(x^3-x^{-3})/(x-x^{-1})$. By adding and subtracting a point of multiplicity $+1$ at each of the positions -1 and $+1$ [Fig. 10(d)], the exact division can be effected. If two sets of points are not exactly divisible, division can still be carried out by adding and subtracting points to the dividend set *ad infinitum*. Figure 10(e) illustrates the geometric method of carrying out the expansion $1/1-x=1+x+x^2+\cdots$. In what follows, we use only exactly divisible point sets.

All of the above manipulations are quite trivial for linear sets of points. However, it is possible to generalize

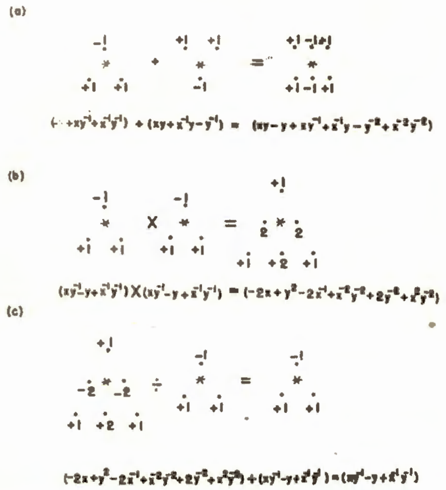


FIG. 11. Algebraic processes on two dimensional sets of points. (a) Addition; (b) multiplication; (c) division.

to an algebra of sets of points in an n-dimensional space, each point being characterized by a coordinate $\mathbf{m}=(m_1m_2\cdots m_l)$ and an assigned multiplicity and the total set being provided with a center. Every such point is again associated with a term in an algebraic expression in n variables. For example, the point at $\mathbf{m}=(m_1m_2\cdots m_l)$ with multiplicity μ_m is the geometric representation of $\mu_m x_1^{m_1}x_2^{m_2}\cdots x_l^{m_l}$. All algebraic processes on algebraic expressions in n variables of the form $\sum_m \mu_m x_1^{m_1}\cdots x_l^{m_l}$ can now be given a geometric analog.

Since our concern is with functions in two variables, we illustrate in Fig. 11 some algebraic processes carried out on sets of points in two dimensions. It is to be remarked that the operations on the sets of points are completely isomorphic to the corresponding algebraic processes and as such are, for example, associative and commutative.

C. Construction of Weights and Multiplicities of Irreducible Representations

Our goal in this section is to assign to every irreducible representation of a group a set of points

TABLE I. Coordinates of points in the set $\xi(\lambda_1\lambda_2)$ for SU_3.

$(6/\sqrt{3})x$	$6y$	Multiplicity
$(\lambda_1+\lambda_2+2)$	$(\lambda_1-\lambda_2)$	$+1$
(λ_1+1)	$(\lambda_1+2\lambda_2+3)$	-1
$-(\lambda_1+1)$	$(\lambda_1+2\lambda_2+3)$	$+1$
$-(\lambda_1+\lambda_2+2)$	$(\lambda_1-\lambda_2)$	-1
$-(\lambda_2+1)$	$-(2\lambda_1+\lambda_2+3)$	$+1$
(λ_2+1)	$-(2\lambda_1+\lambda_2+3)$	-1

(called the representation set from now on) and derive the admissible sets of points constituting a representation. The fundamental observation is the following: The character of the representation is the algebraic expression associated with the representation set.[68] For a group of rank l the algebraic variables associated with the representation set may be selected as $x_i=e^{i\varphi_i}$. Recall now that every representation of a rank two group is characterized by two integers λ_1 and λ_2 where λ_1, λ_2 run over all non-negative integers. The general expressions for the characters of all the groups which interest us have been given by Weyl.[8] Letting $\chi(\lambda_1,\lambda_2)$ denote the set of points constituting the representation, the general expression for $\chi(\lambda_1,\lambda_2)$ is

$$\chi(\lambda_1,\lambda_2)=\xi(\lambda_1,\lambda_2)/\xi(0,0), \qquad \text{(IV.1)}$$

where the algebraic expressions $\xi(\lambda_1,\lambda_2)$ were given in Secs. III A and III B. The set of points $\xi(\lambda_1,\lambda_2)$ is called the *girdle* of points uniquely characterizing a

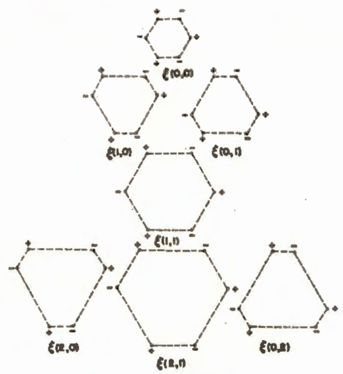


FIG. 12. Some girdles of SU_3.

representation. We thus see that to generate the representation set, the girdle $\xi(\lambda_1,\lambda_2)$ must be divided by the girdle $\xi(0,0)$. Since the $\chi(\lambda_1,\lambda_2)$ form a finite set of points, $\xi(\lambda_1,\lambda_2)$ must be exactly divisible by $\xi(0,0)$.

To illustrate the detailed mechanics of generating representation sets, we turn to the groups SU_3, C_2, and G_2.

SU_3. The coordinates of the six points making up $\xi(\lambda_1,\lambda_2)$ are given in Table I. They are the values of the components of (SK) of Eq. (III.4). For SU_3 the girdle $\xi(\lambda_1,\lambda_2)$ forms the vertices of a hexagon which has the following properties:

(a) Every other side is of the same length, either $\frac{1}{3}\sqrt{3}(\lambda_1+1)$ or $\frac{1}{3}\sqrt{3}(\lambda_2+1)$,
(b) The hexagons are always symmetric about the y axis,
(c) A hexagon is symmetric about the x axis if and only if $\lambda_1=\lambda_2$. In this case, the hexagon is regular (all sides being equal).

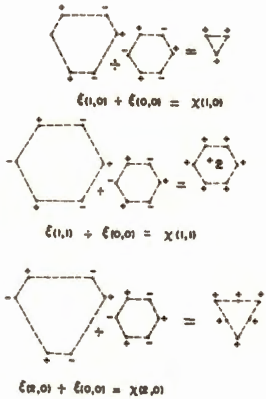


FIG. 13. Some characters of SU_3 obtained by the division process.

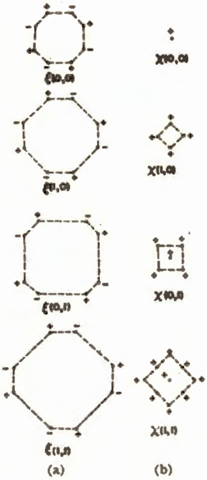


FIG. 14. Some girdles and characters of C_2.

If $\chi(\lambda_1,\lambda_2)$ is a representation set, then the complex conjugate representation set $\chi^*(\lambda_1,\lambda_2)=\chi(\lambda_2,\lambda_1)$ (for SU_3 only) is obtained by inverting the $\chi(\lambda_1,\lambda_2)$ hexagon through the origin, and changing the signs of the multiplicities. An equivalent procedure is to reflect $\chi(\lambda_1,\lambda_2)$ in the x axis and leave the multiplicities unchanged. Thus, the necessary and sufficient condition for equivalence of $D(\lambda_1,\lambda_2)$ and $D^*(\lambda_1,\lambda_2)$ is that the $\xi(\lambda_1,\lambda_2)$ hexagon be regular.

Figure 12 illustrates the girdles of some low-dimensional representations of SU_3. Triangular graph paper is admirably suited for the plot.

The construction of the weights and multiplicities of a representation is now effected by dividing $\xi(\lambda_1,\lambda_2)$ by $\xi(0,0)$ and identifying the quotient points as the representation set. In Fig. 13, we carry out some representative divisions.

C_2. With the use of Table II any girdle can be found; in particular, those illustrated in Fig. 14(a). The points of $\xi(\lambda_1,\lambda_2)$ define the vertices of an octagon symmetric about the x and y axes. Every representation is therefore equivalent to its complex conjugate representation. The sides of the octagon alternate in length between $\frac{1}{2}\sqrt{3}(\lambda_2+1)$ and $(\frac{3}{2})^{\frac{1}{2}}(\lambda_1+1)$.

TABLE II. Coordinates of the point s in the set $\xi(\lambda_1,\lambda_2)$ for C_2.

$2\sqrt{3}\,x$	$2\sqrt{3}\,y$	Multiplicity
$(\lambda_1+\lambda_2+2)$	(λ_2+1)	+1
(λ_2+1)	$(\lambda_1+\lambda_2+2)$	−1
$-(\lambda_2+1)$	$(\lambda_1+\lambda_2+2)$	+1
$-(\lambda_1+\lambda_2+2)$	(λ_2+1)	−1
$-(\lambda_1+\lambda_2+2)$	$-(\lambda_2+1)$	+1
$-(\lambda_2+1)$	$-(\lambda_1+\lambda_2+2)$	−1
(λ_2+1)	$-(\lambda_1+\lambda_2+2)$	+1
$(\lambda_1+\lambda_2+2)$	$-(\lambda_2+1)$	−1

Figure 14(b) gives the result of dividing the $\xi(\lambda_1,\lambda_2)$ of Fig. 14(a) by $\xi(0,0)$.[54]

G_2. Table III specifies the sets $\xi(\lambda_1,\lambda_2)$ as dodecahedrons symmetric about the x and y axis. Thus the complex conjugate representations are equivalent. As in C_2 and SU_3, the sides of the $\xi(\lambda_1,\lambda_2)$ polygon alternate in length, in this case between $\frac{1}{2}(\lambda_2+1)$ and $\frac{1}{2}\sqrt{3}(\lambda_1+1)$. Figure 16 contains the representation sets, $\chi(1,0)$ and $\chi(0,1)$, while Fig. 15 illustrates some girdles.

D. Reduction of Direct Products of Representations

In the previous section, we have shown how to derive all the representation point sets including the multiplicity assignments. However, for the purpose of reducing the direct product of representations, only

TABLE III. Coordinates of points in the set $\xi(\lambda_1,\lambda_2)$ for G_2.

$4\sqrt{3}\,x$	$4y$	Multiplicity
$(2\lambda_1+3\lambda_2+5)$	(λ_2+1)	+1
$(\lambda_1+3\lambda_2+4)$	$(\lambda_1+\lambda_2+2)$	−1
(λ_1+1)	$(\lambda_1+2\lambda_2+3)$	+1
$-(\lambda_1+1)$	$(\lambda_1+2\lambda_2+3)$	−1
$-(\lambda_1+3\lambda_2+4)$	$(\lambda_1+\lambda_2+2)$	+1
$-(2\lambda_1+3\lambda_2+5)$	(λ_2+1)	−1
$-(2\lambda_1+3\lambda_2+5)$	$-(\lambda_2+1)$	+1
$-(\lambda_1+3\lambda_2+4)$	$-(\lambda_1+\lambda_2+2)$	−1
$-(\lambda_1+1)$	$-(\lambda_1+2\lambda_2+3)$	+1
(λ_1+1)	$-(\lambda_1+2\lambda_2+3)$	−1
$(\lambda_1+3\lambda_2+4)$	$-(\lambda_1+\lambda_2+2)$	+1
$(2\lambda_1+3\lambda_2+5)$	$-(\lambda_2+1)$	−1

[54] The method of dividing point sets by point sets turns out to be quite powerful. Further details will be found in a paper by two of the authors (J.D. and C.F.).

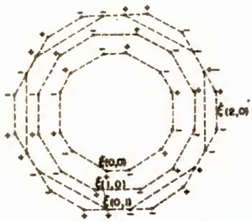


FIG. 15. Some girdles of G_2.

the girdle $\xi(\lambda_1,\lambda_2)$ associated with the representation is needed, as we now prove.

The direct product of two representations of a simple group reduces completely and uniquely into a sum of irreducible representations some of which may occur more than once. Letting $\nu(\mu_1,\mu_2)$ designate the number of times a specific representation $\chi(\mu_1,\mu_2)$ occurs in the reduction of a direct product of irreducible representations, we have the following equality between point sets

$$\chi(\lambda_1,\lambda_2)\otimes\chi(\lambda_1',\lambda_2')=\sum_{\mu_1\mu_2}\nu(\mu_1,\mu_2)\chi(\mu_1,\mu_2). \quad \text{(IV.2)}$$

If we use the fundamental relation Eq. (IV.1), Eq. (IV.2) reduces to

$$\frac{\xi(\lambda_1,\lambda_2)\times\xi(\lambda_1',\lambda_2')}{\xi(0,0)}=\sum_{\mu_1\mu_2}\nu(\mu_1,\mu_2)\xi(\mu_1,\mu_2), \quad \text{(IV.3)}$$

where we have multiplied both sides of Eq. (IV.2) by $\xi(0,0)$. Because only the girdles of the irreducible representations $\chi(\mu_1,\mu_2)$ occur on the right-hand side of Eq. (IV.3), we need only carry out the point set process $\{\xi(\lambda_1,\lambda_2)\times\xi(\lambda_1',\lambda_2')\div\xi(0,0)\}$, and then identify the girdles and their multiplicities $\nu(\mu_1,\mu_2)$ in the resulting set to reduce completely the product representations. Use of one of the several alternative forms of $\xi(\lambda_1,\lambda_2)\times\xi(\lambda_1',\lambda_2')\div\xi(0,0)$, namely

$$\begin{aligned}\chi(\lambda_1,\lambda_2)\times\xi(\lambda_1',\lambda_2')&=\xi(\lambda_1,\lambda_2)\chi(\lambda_1',\lambda_2')\\&=\xi(0,0)\chi(\lambda_1,\lambda_2)\chi(\lambda_1',\lambda_2')\end{aligned}$$

will simplify the computations in some cases.

As examples of the reduction process, we carry out

(a) $\chi(1,0)\times\chi(1,0)$ and $\chi(1,0)\times\chi(0,1)$ in SU_3

(b) $\chi(1,0)\times\chi(1,0)$ for C_2

(c) $\chi(1,0)\times\chi(1,0)$ for G_2.

Figures 17 and 18 illustrate the reduction processes for SU_3 and C_2, respectively. The superimposed girdle

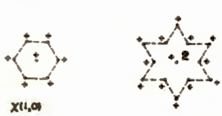


FIG. 16. Some characters of G_2.

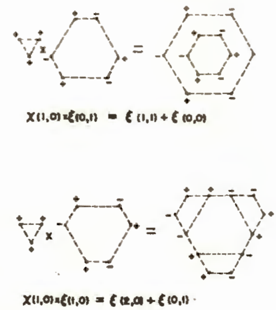


FIG. 17. Geometric derivation of girdles in direct product representation of SU_3.

diagram of Fig. 15 is the product of $\xi(0,0)\times\chi(1,0)\times\chi(1,0)$ for G_2.

V. TENSOR ANALYSIS OF SIMPLE LIE GROUPS

In this section we present some results by an alternative, purely algebraic method, which to a certain extent is complementary to the geometric method. The specific advantages of the algebraic method is that it deals directly with the bases of the representation space (the "wave functions"), and that it gives directly the explicit form of invariants, product representations and transformation matrices.

Let m be the dimensionality of any representation of some simple Lie group. The matrix algebra of that representation consists of Hermitian traceless matrices. Since the matrix algebra of an m-dimensional representation of SU_m is the set of *all* Hermitian traceless matrices, it follows that the group in question is a subgroup of SU_m. For example, C_2 and G_2 are subgroups of SU_4 and SU_7, respectively. Therefore the reduction of a product of several m-dimensional representations is a refinement of the reduction according to SU_m. It is very helpful, then, to begin with a discussion of SU_m for arbitrary m.

A. Group SU_m

Let ψ_a, $a=1, \cdots, m$, be a basis for an m-dimensional representation of SU_m. The matrices representing a basis for the Lie algebra are any set of m^2-1 independent Hermitian traceless matrices. The *contragredient* representation ψ^a is defined by[56]

$$\psi_a\rightarrow(\delta_a^b+i\epsilon^A L_{Aa}{}^b)\psi_b,\quad \psi^a\rightarrow\psi^b(\delta_b^a-i\epsilon^A L_{Ab}{}^a). \quad \text{(V.1)}$$

[For $m=3$, these representations are those labeled $D^{(3)}(1,0)$ and $D^{(3)}(0,1)$ in Secs. III and IV. The weight diagrams are those of Fig. 2(a) and Fig. 2(b).]

Next consider the "tensors" $\psi_{ab\cdots}{}^{ef\cdots}$. These are quantities transforming in the same way as products of the representations ψ_a and ψ^a. Thus ψ_{ab} has m^2 components which transform among themselves like the

[56] This will be recognized as agreeing with the definitions of Eqs. (I.1) and (III.21).

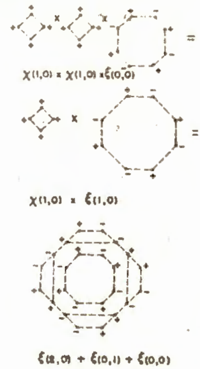


FIG. 18. Geometric derivation of girdles in the direct product representation of C_3.

m^2 quantities $\psi_a\psi_b$, $\psi_a{}^b$ transforms like $\psi_a\psi^b$, etc. The tensors form bases for representations called *product representations*; the present definition agrees with that of Sec. III.

Product representations are usually[56] reducible. The reduction of second-rank tensors according to SU_m is entirely elementary. The tensor $\psi_a{}^b$, for example, transforms according to (1), as follows[57]:

$$\psi_a{}^b \to (\delta_a{}^c+i\epsilon^A L_{Aa}{}^c)(\delta_d{}^b-i\epsilon^B L_{Bd}{}^b)\psi_c{}^d$$
$$=\psi_a{}^b+i\epsilon^A(L_{Aa}{}^c\delta_d{}^b-L_{Ad}{}^b\delta_a{}^c)\psi_c{}^d. \quad \text{(V.2)}$$

In particular, if we put $a=b$ and sum, we find

$$\psi_a{}^a \to \psi_a{}^a.$$

Thus the trace is invariant, meaning that the m^2 dimensional representation $\psi_a{}^b$ may be reduced into a one-dimensional representation and the m^2-1 dimensional representation whose basis is the traceless tensor

$$\psi_a{}^b-\frac{1}{m}\delta_a{}^b\psi_e{}^e=P_a{}^b{}_c{}^d\psi_d{}^c. \quad \text{(V.3)}$$

Here $P_a{}^b{}_c{}^d$ is the projection operator

$$P_a{}^b{}_c{}^d=\delta_a{}^d\delta_c{}^b-\frac{1}{m}\delta_a{}^b\delta_c{}^d, \quad \text{(V.4)}$$

whose rows are labeled by a, b and whose columns are labeled by c, d.

The proof that (3) is the basis of an irreducible representation is instructive. First, we show that (3) is the *regular representation*[47] for SU_m, and that it *contains* the regular representation for any subgroup of SU_m. Let r be the order of the subgroup, and consider the r linearly-independent combinations.

$$\varphi_A=L_{Ab}{}^a\psi_a{}^b=L_{Ab}{}^a(\delta_a{}^c\delta_d{}^b-\frac{1}{m}\delta_a{}^b\delta_d{}^c)\psi_c{}^d. \quad \text{(V.5)}$$

The second equality is a result of the fact that the matrices $L_{Aa}{}^b$ are traceless, and shows that φ_A depend on the traceless tensor (3) only. From (2) and (5) we get

$$\varphi_A \to \varphi_A+i\epsilon^B(L_{Ad}{}^eL_{Be}{}^c-L_{Bd}{}^bL_{Ab}{}^c)\psi_c{}^d$$
$$=\varphi_A+i\epsilon^BC_{AB}{}^DL_{Dd}{}^c\psi_c{}^d$$
$$=\varphi_A+i\epsilon^BC_{AB}{}^D\varphi_D. \quad \text{(V.6)}$$

Hence, the φ_A are the basis of that representation of the r-parameter subgroup in which the operators L_B are represented by the structure constants $C_{AB}{}^D$, and that is the regular representation. Equation (5) shows that this representation is contained in the traceless $\psi_a{}^b$. In the special case of SU_m, $r=m^2-1$, and $L_{Aa}{}^b$ is the set of *all* Hermitian traceless matrices. Hence, in that case the regular representation φ_A is equivalent to the representation whose basis is the traceless $\psi_a{}^b$. Since the former is irreducible[47] (for any simple group), so is the latter.

With the proof that (3) is irreducible, the reduction of $\psi_a{}^b$ has been completed. We can also prove that ψ_a and ψ^a are inequivalent. For suppose that they are equivalent. Then there exists a nonsingular form invariant matrix A^{ab} such that $\psi^a=A^{ab}\psi_b$. This could be used to prove that $\psi_a{}^b$ and ψ^{ab} were equivalent, which is impossible since ψ^{ab} reduces quite differently, as we shall see immediately. Hence, no matrix exists for raising and lowering indices.

The reduction problem for tensors of arbitrary rank, but with all indices either upstairs or downstairs, has a complete and beautiful solution in terms of Young tableaux.[58] We do not present the general theory here, since it is only of marginal interest, and thus do not prove that the representations obtained are irreducible. However, whenever appropriate, we indicate the connection between the representations and the tableaux. The complete reduction of the second-rank tensor ψ_{ab} is given by

$$\psi_{ab}=\psi_{ab,}+\psi_{a,b},$$

where

$$\psi_{ab,}=\tfrac{1}{2}(\psi_{ab}+\psi_{ba}), \quad \psi_{a,b}=\tfrac{1}{2}(\psi_{ab}-\psi_{ba}).$$

The symmetric part $\psi_{ab,}$ has $\frac{1}{2}m(m+1)$ components and corresponds to the Young tableau of Fig. 19(a). The skew part $\psi_{a,b}$, has $\frac{1}{2}m(m-1)$ components and the Young tableau is that of Fig. 19(b).

Roughly, indices appearing in the same row in a Young diagram are subject to symmetrization, while indices appearing in the same column are subject to

[56] The only exception is the case when one of the factors is the identity representation.

[57] The tensor $\psi_a{}^b$ would be written $|(m),a;\ (m),b\rangle$ in the notation of Sec. III E. Equation (V.2) is an application of Eq. (III.14).

[58] A readable exposition is given in D. Rutherford; *Substitutional Analysis* (Edinburgh University Press, Edinburgh, Scotland, 1948).

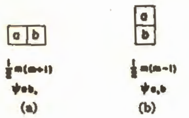


FIG. 19. The Young tableaux related to the reduction of the second-rank tensor in m dimensions.

antisymmetrization. The notation is the following: A comma between the indices separate those of the first row from those of the second row, a second comma separates the indices in the second row from those of the third, and so on. The completely symmetric tensor $\psi_{a\cdots d,}$ is furnished with a comma to distinguish it from the general nonsymmetrized tensor $\psi_{a\cdots d}$.

Corresponding to the reduction of the third-rank tensor there are the four Young tableaux of Fig. 20. The irreducible bases, as well as the dimensionalities, are indicated; the latter, of course, add up to m^3. Whereas $\psi_{abc,}$ and $\psi_{a,b,c}$ are uniquely defined as the completely symmetric and the completely skew parts, respectively, the other two parts have mixed symmetry and their definition is slightly ambiguous.[59] This is due to the fact that they are a pair of equivalent representations of SU_m. A possible choice is:

$$\psi_{ab,c}=\tfrac{1}{3}(\psi_{abc}-\psi_{acb}+\psi_{bac}-\psi_{bca}),$$
$$\psi_{ac,b}=\tfrac{1}{3}(\psi_{acb}-\psi_{bca}+\psi_{cab}-\psi_{bac}).$$

With this choice the four parts are orthogonal. This summarizes the complete reduction of ψ_{abc}.

We have seen how covariant tensors are reduced according to their symmetry, and how the mixed tensor $\psi_a{}^b$ reduces by separating the trace. For a general mixed tensor, judicious use of both operations gives the complete reduction into irreducible representations of SU_m. The theorem that is needed is that a mixed tensor is irreducible if and only if; (1) the symmetry of the lower indices is that of a single Young tableau, (2) the symmetry of the upper indices is that of a single Young tableau, and (3) contraction with respect to one upper and one lower index gives zero. The tensor $\psi_{bc}{}^a$ is easily reduced into the following four parts; the two m-dimensional representations $\psi_b=\psi_{ba}{}^a$ and $\psi_b'=\psi_{ab}{}^a$, the traceless symmetric part

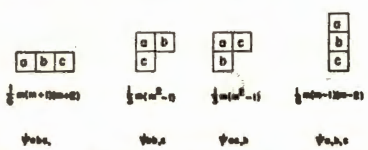


FIG. 20. The Young tableaux related to the reduction of the third-rank tensor in m dimensions.

[59] For tensors of higher rank, the ambiguity is much greater. T. Yamanouchi has prescribed a general procedure which always leads to orthogonal wave functions in Proc. Phys. Math. Soc. Japan, 18, 623 (1936); 19, 436 (1937).

having the $\frac{1}{2}m^2(m+1)-m$ components

$$\psi_{bc,}{}^a-\frac{1}{m+1}(\delta_b{}^a\psi_{dc,}{}^d+\delta_c{}^a\psi_{db,}{}^d), \tag{V.7}$$

and the traceless skew part

$$\psi_{b,c}{}^a-\frac{1}{m-1}(\delta_b{}^a\psi_{d,c}{}^d-\delta_c{}^a\psi_{d,b}{}^d) \tag{V.8}$$

which has $\frac{1}{2}m^2(m-1)-m^2$ components.

B. Group SU_3

We have seen how tensors of rank 2 or 3 reduce under SU_m. A significant simplification occurs in the case $m=3$, because the Levi-Civita tensors ϵ_{abc} and ϵ^{abc}, which equal $+1$ (-1) if abc is an even (odd) permutation of 123 and zero otherwise, have only three indices.

The relation between the above reduction of second-rank tensors and the labeling of representations introduced earlier is (more information in Table IV):

$$\begin{array}{ccc} \psi_a & \psi^a & \psi_{a,b} \\ D^{(3)}(1,0) & D^{(3)}(0,1) & D^{(3)}(0,1) \\ \text{(a)} & \text{(b)} & \text{(c)} \\ \psi^{a,b} & \psi_{ab,} & \psi^{ab,} \\ D^{(3)}(1,0) & D^{(6)}(2,0) & D^{(6)}(0,2) \\ \text{(d)} & \text{(e)} & \text{(f)} \end{array} \tag{V.9}$$

as we now prove. The first relation, identifying ψ_a as the basis for $D^{(3)}(1,0)$, is essentially a definition. Then (9b) follows from the fact that ψ^a is contragredient to ψ_a, and $D^{(3)}(0,1)$ is contragredient to $D^{(3)}(1,0)$. Next consider (9c), according to which $\psi_{a,b}$ is equivalent to ψ^a. This equivalence is exhibited and proved by the relation $\psi^a=\epsilon^{abc}\psi_{b,c}$, which expresses the three components of ψ^a in terms of the three linearly-independent components of $\psi_{b,c}$. In general, the operation of converting two lower indices on a tensor into one upper index by means of ϵ^{abc}, is nonsingular if and only if the tensor is skew in the two lower indices. This follows from the relation

$$\epsilon_{abc}\epsilon^{ade}=\delta_b{}^d\delta_c{}^e-\delta_b{}^e\delta_c{}^d. \tag{V.10}$$

Finally, relation (9e) follows from the fact that $\psi_{ab,}$ is (the highest dimensional) part of ψ_{ab}.

In terms of outer products of representations, (9) shows that[60]

$$\begin{gathered} D^{(3)}(1,0)\otimes D^{(3)}(1,0)=D^{(6)}(2,0)\oplus D^{(3)}(0,1), \\ \psi_a\otimes\psi_b\sim\psi_{ab,}\oplus\psi_{a,b}. \end{gathered} \tag{V.11}$$

A second relation follows from

$$\begin{gathered} [D^{(3)}(1,0)]^*=D^{(3)}(0,1), \\ \psi_a{}^*\sim\psi^a. \end{gathered} \tag{V.12}$$

[60] The symbol $\sim$ reads "transforms like."

TABLE IV. Representations of SU_3. All mixed tensors are supposed to be traceless, e.g., $\psi_{ac}^{a}=0$. The missing representation "64" is $D^{64}(3,3)$ with the basis ψ_{def}^{abc} and the isotopic content 0, ½, ½, 1, 1, 1, $\frac{3}{2}$, $\frac{3}{2}$, $\frac{3}{2}$, $\frac{3}{2}$, 2, 2, 2, $\frac{5}{2}$, $\frac{5}{2}$, 3. The dimension of $D(\lambda_1,\lambda_2)$ is $\frac{1}{2}(\lambda_1+1)\times(\lambda_2+1)(\lambda_1+\lambda_2+2)$. The regular representation is $D^8(1,1)$.

Complete designation	Abbr. design	Highest weight	Fig. no.	Isotopic content	Basic	$\otimes D^3(1,0)$	$\otimes D^6(2,0)$	$\otimes D^8(1,1)$	$\otimes D^{10}(3,0)$
$D^1(0,0)$	1	(0,0)		0	ψ	3	6	8	10
$D^3(1,0)$	3	$\frac{1}{6}(\sqrt{3},1)$	2(a)	0,½	ψ_a	6+3*	10+8	15+6*+3	15′+15
$D^3(0,1)$	3*	$\frac{1}{6}(\sqrt{3},-1)$	2(b)	0,½	ψ^a	8+1	15+3	15*+6+3*	24+6
$D^6(2,0)$	6	$\frac{1}{3}(\sqrt{3},1)$	2(c)	0,½,1	ψ_{ab}	10+8	15′+15+6*	24+15*+6+3*	24+21+15*
$D^6(0,2)$	6*	$\frac{1}{3}(\sqrt{3},-1)$	2(d)	0,½,1	ψ^{ab}	15*+3*	27+8+1	24*+15+6*+3	42+15+3
$D^8(1,1)$	8	$\frac{1}{3}(\sqrt{3},0)$	2(e)	0,½,½,1	ψ_a^b, χ_A	15+6*+3	24+15*+6+3*	27+10+10*+8+8+1	35+27+10+8
$D^{10}(3,0)$	10	$\frac{1}{2}(\sqrt{3},1)$	22	0,½,1,$\frac{3}{2}$	ψ_{abc}	15′+15	24+21+15*	35+27+10+8	35+28+27+10
$D^{10}(0,3)$	10*	$\frac{1}{2}(\sqrt{3},-1)$		0,½,1,$\frac{3}{2}$	ψ^{abc}	24*+6*	42*+15*+3*	35*+27+10*+8	64+27+8+1
$D^{15}(2,1)$	15	$\frac{1}{2}(\sqrt{3},+\frac{1}{3})$		0,½,½,1,1,$\frac{3}{2}$	ψ_{bc}^{a}				
$D^{15}(1,2)$	15*	$\frac{1}{2}(\sqrt{3},-\frac{1}{3})$							
$D^{15}(4,0)$	15′	$\frac{2}{3}(\sqrt{3},+1)$		0,½,1,$\frac{3}{2}$,2	ψ_{abcd}				
$D^{15}(0,4)$	15′*	$\frac{2}{3}(\sqrt{3},-1)$							
$D^{21}(5,0)$	21	$\frac{5}{6}(\sqrt{3},+1)$		0,½,1,$\frac{3}{2}$,2,$\frac{5}{2}$	ψ_{abcde}				
$D^{21}(0,5)$	21*	$\frac{5}{6}(\sqrt{3},-1)$							
$D^{24}(3,1)$	24	$\frac{2}{3}(\sqrt{3},+\frac{1}{2})$		0,½,½,1,1,$\frac{3}{2}$,$\frac{3}{2}$,2	ψ_{bcd}^{a}				
$D^{24}(1,3)$	24*	$\frac{2}{3}(\sqrt{3},-\frac{1}{2})$							
$D^{27}(2,2)$	27	$\frac{2}{3}(\sqrt{3},0)$		0,½,½,1,1,1,$\frac{3}{2}$,$\frac{3}{2}$,2	ψ_{cd}^{ab}				
$D^{28}(6,0)$	28	$(\sqrt{3},+1)$		0,½,1,$\frac{3}{2}$,2,$\frac{5}{2}$,3	ψ_{abcdef}				
$D^{28}(0,6)$	28*	$(\sqrt{3},-1)$							
$D^{35}(4,1)$	35	$\frac{5}{6}(\sqrt{3},+3)$		0,½,½,1,1,$\frac{3}{2}$,$\frac{3}{2}$,2,2,$\frac{5}{2}$	ψ_{bcde}^{a}				
$D^{35}(1,4)$	35*	$\frac{5}{6}(\sqrt{3},-3)$							
$D^{36}(7,0)$	36	$\frac{7}{6}(\sqrt{3},+1)$		0,½,1,$\frac{3}{2}$,2,$\frac{5}{2}$,3,$\frac{7}{2}$	$\psi_{abcdefg}$				
$D^{36}(0,7)$	36*	$\frac{7}{6}(\sqrt{3},-1)$							
$D^{42}(3,2)$	42	$\frac{5}{6}(\sqrt{3},+1)$		0,½,½,1,1,1,$\frac{3}{2}$,$\frac{3}{2}$,$\frac{3}{2}$,2,2,$\frac{5}{2}$	ψ_{cde}^{ab}				
$D^{42}(2,3)$	42*	$\frac{5}{6}(\sqrt{3},-1)$							
$D^{45}(8,0)$	45	$\frac{4}{3}(\sqrt{3},+1)$		0,½,1,$\frac{3}{2}$,2,$\frac{5}{2}$,3,$\frac{7}{2}$,4	$\psi_{abcdefgh}$				
$D^{45}(0,8)$	45*	$\frac{4}{3}(\sqrt{3},-1)$							
$D^{48}(5,1)$	48	$(\sqrt{3},+\frac{2}{3})$		0,½,½,1,1,$\frac{3}{2}$,$\frac{3}{2}$,2,2,$\frac{5}{2}$,$\frac{5}{2}$,3	ψ_{bcdef}^{a}				
$D^{48}(1,5)$	48*	$(\sqrt{3},-\frac{2}{3})$							

The reduction of ψ_a^b was discussed in detail. For $m=3$,

$$D^{(3)}(1,0)\otimes D^{(3)}(0,1)=D^{(8)}(1,1)\oplus D^{(1)}(0,0),\quad \psi_a\otimes\psi^b\sim(\psi_a^b-\tfrac{1}{3}\delta_a^b\psi_c^c)\oplus\delta_a^b\psi_c^c. \tag{V.13}$$

The analogs of (11) and (13) for third-rank tensors are

$$D^{(3)}(1,0)\otimes D^{(3)}(1,0)\otimes D^{(3)}(1,0) =D^{(10)}(3,0)\oplus D^{(8)}(1,1)\oplus D^{(8)}(1,1)\oplus D^{(1)}(0,0),\quad \psi_a\otimes\psi_b\otimes\psi_c\sim\psi_{abc},\oplus\psi_{ab,c}\oplus\psi_{ac,b}\oplus\psi_{a,b,c}, \tag{V.14}$$

and

$$D^{(3)}(1,0)\otimes D^{(3)}(1,0)\otimes D^{(3)}(0,1) =D^{(15)}(2,1)\oplus D^{(6)}(0,2)\oplus D^{(3)}(1,0)\oplus D^{(3)}(1,0)\quad \psi_a\otimes\psi_b\otimes\psi^c\sim\psi_{ab,}{}^c\oplus\psi_{a,b}{}^c\oplus\psi_{ac}{}^c\oplus\psi_{cb}{}^c. \tag{V.15}$$

The equivalence of $\psi_{ab,c}$ with $D^{(8)}(1,1)$ is exhibited by $\psi_a{}^d=\epsilon^{bcd}\psi_{ab,c}$ (obviously $\psi_a{}^d$ is traceless). In (15), by $\psi_{ab,}{}^c$ and $\psi_{a,b}{}^c$, we mean the traceless parts (7) and (8). The equivalence of the latter to $D^{(6)}(0,2)$ is displayed by $\epsilon^{bce}[\psi_{b,c}{}^a-\frac{1}{2}(\delta_b{}^a\psi_{d,c}{}^d-\delta_c{}^a\psi_{d,b}{}^d)]=\psi^{ae}$. We might also argue as follows. Since the traceless part of $\psi_{b,c}{}^a$ is irreducible, and raising of the lower indices by means of ϵ^{bce} is a similarity transformation, the result must be one of the irreducible parts of ψ^{ae}. Since the dimension is $\frac{1}{2}m^2(m-1)-m=6$ the irreducible part in question must be the six-dimensional symmetric part ψ^{ae}.

It is clearly possible to convert, in the manner just illustrated by several examples, any tensor of mixed symmetry into tensors of lower rank, symmetric in all upstairs indices and symmetric in all downstairs indices. For the latter, the reduction is completed by separating out the traceless part. Hence, a complete set of irreducible representations is given by the set of traceless symmetrized tensors $\psi_{ab\cdots,}{}^{cd\cdots}$. If λ_1 is the number of lower indices, and λ_2 is the number of upper indices, the irreducible representations may be labeled $D(\lambda_1,\lambda_2)$. Since this is the highest[61] representation contained in the product of λ_1 factors of $D(1,0)$ and λ_2 factors of $D(0,1)$, the present labeling agrees exactly with that of Sec. III.

Alternatively, all indices may be lowered, converting each upper index into two lower ones. Starting with a symmetrized traceless mixed tensor with λ_1 lower and λ_2 upper indices, this process must give an irreducible representation, i.e., a tensor with the symmetry of a particular Young tableau. It is easily verified that the table in question has two rows, with $\lambda_1+\lambda_2$ boxes in the first row and λ_2 boxes in the second row. The reason why no tableaux with three rows are obtained is that adding a column with three rows means multiplying with the representation $\psi_{a,b,c}$, which is an invariant.

The dimension of $\psi_{ab\cdots,}$, symmetric in λ_1 indices, is $\frac{1}{2}(\lambda_1+1)(\lambda_1+2)$. Hence $\psi_{a\cdots,}{}^{b\cdots}$, symmetric in λ_1 lower and λ_2 upper indices, has $\frac{1}{4}(\lambda_1+1)(\lambda_1+2)(\lambda_2+1)\times(\lambda_2+2)$ components. The tensor obtained by contracting one upper and one lower index has $\frac{1}{4}\lambda_1(\lambda_1+1)\lambda_2\times(\lambda_2+1)$ components. Hence the traceless part has $\frac{1}{2}(\lambda_1+1)(\lambda_2+1)(\lambda_1+\lambda_2+2)$ components, and this is therefore the dimensionality of $D(\lambda_1,\lambda_2)$. The same result was obtained in Sec. IV by the geometric method,[54] which is more suited to that kind of calculation.

The reduction of the product of any two representations is easily calculated by the above methods. The results of Table IV have been obtained by this method as well as independently by the geometric method. In Table IV may also be found the "wave functions" for any representation of SU_3 with dimension less than 50. The projection operators, which effect the symmetrization and subtracts out the trace, are easily written down as in (3) and (4), and allows us to obtain the transformation matrices explicitly. One example may be sufficient to illustrate this. The transformation of the basis (3), obtained from (2) and (4), is given by the representation

$$L_A \to P_a{}^f{}_e{}^b(L_{Ab}{}^d\delta_a{}^c - L_{Aa}{}^c\delta_b{}^d). \qquad \text{(V.16)}$$

C. Group $C_2(B_2)$

This is the group of 4×4 matrices that leaves a nondegenerate skew form h^{ab} invariant.[62] This is evidently a subgroup of SU_4, and the reduction of product representations is merely a refinement of that carried out for SU_m, with $m=4$. The fact that the form-invariant h^{ab} exists, and may be used as a raising and lowering operator if we define h_{ab} by[63]

$$h_{ab}h^{bc}=\delta_a{}^c,$$

means that the two representations ψ_a and ψ^a are equivalent. The equivalence is exhibited and proved by noting that $h^{ab}\psi_b$ transforms like ψ^a. Both ψ_a and ψ^a are (different and equivalent) bases for the representation denoted $D^{(4)}(1,0)$ in a previous section. Clearly a tensor of arbitrary mixed rank can be converted into a tensor with all the indices downstairs. The reduction problem then consists of two steps: First reduce according to SU_4 (that is, split the tensor into its various possible symmetry classes, or Young tables), then separate out the "traces" formed with h^{ab}. Remembering that h^{ab} is skew, so that taking the trace on a pair of symmetrized indices gives zero, we easily find the results of Table V. [The method of the last section is even easier, and for higher representations, it is the only practical one.] As in the case of SU_3, the low dimensionality (4 in this case) allows a simplification. Thus the completely skew tensor $\psi_{a,b,c}$ is equivalent to $\psi^d=\epsilon^{abcd}\psi_{a,b,c}$, where ϵ^{abcd} is the Levi-Civita symbol.

Let $L_{Aa}{}^b$ be the infinitesimal generators of the fundamental representation $D^{(4)}(1,0)$ of C_2. The form invariance of h^{ab} means that

$$h^{ab}\to h^{ab}-i\epsilon^A(L_{Ac}{}^a h^{cb}+L_{Ac}{}^b h^{ac})=h^{ab}.$$

Writing $h^{ac}L_{Ac}{}^b\equiv L_A{}^{ab}$, we get

$$L_A{}^{ab}=L_A{}^{ba}.$$

Hence the infinitesimal generators, with the lower index raised, are symmetric. Hence the number of linearly independent $L_A{}^{ab}$ is 10 which is the order of C_2. In order to obtain a complete set of 16 independent matrices we introduce 5 linearly-independent skew matrices $\sigma_i{}^{ab}$, $i=1, 2, 3, 4, 5$ and choose them so that

$$\sigma_i{}^{ab}h_{ab}=0.$$

We are now able to understand the reduction of ψ_{ab} and the higher tensors in greater detail. We have already noted that $\psi_{a,b}$ contains the invariant $h^{ab}\psi_{a,b}$. The five-dimensional representation, which is the traceless part of the skew part, can now conveniently be written

$$\varphi_i\equiv\sigma_i{}^{ab}\psi_{ab}, \quad i=1,\cdots,5. \qquad \text{(V.17)}$$

The proof of this statement follows. The six skew components of $\psi_{a,b}$ form a basis for a representation,

[61] That is, the one with the highest weight.

[62] Any skew metric may be transformed into the form

$$h^{ab}=\begin{bmatrix} & & & 1\\ & & -1 & \\ & 1 & & \\ -1 & & & \end{bmatrix}.$$

This is the choice we have made in Eq. (III.36).

[63] If h^{ab} is as in reference 62, then

$$h_{ab}=\begin{bmatrix} & & & -1\\ & & 1 & \\ & -1 & & \\ 1 & & & \end{bmatrix}.$$

TABLE V. Representations of $C_2[B_2]$. The bases satisfy the "subsidiary conditions" $h^{ab}\psi_{a,b}=h^{ac}\psi_{ab,c}=0$, $\sigma^i{}_a{}^b\varphi_{bi}=\sigma^i{}_a{}^b\varphi_{ij,}{}^a=g^{ij}\varphi_{ij,}=g^{ij}\varphi_{ijk}=g^{ij}\varphi_{ij,k}=g^{ij}\varphi_{ij,}{}^a=0$.

Complete designation	Abbr. design.	Highest weight	Fig. no.	Isotopic content	Basis	$\otimes D^4(1,0)$	$\otimes D^5(0,1)$	$\otimes D^{10}(2,0)$	$\otimes D^{14}(0,2)$
$D^1(0,0)$	1	(0,0)	0		ψ	4	5	10	14
$D^4(1,0)$	4	$\frac{1}{2\sqrt{3}}(1,0)$	4(a)	0,0,½[½,½]	ψ_a	10+5+1	16+4	20+16+4	40+16
$D^5(0,1)$	5	$\frac{1}{2\sqrt{3}}(1,1)$	4(b)	0,½,½[0,0,1]	$\psi_{a,b},\varphi_i$	16+4	14+10+1	35′+10+5	35′+30+5
$D^{10}(2,0)$	10	$\frac{1}{2\sqrt{3}}(2,0)$	4(c)	0,0,0,½,½,1 [0,1,1,1]	$\psi_{ab},\varphi_{i,j},\chi_A$	20+16+4	35′+10+5	35+35′+14+10+5+1	
$D^{14}(0,2)$	14	$\frac{1}{2\sqrt{3}}(2,2)$		0,½,½,1,1,1 [0,0,0,1,1,2]	$\varphi_{ij,}$	40+16	35+30+5		
$D^{16}(1,1)$	16	$\frac{1}{2\sqrt{3}}(2,1)$		0,0,½,½,½,½,1,1 [½,½,½,½,½,½]	$\varphi_{ai},\psi_{ab,c}$	35+14+10+5	40+20+16+4		
$D^{20}(3,0)$	20	$\frac{1}{2\sqrt{3}}(3,0)$		0,0,0,0,½,½,½, 1,1,½[½,½,½, ½,½,½]	ψ_{abc}				
$D^{30}(0,3)$	30	$\frac{1}{2\sqrt{3}}(3,3)$		0,½,½,1,1,1,½,½, ½,½[0,0,0,0, 1,1,1,2,2,3]	φ_{ijk}				
$D^{35}(4,0)$	35	$\frac{1}{2\sqrt{3}}(4,4)$		0,0,0,0,0,½,½,½,½, 1,1,1,½,½,2[0,1, 1,1,2,2,2,2,2]	ψ_{abcd}				
$D^{35}(2,1)$	35′	$\frac{1}{2\sqrt{3}}(3,1)$		0,0,0,½,½,½,½, ½,½,1,1,1,1,½,½ [0,0,1,1,1,1,1, 1,2,2,2]	$\varphi_{ij,k}$				
$D^{40}(1,2)$	40	$\frac{1}{2\sqrt{3}}(3,2)$		0,0,½,½,½,½,1,1, 1,1,1,1,½,½,½ [½,½,½,½,½,½, ½,½,½,½,½,½]	$\varphi_{ij,}{}^a$				

that is, they transform among themselves. Therefore the six linearly-independent combinations φ_i, $i=1, 2, \cdots, 5$ and $h^{ab}\psi_{ab}$ transform among themselves. But $h^{ab}\psi_{ab}$ is invariant and orthogonal to φ_i. Therefore, the φ_i transform among themselves; that is, the φ_i form the basis for a five-dimensional representation. We do not prove here that this representation is irreducible, but it can easily be seen to be the representation $D^{(5)}(0,1)$ discussed in preceding sections. The way that the φ_i transform among themselves is given by

$$\varphi_i=\sigma_i{}^{ab}\psi_{ab}\rightarrow\sigma_i{}^{ab}(\delta_a{}^c+i\epsilon^A L_{Aa}{}^c)(\delta_b{}^d+i\epsilon^B L_{Bb}{}^d)\psi_{cd}$$
$$\equiv(\delta_i{}^j+i\epsilon^A L_{Ai}{}^j)\sigma_j{}^{ab}\psi_{ab}. \quad (V.18)$$

As is the usual treatment of the Pauli σ matrices, we interpret $\sigma_i{}^{ab}$ as a constant tensor. This nomenclature is justified by noting that the above definition of $L_{Ai}{}^j$ gives

$$\sigma_i{}^{ab}\rightarrow(\delta_c{}^a-i\epsilon^A L_{Ac}{}^a)(\delta_d{}^b-i\epsilon^B L_{Bd}{}^b)\times(\delta_i{}^j+i\epsilon^c L_{ci}{}^j)\sigma_j{}^{cd}=\sigma_i{}^{ab}.$$

That is, $\sigma_i{}^{ab}$ is form invariant.

This representation $D^{(5)}(0,1)$ may appropriately be called the *vector representation*. The form

$$g_{ij}=\sigma_i{}^{ab}\sigma_{jba} \quad (V.19)$$

is clearly symmetric, nonsingular and constant (form invariant). It may be used to raise and lower vector indices. For example, we have from (18):

$$L_{Ai}{}^j=\sigma_i{}^{ab}(L_{Aa}{}^c\delta_b{}^d+\delta_a{}^c L_{Ab}{}^d)\sigma^j{}_{cd}.$$

Clearly $L_A{}^{ij}$ are the 10 skew 5×5 matrices, and their skewness is equivalent to the form invariance of g_{ij}. Hence, this representation of C_2 is B_2, the orthogonal group in five dimensions. (The isomorphism between C_2 and B_2 was pointed out by Cartan.)

To complete this discussion of the reduction of ψ_{ab}, we note that the ten-dimensional representation $D(2,0)$, which is the symmetric part of ψ_{ab}, is just the regular representation:

$$\chi_A=L_A{}^{ab}\psi_{ab}.$$

The $\sigma_i{}^{ab}$ play the same role here as in ordinary spinor analysis, providing the link between the "spinor" indices a, b, $\cdots$, and the "vector indices," i, j, $\cdots$. For tensors of higher rank, it may be convenient to employ a mixed notation. Thus, the basis ψ_{ab}, is equivalent to $\varphi_{i,j}$ and the basis for $D(1,1)$ is either

$$\psi_{ab,c} \quad \text{with} \quad h^{bc}\psi_{ab,c}=0,$$

or

$$\varphi_{i,j}{}^{a} \quad \text{with} \quad \sigma^{i}{}_{ab}\varphi_{i,j}{}^{a}=0.$$

D. Group G_2

Because G_2 is a subgroup[64] of O_7 it is helpful first to discuss the latter group. The spinor-representation of O_7 is eight-dimensional. Let g^{ab} be a symmetric nonsingular matrix and g_{ab} its inverse:

$$g^{ab}g_{bc}=\delta_c{}^a.$$

We use this tensor to raise or lower indices in spinor space (i.e., latin indices a, b, c). Let $\gamma_{ia}{}^{b}$ be a set of seven 8×8 matrices that satisfy

$$\begin{aligned} g^{ac}\gamma_{ic}{}^{b}=\gamma_i{}^{ab}&=-\gamma_i{}^{ba}, \\ (\gamma_i\gamma_j+\gamma_j\gamma_i)_a{}^b&=-2g_{ij}\delta_a{}^b. \end{aligned} \tag{V.20}$$

The numbers g_{ij} are defined by these equations, once a fixed g^{ab} and a fixed set of γ matrices have been chosen. The latter should be taken to be linearly independent; then g_{ij} is a nonsingular quadratic form with an inverse defined by

$$g^{ij}g_{jk}=\delta_k{}^i.$$

These matrices are used to raise and lower latin indices i, j, k, henceforth called vector indices. Hence g^{ab} and g^{ij} are the metric tensors in spinor space and in vector space, respectively.

The γ matrices may be used to construct a complete set of matrices (64 independent ones) in spinor space. The 21 independent matrices

$$G_{ija}{}^{b}\equiv\tfrac{1}{4}[\gamma_i,\gamma_j]_a{}^b$$

are of particular interest. First we note that $G_{ij}{}^{ab}$ are skew and independent of $\gamma_i{}^{ab}$:

$$G_{ij}{}^{ab}=-G_{ij}{}^{ba}, \quad G_{ij}{}^{ab}\gamma_{kab}=0.$$

Therefore, the 21 $G_{ij}{}^{ab}$ and the 7 $\gamma_i{}^{ab}$ form a complete set of 28 linearly independent skew matrices. Next, defining the 35 matrices

$$G_{ijka}{}^{b}\equiv\tfrac{1}{6}[\gamma_i\gamma_j\gamma_k]_a{}^b,$$

where $[\gamma_i\gamma_j\gamma_k]$ is the antisymmetrized product of γ_i, γ_j, γ_k, we note that $G_{ijk}{}^{ab}$ are symmetric and independent of g^{ab}:

$$G_{ijk}{}^{ab}=G_{ijk}{}^{ba}, \quad G_{ijk}{}^{ab}g_{ab}=0.$$

Hence, the $G_{ijk}{}^{ab}$ and g^{ab} form a complete set of 36 independent symmetric matrices.

As a simple consequence of the "anticommutativity" of the γ matrices, we find

$$G_{ij},G_{kl}=g_{il}G_{kj}-g_{ik}G_{lj}+g_{lj}G_{ik}-g_{kj}G_{il},$$

which are the correct commutation relations for the group of rotations in seven dimensions.[65] Therefore the $(G_{ij})_a{}^b$ are the infinitesimal generators of that group.

The group G_2 may now be obtained as a subgroup of O_7 in the following way.[66] Let η^a be a constant spinor. Of course O_7 does not admit such an object, and η^a is not constant under O_7. However, there exists a subgroup of O_7 that does leave η^a invariant, and it turns out that this subgroup is G_2. Hence η^a is constant with respect to G_2 only. The subspace of spinor space which is normal to η^a is seven dimensional, and there exists a very convenient way of labeling the seven components of ψ_a which span this subspace. For, let $\eta_i{}^a$ be defined by

$$\eta_i{}^a\equiv-\gamma_{ib}{}^a\eta^b, \quad \eta_{ia}=\gamma_{iab}\eta^b=\eta_i{}^bg_{ba}.$$

Then clearly $\eta_i{}^a\eta_a=0$. Hence the seven components $\eta_i{}^a\psi_a$ of ψ_a are the basis for a representation of that subgroup of O_7 that leaves η_a invariant.

In order to find the matrices of this group, let us define

$$\Gamma_{ijk}=\gamma_{iab}\eta_j{}^a\eta_k{}^b.$$

It is not at first obvious how this can be solved for γ_{iab}, since the $\eta_i{}^a$ are singular. It is clear, however, that γ_{iab} is of the form

$$\gamma_{iab}=A\Gamma_{ijk}\eta_a{}^j\eta_b{}^k+B(\eta_{ia}\eta_b-\eta_{ib}\eta_a).$$

From the commutation relations we find, with the normalization

$$\eta^a\eta_a=1, \quad \text{that} \quad \eta_i{}^a\eta_{ja}=g_{ij},$$

and this immediately gives $A=B=1$, or

$$\gamma_{iab}=\Gamma_{ijk}\eta^{aj}\eta^{bk}+\eta_{ia}\eta_b-\eta_a\eta_{ib}.$$

Using this formula in the anticommutation relations, we find that the necessary and sufficient conditions for the Γ_{ijk} to yield $\gamma_{ia}{}^b$ with the defining properties (a) that the Γ_{ijk} be totally skew, and (b) that

$$\Gamma_i{}^{ln}\Gamma_{jn}{}^k+\Gamma_j{}^{ln}\Gamma_{in}{}^k=\delta_i{}^l\delta_j{}^k+\delta_j{}^l\delta_i{}^k-2g_{ij}g^{lk}. \tag{V.21}$$

Some simple consequences are

$$\Gamma_{il}{}^l=0, \quad \Gamma_{ikl}\Gamma^{jkl}=6\delta_i{}^j.$$

Although not obvious, it is nevertheless true that the above properties suffice to reduce the product of any three Γ matrices to a sum of terms that are linear in

[64] The first physical application of this fact appears to have been made by G. Racah, Phys. Rev. 76, 1352 (1949).

[65] The differential operators $(l/i)(x_i\partial_j-x_j\partial_i)$ are a realization of these commutation relations.

[66] The following development was suggested by the calculations of reference 19.

Γ's. The formula is

$$\begin{aligned} \Gamma_{mn}{}^s\Gamma_s{}^{il}\Gamma_l{}^{jk}=&-\delta_m{}^i\Gamma_n{}^{jk}-\delta_m{}^j\Gamma_n{}^{ki}-\delta_m{}^k\Gamma_n{}^{ij}-g^{ik}\Gamma_{mn}{}^j \\ &+\delta_n{}^i\Gamma_m{}^{jk}+\delta_n{}^j\Gamma_m{}^{ki}+\delta_n{}^k\Gamma_m{}^{ij}+g^{ij}\Gamma_{mn}{}^k. \end{aligned}$$

The generators of G_2 are those linear combinations

$$S^{ij}G_{ija}{}^b,$$

that satisfy

$$S^{ij}G_{ija}{}^b\eta_b=0.$$

This is easily reduced to

$$S^{ij}\Gamma_{ijk}=0.$$

The general solution (taking $S^{ij}=-S^{ji}$) is a linear combination of the following matrices

$$P(14)_{(mn)}{}^{ij}\equiv\tfrac{1}{2}(\delta_m{}^i\delta_n{}^j-\delta_m{}^j\delta_n{}^i)-\tfrac{1}{6}\Gamma_{kmn}\Gamma^{kij},$$

of which 14 are linearly independent. Hence G_2 has 14 parameters, and the generators are

$$L_{(mn)a}{}^b\equiv P(14)_{(mn)}{}^{ij}G_{ija}{}^b.$$

In vector space this becomes

$$L_{(mn)k}{}^l\equiv L_{(mn)a}{}^b\eta_b{}^l\eta_k{}^a.$$

Hence, the generators of G_2 is the set of skew matrices orthogonal to Γ_{ijk}:

$$L_{(mn)}{}^{ij}\Gamma_{ijk}=0.$$

The reduction problem for G_2 can now be solved. Let $D^{(7)}(1,0)$ stand for the representation ψ_i. The second-rank tensor is first split into the symmetric and the skew part. The symmetric part $\psi_{ij,}$ contains the invariant $g^{ij}\psi_{ij}$ and the remaining 27 components form an irreducible representation that we label $D^{(27)}(2,0)$. The 21 skew components $\psi_{i,j}$ break up into the $D(1,0)$ $\Gamma^{ijk}\psi_{ij}\equiv\varphi^k$ and a remainder with 14 components. The latter make up the regular representation $L_{(mn)}{}^{ij}\psi_{ij}$, which we call $D^{(14)}(0,1)$. (These labels agree with those of the previous sections.)

The reduction of the third-rank tensor ψ_{ijk} is nontrivial. First write down all the operators that exist for reducing the number of indices, that is, all the form-invariant matrices with 3 to 5 indices:

$$\begin{aligned} A&=\Gamma^{ijk}, \quad B_m=g^{ij}\delta_m{}^k, \\ A_m&=\tfrac{1}{3}(\Gamma^{ijl}\Gamma_{lm}{}^k+\Gamma^{jkl}\Gamma_{lm}{}^i+\Gamma^{kil}\Gamma_{lm}{}^j), \\ A_{mn,}&=\tfrac{1}{2}(\Gamma_m{}^{ij}\delta_n{}^k+\Gamma_n{}^{ij}\delta_m{}^k)-1/7g_{mn}\Gamma^{ijk}, \\ A_{m,n}&=\tfrac{1}{2}(\Gamma_m{}^{ij}\delta_n{}^k-\Gamma_n{}^{ij}\delta_m{}^k)-\tfrac{1}{6}\Gamma_{mn}{}^l\Gamma_l{}^{0k}\Gamma_0{}^{ij}. \end{aligned}$$

Here A_m has been made completely skew in i, j, k, since any symmetric part would reduce to B_m by use of (21). We have subtracted the trace A from $A_{mn,}$ and the part $A_m\sim\Gamma_m{}^{nl}A_{nl}$ from $A_{m,n}$. In this way we are assured that $A_{mn,}{}^{ijk}\psi_{ijk}$ and $A_{m,n}{}^{ijk}\psi_{ijk}$ are irreducible. These operators are then applied to each of the symmetry classes of Fig. 20. We start with the skew part $\psi_{i,j,k}$ which has 35 components. Applying B_m gives zero trivially; $A_{m,n}$ also yields zero after some calculation. Thus we are left with

$$\begin{aligned} \psi_{i,j,k}&=A_{mn,}{}^{ijk}\psi_{i,j,k}\oplus A_m{}^{ijk}\psi_{ijk}\oplus A^{ijk}\psi_{ijk}, \\ &\therefore \quad D^{(27)}(2,0)\oplus D^{(7)}(1,0)\oplus D^{(1)}(0,0). \end{aligned} \tag{V.22}$$

To each of the two parts $\psi_{ij,k}$ and $\psi_{ik,j}$ with mixed symmetry (having 112 components each), B_m, $A_{mn,}$ and $A_{m,n}$ gives $D^{(7)}$, $D^{(27)}$, and $D^{(14)}$, respectively. The remaining components, of which there are $112-7-27-17=64$, are irreducible. Since this is the highest representation[61] in $D(1,0)\otimes D(0,1)$ it must be $D(1,1)$. Thus

$$\begin{aligned} \psi_{ij,k}=B_m{}^{ijk}\psi_{ij,k}&\oplus A_{mn,}{}^{iki}\psi_{ij,k} \\ &\oplus A_{m,n}{}^{iki}\psi_{ij,k}\oplus\text{"remainder"}, \\ \therefore \quad D^{(7)}(1,0)\oplus D^{(27)}&(2,0)\oplus D^{(14)}(0,1) \\ &\oplus D^{(64)}(1,1). \end{aligned} \tag{V.23}$$

The "remainder" is the tensor $\psi_{ij,k}$, that satisfies the "subsidiary conditions"

$$B_m{}^{ijk}\psi_{ij,k}=A_{mn,}{}^{iki}\psi_{ij,k}=A_{m,n}{}^{iki}\psi_{ij,k}=0.$$

The result for $\psi_{ik,j}$ is, of course, exactly similar. The completely symmetric part $\psi_{ijk,}$, can be contracted with B_m only. Therefore, the remaining 77 components are irreducible. Since this is the highest representation in $[D(1,0)]^3$, it must be $D^{(77)}(3,0)$. Thus

$$\begin{aligned} \psi_{ijk,}&=B_m{}^{ijk}\psi_{ijk,}\oplus\text{"remainder"} \\ &\therefore \quad D^{(7)}(1,0)\oplus D^{(77)}(3,0). \end{aligned}$$

The complete results for ψ_{ijk} are listed in Table VI.

It may be helpful, to support our claim that our method supplies explicit matrices of the transformations for each representation, to write them down for some of the representations that are listed in Table VI.

We found the transformation matrices $L_{(mn)i}{}^j$ for the representation $D^{(7)}(1,0)$. These are given explicitly in terms of the Γ^{ijk}. For the present purpose the tensor character of the label (mn) on $L_{(mn)i}{}^j$ is irrelevant, and it is perhaps less confusing to replace it by a single index A running from 1 to 14. Then the transformations of the tensors before symmetrization are

$$\psi_{ij}\rightarrow\psi_{ij}+\epsilon^A(L_{Ai}{}^k\delta_j{}^l+L_{Aj}{}^l\delta_i{}^k)\psi_{kl},$$

$$\psi_{ijk}\rightarrow\psi_{ijk}+\epsilon^A(L_{Ai}{}^l\delta_j{}^m\delta_k{}^n+L_{Aj}{}^m\delta_i{}^l\delta_k{}^n+L_{Ak}{}^n\delta_i{}^l\delta_j{}^m)\psi_{lmn}.$$

The representation $D^{(27)}(2,0)$ is obtained from ψ_{ij} by the projection operator that symmetrizes and makes traceless, namely,

$$P(27)_{ij}{}^{kl}=\tfrac{1}{2}(\delta_i{}^k\delta_j{}^l+\delta_i{}^l\delta_j{}^k)-(1/7)g_{ij}g^{kl}. \tag{V.24}$$

Hence the matrices for $D^{(27)}(2,0)$ are simply,

$$L_A\rightarrow P(27)_{ij}{}^{kl}(L_{Ak}{}^m\delta_l{}^n+L_{Al}{}^n\delta_k{}^m).$$

We already found the projection operator $P(14)_{ij}{}^{kl}$ similarly:

$$\begin{aligned} P(1)_{ij}{}^{kl}&=(1/7)g_{ij}g^{kl}, \\ P(7)_{ij}{}^{kl}&=\tfrac{1}{6}\Gamma_{ijm}\Gamma^{klm}. \end{aligned} \tag{V.25}$$

TABLE VI. Representations of G_2. The bases satisfy $\Gamma^{ijk}\psi_{i,j}=g^{ij}\psi_{ij,}=g^{ij}\psi_{ij,k}=\Gamma^{ikl}\psi_{ij,k}=g^{ij}\psi_{ijk,}=g^{ij}\psi_{ij,k,l}=\Gamma^{ikm}\psi_{ij,kl}=0$.

Complete designation	Abbr. design.	Highest weight	Fig. no.	Isotopic content	Basis	$\otimes D^7(1,0)$	$\otimes D^{14}(0,1)$	$\otimes D^{27}(2,0)$
$D^7(1,0)$	7		3(a)	$\frac{1}{2},\frac{1}{2},1$	ψ_i	27+14+7+1	64+27+7	77+64+27+14+7
$D^{14}(0,1)$	14		3(b)	$0,0,0,1,\frac{3}{2}$	$\psi_{i,j},\chi_A$	64+27+7	77+77′+27+14+1	
$D^{27}(2,0)$	27			$0,\frac{1}{2},\frac{1}{2},1,1,1,\frac{3}{2},\frac{3}{2},2$	$\psi_{ij,}$	77+64+27+14+7		
$D^{64}(1,1)$	64			$\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{3}{2},\frac{5}{2},$ 2,2,2,2,$\frac{3}{2},\frac{5}{2}$	$\psi_{ij,k}$			
$D^{77}(3,0)$	77			$0,0,0,\frac{1}{2},\frac{1}{2},1,1,1,1,\frac{3}{2},$ $\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},2,2,2,\frac{5}{2},\frac{5}{2}$	$\psi_{ijk,}$			
$D^{77}(0,2)$	77′			$0,0,0,0,0,0,1,1,1,\frac{3}{2},\frac{3}{2},$ $\frac{3}{2},\frac{3}{2},\frac{3}{2},\frac{3}{2},2,\frac{5}{2},\frac{5}{2},3,3,3$	$\psi_{ij,k,l}$			

It is easily verified that these operators are indeed projection operators, and that they add up to $\delta_i^k\delta_j^l$. The reader is now able to write down the matrices for any one of the representations in Table VI.

VI. APPLICATIONS

A. Introductory Remarks

We have presented the tools that are needed to construct a physical theory of the strongly interacting particles. The important question now at hand is to select the group which is appropriate, make the proper identification between the basis of the representations and the physical states (particles, resonances, etc.) and determine the experimental predictions that ensue.

The myriad of schemes that may be constructed is limited only by the imagination of the inventor. Consequently, we have been unable to find a course between the Scylla of being too abstract and the Charybdis of leaving out many logical possibilities. It would seem then, that our purpose is best served by giving illustrative *examples*, from which the general pattern of procedure may be gleaned. It cannot be emphasized too strongly, however, that ***all of the remainder of this section is offered for illustration only.***

We identify the components of the basis of an irreducible representation of a group with a set of physical states (particles, resonances, scattering states, etc.). Since we have only considered groups of "charge"-space transformations, which commute with every space-time transformation, each of the physical states within one irreducible multiplet must have the same space-time properties, i.e., spin, relative parity, baryon number, etc. In particular, the square of the total four momentum, or the mass, of each of these states must be the same. But a cursory examination of the mass spectrum of the known baryons and mesons tends to preclude the possibility that these particles could form, in any meaningful way, the components of the basis of an irreducible representation of any group larger than the four-parameter semi-simple group of isotopic spin and hypercharge conservation. Namely, the only apparent approximate multiplet structure seems to be that associated with isotopic spin. This is the basis for one possible point of view.

Another point of view can be based on an analogy, which, if fruitful, would allow us to consider meaningfully the baryons as members of a "supermultiplet." That the analogy may be misleading, in whole or in part, is understood; we cite it only as one possible flight of fancy.

Let us consider again the concept of isotopic spin. As commonly conceived, the particle interactions break into an isotopic invariant part and a much weaker symmetry-breaking part, most likely due to the electromagnetic field. In the absence of the latter, the neutron and proton are identified as the degenerate members of an isotopic spin doublet, a spin 1/2 basis for an irreducible representation of the isotopic spin rotation group. Being members of such a multiplet, all their space-time properties are the same, including their masses. The main effect of the symmetry-breaking interaction, in this case, is to remove the degeneracy in the masses and interactions of the proton and neutron. This is so, because electromagnetic interactions conserves parity and baryon number, and is Lorentz invariant, and hence does not change the other space-time properties of the states, such as spin, relative parity, and baryon number. That these statements are independent of the symmetry-breaking coupling strength is obvious. It is conceivable that the interaction could change the number of states by giving rise to some new resonant states. In the isotopic spin case, it would seem that these new states either do not exist or are far removed in mass from the perturbed doublet. Thus, it has been found, since the proton and neutron are related states even in the presence of the symmetry-breaking interaction, that to some extent it is still meaningful to consider them as members of a doublet.

One may now carry the analogy over to the case of some higher symmetry. That is, one might speculate that the particle interactions split into a symmetry-preserving and a symmetry-destroying part, the latter involving some "fields" not contained in the former.

One such possibility is the global symmetry scheme[67] where the bosons involved in the symmetry-preserving part are only pions, while in the violating part, they are only K mesons.[68] Alternatively, the symmetric interaction could involve both K and π mesons, while the symmetry-breaking interaction could be due to some other, as yet unknown, field. It is this latter interaction, then, which would be responsible for the observed mass splittings and other mischief. If the analogy is not misleading, we may, even in the presence of this interaction, still speak in a meaningful way of the "supermultiplets," the components of the basis of an irreducible representation. Namely, we are able to say that all the components must have the same spin, relative parity, and baryon number. In addition, as before, we assume that if any new states arise as a result of the symmetry-breaking interaction, they are separated by a large mass from the presently known resonances.[69] This guarantees that the number of states will be preserved in the presence of the interaction. Thus, by completely describing one component of a multiplet, we have completely specified all the other components, except as to their masses and widths. If, in addition, we adopt the viewpoint of Lee and Yang[17] with regard to relating the widths of the various components, we are left with the masses as the main quantum number perturbed by the symmetry-breaking interaction.

According to the methods developed in the preceding sections, the components of the basis of an irreducible representation are identified by the weights $\mathbf{m}$ to which they belong.

For representations of simple group of rank two, it turns out that at most two linear combinations of H_1 and H_2 can be interpreted as I_3. This is because the spectrum of I_3 must be symmetric about $I_3=0$. For both SU_3 and G_2 only one of the two possibilities, namely $I_3\propto H_1$, are considered; but for $C_2(B_2)$ either choice gives rise to reasonable physical models (see below). In Tables IV–VI the *isotopic content* of many representations are recorded. This is the number of isotopic spin singlets, doublets, triplets, etc., contained in a representation. It may most easily be read off the weight diagram. The number of times that the total isospin I' is contained is equal to the number of states with $I_3=I'$ minus the number of states with $I_3=I'+1$.

B. Analysis of Invariant Amplitudes

In most cases, an attempt at a physical theory will begin with associating a particular representation D_B of some group G with a set of particles called the "fundamental baryons." Which baryons are fundamental depends on the model; it is not even necessary that the "fundamental baryons" be stable baryons. To fix the ideas, however, we assume that this is the case, and refer unambigously to "the baryons." Let ψ_a be the wave function for these baryons, that is, the basis for D_B, and let $\bar{\psi}^a$ be the wave function for the "antibaryons." Clearly $\bar{\psi}^a$ is contragredient to ψ_a and is the basis for $D_{B^*}=(D_B)^*$. The first experiment that may be discussed, even before introducing the bosons, is baryon-baryon or baryon-antibaryon scattering. The relevant four point function is of the form (suppressing the space-time variables)

$$\mathcal{A}=A_{ac}{}^{bd}\langle T(\psi^{\dagger a}\psi_b\psi^{\dagger c}\psi_d)\rangle, \tag{VI.1}$$

where the coefficients $A_{ac}{}^{bd}$ must be chosen so as to make $\mathcal{A}$ invariant.

Consider first the four-point function for a specified set of baryons, one of the terms in the sum $\mathcal{A}$:

$$\langle T(\psi^{\dagger a}\psi_b\psi^{\dagger c}\psi_d)\rangle.$$

A knowledge of the effect of the E_α on a basis, ψ_a, of an irreducible representation can be used to find relations among the four-point functions for different processes. First of all, we insert the operator form of the commutation relations $[\hat{E}_\alpha,\hat{E}_{-\alpha}]=r^i(\alpha)\hat{H}_i$ into the four-point function above to obtain

$$\langle T(\psi^{\dagger a}\psi_b[E_\alpha,E_{-\alpha}]\psi^{\dagger c}\psi_d)\rangle=r^i(\alpha)\langle T(\psi^{\dagger a}\psi_b\hat{H}_i\psi^{\dagger c}\psi_d)\rangle.$$

The $\hat{E}_\alpha$ and the $\hat{H}_i$ are linear operators acting on the product basis as in (III.14). By remembering that when they act on the vacuum they give zero and that $E_\alpha{}^\dagger=E_{-\alpha}$, we find

$$\begin{aligned}\langle T\{&[(E_{-\alpha}\psi)^{\dagger a}\psi_b-\psi^{\dagger a}(E_\alpha\psi)_b]\\&\times[-(E_\alpha\psi)^{\dagger c}\psi_d+\psi^{\dagger c}(E_{-\alpha}\psi)_d]\}\rangle\\&-\langle T\{[(E_\alpha\psi)^{\dagger a}\psi_b-\psi^{\dagger a}(E_{-\alpha}\psi)_b]\\&\times[-(E_{-\alpha}\psi)^{\dagger c}\psi_d+\psi^{\dagger c}(E_\alpha\psi)_d]\}\rangle\\&=r^i(\alpha)[-m_i(c)+m_i(d)]\langle T(\psi^{\dagger a}\psi_b\psi^{\dagger c}\psi_d)\rangle,\end{aligned} \tag{VI.2}$$

where $m_i(c)$ and $m_i(d)$ are the weights of ψ^c and ψ_d, respectively. Thus, by knowning the effect of the E_α on the basis of the baryon representation, we can determine an equality between the four-point functions of two different processes.

As an obvious example of the relationship just found, consider the scattering $p+\bar{n}\rightarrow p+\bar{n}$ and let the $E_{-\alpha}$ be the isotopic spin lowering operator. The above relation then becomes

$$\begin{aligned}\langle T(&\psi_p{}^\dagger\psi_n\psi_n{}^\dagger\psi_p)\rangle\\&=\tfrac{1}{2}\langle T[(\psi_p{}^\dagger\psi_p-\psi_n{}^\dagger\psi_n)(\psi_p{}^\dagger\psi_p-\psi_n{}^\dagger\psi_n)]\rangle,\end{aligned} \tag{VI.3}$$

which states that the $I=1$, $I_3=1$ four-point function is equal to the $I=1$, $I_3=0$ function, i.e., that the four-point function depends only upon I and not upon I_3, a well-known result. The remaining equalities can obviously be obtained by repeating the procedure of inserting these commutation relations into the newly formed four-point functions; this procedure clearly

[67] See, M. Gell-Mann, reference 2.

[68] Or vice versa, see J. Schwinger, reference 2.

[69] This assumption, or a similar one, is a *sine qua non* of any theory of higher symmetries.

terminates when we reach the $I=1$, $I_3=1$ state in this example. In the general case, when the E_α are not just restricted to the isotopic spin operators, we proceed in the same manner. Namely, with repeated use of the general relation we can generate a string of equal four-point functions. From the example above, it is clear that this string will terminate after a finite number of steps since there can only be a finite number of independent relations. This procedure determines all the four-point functions which are equal to the one we started with. Similar statements, of course, can be made for the n-point function.

If we choose our original four-point function such that $\psi^{\dagger c}\psi_d$ is a component of the basis of an irreducible representation (as in the above example), then all the related four-point functions may be completely characterized by this irreducible representation, which in turn is characterized by its highest weight. They will be independent of the other weights (in much the same way as in the above example, they were characterized by I and independent of I_3). If $\psi^{\dagger c}\psi_d$ and $\psi^{\dagger c'}\psi_{d'}$ belong to two different irreducible representations, then the four-point functions in which they appear are, of course, unrelated (just as the $I=0$ amplitude is unrelated to the three $I=1$ amplitudes). We now show that it is possible to gain a much deeper insight into the structure and interrelations of four-point functions after we have found the most general matrix $A_{ac}{}^{bd}$ that makes (1) invariant.

To find all possible solutions of this problem is the same as determining the one-dimensional representations contained in $D_{B^*}\otimes D_{B^*}\otimes D_B\otimes D_B$. It is both convenient and traditional to do this in two steps. For example, for baryon-antibaryon scattering, one first decomposes $D_{B^*}\otimes D_B$:

$$D_{B^*}\otimes D_B=\sum\oplus\,\nu_\sigma D_\sigma, \qquad \text{(VI.4)}$$

where the sum is over inequivalent irreducible representations, and the ν_σ are integers. The invariants in (1) are then the invariants in

$$\sum\oplus\,\nu_\sigma^2(D_\sigma\otimes D_\sigma^*), \qquad \text{(VI.5)}$$

where each $D_\sigma\otimes D_\sigma^*$ contains exactly one invariant. Techniques for finding the ν_σ in (2) were amply discussed in Secs. IV and V, and many examples were listed in Tables IV–VI. Although the ν_σ contain some information that is quite important in applications to follow, we need a more explicit form of the reduction for the present purpose.

Suppose that a particular N_1-dimensional, irreducible representation D_1, whose basis we label by the letters $\mu, \nu, \rho, \cdots$, is contained in the product $D_{B^*}\otimes D_B$ or $\bar\psi^c\psi_d$. This means that there exists linear combinations

$$(\Omega_\mu{}^{(1)})_c{}^d\bar\psi^c\psi_d,\quad \mu=1,2,\cdots,N_1, \qquad \text{(VI.6)}$$

which transform among themselves according to D_1. The numbers $(\Omega_\mu{}^{(1)})_c{}^d$ may be regarded as the components of a constant (=form invariant) tensor, and will be called, after proper normalization, an *isometry*. Although the name may be new, the concept is well known, and several examples have already appeared in previous sections: (1) The Pauli $\sigma_{ia}{}^b$ matrices connect the product of two spinors to a vector $(\psi^\dagger\sigma_i\psi)$, (2) The matrices 1, γ_μ, $\frac{1}{4}\sqrt{2}(\gamma_\mu\gamma_\nu-\gamma_\nu\gamma_\mu)$, $\gamma_5\gamma_\mu$, γ_5 used for writing down Lorentz invariant couplings connect the product of two four-spinors to tensors, (3) The matrices of any representation D of a Lie algebra connects the product $D\otimes D^*$ to the regular representation (as was emphasized in Sec. V), and (4) Matrices [illegible] and Γ_{ijk} were introduced in Sec. V.

The normalization that qualifies these operators for the title isometry is

$$(\Omega_\mu{}^{(1)})_c{}^d(\Omega^{(1)\nu})_d{}^c=\delta_\mu{}^\nu,$$
$$(\Omega_\mu{}^{(1)})_c{}^d(\Omega^{(1)\mu})_e{}^f=P(1)_{ce}{}^{df}. \qquad \text{(VI.7)}$$

Here $(\Omega^{(1)\nu})_c{}^d\equiv g^{\nu\mu}(\Omega_\mu{}^{(1)})_c{}^d$ when $g^{\nu\mu}$ exists; in general it is the isometry of the representation D_1^* contragredient to D_1. [It may be proved that $D_1\otimes D_1^*$ contains D_1^* if it contains D_1.] The matrix $P(1)_{ce}{}^{df}$, in which c, d labels the rows and e, f labels the columns, is the projection operator associated with D_1. Several examples of (7) are well known:

(1). $\frac{1}{2}\sqrt{2}\epsilon_{abc}\frac{1}{2}\sqrt{2}\epsilon^{bcd}=\delta_a{}^d$,

$\frac{1}{2}\sqrt{2}\epsilon_{abc}\frac{1}{2}\sqrt{2}\epsilon^{ade}=\frac{1}{2}(\delta_b{}^d\delta_c{}^e-\delta_b{}^e\delta_c{}^d)=P_{bc}{}^{de}$

where $P_{bc}{}^{de}$ is the antisymmetrization operator;

(2). $\frac{1}{2}\sqrt{2}(\sigma_i)_a{}^b\frac{1}{2}\sqrt{2}(\sigma^j)_b{}^a=\delta_i{}^j$,

$\frac{1}{2}\sqrt{2}(\sigma_i)_a{}^b\frac{1}{2}\sqrt{2}(\sigma^i)_c{}^d=\delta_a{}^d\delta_c{}^b-\frac{1}{2}\delta_a{}^b\delta_c{}^d=P_a{}^b{}_c{}^d$

where the σ_i are the Pauli matrices and $P_a{}^b{}_c{}^d$ is projection operator that separates out the trace;

(3). In Sec. V, Eq. (V.19) we found that

$$(\tfrac{1}{6})^{\frac{1}{2}}\Gamma_{ijk}(\tfrac{1}{6})^{\frac{1}{2}}\Gamma^{ljk}=\delta_i{}^l,$$
$$(\tfrac{1}{6})^{\frac{1}{2}}\Gamma_{ijk}(\tfrac{1}{6})^{\frac{1}{2}}\Gamma^{ilm}=P(7)_{jk}{}^{lm}$$

where $P(7)_{jk}{}^{lm}$ is the projection operator (V.25) that projects out the 7-dimensional representation of G_2 from $D^{(7)}\otimes D^{(7)}$.

As in (4), let σ label the inequivalent irreducible representations, and write $\Omega^{(\sigma,\kappa)}$, $\kappa=1,\cdots,\nu_\sigma$ for the ν_σ isometries associated with each of the ν_σ equivalent representations D_σ. Then the equivalence between $(\Omega_\mu{}^{(\sigma,1)})_c{}^d$ and $(\Omega_\mu{}^{(\sigma,2)})_c{}^d$ means that there exists a nonsingular matrix $(P^{(\sigma,1,2)})_c{}^d{}_e{}^f$ such that

$$(P^{(\sigma,1,2)})_c{}^d{}_e{}^f(\Omega_\nu{}^{(\sigma,2)})_f{}^e=(\Omega_\mu{}^{(\sigma,1)})_c{}^d.$$

Using (7) we get

$$(P^{(\sigma,1,2)})_c{}^d{}_e{}^f=(\Omega_\mu{}^{(\sigma,1)})_c{}^d(\Omega^{(\sigma,2)\mu})_e{}^f. \qquad \text{(VI.8)}$$

From this we see that $P^{(\sigma,1,2)}$ is an isometry. In particular, if the indices are the same as in $P^{(\sigma,1,1)}$ we get back the projection operators. Thus we label the

projection operators associated with one of the D_σ by $P^{(\sigma,1,1)}$. Then the properties of the isometries, and in particular the projection operators, may be summarized by

$$(\Omega_\mu{}^{(\sigma,\kappa)})_c{}^d(\Omega^{(\sigma',\kappa')\nu})_d{}^c=\delta_{\sigma\sigma'}\delta_{\kappa\kappa'}\delta_\mu{}^\nu, \qquad \text{(VI.9)}$$

$$(\Omega_\mu{}^{(\sigma,\kappa)})_c{}^d(\Omega^{(\sigma,\kappa')\mu})_e{}^f=(P^{(\sigma,\kappa,\kappa')})_c{}^d{}_e{}^f, \qquad \text{(VI.10)}$$

$$(P^{(\sigma,\kappa,\kappa')})_c{}^d{}_e{}^f(P^{(\sigma',\kappa'',\kappa''')})_{fg}{}^{eh}$$
$$=\delta_{\sigma\sigma'}\delta_{\kappa'\kappa''}(P^{(\sigma,\kappa,\kappa''')})_c{}^d{}_g{}^h. \qquad \text{(VI.11)}$$

A direct result of Schur's lemma[70] is that the most general form of $A_{ac}{}^{bd}$ that makes (1) invariant is given by

$$A_{ac}{}^{bd}=\sum_{\sigma,\kappa,\kappa'}F^{\sigma,\kappa,\kappa'}(P^{(\sigma,\kappa,\kappa')})_a{}^b{}_c{}^d,$$

$$\mathfrak{A}=\sum_{\sigma,\kappa,\kappa'}F^{\sigma,\kappa,\kappa'}(\bar\psi^a\psi_b)(P^{(\sigma,\kappa,\kappa')})_a{}^b{}_c{}^d(\bar\psi^c\psi_d). \qquad \text{(VI.12)}$$

where $F^{\sigma,\kappa,\kappa'}$ are arbitrary and include all references to space-time coordinates or transformation properties. Using (10):

$$\mathfrak{A}=\sum_{\sigma,\kappa',\kappa,\mu}F^{\sigma,\kappa,\kappa'}(\bar\psi\Omega_\mu{}^{(\sigma,\kappa)}\psi)(\bar\psi\Omega^{(\sigma,\kappa')\mu}\psi). \qquad \text{(VI.13)}$$

This is the explicit realization of (5). The number of terms with the same σ is ν_σ^2.

The number of terms in (13) is $\sum\nu_\sigma^2$ and depends, of course, on the choice of the group G and the representation D_B. The procedure that we have outlined is a direct generalization of the well known treatment of isotopic spin. In that case, the index κ is superfluous, since the ν_σ in (4) are always zero or one. Thus, the summation over σ, κ, κ' reduces to a sum over I, the total isotopic spin. If all the ψ_a have isotopic spin 1/2, (13) reduces to

$$\mathfrak{A}=F^1(\bar\psi\tfrac{1}{2}\sqrt{2}\sigma_i\psi)(\bar\psi\tfrac{1}{2}\sqrt{2}\sigma^i\psi)$$
$$+F^0[\bar\psi\tfrac{1}{2}\sqrt{2}(\delta_i{}^j-\tfrac{1}{3}\sigma_i\sigma^j)\psi]$$
$$\times[\bar\psi\tfrac{1}{2}\sqrt{2}(\delta_j{}^i-\tfrac{1}{3}\sigma_j\sigma^i)\psi]. \qquad \text{(VI.14)}$$

The process of applying the generator $\hat{E}_\alpha$ and $\hat{H}_i$ to a basis $\bar\psi^c\psi_d$ of an irreducible representation in (12) or (13) clearly can lead to any other basis of the same irreducible representation, but cannot lead out of that representation. Thus the method that was outlined following (1) relates four-point functions within each term of the σ, κ', κ sum in (13). In fact, that method is simply a way of calculating the isometries. For example, the relation (3) expresses the fact that the right-hand side and the left-hand side occur with equal weight F^1 in (14).

C. Resonances and Mesons

Scattering in one or more states of σ, κ, κ' may exhibit resonances. The resonant states are then multiplets transforming according to D_σ. In order to determine the possible resonance multiplets and their transformation properties, it is sufficient to know the Clebsch-Gordan Series (4). For simple groups of rank two, and low-dimensional representations, this information is contained in Tables IV–VI.

Nothing in our development thus far distinguishes between stable and unstable resonant states. Therefore, it is impossible to make any definite predictions about the number of mesons in a given model. However, in the limit in which the invariance is exact, the various resonance states within one multiplet will have the same mass, width, etc. This might lead one to expect that if one member of a multiplet is stable, so are all the other members of that multiplet. If this is true, the number of mesons will be related to the dimensionalities of the representations occurring in the decomposition (2).[22]

If one likes to write an unrenormalized Lagrangian involving Yukawa couplings, it is necessary to find the trilinear invariants involving ψ_a, $\bar\psi^b$, and the meson field. If stable mesons are indeed possible intermediary states in $B-\bar{B}$ scattering, then these same trilinear forms are needed to write the vertex function. This remains true even if the mesons are regarded as bound states of the $B-\bar{B}$ system. From a mathematical point of view, these trilinear couplings are already known. All that is needed is to reinterpret the quantities $(\bar\psi^a\psi_b)$ appearing in (12) as the components of the meson field. For practical purposes, however, it is convenient to label the mesons by a single index, as for example φ^μ, such that each component corresponds to one physical meson. Let D_M be the representation for which φ^μ is the basis. In order for a trilinear invariant to exist, D_M must be equivalent to one of the terms in (4). That is, an isometry $(\Omega_\mu{}^{(M)})_a{}^b$ must exist such that φ^μ transforms contragrediently to $(\bar\psi\Omega_\mu{}^{(M)}\psi)$. Then the trilinear invariants are of the desired form, namely

$$(\bar\psi\Omega_\mu{}^{(M)}\psi)\varphi^\mu. \qquad \text{(VI.15)}$$

In the manner of Eq. (1), consider the three-point function for a specific set of two baryons and a meson, one component of the general invariant three-point function (15),

$$\langle T(\psi^{\dagger a}\psi_b\varphi^\mu)\rangle.$$

Again, insert the operator commutation relation to obtain

$$\langle T(\psi^{\dagger a}\psi_b[\hat{E}_\alpha,\hat{E}_{-\alpha}]\varphi^\mu)\rangle=r^i(\alpha)\langle T(\psi^{\dagger a}\psi_b\hat{H}_i\varphi^\mu)\rangle.$$

Proceeding as previously, we find

$$\langle T\{[(E_{-\alpha}\psi)^{\dagger a}\psi_b-\psi^{\dagger a}(E_\alpha\psi_b)][-(E_{-\alpha}\varphi^\mu)]\}\rangle$$
$$-\langle T\{[(E_\alpha\psi)^{\dagger a}\psi_b-\psi^{\dagger a}(E_{-\alpha}\psi_b)][-(E_\alpha\varphi^\mu)]\}\rangle$$
$$=-r^i(\alpha)m_i(\mu)\langle T(\psi^{\dagger a}\psi_b\varphi^\mu)\rangle.$$

A trivial example is afforded by the pion-nucleon

[70] I. Schur, Sitzber. preuss. Akad. Wiss., Physik.-math. Kl. 1905, p. 406.

vertex. Consider

$$\langle T(\psi_p{}^\dagger\psi_n\varphi_{\pi^+})\rangle$$

and $\hat{E}_\alpha$ as the isotopic spin raising operator. The well-known result follows

$$\langle T(\psi_p{}^\dagger\psi_n\varphi_{\pi^+})\rangle=\frac{1}{\sqrt{2}}\langle T[(\psi_p{}^\dagger\psi_p-\psi_n{}^\dagger\psi_n)\varphi_{\pi^0}]\rangle.$$

Since we have demonstrated the method both in the case of the three-point and four-point functions, it should be obvious that this method can be generalized to n-point functions involving both mesons and baryons.

Let us now proceed to the specific cases of SU_3, B_2, C_2, and G_2. In the examples contrived for SU_3 and G_2, we follow a line of reasoning according to which the eight known baryons are more fundamental physical states than are the baryon resonances, (or baryon excited states). Specifically, no resonance state or unobserved baryon is to appear in the same multiplet with any of the eight observed baryons. Such a distinction is quite unfounded, even though it seems to be the most fashionable procedure at present. We remove this restriction in our examples of theories built on B_2 and C_2.

D. Model Built on SU_3

If we assume that the eight baryons can form the bases for one or more representations, then the dimensionality of these representations must add up to eight. An inspection of Table IV for SU_3 shows that there is only one possibility with the correct isotopic content; the eight-dimensional representation $D^{(8)}(1,1)$. This implies that all the baryons must have the same space-time properties. If we assume that there are only the seven known mesons, it is impossible to assign the correct isotopic content under SU_3. In addition, if we require that the meson-baryon vertex function does not vanish, which incidentally corresponds to the existence of pole terms in dispersion relations, the dimensionality of the meson representations must be either 1, 8, 10, or 27. This follows from the fact that the Kronecker product of two eight-dimensional representations of baryons contains representations of only those dimensions (Table IV). One possible way out of the dilemma is to postulate the existence of an eighth meson which has not been experimentally detected as yet.[71] This is the approach of Gell-Mann,[18] which we follow here. It then follows that the meson representation is also eight-dimensional, and that all 8 mesons have the same space-time properties. [For example, Σ and Λ have the same parities, and the parity of $(K\Sigma)$ is the same as that of (πN).]

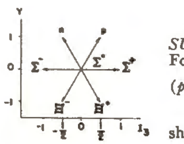


FIG. 21. Weight diagram for D^8 (1,1) of SU_3 with bases associated with baryons. For meson bases, the substitution, $(p,n,\Xi^0,\Xi^-,\Sigma^+,\Sigma^0,\Sigma^-,\Lambda)\to(K^+,K^0,-\bar{K}^0,\bar{K}^+,\pi^+,\pi^0,\pi^-,\pi^{00})$, should be made.

[71] See, for example, M. Gettner and W. Selove, Phys. Rev. 120, 593 (1960); J. Poirer and M. Pripstein, Phys. Rev. 122, 1917 (1961).

Because $D^{(8)}$ occurs twice in the product $D^{(8)}\otimes D^{(8)}$, there are two of the isometries in (15). To find them is to make a very slight extension of the tensor analysis developed for SU_3 in Sec. V. The baryon wave function is written ψ_A, in keeping with our convention to use capital Latin indices for the regular representation. The antibaryons are labeled $\bar{\psi}^B$. Clearly the structure constants $C_{BD}{}^A$ supply one of the two isometries. The normalization is fixed by the usual definition

$$g_{DE}=C_{BD}{}^AC_{AE}{}^B. \qquad \text{(VI.16)}$$

From the commutation relations (II.3) we find

$$C_{BD}{}^A=\text{trace}[L_BL_DL^A-L_DL_BL^A]. \qquad \text{(VI.17)}$$

We can define the second isometry by

$$C'_{BD}{}^A=\text{trace}[L_BL_DL^A+L_DL_BL^A]. \qquad \text{(VI.18)}$$

Although these relations are true regardless of which representation L_A occurs on the right, the most convenient choice is $D^{(3)}(1,0)$, given in (III.20). Both g_{DE} and $C_{BD}{}^A$ were calculated in Sec. III F.

The most general three-point function is

$$\langle F^1(\bar{\psi}^BC_{BD}{}^A\psi_A)\varphi^D+F^2(\bar{\psi}^BC'_{BD}{}^A\psi_A)\varphi^D\rangle, \qquad \text{(VI.19)}$$

where $\varphi_E=g_{DE}\varphi^D$ is the meson field.

In Fig. 21 we have furnished the weight diagram for $D^{(8)}(1,1)$ with the appropriate baryon symbols. We associate I_3 with $\sqrt{3}m_1$, and Y with $2m_2$, and summarize the relations between the four different labels that we have used:

$$\begin{aligned}
&|A\rangle: |1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle, |6\rangle, \\
&\qquad\qquad |7\rangle, |8\rangle; \\
&|\alpha\rangle: |-1\rangle, |-2\rangle, |-3\rangle, -|+1\rangle, \\
&|i\rangle: \qquad -|+2\rangle, |+3\rangle, -|1\rangle, -|2\rangle; \\
&\text{Baryons}: -|\Sigma^+\rangle, |p\rangle, |n\rangle, |\Sigma^-\rangle, |\Xi^-\rangle, |\Xi^0\rangle, \\
&\qquad\qquad |\Sigma^0\rangle, |\Lambda\rangle; \\
&\text{Mesons}: -|\pi^+\rangle, |K^+\rangle, |K^0\rangle, |\pi^-\rangle, |K^-\rangle, \\
&\qquad\qquad -|\bar{K}^0\rangle, |\pi^0\rangle, |\pi^{00}\rangle.
\end{aligned} \qquad \text{(VI.20)}$$

The action of the operators $\hat{H}_i$ and $\hat{E}_\alpha$ was given both in (III.26, 27) and in (III.30, 31). Using the dictionary (20), this is easily translated. The result for baryons is given in Table VII.

We are now in a position to make the predictions of the theory. Consider the scattering of a meson M and a baryon B, $M+B\to M'+B'$. The pertinent four-point function (suppressing the space-time variables) is

$$\langle T(\psi_{B'}{}^\dagger\psi_{M'}{}^\dagger\psi_B\psi_M)\rangle.$$

The combination $\psi_B\psi_M$ is the Kronecker product of two eight-dimensional representations, one for the meson and one for the baryon. This reduces, according to Table IV, as follows:

$$8\otimes 8=1\oplus 8\oplus 8\oplus 10\oplus 10^*\oplus 27.$$

There are eight $(=1^2+2^2+1^2+1^2+1^2)$ different four-point invariants, or equivalently 8 independent amplitudes.[72]

So far we have considered the representations for the known baryons and particles. It is conceivable that the other representations of this group might also be realized, not for stable particles, but perhaps for what we might call unstable particles, i.e., the resonances, excited isobaric states, or whatever. In particular, we concentrate on the well-known (3,3) resonance in pion-nucleon scattering and its possible analogs in other baryon-meson scattering processes. We have emphasized before the limitations of such a procedure (see the general discussion of this Section). We note again that the product representation of one baryon and one meson decomposes into irreducible representations of dimensions 1, 8, 8, 10, 10*, and 27. The weight (m_1,m_2) of the compound state π^+p which is a member of the (3,3) resonance, is $\frac{1}{2}(\sqrt{3},1)$. This is the highest weight of the 10-dimensional representation and one of the weights in the 27-dimensional representation. We assume that the (3,3) isobar states are members of the 10-dimensional multiplet.

The weight diagram for the 10-dimensional representation is shown in Fig. 22. Besides the $T=3/2$, $Y=1$, multiplet, which we identify as the (3,3) isobar states (N^*), we have a $T=1$, $Y=0$ triplet, a $T=1/2$, $Y=-1$ doublet, and a $T=0$, $Y=-2$ singlet. The triplet $T=1$, $Y=0$ has the same charge quantum numbers as the excited states Y^* of the $\Lambda\pi$ system.[73] It is very attractive to consider the Y^* as an analog of the N^*. In order for them to belong to the same supermultiplet, these two multiplets must have the same space-time quantum numbers. We therefore assume Y^* to have spin 3/2 and negative orbital parity.

In order to compare these two states and make certain predictions which can be verified by experiments, we must assume certain features of the symmetry-breaking forces. We may assume, after Lee and Yang,[17] that the symmetry-breaking forces are short-range in character and that long-range phenomena are relatively insensitive to them, even though they must be strong enough to account for the mass splittings. Then the same cause that splits the baryon masses is responsible for the difference of the energy levels of N^* and Y^*, while the resonance widths should be predictable from the symmetry. This is because the width of a resonance is proportional to the overlap of the resonance-state wave function and the initial- (or final-) state wave function at the "channel entrance," as we know from nuclear physics,[18,74] so that the relative widths are essentially independent of short range effects.

TABLE VII. Action of E_α on baryons for $D^{(8)}(1,1)$ in SU_3. Table for bosons is obtained by substitution $(p,n,\Xi^0,\Xi^-,\Sigma^+,\Sigma^0,\Sigma^-,\Lambda)\to(K^+,K^0,-\bar{K}^0,\bar{K}^+,\pi^+,\pi^0,\pi^-,\pi^{00})$.

$E_\alpha \backslash \psi$	p	n	Ξ^0	Ξ^-	Σ^+	Σ^0	Σ^-	Λ
$6^{\frac{1}{2}}E_1$		p		Ξ^0		$-\sqrt{2}\Sigma^+$	$\sqrt{2}\Sigma^0$	
$6^{\frac{1}{2}}E_{-1}$	n		Ξ^-		$-\sqrt{2}\Sigma^0$	$\sqrt{2}\Sigma^-$		
$2\sqrt{3}E_2$			$-\sqrt{2}\Sigma^+$	$\Sigma^0+\sqrt{3}\Lambda$		p	$\sqrt{2}n$	$\sqrt{3}p$
$2\sqrt{3}E_{-2}$	$\Sigma^0+\sqrt{3}\Lambda$	$\sqrt{2}\Sigma^-$			$-\sqrt{2}\Xi^0$	Ξ^-		$\sqrt{3}\Xi^-$
$2\sqrt{3}E_3$			$\Sigma^0-\sqrt{3}\Lambda$	$\sqrt{2}\Sigma^-$	$+\sqrt{2}p$	$-n$		$\sqrt{3}n$
$2\sqrt{3}E_{-3}$	$+\sqrt{2}\Sigma^+$	$-\Sigma^0+\sqrt{3}\Lambda$				Ξ^0	$\sqrt{2}\Xi^-$	$-\sqrt{3}\Xi^0$
$\sqrt{3}H_1$	$\frac{1}{2}p$	$-\frac{1}{2}n$	$\frac{1}{2}\Xi^0$	$-\frac{1}{2}\Xi^-$	Σ^+		$-\Sigma^-$	
$2H_2$	p	n	$-\Xi^0$	$-\Xi^-$				

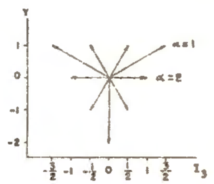


FIG. 22. Weight diagram for $D^{(10)}(3,0)$ for SU_3. The weight labeled as $\alpha=1$ corresponds to the isobar state $(N^*)^{++}$; $\alpha=2$ to $(Y^*)^+$.

[72] It is possible to distinguish between the two equivalent 8-dimensional representations by adding a discrete element (reflection) to the group. Invariance under this operation would prohibit transitions between the two octets and reduce the number of invariant amplitudes to six. See M. Gell-Mann, reference 18.

[73] M. Alston, L. Alvarez, P. Eberhard, M. Good, W. Graziano, H. Ticho, and S. Wojcichi, Phys. Rev. Letters 5, 520 (1960).

[74] R. G. Sachs, *Nuclear Theory* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1953), Chap. 10.

The state corresponding to the highest weight $(I_3,Y)=(\frac{3}{2},1)$ of the 10-dimensional representation is that linear combination

$$\alpha|p\pi^+\rangle+\beta|\Sigma^+K^+\rangle,$$

which is annihilated by E_1, E_2, E_3, and E_{-3}, as discussed in Sec. III E. Therefore, the normalized state $|\{10\},1\rangle$ can be chosen to be

$$|\{10\},1\rangle=\tfrac{1}{2}\sqrt{2}[|p\pi^+\rangle-|\Sigma^+K^+\rangle]. \qquad \text{(VI.21)}$$

The state of interest, consisting of $\Lambda\pi^+$, $\Sigma^+\pi^0$, $\cdots$, can be obtained by operating with E_{-2} on $|\{10\},1\rangle$, i.e.,

$$|\{10\},2\rangle=\tfrac{1}{6}\sqrt{3}[\sqrt{3}|\Lambda\pi^+\rangle+|\Sigma^0\pi^+\rangle+\sqrt{2}|p\bar{K}^0\rangle +\sqrt{2}|\Xi^0K^+\rangle-|\Sigma^+\pi^0\rangle-\sqrt{3}|\Sigma^+\pi^{00}\rangle]. \qquad \text{(VI.22)}$$

The partial width for the transition from a $|\{10\},\alpha\rangle$ multiplet to a $|BM\rangle$ state is given by

$$\Gamma_{BM}=\left(q^3\frac{E_B}{E_B+E_M}\right)|C|^2|\langle\{10\},\alpha|BM\rangle|^2, \qquad \text{(VI.23)}$$

where q= c.m. momentum, E_B= baryon energy in c.m., E_M= meson energy in c.m., and the first bracket on the right is the kinematical factor arising from the phase-space and the centrifugal barrier for the p-wave; and C is a quantity independent of the "magnetic quantum number" α. The generalized Clebsch-Gordan coefficient $\langle\{10\},\alpha|BM\rangle$ can be read off directly from the foregoing expressions. We list in Table VIII the relative partial widths predicted by the SU_3 symmetry. It is interesting to note that, if the mass of π^{00} is near that of the π, the decay process of the Y^* can produce π^{00} copiously, since the branching ratio of $Y^*\to\Lambda+\pi^+$ to $Y^*\to\Sigma^++\pi^{00}$ is approximately unity. This does not seem to agree with experimental findings, however.

E. Model Built on C_2

For this example we discard the assumption that different components of the same basis of an irreducible representation must be identified with the baryons only or with the resonances only. This allows us a good deal more flexibility in making an identification of the particles with a basis. For the purpose of illustration, we have chosen one of the many schemes which might be devised.

Upon inspection of the lower dimensional weight diagrams for C_2 in Fig. 4, we see that the N, Λ, and Ξ can be identified as the basis of the five-dimensional "vector" representation, $D^{(5)}(0,1)$, where $I_3=\sqrt{3}m_1$, and $Y=2\sqrt{3}m_2$. By making the association from Sec. III, $(1,2,3,4,5)\to(p,n,\Lambda,\Xi^0,\Xi^-)$, (compare Figs. 8 and 23), we can use Eqs. (III.37, 38) to construct Table IX. With this assignment, the Σ must be components of a basis for another irreducible representation and as such could have space-time quantum numbers which differ from those assigned to the $N-\Lambda-\Xi$ set. Specifically, this scheme would admit an odd relative $\Sigma\Lambda$ parity and an odd $K\Sigma$ parity relative to πN.[75] From the weight diagrams (Fig. 4), we see that the lowest dimensional representation in which the isotopic spin and hypercharge content allows both the π and K mesons is $D^{(10)}$ (Fig. 24). This is a representation which admits the existence of an invariant effective Yukawa interaction, because, as may be seen from Table V, $D^{(5)}\otimes D^{(5)}=D^{(1)}\oplus D^{(10)}\oplus D^{(14)}$.

In addition to the K and π, however, $D^{(10)}$ requires three isotopic spin zero mesons, D, with $Y=2$, 0, -2 (charge, $Q=1$, 0, -1, respectively). Of the three, the existence of the charged ones, $D^\pm$, and the consequences thereof, have been discussed by Yamanouchi.[76] The prediction of the existence of a neutral particle, D^0, is a novel feature of the C_2 scheme. Although it is a neutral isotopic scalar meson, it differs from the π^{00} of SU_3 in that it is a member of a hypercharge rotation triplet. If the mass of the D^0 were near that of the $D^\pm$, about 730 Mev as suggested by Yamanouchi,[76] it would have sufficient energy to decay into either 2π or 3π via the strong interactions. The 2π mode, however, can be shown to be forbidden because of parity while the 3π mode is allowed only insofar as the symmetry of C_2 is broken (for such a low-energy process, one would expect the symmetry to be violated to a rather large extent). If it were energetically possible for the D^0 to decay into $K+\bar{K}$, such a mode would again be ruled out by parity conservation.

So far, we have not assigned the Σ's to an irreducible representation. The lowest dimensional representation that can contain them is easily seen to be $D^{(10)}$. This implies the existence of baryon resonances, associated in the same irreducible representation with the Σ's, which have the same space-time properties, e.g., $J=1/2$, and the following isotopic spin and hypercharge assignments: $I=1/2$, $Y=1$; $I=1/2$, $Y=-1$; and $I=0$, $Y=2$, 0, -2. The first isotopic doublet would appear as a nucleon-pion resonance, the second as a $\Xi\pi$ resonance, in the $J=1/2$ state. The hypercharge triplet would appear as a resonance in the NK, the $N\bar{K}$ and ΞK, and the $\Xi\bar{K}$ scattering states. As pointed out before, the masses of such states remain theoretically unknown.

For demonstration purposes, let us use a combination of the techniques developed in Secs. III and V to analyze the product representation $\psi_i{}^j=\psi_i\bar{\psi}^j$, where ψ_i is the basis of the five-dimensional representation. We choose its components as $(p,n,\Lambda,\Xi^0,\Xi^-)$. According to Sec. V, there exists a symmetric metric, g^{ij}, which relates ψ^i with ψ_k. In order to determine the form of g^{ij} we first formally form the invariant

$$\chi=g^{ij}\psi_i\psi_j.$$

By remembering that this invariant must have a weight (0,0), it must be a linear combination

$$\chi=a\Xi^-p+b\Xi^0n+c\Lambda\Lambda+dp\Xi^-+en\Xi^0.$$

In order to determine the coefficients a, b, $\cdots$, we use the fact that $E_\alpha\chi=0$ for any E_α. The immediate result is

$$\chi=a(\Xi^-p-\Xi^0n+\Lambda\Lambda+p\Xi^--n\Xi^0).$$

With a normalization such that $g^2=1$, g^{ij} may now be written as

$$g_{ij}=g^{ij}=\begin{pmatrix} & & & & 1\\ & & & -1 & \\ & & 1 & & \\ & -1 & & & \\ 1 & & & & \end{pmatrix}, \qquad \text{(VI.24)}$$

so that

$$(\psi_i)=\begin{pmatrix}p\\ n\\ \Lambda\\ \Xi^0\\ \Xi^-\end{pmatrix},\quad (\psi^i)=\begin{pmatrix}\Xi^-\\ -\Xi^0\\ \Lambda\\ -n\\ p\end{pmatrix},$$
$$(\bar{\psi}^i)=\begin{pmatrix}\bar{p}\\ \bar{n}\\ \bar{\Lambda}\\ \bar{\Xi}^0\\ \bar{\Xi}^-\end{pmatrix},\quad (\bar{\psi}_i)=\begin{pmatrix}\bar{\Xi}^-\\ -\bar{\Xi}^0\\ \bar{\Lambda}\\ -\bar{n}\\ \bar{p}\end{pmatrix}. \qquad \text{(VI.25)}$$

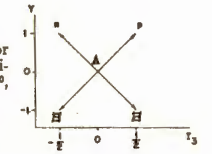


FIG. 23. Weight diagram for $D^{(5)}(0,1)$ of C_2 with bases associated with baryons (p, n, Λ, Ξ^0, and Ξ^-).

This matrix g_{ij} is the same as that introduced under the pseudonym C in Eq. (III.39). It is now possible to construct the bilinear forms $\psi_i{}^j$ for the 10- and 14-dimensional representations:

$$\psi(10)_i{}^j=\bar{\psi}^j\psi_i-\bar{\psi}_i\psi^j;$$
$$\psi(14)_i{}^j=\bar{\psi}^j\psi_i+\bar{\psi}_i\psi^j-\tfrac{2}{5}\delta_i{}^j\bar{\psi}^k\psi_k. \qquad \text{(VI.26)}$$

Specifically, for the 10-dimensional representation, in terms of $B\bar{B}$,

$$\chi_1=-\psi_1{}^2=-\bar{n}p-\bar{\Xi}^-\Xi^0,$$
$$\chi_2=\frac{1}{\sqrt{2}}(\psi_1{}^1-\psi_2{}^2)=\frac{1}{\sqrt{2}}(\bar{p}p-\bar{n}n+\bar{\Xi}^0\Xi^0-\bar{\Xi}^-\Xi^-),$$
$$\chi_3=\psi_2{}^1=\bar{p}n+\bar{\Xi}^0\bar{\Xi}^-,\qquad \chi_4=\psi_1{}^4=\bar{\Xi}^0p+\bar{\Xi}^-n,$$
$$\chi_5=\frac{1}{\sqrt{2}}(\psi_1{}^1+\psi_2{}^2)=\frac{1}{\sqrt{2}}(\bar{p}p+\bar{n}n-\bar{\Xi}^0\Xi^0-\bar{\Xi}^-\Xi^-),$$
$$\chi_6=\psi_4{}^1=\bar{p}\Xi^0+\bar{n}\Xi^-,\qquad \chi_7=\psi_3{}^1=\bar{p}\Lambda-\bar{\Lambda}\Xi^-,$$
$$\chi_8=-\psi_2{}^3=-\bar{n}\Lambda-\bar{\Lambda}\Xi^0,\qquad \chi_9=\psi_1{}^3=\bar{\Lambda}n+\bar{\Xi}^0\Lambda,$$
$$\chi_{10}=\psi_2{}^3=\bar{\Lambda}p-\bar{\Xi}^-\Lambda.$$

Since the 10-dimensional representation is the regular representation, these χ_A, if assigned the space-time properties of a four-vector (i.e., by inserting a γ_μ into each term, e.g., $\bar{p}n\to\bar{p}\gamma_\mu n$), form the baryon part of the current which is conserved due to the group C_2. If the spin zero mesons, K, π, etc., were considered compound baryon-antibaryon systems, these χ_A would, of course, be the complete conserved currents in the interaction representation.[77] In order to avoid being quoted as not having considered strongly-interacting

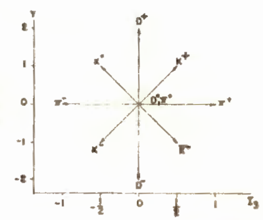


FIG. 24. Weight diagram for $D^{(10)}(2,0)$ of C_2 with bases associated with mesons.

TABLE VIII. Comparison of relative partial widths of the N^* and Y^* resonances. In computing the relative partial widths $q^3E_B/(E_B+E_M)$ in Eq. (VI.23) are taken from reference 17.

Isobar	Resonance energy (experimental) in Mev	Disintegration products, BM	$\lvert\langle\{10\},\alpha\lvert BM\rangle\rvert^2$ $\alpha=1$	$\alpha=2$	Relative partial width
$(N^*)^{++}$	1237	$p\pi^+$	1/2		1
		$\Lambda\pi^+$		1/4	0.38
$(Y^*)^+$	1385	$\Sigma^+\pi^0$		1/12	0.03
		$\Sigma^0\pi^+$		1/12	0.03
		$\Sigma^+\pi^{00}$		1/4	?

TABLE IX. Action of E_α on baryons for $D^{(5)}(0,1)$ in C_2.

E_α \ ψ	p	n	Λ	Ξ^0	Ξ^-
$6^{\frac{1}{2}}E_1$		p			Ξ^0
$6^{\frac{1}{2}}E_{-1}$	n			Ξ^-	
$6^{\frac{1}{2}}E_2$			p		$-\Lambda$
$6^{\frac{1}{2}}E_{-2}$	Λ		$-\Xi^-$		
$6^{\frac{1}{2}}E_3$				p	n
$6^{\frac{1}{2}}E_{-3}$	Ξ^0	Ξ^-			
$6^{\frac{1}{2}}E_4$			n	Λ	
$6^{\frac{1}{2}}E_{-4}$		Λ	Ξ^0		
$\sqrt{3}H_1$	$\frac{1}{2}p$	$-\frac{1}{2}n$		$\frac{1}{2}\Xi^0$	$-\frac{1}{2}\Xi^-$
$2\sqrt{3}H_2$	p	n		$-\Xi^0$	$-\Xi^-$

[75] S. Barshay, Phys. Rev. Letters 1, 97 (1958). Recent experimental evidence is compared with this conjecture in Y. Nambu and J. J. Sakurai, Phys. Rev. Letters 6, 377 (1961).

[76] T. Yamanouchi, Phys. Rev. Letters 3, 480 (1959).

[77] The currents can easily be written down in interaction representation. The transformation to Heisenberg representation will introduce extra terms in the current, if there are derivatives of the fields in the interaction Lagrangian.

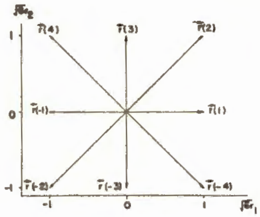


FIG. 25. Root diagram for B_2.

intermediate vector mesons,[78,79] we point out here that by the extension to space-time dependent transformations, these ten currents would be coupled to ten such mesons; this technique is trivially extended to the other groups.

On the other hand, if the χ_A are given the space-time property of a pseudoscalar, the effective Yukawa coupling between the baryons (N,Λ,Ξ) and the pseudoscalar mesons can be written down. Writing the ten mesons as a 10-component $M_A=(-\pi^+,\pi^0,\pi^-,D^+,D^0,D^-,\bar{K}^+,-\bar{K}^0,K^0,K^+)$, the coupling becomes[80]

$$I=g\chi_A M^A, \tag{VI.27}$$

where

$$M^A=(-\pi^-,\pi^0,\pi^+,D^-,D^0,D^+,K^+,-K^0,\bar{K}^0,\bar{K}^+),$$
$$M^A=g^{AB}M_B.$$

A simpler method than the above exists for finding the χ_A. Since they form the basis of the regular representation, they are given by $\chi_A=\bar{\psi}^a L_{Aa}{}^b\psi_b$ where the $L_{Aa}{}^b$ can be read directly from Table IX. The advantage of the method described above is that we can now also immediately write down the 14-dimensional basis.

Model Built on B_2

Another possible scheme based on the symmetry of C_2 $(=B_2)$ is obtained by rotating the coordinates of the root diagram, Fig. 1(b) of Sec. II by 45°. We recapitulate the procedure of constructing the Lie algebra, using the root diagram, Fig. 25. In this basis of the algebra

$$[\hat{H}_1,\hat{E}_1]=6^{-\frac{1}{2}}\hat{E}_1,\quad [\hat{H}_1,\hat{E}_2]=6^{-\frac{1}{2}}\hat{E}_2,\ \cdots,$$
$$[\hat{H}_2,\hat{E}_1]=0,\quad [\hat{H}_2,\hat{E}_2]=6^{-\frac{1}{2}}\hat{E}_2,\ \cdots, \tag{VI.28}$$

and we choose the $N_{\alpha\beta}$ to be

$$N_{1,4}=N_{-1,3}=N_{3,-4}=N_{1,3}=N_{-1,2}=N_{2,-3}=6^{-\frac{1}{2}}. \tag{VI.29}$$

The highest weight of the representation (λ_1,λ_2) is, in this case,

$$\mathbf{M}=\lambda_2/6^{-\frac{1}{2}}(1,0)+\lambda_1/6^{-\frac{1}{2}}(\tfrac{1}{2},\tfrac{1}{2}). \tag{VI.30}$$

The dimensionality is given by

$$N=(1+\lambda_1)(1+\lambda_2)[1+\tfrac{1}{2}(\lambda_1+\lambda_2)]\times[1+\tfrac{1}{3}(\lambda_1+2\lambda_2)]. \tag{VI.31}$$

The dimensionalities of the representations $D(\lambda_1,\lambda_2)=D(0,0)$, $D(1,0)$, $D(0,1)$, $D(2,0)$, $D(0.2)$, $\cdots$, are 1, 4, 5, 10, 14, $\cdots$, just as before.

We can identify the Λ particle as the basis of the one-dimensional representation. Inspection of the weight diagram, Fig. 26(a), shows that (p,n,Ξ^0,Ξ^-) and $(K^+,K^0,-\bar{K}^0,+K^-)$ can be chosen as the bases of the four-dimensional representation.

The isotopic content of the five dimensional representation of Fig. 26(b) requires, in addition to the isotopic triplet with $Y=0$, which we identify with $\Sigma^+,\Sigma^0,\Sigma^-$ (π^+,π^0,π^-), two more charged baryons $X^\pm$ ($D^\pm$ for bosons) with $T_3=0$, $Y=0$.

We now illustrate the tensor analysis of Sec. V on the basis of this model. Since we have identified the H_i differently than in the previous case (Sec. III G; Sec. VI E), the matrices derived below are not the same as before. The ten operators may be represented by 4×4 traceless matrices:

$$(H_1)_a{}^b=2(6)^{\frac{1}{2}}\begin{bmatrix}1&&&\\&-1&&\\&&1&\\&&&-1\end{bmatrix},\quad (H_2)_a{}^b=2(6)^{\frac{1}{2}}\begin{bmatrix}1&&&\\&1&&\\&&-1&\\&&&-1\end{bmatrix},$$
$$(E_1)_a{}^b=2(3)^{\frac{1}{2}}\begin{bmatrix}0&1&&\\&0&&\\&&0&1\\&&&0\end{bmatrix},\quad (E_3)_a{}^b=2(3)^{\frac{1}{2}}\begin{bmatrix}0&&-1&\\&0&&1\\&&0&\\&&&0\end{bmatrix}, \tag{VI.32}$$
$$(E_2)_a{}^b=6^{-\frac{1}{2}}\begin{bmatrix}0&&&1\\&0&&\\&&0&\\&&&0\end{bmatrix},\quad (E_4)_a{}^b=6^{-\frac{1}{2}}\begin{bmatrix}0&&&\\&0&-1&\\&&0&\\&&&0\end{bmatrix},\quad (\psi_a)=\begin{bmatrix}p\\n\\\Xi^0\\\Xi^-\end{bmatrix}$$
$$(E_{-\alpha})_a{}^b=(E_\alpha{}^+)_a{}^b=(E_\alpha)_b{}^a.$$

[78] J. J. Sakurai, Ann. Phys. 11, 1 (1960).
[79] R. Utiyama, Phys. Rev. 101, 1597 (1956); S. L. Glashow and M. Gell-Mann (to be published). The latter authors have independently suggested the B_2 and C_2 models discussed in this section.
[80] The relation between M_A and $M^A=g^{AB}M_B$ is most easily determining by requiring $\chi^A\chi_A=g^{AB}\chi_A\chi_B$ to be invariant.

These matrices may be derived by the method developed in Sec. IV. The metric h^{ab} is defined to be the skew matrix that makes $h^{ab}\psi_a\psi_b$ form invariant. One can easily verify that

$$\Xi^-p-\Xi^0n+n\Xi^0-\Xi^-p$$

is form invariant since the $\hat{E}_\alpha$ operating on it annihilate it. Therefore, we choose h^{ab} to be

$$h^{ab}=\begin{bmatrix} & & & -1\\ & & 1 & \\ & -1 & & \\ 1 & & & \end{bmatrix}=-h_{ab},\quad h_{ac}h^{cb}=\delta_a{}^b. \qquad \text{(VI.33)}$$

Note that $h^{ac}(L_i)_c{}^b=(L_i)_c{}^a h^{bc}$, i.e., $L_i{}^{ab}=L_i{}^{ba}$. The contragredient bases are

$$\psi^a=h^{ac}\psi_c=\begin{bmatrix}-\Xi^-\\ \Xi^0\\ -n\\ p\end{bmatrix},\quad \bar{\psi}^a=\begin{bmatrix}\bar{p}\\ \bar{n}\\ \bar{\Xi}^0\\ \bar{\Xi}^-\end{bmatrix}. \qquad \text{(VI.34)}$$

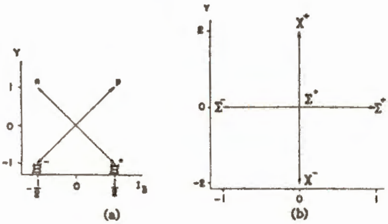


FIG. 26. (a) Weight diagram for $D(1,0)$ of B_2. For meson bases, the substitution, $(p,n,\Xi^0,\Xi^-)\rightarrow(K^+,K^0,-\bar{K}^0,\bar{K}^+)$, should be made. (b) Weight diagram for $D(0,1)$ of C_2. For meson bases, the substitution, $(\Sigma^+,\Sigma^0,\Sigma^-,X^+,X^-)\rightarrow(\pi^+,\pi^0,\pi^-,D^+,D^-)$, should be made.

We take the five skew 4×4 matrices $\sigma_i{}^{ab}$ just introduced in Eq. (V.17), satisfying $h_{ab}\sigma_i{}^{ba}=\text{trace}\,h\sigma_i=0$, to be

$$\sigma_1{}^{ab}=\tfrac{1}{2}\begin{bmatrix} & & & -1\\ & & -1 & \\ & 1 & & \\ 1 & & & \end{bmatrix},\quad \sigma_2{}^{ab}=1/\sqrt{2}\begin{bmatrix}0 & 0 & -1 & 0\\ 0 & & & \\ 1 & & & \\ 0 & & & \end{bmatrix},\quad \sigma_3{}^{ab}=1/\sqrt{2}\begin{bmatrix} & & & 0\\ & & & -1\\ & & & 0\\ 0 & 1 & 0 & 0\end{bmatrix},$$
$$\sigma_4{}^{ab}=1/\sqrt{2}\begin{bmatrix}0 & -1 & 0 & 0\\ 1 & & & \\ 0 & & & \\ 0 & & & \end{bmatrix},\quad \sigma_5{}^{ab}=1/\sqrt{2}\begin{bmatrix} & & & 0\\ & & & 0\\ & & & 1\\ 0 & 0 & -1 & 0\end{bmatrix}. \qquad \text{(VI.35)}$$

These matrices are chosen such that $\chi_i\equiv\bar{\psi}^a(\sigma_i)_a{}^b\psi_b=\bar{\psi}^a h_{ac}(\sigma_i)^{cb}\psi_b$ are normalized bases of the five-dimensional representation that transform as the

$$(\psi_i)=(\Sigma^0,\,-\Sigma^+,\Sigma^-,\,-X^+,X^-)$$

or the

$$(M_i)=(\pi^0,\,-\pi^+,\pi^-,\,-D^+,D^-);$$

$$\chi_1=\tfrac{1}{2}(\bar{p}p-\bar{n}n-\bar{\Xi}^0\Xi^0+\bar{\Xi}^-\Xi^-),$$

$$\chi_2=\frac{1}{\sqrt{2}}(-\bar{n}p+\bar{\Xi}^-\Xi^0),\quad \chi_4=\frac{1}{\sqrt{2}}(\bar{\Xi}^-n+\bar{\Xi}^0p),$$

$$\chi_3=\frac{1}{\sqrt{2}}(\bar{p}n-\bar{\Xi}^0\Xi^-),\quad \chi_5=\frac{1}{\sqrt{2}}(-\bar{n}\Xi^--\bar{p}\Xi^0).$$

The symmetric five-dimensional metric g_{ij} is defined as in Eq. (V.19);

$$g_{ij}=\sigma_i{}^{ab}\sigma_{jba}=\text{trace}\,\sigma_i\sigma_j$$

$$=\begin{bmatrix}1 & & & & \\ & 0 & -1 & & \\ & -1 & 0 & & \\ & & & 0 & -1\\ & & & -1 & 0\end{bmatrix}=g^{ij};\quad g^{il}g_{lj}=\delta^i{}_j. \qquad \text{(VI.36)}$$

The contragredient bases M^i of the five-dimensional representation are obtained by

$$M^i=g^{ij}M_j=\begin{bmatrix}\pi^0\\ -\pi^-\\ \pi^+\\ -D^-\\ D^+\end{bmatrix}. \qquad \text{(VI.37)}$$

The explicit form of the L_A in the five-dimensional representation is obtained from $(L_A)_i{}^j=2\sigma_i{}^{ab}(L_A)_b{}^c\sigma^j{}_{ca}$. We can go on to construct explicit forms of tensors *ad infinitum*. The above examples suffice to illustrate the method.

Let us now turn back to physics. As an example, let us consider the invariant Yukawa type coupling of the (π,D) to the (N,Ξ). It is clear that, by construction, the $\sigma_{ia}{}^b$ are just the $(\Omega_\mu{}^{(\tau)})_a{}^b$ discussed in the early part of this section where τ refers to the five-dimensional representation and $i=\mu$. The invariant coupling is, therefore,

$$I=\bar{\psi}^a\sigma g_{ia}{}^b\psi_b M^i,$$

$$=\frac{g}{2}\{(\bar{p}\gamma_5p-\bar{n}\gamma_5n-\bar{\Xi}^0\gamma_5\Xi^0+\bar{\Xi}^-\gamma_5\Xi^-)\pi^0$$
$$+\sqrt{2}[(\bar{n}\gamma_5p-\bar{\Xi}^-\gamma_5\Xi^0)\pi^-$$
$$-(\bar{\Xi}^0\gamma_5p+\bar{\Xi}^-\gamma_5n)D^-+\text{h.c.}]\}. \qquad \text{(VI.38)}$$

In this case, the number of independent coupling constants required is one, because the product repre-

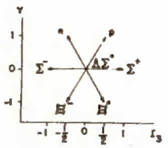


FIG. 27. Weight diagram for $D^{(7)}(1,0)$ with bases associated with baryons. For meson bases, substitute

$$(K^+,K^0,-\bar{K}^0,\bar{K}^+,\pi^+,\pi^0,\pi^-)$$

for

$$(p,n,\Xi^0,\Xi^-,\Sigma^+,\Sigma^0,\Sigma^-).$$

sentation $D^{(4)*}\otimes D^{(4)}$ contains the irreducible representation $D^{(5)}$ only once.

There are three independent amplitudes for scattering of (N,Ξ) and $(\bar{N},\bar{\Xi})$, corresponding to the decomposition (Clebsch-Gordan series) $D^{(4)*}\otimes D^{(4)}=D^{(1)}\oplus D^{(5)}\oplus D^{(10)}$. Let us consider the process

$$a+\bar{b}\to a'+\bar{b}',$$

where a, b, a', and b' label members of the (N,Ξ) multiplet. The T matrix for this process can be written

$$(a'\bar{b}'|T|a\bar{b})=F^{\prime}\delta_{a'}{}^{b'}\delta_b{}^a+F^2\sigma_{ia'}{}^{b'}\sigma^i{}_b{}^a+F^3(\Omega_A{}^{(10)})_{a'}{}^{b'}(\Omega_A{}^{(10)})_b{}^a,\quad \text{(VI.39)}$$

where the isometric operators $(\Omega_A{}^{(10)})_b{}^a$ are proportional to $L_{Ab}{}^a$:

$$(\Omega_A{}^{(10)})_b{}^a=\pm\frac{10}{(\text{trace } L_BL^B)^{\frac{1}{2}}}L_{Ab}{}^a.$$

Finally, let us discuss the Y^* isobar states in this scheme. Since the five-dimensional representation is the lowest dimensional one with the right isotopic spin and hypercharge content to accommodate the triplet, $(Y^*)^+$, $(Y^*)^0$, and $(Y^*)^-$, we propose to identify the Y^* with components of a basis of $D^{(5)}(0,1)$. We have already assigned the Σ and $X^{\pm}$ to a five-dimensional representation. Therefore, the Y^* and the Σ would have to have the same transformation properties in charge space, while their differences would be described by space-time quantum numbers. An interesting feature of this model is that the decay process $Y^*\to\Sigma+\pi$ is forbidden by symmetry while $Y^*\to\Lambda+\pi$ is allowed,[81] independent of spin of the Y^* or the relative $\Lambda\Sigma$ parity. The reason is as follows. The π and the Σ are members of five dimensional representations. Since the product representation $D^{(5)}\otimes D^{(5)}$ does not contain any irreducible $D^{(5)}$, it is impossible for a Y^* to decay into a π and a Σ. On the other hand, since the Λ particle is a basis of the one dimensional representation, the product representation of the Λ and the π multiplet naturally gives rise to a five dimensional representation.

G. Model Built on G_2

If we assume that there are eight baryons which can form the bases for one or more representations, then the dimensionality of these representations must add up to eight. An inspection of Table VI for G_2 shows that the only possibility involves two representations, $D^{(1)}(0,0)$ and $D^{(7)}(1,0)$. This implies that seven of the baryons must have the same space-time quantum numbers; the eighth baryon may have a different set of these quantum numbers. In contrast with SU_3, we see that there is a seven-dimensional representation which allows the possibility of using only the seven known mesons, i.e., $D^{(7)}(1,0)$.[82] In this scheme, of course, it would also be possible to accommodate an eighth meson, the π^{00} which would then correspond to the one-dimensional representation, $D^{(1)}(0,0)$. Until such time as this meson is experimentally detected, we shall consider only the known particles. It is again clear that these seven mesons must have the same space-time quantum numbers.

In Fig. 27 we have drawn the weight diagram for the seven-dimensional representation of G_2. From this, it is clear that if we associate $2\sqrt{3}\hat{H}_1$ with the operator for I_3 and $4\hat{H}_2$ with the operator for Y (hypercharge), then each of the baryons has a specific weight associated with it. There remains only the question of the Σ^0 and Λ which both have zero eigenvalues for these two operators. Since we want charge independence to hold for the strong interactions (this implies the existence of the isotopic spin lowering operator as one of the E_α), the Σ^0 must belong to the seven-dimensional representation and that the Λ is the basis for the one-dimensional representation. Because the Σ and Λ belong to different representations, we see that G_2 can accommodate opposite parities for the Σ and Λ. It predicts specifically, however, that the ΣK parity must be the same as that of $N\pi$

In order to give the usual isotopic spin and hypercharge assignment to the N, Σ, and Ξ, in accordance with the association of H_1 and H_2 with I_3 and Y given above, we must make the following connection between the states given in Sec. III B and the particles

$\|A\rangle$:	$\|1\rangle$	$\|2\rangle$	$\|3\rangle$	$\|4\rangle$	$\|5\rangle$	$\|6\rangle$	$\|7\rangle$	
Baryons:	p	n	Ξ^0	Ξ^-	$-\Sigma^+$	Σ^0	Σ^-	(VI.40)
Mesons:	K^+	K^0	$-\bar{K}^0$	$\bar{K}^+$	$-\pi^+$	π^0	π^-.	

With the aid of this dictionary, it is easy to construct Table X for the particles from the results of Sec. III H.

We proceed now to analyze the scattering $B+M\to B'+M'$ in the same manner as described in the general part of this section. The pertinent four-point function is

$$\langle T[\psi_{B'}{}^{\dagger}\psi_{M'}{}^{\dagger}\psi_B\psi_M]\rangle.$$

The combination $\psi_B\psi_M$ is the Kronecker product of two seven-dimensional representations, one for the meson and one for the baryon. This reduces, following Table VI, according to $7\otimes7=1\otimes7\otimes14\otimes27$. In the manner described previously, we conclude that there are only four different four-point functions or amplitudes for the scattering of the seven baryons by the bosons.

[81] This is in accord with experimental observation (see reference 73).

[82] The model built on G_2 was first suggested by Behrends and Sirlin (reference 19) and, independently, by another of the authors (C.F.) (unpublished).

For further physical predictions, it is necessary to find the metric tensor. This can be done by noting that the invariant part of the compound representation $\chi=g^{ij}\psi_i\varphi_j$ must be a linear combination of the form

$$\chi=ap\bar{K}^++bn\bar{K}^0+c\Sigma^+\pi^-+d\Sigma^-\pi^++e\Sigma^0\pi^0+f\Xi^-K^++g\Xi^0K^0.$$

By use of the fact that $E_\alpha\chi=0$, we can readily find the coefficients a, b, $c\cdots$, and thereby find

$$(g^{ij})=(g_{ij})=\begin{pmatrix} 0 & & & 1 & & & \\ & 0 & -1 & & & & \\ & -1 & 0 & & & & \\ 1 & & & 0 & & & \\ & & & & 0 & & -1 \\ & & & & & 1 & \\ & & & & -1 & & 0 \end{pmatrix},\quad \text{(VI.41)}$$

where

$$(\psi_i)=(p,n,\Xi^0,\Xi^-,-\Sigma^+,\Sigma^0,\Sigma^-).$$

It follows that

$$(\psi^i)=(\Xi^-,-\Xi^0,-n,p,-\Sigma^-,\Sigma^0,+\Sigma^+).$$

[This matrix was introduced in Eq. (III.42).]

We further need to determine the isometries $6^{-\frac{1}{2}}\Gamma^{ijk}$. This is most easily done by noting that the basis for the seven-dimensional compound representation is given by $\chi^i=\Gamma^{ijk}\psi_j\varphi_k$. We proceed by noting that the χ^i with the fundamental dominant weight $\mathbf{M}^{(1)}$ is a linear combination of the form:

$$-\chi_6=\chi^7=ap\bar{K}^0+b\Xi^0K^++c\Sigma^+\pi^0+d\Sigma^0\pi^+.$$

Since $E_1\chi_6$ and $E_3\chi_6$ would have weights that cannot belong to $D^{(7)}$, they must vanish. These conditions are sufficient to determine the constants a, b, c, and d. The result is

$$\chi^7=\sqrt{2}(p\bar{K}^0+\Xi^0K^+)-\Sigma^+\pi^0+\Sigma^0\pi^+.$$

The matrix $\Gamma^{7jk}=-\Gamma_6{}^{jk}$ may readily be found from this expression. By operating on χ_6 with E_6, we find, [remembering that

$$E_6\psi_6=-E_6(\Sigma^+)=-\tfrac{1}{2}\sqrt{\tfrac{3}{2}}p=-\tfrac{1}{2}\sqrt{\tfrac{3}{2}}\psi_4]$$

$$p\pi^0+\sqrt{2}n\pi^+-\Sigma^0K^+-\sqrt{2}\Sigma^+K^0=\chi_1=\chi^4.$$

Proceeding successively in this manner, we can determine all the χ^i, and thereby all the (Γ^{ijk}). For convenience, we list these matrices

$$\Gamma^1=\Gamma_4=\left(\begin{array}{cccc|ccc} & & & & & & 0 \\ & & & & & & 0 \\ & & & & & & \sqrt{2} \\ & & & & 0 & -1 & 0 \\ \hline & & & & 0 & & \\ & & & 1 & & & \\ & & -\sqrt{2} & 0 & & & \end{array}\right);\quad \Gamma^2=-\Gamma_3=\left(\begin{array}{cccc|ccc} & & & & & & \\ & & & & & & \\ & & & & & -1 & \\ & & & & \sqrt{2} & 0 & \\ \hline & & & -\sqrt{2} & & & \\ & & 1 & 0 & & & \\ & & & 0 & & & \end{array}\right),$$

$$\Gamma^3=-\Gamma_2=\left(\begin{array}{cccc|ccc} & & & & & & -\sqrt{2} \\ & & & & & 1 & \\ & & & & & & \\ & & & & & & \\ \hline & & & & & & \\ & -1 & & & & & \\ \sqrt{2} & & & & & & \end{array}\right);\quad \Gamma^4=\Gamma_1=\left(\begin{array}{cccc|ccc} & & & & 0 & 1 & 0 \\ & & & & -\sqrt{2} & & \\ & & & & & & \\ & & & & & & \\ \hline 0 & \sqrt{2} & & & & & \\ -1 & & & & & & \\ 0 & & & & & & \end{array}\right),\quad \text{(VI.42)}$$

$$\Gamma^5=-\Gamma_7=\left(\begin{array}{cccc|ccc} & & & 0 & & & \\ & & & \sqrt{2} & & & \\ & & & & & & \\ 0 & -\sqrt{2} & & & & & \\ \hline & & & & & & \\ & & & & & & 1 \\ & & & & 0 & -1 & 0 \end{array}\right);\quad \Gamma^6=\Gamma_5=\left(\begin{array}{cccc|ccc} & & & -1 & & & \\ & & -1 & & & & \\ & 1 & & & & & \\ 1 & & & & & & \\ \hline & & & & & & -1 \\ & & & & & 0 & \\ & & & & 1 & & \end{array}\right);$$

$$\Gamma^7=-\Gamma_6=\left(\begin{array}{cccc|ccc} & & +\sqrt{2} & 0 & & & \\ & & & & & & \\ -\sqrt{2} & & & & & & \\ 0 & & & & & & \\ \hline & & & & 0 & -1 & \\ & & & & +1 & 0 & \\ & & & & & & 0 \end{array}\right).$$

TABLE X. Action of E_α on baryons for $D^{(7)}(1,0)$ in G_2. Table for bosons is obtained by substitution $(p,n,\Xi^0,\Xi^-,\Sigma^+,\Sigma^0,\Sigma^-)\rightarrow(K^+,K^0,-\bar{K}^0,\bar{K}^+,\pi^+,\pi^0,\pi^-)$.

$E_\alpha \backslash \psi$	p	n	Ξ^0	Ξ^-	Σ^+	Σ^0	Σ^-
$2(6)^{1/2}E_1$		p		Ξ^0		$-\sqrt{2}\Sigma^+$	$\sqrt{2}\Sigma^0$
$2(6)^{1/2}E_{-1}$	n		Ξ^-		$-\sqrt{2}\Sigma^0$	$\sqrt{2}\Sigma^-$	
$2\sqrt{2}E_2$				$-\Sigma^+$			p
$2\sqrt{2}E_{-2}$	Σ^-				$-\Xi^-$		
$2(6)^{1/2}E_3$			$-\Sigma^+$	$-\sqrt{2}\Sigma^0$		$\sqrt{2}p$	$-n$
$2(6)^{1/2}E_{-3}$	$\sqrt{2}\Sigma^0$	$-\Sigma^-$			$-\Xi^0$	$-\sqrt{2}\Xi^-$	
$2\sqrt{2}E_4$			p	n			
$2\sqrt{2}E_{-4}$	Ξ^0	Ξ^-					
$2(6)^{1/2}E_5$			$\sqrt{2}\Sigma^0$	$-\Sigma^-$	$+p$	$\sqrt{2}n$	
$2(6)^{1/2}E_{-5}$	$+\Sigma^+$	$\sqrt{2}\Sigma^0$				$\sqrt{2}\Xi^0$	$-\Xi^-$
$2\sqrt{2}E_6$			Σ^-		$+n$		
$2\sqrt{2}E_{-6}$		$-\Sigma^+$					Ξ^0
$2\sqrt{3}H_1$	$\frac{1}{2}p$	$-\frac{1}{2}n$	$\frac{1}{2}\Xi^0$	$-\frac{1}{2}\Xi^-$	Σ^+		$-\Sigma^-$
$4H_2$	p	n	$-\Xi^0$	$-\Xi^-$			

It is now a trivial matter to list various amplitudes in a compact notation. For example, the invariant three-point function is

$$\Gamma^{ijk}\langle T(\bar{\psi}_i\psi_j\varphi_k)\rangle. \tag{VI.43}$$

Another simple example is afforded by Λ production mesons on baryons, $M+B\rightarrow M'+\Lambda$. The four-point function is

$$\Gamma^{ijk}\langle T(\bar{\psi}_\Lambda\psi_i\varphi_j\varphi_k)\rangle. \tag{VI.44}$$

With regard to G_2, it might be interesting to play again the game of finding the processes which might have resonances corresponding to the (3,3) pion-nucleon resonance. At this point, we re-emphasize, the limitations of this game (see the general discussion above). First, the product representation of one baryon and one meson decomposes into representations with dimensionalities of 1, 7, 14, and 27. But the weight of the π^+p state, say, which is a member of the (3,3) resonance, is $(1/4\sqrt{3})(3,\sqrt{3})$. This is just the highest weight for the 14-dimensional representation $D(0,1)$ and it is one of the weights for the 27-dimensional representation. Thus the (3,3) resonance must belong to either the 14 or 27 dimensional representation.

Again, as an example, we have drawn the weight diagram for the 14-dimensional representation in Fig. 3(b). From this, it is clear that besides the $I=\frac{3}{2}$, $Y=1$ multiplet, which we might identify as the 3,3 resonance, the isotopic content includes an $I=\frac{3}{2}$, $Y=-1$ multiplet, an $I=1$, $Y=0$ multiplet, and three singlets, $I=0$, $Y=2$, 0, -2. All of these multiplets must have $J=\frac{3}{2}$. The actual product representation written in terms of the product MB may be found in the manner illustrated above. Namely, the basis for the highest weight of the 14-dimensional representation must be of the form $ap\pi^+ + b\Sigma^+K^+$. But E_α, for a positive root $r(\alpha)$, acting on this basis must be zero. Specifically, application of E_{-6} gives $a=-b$, so that the basis for the highest weight is $\frac{1}{2}\sqrt{2}(p\pi^+ - \Sigma^+K^+)$. The bases for the other weights can be obtained by repeated use of all the E_α. In contrast with SU_3, the $\pi\Lambda$ resonance cannot be associated with the (3,3) pion nucleon resonance, since the $\pi\Lambda$ resonance must be 7-dimensional which does not contain an $I=\frac{3}{2}$ multiplet.

If the (3,3) resonance were identified with the 27-dimensional representation, we would proceed in the same manner. The result would be that the (3,3) resonance would be associated with a different set of isotopic spin multiplets.

Классификация простых групп Ли

Е. Б. Дынкин (Москва)

1. Простые группы Ли перечислил впервые Киллинг в 1890 г. Первое полное доказательство результата Киллинга принадлежит Картану (1894 г.). В 1933 г. Ван-дер-Варден [1] предложил, опираясь на работу Г. Вейля [2], новый более геометричный метод классификации простых групп Ли. В настоящей заметке доказывается, что полупростая группа определяется системой своих простых корней, и задача перечисления всех простых групп Ли сводится этим к простой геометрической задаче: построить в n-мерном эвклидовом пространстве всевозможные реперы такие, что для любых двух векторов a и b $\frac{2(a, b)}{(a, a)}$ — целое неположительное число ((a, b) — скалярное произведение a и b).

2. Г. Вейль относит всякой полупростой группе Ли $\mathfrak{G}$ с комплексными параметрами систему $\Sigma(\mathfrak{G})$ ее корневых векторов, заданием которой группа $\mathfrak{G}$ полностью определяется. $\Sigma(\mathfrak{G})$ — конечное множество векторов n-мерного вещественного эвклидова пространства R^n, обладающее следующими свойствами:

2 (1). Если $a \in \Sigma$, то $-a \in \Sigma$, но для $k = 2, 3, \ldots$ $ka \bar{\in} \Sigma$.

2 (2). Пусть a и b — различные корни. Если для $-p \leqslant l \leqslant q$, $b + la \in \Sigma$, но $b-(p+1)a \bar{\in} \Sigma$ и $b+(q+1)a \bar{\in} \Sigma$, то $p - q = \frac{2(b, a)}{(a, a)}$.

2 (3). Если системы $\Sigma(\mathfrak{G}_1)$ и $\Sigma(\mathfrak{G}_2)$ подобны, т. е. переходят одна в другую при некотором растяжении пространства R^n, то они равны.

Если, в частности, $\mathfrak{G}$ — простая группа, то

2 (4). $\Sigma(\mathfrak{G})$ не распадается на две взаимно ортогональные подсистемы Σ_1 и Σ_2.

3. Приведем примеры простых групп и выпишем системы их корневых векторов. Эти примеры подробно изучены Вейлем [2].

A_n — группа линейных преобразований с детерминантом 1 пространства L^{n+1} — $(n+1)$-мерного пространства над полем комплексных чисел. $\sum(A_n)$: $\{e_p - e_q\}_{p,q=1}^{n+1}$ ($p \neq q$; $e_1, \ldots, e_{n+1}$ — ортогональный нормированный базис R^{n+1}).

B_n — группа ортогональных преобразований L^{2n+1}.

$$\sum(B_n): \quad \{\pm e_p, \pm e_p \pm e_q\}_{p,q=1}^{n} \quad (p \neq q).$$

C_n — комплекс-группа, т. е. группа линейных преобразований L^{2n}, оставляющих инвариантной дифференциальную форму

$$\sum_{k=1}^{n}(x_k\,dx_{n+k}-x_{n+k}\,dx_k).$$

$$\Sigma(C_n):\quad \{\pm 2e_p,\ \pm e_p\pm e_q\}_{p,\,q=1}^{n}\quad (p\neq q).$$

D_n — группа ортогональных преобразований L^{2n}.

$$\Sigma(D_n):\quad \{\pm e_p\pm e_q\}_{p,\,q=1}^{n}\quad (p\neq q).$$

4. Мы назовем вектор из R^n **положительным**, если его первая координата, отличная от нуля, положительна. Множество P всех положительных векторов удовлетворяет следующим условиям:

4 (1). Пусть $a\neq 0$. Тогда либо $a\in P$, либо $-a\in P$, но невозможно, чтобы $a\in P$ и $-a\in P$.

4 (2). Если $a\in P$, $b\in P$, $\lambda>0$, $\mu\geqslant 0$, то $\lambda a+\mu b\in P$.

Мы условимся писать $a>0$, если $a\in P$, и $a<0$, если $-a\in P$.

Лемма I. *Если векторы $a_1, a_2, \ldots, a_p$ положительны и $(a_i, a_k)\leqslant 0$ $(i, k=1,\ldots,p;\ i\neq k)$, то эти векторы линейно независимы.*

Пусть, в самом деле, $a_p=\sum_{i=1}^{p-1}\lambda_i a_i=\sum'\lambda_i a_i+\sum''\lambda_i a_i$, где к $\sum'$ отнесены слагаемые с положительными коэффициентами λ_i, а к $\sum''$ — слагаемые с отрицательными λ_i. Положим $b=\sum'\lambda_i a_i$, $c=\sum''\lambda_i a_i$. Тогда $(b, c)\geqslant 0$, $a_p=b+c$, причем $c\leqslant 0$, так что $b\neq 0$. Мы имеем $(a_p, b)=(b, b)+(c, b)>0$, но, с другой стороны, $(a_p, b)=\sum'\lambda_i(a_p, a_i)\leqslant 0$.

5. Положительный корень a называется **простым**, если его нельзя разложить на сумму двух положительных корней. Всякий положительный корень можно представить в виде суммы простых корней.

Если b — положительный корень и a — простой корень, то $a-b$ не будет положительным корнем. Поэтому разность двух простых корней a_1 и a_2 не будет корнем, и формула 2 (2) дает для них $\frac{2(a_1, a_2)}{(a_1, a_1)}=-q\leqslant 0$. Следовательно, $(a_1, a_2)\leqslant 0$ и, в силу леммы I, простые корни линейно независимы. Произвольный положительный корень однозначно разлагается на простые.

Положительный корень, являющийся суммой k простых корней, назовем **корнем порядка** k. Покажем, что всякий корень c порядка k имеет вид $a+b$, где a — простой корень, b — корень порядка $k-1$. В самом деле, если $a_1, a_2, \ldots, a_n$ — система всех простых корней, то система $c, a_1, a_2, \ldots, a_n$ — линейно зависима и, в силу леммы I, одно из произведений (c, a_i) положительно. Это означает, что в формуле 2 (2) $p\neq 0$ и $c-a_i$ — корень.

6. Теорема I. *Полупростая группа $\mathfrak{G}$ определяется системой $\Pi(\mathfrak{G})$ своих простых корней.*

Для доказательства достаточно построить по простым корням группы $\mathfrak{G}$ все ее корни. В силу 2 (1), можно ограничиться построением положительных корней. Все корни первого порядка нам даны, ибо это — простые корни. Пусть мы уже построили все корни порядка, меньшего k. Корни порядка k имеют вид $b+a$, где b — корень порядка $k-1$, a — простой корень (n° 5). Формула $q=p-\frac{2(b, a)}{(a, a)}$ (см. 2 (2)) позволяет решить вопрос о том, будет ли сумма простого корня a и корня b порядка $k-1$ корнем. Действительно, все корни серии $b, b-a, b-2a, \ldots$ положительны и порядка, меньшего k, так что p известно по предположению индукции. Таким образом мы можем построить все корни порядка k.

7. Не представляет труда определить системы простых корней для групп из n° 3.

$$\Pi(A_n):\quad \{e_p-e_{p+1}\}_1^n;\qquad \Pi(B_n):\quad \{e_p-e_{p+1},\ e_n\}_{p=1}^{n-1};$$

$$\Pi(C_n):\quad \{e_p-e_{p+1},\ 2e_n\};\qquad \Pi(D_n):\quad \{e_p-e_{p+1},\ e_{n-1}+e_n\}_{p=1}^{n-1}.$$

8. Назовем конечную систему Γ векторов пространства R^n (П)-системой, если она удовлетворяет следующим условиям:

8 (1). Если $a\in\Gamma$ и $b\in\Gamma$, $a\neq b$, то $\frac{2(a, b)}{(a, a)}$ — целое неположительное число.

8 (2). Γ — линейно независимая система.

8 (3). Γ не распадается на две взаимно ортогональные подсистемы.

В силу 2 (2), 2 (4) и n° 5, имеет место

Теорема II. *Система $\Pi(\mathfrak{G})$ простых корней простой группы Ли $\mathfrak{G}$ есть (П)-система.*

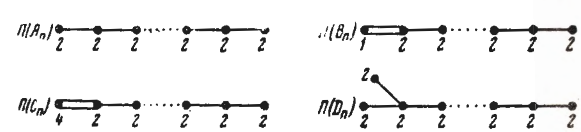
Теоремами I и II задача о классификации простых групп Ли сведена к задаче о построении всевозможных (П)-систем.

9. Пусть a и b — два различные вектора (П)-системы Γ. Тогда угол $\widehat{(a, b)}$ между a и b равен либо 90°, либо 120°, либо 135°, либо 150°.

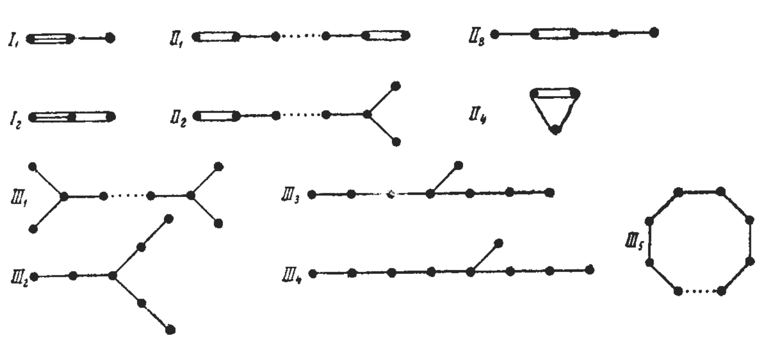
Действительно, поскольку $\frac{2(a, b)}{(a, a)}$ и $\frac{2(a, b)}{(b, b)}$ — целые числа, то $4\cos^2\widehat{(a, b)}=\frac{2(a, b)}{(a, a)}\cdot\frac{2(a, b)}{(b, b)}$ также целое число; стало быть, 0, 1, 2 или 3. Таким образом, единственно возможные значения для $\cos\widehat{(a, b)}$ суть $0,\ -\frac{1}{2},\ -\frac{\sqrt{2}}{2},\ -\frac{\sqrt{3}}{2}$.

10. Отнесем каждому элементу (П)-системы Γ точку на чертеже. Соединим две точки одним, двумя или тремя отрезками, смотря по тому, образуют ли соответствующие векторы угол, равный 120°, 135° или 150°. Пару точек, соответствующих ортогональным векторам, не будем соединять вовсе. Построенную таким образом схему мы будем называть **схемой углов** системы Γ. Если под каждой точкой схемы выписать квадрат длины (a, a) соответствующего вектора a, то получим схему, полностью определяющую систему Γ — схему системы Γ.

В качестве примера построим схемы систем $\Pi(A_n)$, $\Pi(B_n)$, $\Pi(C_n)$, $\Pi(D_n)$.



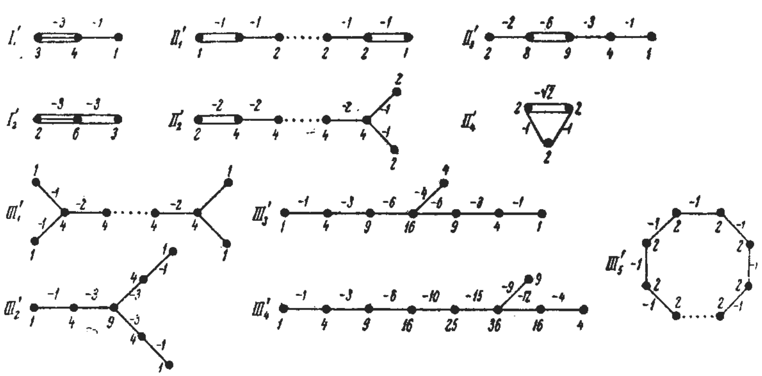
11. Лемма II. *Схема углов (II)-системы не может иметь вид* I_1-I_2, II_1-II_4, III_1-III_5.



Допустим, что некоторая (П)-система Γ имеет схемой углов одну из этих схем. Пусть $a_1, \dots, a_p$ — векторы системы Γ. Положим $b_i=\lambda_i a_i$, где $\lambda_i \neq 0$ $(i=1, 2, \dots, p)$. Тогда

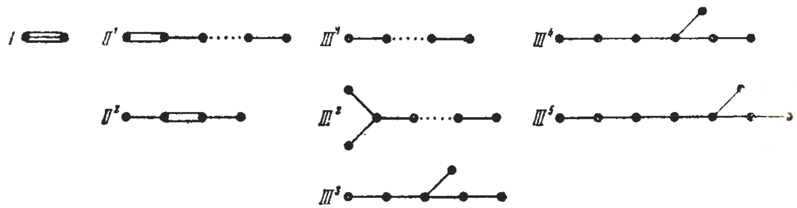
$$\sum_{i=1}^{p}\sum_{k=1}^{p}(b_i, b_k)=\left(\sum_{i=1}^{p} b_i,\ \sum_{i=1}^{p} b_i\right)>0.$$

Мы придем к противоречию, подобрав длины b_i так, чтобы $\sum_{i=1}^{p}\sum_{k=1}^{p}(b_i, b_k)\leqslant 0$. Как это сделать, видно из схем $I_1'-I_2'$, $II_1'-II_4'$, $III_1'-III_5'$, где, кроме величин (b_i, b_i), подписанных под соответствующими точками, вычислены и надписаны над соответствующими отрезками величины (b_i, b_k).



Лемма II допускает, очевидно, следующее усиление: схема углов (II)-системы не содержит подсхемы вида I_1-III_5.

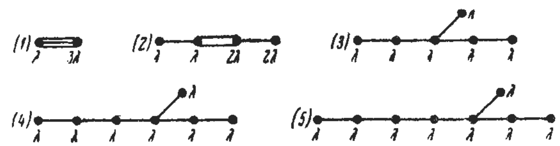
12. Лемма III. *Произвольная (II)-система имеет схемой углов одну из схем* I, II^1-II^2, III^1-III^5:



В самом деле, схема, содержащая тройной отрезок и отличная от схемы I, необходимо содержит подсхемой одну из схем I_1-I_2 леммы II, что невозможно. Аналогично, если схема содержит двойной отрезок, то в силу II_1-II_4 она совпадает с одной из схем II^1-II^2. Наконец, III_1-III_5 исключают для схемы, не содержащей ни тройных, ни двойных отрезков, все возможности, кроме III^1-III^5.

13. Пусть a и b — векторы (II)-системы Γ, делающие угол в 120°. Тогда $\frac{2(a, b)}{(a, a)}\cdot\frac{2(a, b)}{(b, b)}=4\cos^2(\widehat{a, b})=1$. В силу 8 (1) $\frac{2(a, b)}{(a, a)}=\frac{2(a, b)}{(b, b)}=-1$. Следовательно, $(a, a)=(b, b)$. Точно так же мы получим, что при $(\widehat{a, b})=135^\circ$ $(a, a)=2(b, b)$ и при $(\widehat{a, b})=150^\circ$ $(a, a)=3(b, b)$ (предполагая, что $(a, a)\leqslant(b, b)$). Из сопоставления этого замечания и леммы III немедленно получается

Теорема III. *Произвольная (П)-система либо подобна одной из систем* $\Pi(A_n)$, $\Pi(B_n)$, $\Pi(C_n)$, $\Pi(D_n)$ (nn° 7 и 12), *либо имеет схемой одну из схем*



(*множитель пропорциональности* λ — *произвольное положительное число*).

14. Из теорем I, II, III вытекает, что если простая группа $\mathfrak{G}$ не входит ни в одну из серий A_n, B_n, C_n, D_n, то система $\Pi(\mathfrak{G})$ ее простых корней имеет схемой одну из схем (1)—(5) n° 13. (Множитель λ однозначно определен в силу 2(3)). Сославшись на существование пяти различных простых групп, не входящих в серии A_n, B_n, C_n, D_n, мы можем формулировать окончательную теорему:

Теорема IV. *Все простые группы Ли исчерпываются четырьмя бесконечными сериями* A_n, B_n, C_n, D_n *и пятью изолированными группами* $\mathfrak{G}_2$, F_4, E_6, E_7, E_8. *Системы простых корней изолированных пяти групп даются, соответственно, схемами* (1)—(5) n° 13.

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Classification of the simple Lie groups

E. Dynkin (Moscow)

(Résumé)

Following H. Weyl, the structure of a semi-simple Lie group is completely described by the system $\sum(\mathfrak{G})$ of its root vectors. $\sum(\mathfrak{G})$ is a finite set of vectors of an n-dimensional Euclidean space R^n.

We shall say that a vector from R^n is positive if its first coordinate not equal to zero is positive. We shall call a positive root simple, if it cannot be resolved into positive roots.

The present paper contains the following precise version of Weyl's result: a semi-simple Lie group is completely determined by the system of its simple roots. The problem of classification of simple Lie groups is thus reduced to a simple geometrical problem, namely to find in the space R^n all possible systems of vectors Γ such that.

1. If $a \in \Gamma$, $b \in \Gamma$ and $a \neq b$, $\frac{2(a,b)}{(a,a)}$ is a non-positive integer, where (a,b) denotes the scalar product of the vectors a and b.

2. Γ is the bilineary independent system of vectors.

3. Γ cannot be decomposed into orthogonal subsystems Γ_1 and Γ_2.

An elementary study shows that all solutions of this problem are given by the system of simple roots of the groups $A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7, E_8$.

THE FORMALISM OF LIE GROUPS

A. SALAM
IMPERIAL COLLEGE OF SCIENCE AND TECHNOLOGY, LONDON, UNITED KINGDON

1. INTRODUCTION

Throughout the history of quantum theory, a battle has raged between the amateurs and professional group theorists. The amateurs have maintained that everything one needs in the theory of groups can be discovered by the light of nature provided one knows how to multiply two matrices. In support of this claim, they of course, justifiably, point to the successes of that prince of amateurs in this field, Dirac, particularly with the spinor representations of the Lorentz group.

As an amateur myself, I strongly believe in the truth of the non-professionalist creed. I think perhaps there is not much one has to learn in the way of methodology from the group theorists except caution. But this does not mean one should not be aware of the riches which have been amassed over the course of years particularly in that most highly developed of all mathematical disciplines - the theory of Lie groups.

My lectures then are an amateur's attempt to gather some of the fascinating results for compact simple Lie groups which are likely to be of physical interest. I shall state theorems; and with a physicist's typical unconcern rarely, if ever, shall I prove these. Throughout, the emphasis will be to show the close similarity of these general groups with that most familiar of all groups, the group of rotations in three dimensions.

In 1951 I had the good fortune to listen to Prof. Racah lecture on Lie groups at Princeton. After attending these lectures I thought this is really too hard; I cannot learn this; one is hardly ever likely to need all this complicated matter. I was completely wrong. Eleven years later the wheel has gone full cycle and it is my turn to lecture on this subject. I am sure many of you will feel after these lectures that all this is too damned hard and unphysical. The only thing I can say is: I do very much hope and wish you do not have to learn this beautiful theory eleven years too late.

2. SOURCES

A word about the sources [1] and the scheme I wish to follow. The chief sources in this theory are the famous thesis of Cartan in which most of this subject was created Hermann Weyl and his classical text on "Classical Groups" and Racah's Princeton lectures [2]. However, I believe conceptually the most concise existing treatment of the subject is in the works of DYNKIN [3]. Dynkin's paper has a magnificent appendix which gives a review of the known results and this appendix is my major source. From the point of view of a physicist working on symmetry problems perhaps the best reference is to the review paper of BEHRENDS, LEE, FRONSDAL and DREITLEIN [4]. I have checked with Lee that apparently while these authors knew of Dynkin's work they did not have it accessible when they were writing their review. Thus their treatment of the fundamentals resembles Cartan and Racah more closely rather than Dynkin. Another excellent paper for physicists is SPEISER and TARSKI [5]. For a fuller exposition of Dynkin, reference may also be made to two Imperial College theses - those of NE'EMAN [6] and IONIDES [7].

3. DEFINITIONS

The general theory of Lie groups follows closely the pattern of the one group we are all thoroughly familiar with, the theory of the three-dimensional rotation group O_3. It is indeed a matter of deep regret that the elementary expositions of this familiar case do not employ the same terminology as that of the general theory. Half the conceptual difficulties of the subject would simply disappear if this had consistently been done in our undergraduate courses. To illustrate and to anticipate notation we summarize known facts about the rotation group O_3. (All statements made here will be formalized later.) We know that this group is completely determined by three infinitesimal generators:

$$J^{\pm} = 1/\sqrt{2}\,(J_1 \pm i\,J_2),\ J_3$$

and their commutation relations:

$$[J^+, J_3] = J^+,\ [J^-, J_3] = -J^-,\ [J^+, J^-] = J_3.$$

The commutation relations tell us that

(i) The number of operators (out of these three) which can be diagonalized is one (J_3). Call this number the "rank" of the group. Thus the rank of O_3 = 1.

(ii) Call the eigenvalues of J_3 (i.e. the magnetic quantum numbers) by the name "weights". The highest eigenvalues j of J_3 uniquely labels a representation. We shall call this "the highest weight".

(iii) The commutation relations tell us (from $[J^{\pm}, J_3] = \pm J_3$) that, irrespective of what the weights are, the difference of two consecutive weights is ± 1. These numbers ± 1 which are characteristic of the commutation relations of the group and not of any particular representation are called "roots". In the subsequent general study of Lie groups these three concepts, "rank" of the group, "roots" of the group and "weights" (and particularly the highest weight) will be generalized and will play crucial roles.

(iv) Another way of labelling the representations of O_3 is to use the operator $\underline{J}^2$. This operator commutes with all other operators and thus for a given representation equals a constant multiple of unity. If j is the highest weight, $\underline{J}^2 = j(j+1)\,\underline{I}$. This operator is called the "Casimir operator". We shall find that the concept of a general "Casimir operator" is not as highly developed, and for this reason we shall treat this concept at an early stage (section 5) and then not mention it at all later.

4. MATHEMATICAL PRELIMINARIES

4.1. A group G is a set of elements a, b.... with a composition law (multiplication) such that the following conditions are fulfilled:

(i) if a and b are elements of the set, then also the product c = ab belongs to the set,

(ii) the composition is associative: a (b c) = (a b) c,

(iii) the set contains a unit element e such that a e = ea = a,

(iv) to any element a of the set, there exists one and only one element a^{-1} of the set such that $a^{-1}a = a\,a^{-1} = e$.

The definition of a group does not imply that the two elements ab and ba are equal; i.e., the composition is not necessarily commutative. A group in which all elements commute is called abelian.

A sub-group H of a group G is a sub-set of elements of G, which again fulfils the group postulates. G and the group consisting of the unit element, e, are called trivial sub-groups of G. A sub-group N is called an invariant sub-group of G if for any element n of N ($n \in N$), sns^{-1} is again an element of N where s is any element of G($s \in G$).

A group is called simple if it contains no non-trivial invariant sub-groups, except possibly discrete ones.

A group is called semi-simple if it contains no non-trivial invariant abelian sub-groups, except possibly discrete ones.

4.2. A representation of a group G is a mapping of the group into a set of linear transformations D of a vector space R such that

if ab = c

then

$$D(a)\,D(b) = D(c),$$
$$D(a^{-1}) = D^{-1}(a),$$
$$D(e) = I,$$

where I is the unit operator.

A representation is reducible if it leaves a sub-space of R invariant. Then every transformation matrix can be brought into form:

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix}$$

A representation is fully reducible if every transformation matrix can be written as

$$\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$$

4.3. A Lie group is a group whose elements form an analytic manifold in such a way that the composition ab = c is an analytic mapping of the manifold G × G into G and the inverse $a \to a^{-1}$ is an analytic mapping of G into G. A Lie group can thus be viewed from an algebraic, topological or analytical point of view. The topological concepts of importance are connectedness, compactness and invariant integral on the group (see SPEISER and TARKSI [5]).

A group G is compact if every infinite sequence in G has a limit point in G. For a compact group one can define a finite total volume which is invariant under the group.

For example, the group of rotation in three dimensions O_3 without reflections is a connected and compact group. The proper Lorentz group is connected but not compact and the improper Lorentz group is neither connected nor compact.

The study of simple groups is important because every semi-simple connected group is essentially a direct product of simple groups, and any connected compact Lie group is essentially a product of a semi-simple and a one-parameter (abelian) compact group.

Ex. $O_4 \approx O_3 \times O_3$; O_3 simple; O_4 semi-simple.

The symbol $\approx$ means locally isomorphic. From now on we consider only simple compact Lie groups.

5. SIMPLE COMPACT LIE GROUPS

So far as a physicist is concerned, a Lie group is a group of transformation of variables which depend analytically on a finite set of N parameters. The fundamental idea of Lie was to consider not the whole group but that part of it which lies close to the identity consisting of the so-called infinitesimal transformations. To formalize this, we have Theorem I.

Theorem 1

Every representation of a compact Lie group is equivalent to a unitary representation and is fully reducible (RACAH, WEYL [2]). Thus, since the matrices D(g) can be taken as unitary, they can be put into the form:

$$D = \exp(i\epsilon^{\alpha} X_{\alpha}),$$

where X_α are constant hermitian matrices ($X_\alpha^+ = X_\alpha$), which are called infinitesimal generators of the group. $\epsilon^\alpha (\alpha = 1, 2, \ldots, N)$ are N real parameters on which the set of transformations D depend.

The group is called unimodular if for any D(S), $\det[D(s)] = 1$.

Then $\operatorname{tr} X = 0$.

Theorem 2

Fundamental Theorem of Lie

The local structure of a Lie group is completely specified by the commutation relations between the operators X_α:

$$[X_\alpha, X_\beta] = C^\gamma_{\alpha\beta} X_\gamma \; ; \; \alpha, \beta, \gamma = 1, 2, \ldots, N, \qquad (5.1)$$

where the coefficients $C^\gamma_{\alpha\beta}$ which are independent of the representations of

the group are numbers (called the structure constants of the group). These numbers satisfy two requirements:

(a) antisymmetry in the two lower indices

$$C^\gamma_{\alpha\beta} = -C^\gamma_{\beta\alpha},$$

(b)

$$C^\delta_{\alpha\beta} C^\epsilon_{\delta\gamma} + C^\delta_{\gamma\alpha} C^\epsilon_{\delta\beta} + C^\delta_{\beta\gamma} C^\epsilon_{\delta\alpha} = 0.$$

Note that conditions (a) and (b) are equivalent to the antisymmetry of the Commutator bracket $[X_\alpha, X_\beta]$ and the Jacobi identity:

$$[[X_\alpha, X_\beta], X_\gamma] + [[X_\gamma, X_\alpha], X_\beta] + [[X_\beta, X_\gamma], X_\alpha] = 0.$$

Rewrite (b) in the form:

$$(C_\alpha)^\epsilon_\delta (C_\beta)^\delta_\gamma - (C_\beta)^\epsilon_\delta (C_\alpha)^\delta_\gamma = C^\delta_{\alpha\beta} (C_\delta)^\epsilon_\gamma.$$

Thus, we have shown the following:

Theorem 3

The N matrices C_α with matrix elements $(C_\alpha)^\epsilon_\gamma$ form the so-called regular or adjoint representation of the Lie algebra*.

The problem of classification of Lie groups is the problem of finding the numbers c's which satisfy (a) and (b) and then of finding N constant matrices which satisfy the fundamental commutation relation of Theorem 1. This problem was completely solved by Cartan in 1913. Before however we state Cartan's results, we first wish to recast the fundamental commutation relation (5.1) in a "canonical" form and also get over a number of auxiliary results connected with Casimir operators.

6. CASIMIR OPERATORS

From the structure constants we can define a metric tensor:

$$g_{\mu\nu} = C^\beta_{\mu\alpha} C^\alpha_{\nu\beta}.$$

Theorem 4

The necessary and sufficient condition for a Lie group to be semi-simple is that

* The set of N matrices X_α span a linear vector space over the field of complex numbers and define a Lie Algebra; the sum of two matrices is an element of the algebra and so is their commutator. Lie algebras and Lie groups possess a one-one correspondence, and it is possible to go freely from Lie groups to Lie algebras. The study of Lie algebras (first introduced by Weyl) is in effect the study of the infinitesimal aspect of Lie group theory. Even though it is galling to bring in a new concept (of a Lie algebra) at this stage, this apparently improves the mathematical rigour of the statements made in these lectures!

$$\det [g_{\mu\nu}] \neq 0 \quad \text{(Cartan)}.$$

Thus for a semi-simple group we can define an inverse metric $g^{\mu\nu}$ such that

$$g^{\mu\nu} g_{\nu\rho} = \delta^{\mu}_{\rho},$$

and we can use the metric tensors for raising and lowering indices.

Now define an operator $F = g_{\alpha_1 \alpha_2} X^{\alpha_1} X^{\alpha_2}$. This is called the Casimir operator and has the property that it commutes with all the generators of the group:

$$[F, X_{\alpha}] = 0.$$

The proof of the result is trivial. The significance of the Casimir operator lies in recalling that by Schur's Lemma any operator which commutes with all the generators of the group must be a multiple of the identity.

For O_3 this operator is the total angular momentum $\underline{J}^2$. One can define generalized Casimir operators:

$$F^n = C^{\beta_2}_{\alpha_1 \beta_1} C^{\beta_3}_{\alpha_2 \beta_2} \ldots\ldots C^{\beta_1}_{\alpha_n \beta_n} X^{\alpha_1} X^{\alpha_2} \ldots\ldots X^{\alpha_n}.$$

It is easy to see that all these commute with X^{α}.

For O_3 all inequivalent irreducible representations can be characterized by giving different values of λ where $\lambda I = J^2$. The question arises if this is true in general. Racah gives the following partial answer: Write the set $\{\lambda^k\}$ defined by $\lambda^k I = F^k$. For simple groups if the representation D and $(D^{-1})^T$ are equivalent representations, then the set $\{\lambda^k\}$ gives an unequivocal characterization of all the inequivalent representations.

7. CANONICAL FORMS OF THE COMMUTATION RELATIONS AND RANK OF A GROUP

Theorem 6 (P. Ionides)

By a suitable choice of linear combination of the X's, the $C^{\alpha}_{\beta\gamma}$ can be made antisymmetric in all three indices and pure imaginary; i.e. one can write the commutation relations in the form:

$$[X_{\alpha}, X_{\beta}] = i f_{\alpha\beta\gamma} X_{\gamma},$$

with $f_{\alpha\beta\gamma}$ purely antisymmetric and real.

In the usual theory of angular momentum, the first step is to rewrite (the Ionides type of) commutation relations,

$$[J_{\alpha}, J_{\beta}] = i \epsilon_{\alpha\beta\gamma} J_{\gamma} \quad , \quad \alpha, \beta, \gamma = 1, 2, 3, \tag{7.1}$$

in the so-called "canonical form". Defining the non-hermitian operators,

$$J_{\pm} = (J_1 \pm i J_2)/\sqrt{2},$$

we rewrite (7.1) as

$$\begin{aligned} [J_{\pm}, J_3] &= \pm J_{\pm}, \\ [J_+, J_-] &= J_3. \end{aligned} \tag{7.2}$$

There are two virtues of this canonical form:

(1) If J_3 is diagonalized $(J_3 | m) = m | m)$, we infer from (7.2) that the operators $J_{\pm}$ act as "creation" and "annihilation" operators.

(2) (7.2) shows that the consecutive eigenvalues m of J_3 differ by ± 1. Our first task is to cast the commutation relations (5.1) in the "canonical form".

Assume that among the N generators, there are ℓ which mutually commute and can thus be simultaneously diagonalized. This number ℓ is called the rank, and we shall designate these ℓ (hermitian) operators as $H_1, H_2 \ldots \ldots H_{\ell}$. (For O_3, $\ell = 1$). These operators have a direct physical meaning since their eigenvalues for any representation provide us the quantum numbers.

Let us consider $H_1, H_2 \ldots . H_{\ell}$ as the components of an ℓ-dimensional operator-valued vector $\underline{H}$. The components of $\underline{H}$ clearly satisfy the commutation relations:

$$[H_i, H_j] = 0 \qquad \text{for } i, j = 1, 2, \ldots\ldots, \ell.$$

If the dimension of the algebra is N (i.e. the number of parameters of the corresponding group is N), we still need $(N - \ell)$ elements to complete a basis of the algebra. A suitable choice of these is provided by the following:

Theorem 7

There exists a basis of the Lie algebra consisting of the elements $H_1, H_2 \ldots, H_{\ell}$; $E_{\pm 1}, E_{\pm 2} \ldots E_{\pm (N-\ell)/2}$ such that the following commutation relations hold:

$$[\underline{H}, E_{\alpha}] = \underline{r}(\alpha) E_{\alpha}, \tag{7.3}$$

$$[E_{\alpha}, E_{-\alpha}] = \underline{r}(\alpha) \underline{H}, \tag{7.4}$$

$$[E_{\alpha}, E_{\beta}] = N_{\alpha\beta} E_{\gamma} \text{ for } \alpha \neq -\beta, \tag{7.5}$$

with $\alpha, \beta = \pm 1, \pm 2, \ldots \pm (N-\ell)/2$. E's are non-hermitian matrices and $\underline{r}(\alpha)$ are real vectors in an ℓ-dimensional space. The $\underline{r}$'s are called roots of the algebra; they have the property that

$$\underline{r}(\alpha) = -\underline{r}(-\alpha). \tag{7.6}$$

Clearly the total number of the roots is $(N - \ell)$.

The scalar product appearing in (7.4) is the usual Euclidean scalar product provided the H's are chosen in such a way that the following normalization conditions hold:

$$\sum_{\alpha} r_i(\alpha) r_j(\alpha) = R \delta_{ij}; \; ij = 1, 2 \ldots \ell, \tag{7.7}$$

with an arbitrary scale constant. Finally, $N_{\alpha\beta}$ are real numbers which are different from zero if and only if $\underline{r}(\alpha) + \underline{r}(\beta)$ is also a root.

The roots, being essentially our old friends the structure constants, specify completely the group (at least in the local sense). They possess a twin role in the theory. First, as may be inferred from (7.3), the roots are the differences of the eigenvalues of $\underline{H}$. Second and more important for our present purposes, the roots allow us to classify Lie groups. In terms of the roots we can state Cartan's solution of the problem of finding all simple Lie groups. The crucial theorem here is Theorem 8 which lists further properties of the roots and in terms of these gives a complete classification of Lie groups.

8. CLASSIFICATION OF LIE GROUPS

A root is said to be positive if its first non-vanishing component (in an arbitrary basis) is positive. A root is called simple if it is a positive root and in addition it cannot be decomposed into the sum of two positive roots.

Theorem 8

(i) For a simple group of rank ℓ there exist ℓ simple roots and they are all linearly independent. (We shall call the set of simple roots the π-system.)

(ii) Every positive non-simple root can be expressed as a linear combination $\sum_{\underline{r}(\alpha)\in\pi} R_\alpha \underline{r}(\alpha)$ where R_α are non-negative integers.

(iii) If $\underline{r}(\alpha)$ and $\underline{r}(\beta)$ are two simple roots, the angle $\theta_{\alpha\beta}$ between these can take only the following values:

$$90^\circ \qquad 120^\circ \qquad 135^\circ \quad \text{and} \quad 150^\circ,$$

so that $2\,\underline{r}(\alpha)\cdot\underline{r}(\beta)/\underline{r}(\alpha)\cdot\underline{r}(\alpha)$ and $2\,\underline{r}(\alpha)\cdot\underline{r}(\beta)/\underline{r}(\beta)\cdot\underline{r}(\beta)$ are both integers.

(iv) For every simple group, all the simple roots either have the same length or their length ratios assume simple values. More explicitly one has

$$\frac{|r(\alpha)|^2}{|r(\beta)|^2} = \begin{cases} 1 & \text{if } \theta_{\alpha\beta} = 120^\circ \\ 2 & \text{if } \theta_{\alpha\beta} = 135^\circ \\ 3 & \text{if } \theta_{\alpha\beta} = 150^\circ. \end{cases}$$

If $\theta_{\alpha\beta} = 90^\circ$, the ratio of lengths is undetermined.

Dynkin diagrams

As we shall see in a moment, the geometrical properties of the simple roots in the π-system characterize in a unique manner the corresponding Lie groups. Therefore it is most convenient to incorporate them in a schematic diagram. These diagrams (the so-called Schouten-Dynkin diagrams) are drawn in Fig. 1.

From Theorem 8, the lengths of the simple roots of a given simple Lie group can assume at most two different values. This fact together with the

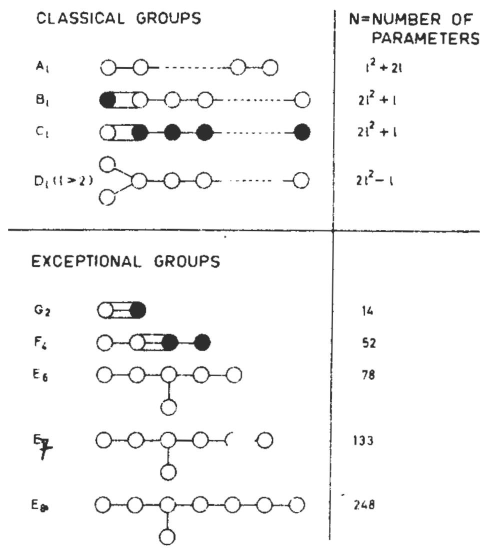


Fig. 1

Cartan solution of all possible single Lie groups.

properties about the angles enumerated above can be symbolically described by associating with each simple root a small circle. For the roots of greatest length the circle is marked in black. If the angle between two consecutive simple roots is equal to 120°, 135° or 150°, the corresponding circles are joined by simple, double or triple lines respectively. If the angle is 90°, the circles are not joined. For a group of rank ℓ there are ℓ simple roots and therefore ℓ circles (black or white).

In terms of these diagrams we give now the Cartan solution of all possible simple Lie groups. Broadly these fall into two categories: the so-called "classical groups" and the five "exceptional groups".

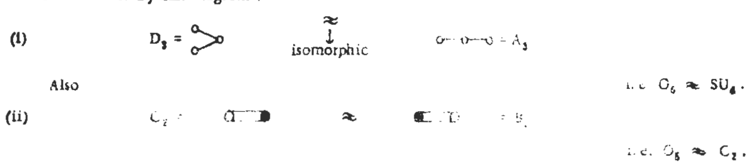
To anticipate we shall find that the classical Lie groups are some of the well known objects:

A_ℓ is the group of unitary unimodular matrices in complex space of $(\ell + 1)$ dimensions $(SU_{\ell+1})$.

B_ℓ and D_ℓ are groups of orthogonal transformations (rotations) in real spaces of $2\ell + 1$ and 2ℓ dimensions respectively $(O_{2\ell+1}$ and $O_{2\ell})$.

C_ℓ is the group of unitary matrices U in complex space of 2ℓ dimensions which fulfil the condition $U^T J U = J$ where J is a non-singular antisymmetric matrix (the symplectic group)*.

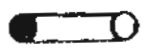
* Note from the Dynkin diagrams:

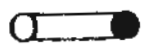


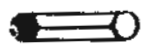
To take simple examples of root structures:

For $\ell = 1$ (i.e. group O_3) there is just one simple root + 1. The space spanned by simple roots (the π-space) is {1}. For $\ell = 2$, the space is a plane, the relevant groups being

A_2:  Two simple roots of equal length, and the angle between them is 120°.

B_2:  Two simple roots. Their length ratio is 2. The angle between them is 135°.

C_2: 

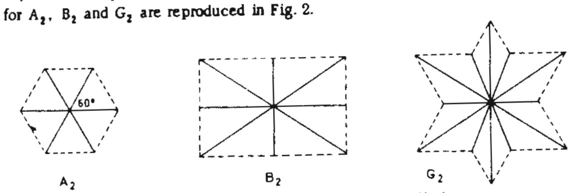
G_2:  Two simple roots with length ratio equal to $\sqrt{3}$, and angle 150°.

D_2: is semi-simple, $D_2 \approx A_1 \times A_1$

Summarizing this section then, from the Dynkin diagrams we read off immediately the rank ℓ of the group, the lengths of the simple roots and their mutual angles (and of course the dimensionality of the Euclidean space (π) spanned by these ℓ independent vectors)*,**. The simple roots $\underline{r}(1)$, $\underline{r}(2), \ldots, \underline{r}(\ell)$, are given by the following formulae:

---

* It is perhaps worthwhile to make the reminder at this stage that not all roots are simple. In fact the total number of roots is $(N-\ell)$, the distinct ones being $(N-\ell)/2$ in virtue of $\underline{r}(\alpha) = -\underline{r}(-\alpha)$, $\alpha = 1, 2, \ldots, (N-\ell)/2$. The remaining $(N-3\ell)/2$ distinct non-simple roots can easily be constructed, and in Footnote ** we give a complete ansatz for drawing a complete root diagram (for $\ell = 2$ for example in a plane; for $\ell = 3$ in {3} space and so on). Personally, I consider these diagrams pointless. However, to satisfy current prejudice the root diagrams for A_2, B_2 and G_2 are reproduced in Fig. 2.



A_2 N = 8 B_2 N = 10 G_2 N = 14

Fig. 2

Root diagrams for A_2, B_2 and G_2

** The following scheme incorporates all the requirements about angles and lengths of simples roots specified by the diagrams.

For A_ℓ define the following vectors:

$$\underline{\lambda}_1, \underline{\lambda}_2, \ldots \underline{\lambda}_{\ell+1}$$

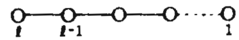
by the conditions

$$\underline{\lambda}_1 + \underline{\lambda}_2 + \ldots + \underline{\lambda}_{\ell+1} = 0,$$

$$\underline{\lambda}_1^2 = \underline{\lambda}_2^2 = \ldots = \underline{\lambda}_{\ell+1}^2 = \ell A,$$

$$\underline{\lambda}_p \cdot \underline{\lambda}_q = -A, \quad p \neq q = 1, 2, \ldots \ell+1.$$

$$\underline{r}(\ell) = \underline{\lambda}_\ell - \underline{\lambda}_{\ell+1},$$
$$\underline{r}(\ell-1) = \underline{\lambda}_{\ell-1} - \underline{\lambda}_\ell, \quad (8.1)$$
$$\vdots$$
$$\underline{r}(1) = \underline{\lambda}_1 - \underline{\lambda}_2.$$

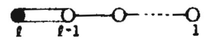


For B_ℓ: the simple root structure is as follows:

$$\underline{r}(\ell) = \underline{\lambda}_\ell, \text{ (This is the smallest root)}$$
$$\underline{r}(\ell-1) = \underline{\lambda}_{\ell-1} - \underline{\lambda}_\ell, \quad (8.2)$$
$$\underline{r}(1) = \underline{\lambda}_1 - \underline{\lambda}_2,$$

where

$$\underline{\lambda}_1^2 = \underline{\lambda}_2^2 = \ldots\ldots \underline{\lambda}_\ell^2 = A,$$
$$\underline{\lambda}_p \underline{\lambda}_q = 0, \; p \neq q, \quad (8.3)$$



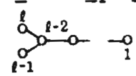
For C_ℓ: the simple roots are given by:

$$\underline{r}(\ell) = 2\underline{\lambda}_\ell, \text{ (This is the greatest root.)}$$
$$\underline{r}(\ell-1) = \underline{\lambda}_{\ell-1} - \underline{\lambda}_{\ell-2}, \quad (8.4)$$
$$\vdots$$
$$\underline{r}(1) = \underline{\lambda}_2 - \underline{\lambda}_1.$$

where the $\underline{\lambda}$'s satisfy (8.3).

For D_ℓ: the simple roots are given by:

$$\underline{r}(\ell) = \underline{\lambda}_{\ell-1} + \underline{\lambda}_\ell,$$
$$\underline{r}(\ell-1) = \underline{\lambda}_{\ell-1} - \underline{\lambda}_\ell \quad (8.5)$$
$$\vdots$$
$$\underline{r}(1) = \underline{\lambda}_1 - \underline{\lambda}_2$$



The λ's satisfy (8.3). So much for simple roots. All roots are given for the classical groups by the following expressions:

A_ℓ: $(\underline{\lambda}_p - \underline{\lambda}_q)$; p, q = 1, 2, ..., $\ell + 1$

$$\left.\begin{array}{ll} B_\ell : & \pm \underline{\lambda}_p,\ \pm\underline{\lambda}_p \pm \underline{\lambda}_q\ ;\ p,\ q = 1, 2, \ldots\ldots, \ell \\ C_\ell : & \pm 2\underline{\lambda}_p,\ \pm\underline{\lambda}_p \pm \underline{\lambda}_q\ ;\ p, q = 1, 2, \ldots\ldots, \ell \\ D\ : & \pm \underline{\lambda}_p \pm \lambda_q\ ;\ p, q = 1, 2, \ldots\ldots, \ell. \end{array}\right\}$$

The ± signs are to be taken in arbitrary combinations.

Similar expressions can be given for the exceptional groups. Also one can give a full correspondence between the "canonical" expressions for the commutation relations and the more familiar manner in which one writes the commutation relations for the orthogonal, symplectic groups, etc.

Thus, for the orthogonal group in $(2\ell + 1)$ dimensions which leaves invariant the quadratic form

$$\sum_{p=-\ell}^{\ell} X^p X^{-p}$$

one may write the infinitesimal operators:

$$X_{pq} = -X_{qp} = X^p \frac{\delta}{\delta X^{-q}} - X^q \frac{\delta}{\delta X^{-p}},$$

with the commutation relations:

$$[X_{ik}, X_{mn}] = \delta_{k+m} X_{in} - \delta_{k+n} X_{im} - \delta_{\ell+m} X_{kn} - \delta_{\ell+n} X_{km}$$

where $\delta q = 1$ if $q = 0$ and zero otherwise. These operators correspond to the E's and the H's of B_ℓ if we make the following identifications:

$$X_{p-p} \equiv H_p,\ X_{\pm p\pm q} \equiv E_{\pm\lambda p \pm\lambda q},\ X_{0\pm p} = E_{\pm\lambda p}\ ;\ p, q > 0.$$

Similar correspondence can be stated for A_ℓ, C_ℓ, D_ℓ etc. (Racah's notes).

9. REPRESENTATIONS OF LIE GROUPS: WEIGHTS

9.1. Now we come to physically the most important problem of all - the problem of finding representations of the group, i.e. the matrices corresponding to $\underset{\sim}{H}$ and E_α.

Consider a representation of dimension (or degree) d. Since $H_1, H_2, \ldots, H_\ell$ are hermitian matrices, and since they commute with each other, we can simultaneously diagonalize these. Let $|m\rangle$ be a simultaneous eigenket:

$$\underset{\sim}{H}\ |m\rangle = \underline{m}\ |m\rangle. \tag{9.1}$$

Since H's are d × d matrices, the total number of such eigenkets $|m\rangle$ is d.

The $\underline{m}$'s in Eq. (9.1) are real numbers and are called "weights". They form ℓ-dimensional vectors in a Euclidean space for whose basis one may take the π-space of the group (the space spanned by the ℓ simple roots). Summarizing, for the case of a group of rank ℓ and for a given representation of dimensionality d, there are

- ℓ : simple root vectors
- $(N-3\ell)/2$: distinct non-simple root vectors
- d : weight vectors (provided we count each weight vector as many times as its multiplicity indicates, the multiplicity being defined as the number of independent eigenkets $|m\rangle$ corresponding to a given weight $\underset{\sim}{m}$).

Note that root vectors are characteristic of the group. They are really the structure constants. The weight vectors on the other hand are characteristic of the representation. There are only ℓ linearly independent roots (simple roots). There are also only ℓ linearly independent weight vectors. The simplest (oblique axis) basis for the weight vectors is that provided by the simple root vectors.

All this intertwining of weights and roots is exciting enough, but still further and the more exciting result comes when we look for the analogue of the result in O_3 that all weights are either integers or half-integers. The analogous result is Theorem 9, which gives the "component" of any weight-vector along a simple root-vector.

Theorem 9

For every weight $\underline{m}$, the number $\underline{m}\cdot\underline{r}(\alpha)/\underline{r}(\alpha)\cdot\underline{r}(\alpha)$, where $\underline{r}(\alpha)\in\pi$, is an integer or a half-integer, $\gtrless 0$.

Theorem 9 provides the justification for Dynkin's insistence on simple roots as the primary entities on which all conceptual emphasis should be placed. Dynkin cares neither for the non-simple roots nor for the weight vectors. Given the simple roots, Theorem 9 tells us what the weights look like through the simplest possible generalization of the familiar results for the (3) rotation group*. In this insistence on simple roots possibly lies the superiority of Dynkin's presentation of Lie group theory.

10. IRREDUCIBLE REPRESENTATIONS AND THEIR DIMENSIONALITY

Definition: A weight $\underline{m}$ is said to be higher than $\underline{m}'$ if $\underline{m}-\underline{m}'$ has a positive number for its first non-vanishing component in an arbitrary basis. The weight $\underline{\Lambda}$ which is higher than all the others is called the highest (or greatest) weight.

Theorem 10

A representation is uniquely characterized by its highest weight $\underline{\Lambda}$, and the highest weight always has multiplicity one.

---

* Earlier it was mentioned that roots are differences of weights. The formal result is: If $|\underline{m}\rangle$ is an eigenket of $\underset{\sim}{H}$ corresponding to a weight $\underline{m}$, $E_\alpha|\underline{m}\rangle$ is also an eigenket with weight $\underline{m} + \underline{r}(\alpha)$. The result follows from

$$[E_\alpha, \underset{\sim}{H}] = \underline{r}(\alpha) E_\alpha.$$

Note the role of E_α as a creation operator.

Theorem 11

In order that a vector $\underline{\Lambda}$ be the highest weight of some irreducible representation, it is necessary and sufficient that j_α, defined as $j_\alpha = \underline{\Lambda}\cdot\underline{r}(\alpha)/\underline{r}(\alpha)\cdot\underline{r}(\alpha)$, is a non-negative integer or half-integer.

Thus to get the irreducible representations of any Lie group, we should mark each circle in the Dynkin diagram with a non-negative integer or half-integer j_α. These numbers characterize uniquely the irreducible representation with $\underline{\Lambda}$ as its highest weight, the "components" $\underline{\Lambda}\cdot\underline{r}(\alpha)/\underline{r}(\alpha)\cdot\underline{r}(\alpha)$ of $\underline{\Lambda}$ being just $(j_1, j_2 \ldots\ldots)$. The dimensionality of this representation is given by the following theorem of Weyl:

Weyl's Theorem: Theorem 12

Let Σ_+ be the system of all positive roots of a semi-simple Lie algebra, and let an irreducible representation be uniquely characterized by the highest weight $\underline{\Lambda}$. Then its dimensionality d is given by the formula:

$$d = \prod_{\underline{r}(\alpha)\in\Sigma_+}\left[1 + \underline{\Lambda}\cdot\underline{r}(\alpha)\right]/\underline{g}\cdot\underline{r}(\alpha),$$

where

$$\underline{g} = \tfrac{1}{2}\sum_{\underline{r}(\beta)\in\Sigma_+}\underline{r}(\beta).$$

If one writes the vectors $\underline{\Lambda}$ and $\underline{g}$ in terms of the auxiliary quantities λ's previously introduced in the third footnote of section 8,

$$\underline{\Lambda} = \Sigma f_i\,\underline{\lambda}_i,$$
$$\underline{g} = \Sigma g_i\,\underline{\lambda}_i.$$

The Weyl formula above gives the explicit expressions listed in Table I.

As examples consider some of the interesting physical cases, namely, the case of rank $\ell = 2$. In this case the number of commuting matrices in the algebra is two, and we can associate them, for example, with the third component of the isotopic spin and the hypercharge. The only simple compact Lie groups of rank 2 are A_2, B_2, C_2 and G_2. Any irreducible representation of these groups can be labelled by means of two non-negative integers j_1, j_2. The formulae for the dimensionality given in Table I can be written explicitly in a simple way and is shown in Table II.

For instance, for the simplest choices of the arrays j_1, j_2 one gets the following dimensions:

A_2 :		$B_2(\approx C_2\approx O_5)$:		G_2 :	
d(0, 0) =	1	d(0, 0) =	1	d(0, 0) =	1
d(1, 0) =	3	d($\frac{1}{2}$, 0) =	4	d($\frac{1}{2}$, 0) =	7
d(0, $\frac{1}{2}$) =	3	d(0, $\frac{1}{2}$) =	5	d(0, $\frac{1}{2}$) =	14
d(1, 0) =	6	d(1, 0) =	10	d(1, 0) =	27
d($\frac{1}{2}$, $\frac{1}{2}$) =	8	d(0, 1) =	14		
d(1, $\frac{1}{2}$) =	15	d($\frac{1}{2}$, $\frac{1}{2}$) =	16		
d(1, 1) =	27				

TABLE I

(j_i's are non-negative integers or half-integers)

Group	number of parameters*	Dynkin diagrams	Dimension of the irred. represent.		Expressions for f and g
A_ℓ	$\ell^2 + 2\ell$		$\prod_{p,q}(1+\alpha_{pq})$	where $\alpha_{pq} = \frac{f_p - f_q}{g_p - g_q}$, $\beta_{pq} = \frac{f_p + f_q}{g_p + g_q}$, $\gamma_p = f_p/g_p$	$f_k = f_{\ell+1} + 2\sum_{i=k}^{\ell} j_i$; $f_{\ell+1} = \frac{-4}{\ell+1}\sum_{i=1}^{\ell} i j_i$; $g_k = \frac{-\ell}{2} + (\ell-k+1)$; $g_{\ell+1} = -\ell/2$; Note $f_1 + \ldots + f_{\ell+1} = 0$
B_ℓ	$2\ell^2 + \ell$		$\prod_{p,q}(1+\gamma_p)(1+\alpha_{pq})(1+\beta_{pq})$	"	$f_k = j_\ell + 2\sum_{i=k}^{\ell-1} j_i$; $g_k = (\ell - k + \frac{1}{2})$
C_ℓ	$2\ell^2 + \ell$		"	"	$f_k = 2\sum_{i=k}^{\ell} j_i$; $g_k = (\ell-k+\frac{1}{2})$
D_ℓ	$2\ell^2 - \ell$		$\prod_{p,q}(1+\gamma_p)(1+\alpha_{pq})$	"	$f_k = j_{\ell-1} + j_\ell + 2\sum_{k}^{\ell-2} j_i$; $f_\ell = j_{\ell-1} - j_\ell$, $g_\ell = 0$. $g_k = \ell - k$ for $k \leq \ell - 1$

* The products here range over all possible values of p and q; the indices denoted by distinct letters must have distinct values, and of all sets of values obtained from one another by permutations of indices only one must be chosen.

TABLE II

Group	Number of parameters N	Dimension of the irr. rep.
A_2	8	$\frac{1}{2!}(J_1)(J_2)[J_1+J_2]$
B_2 } C_2 }	10	$\frac{1}{3!}(J_1)(J_2)[J_1+J_2](2J_2+J_1)$
G_2	14	$\frac{1}{5!}(J_1)(J_2)(J_1+J_2)[2J_2+J_1]\times$ $[3J_2+J_1][3J_2+2J_1]$

{ Note: Here $J_1 = (2j_1+1)$ and $J_2 = (2j_2+1)$ }

These numbers $d(j_1, j_2)$* represent the number of particles which can be accomodated in any given multiplet in physical applications.

The adjoirt (or regular) representation R plays a very important role in vector meson theories. For the case of $\ell = 2$, these representations are the following:

$$A_2 \quad : \quad d_R = d(\tfrac{1}{2}, \tfrac{1}{2}) = 8,$$
$$B_2(C_2): \quad d_R = d(1, 0) = 10,$$
$$G_2 \quad : \quad d_R = d(0, \tfrac{1}{2}) = 14.$$

These groups, therefore, can accommodate 8, 10 and 14 vector gauge mesons respectively if these mesons correspond to the adjoint representation.

11. COMPUTATION OF ALL WEIGHTS OF A GIVEN IRREDUCIBLE REPRESENTATION

Notwithstanding the fact that the greatest weight uniquely characterizes an irreducible representation, it is important for physical applications to be able to compute all the weights of an irreducible representation. Later we shall construct weight diagrams for some irreducible representation of low dimensionality for the case of rank 2 groups (A_2, B_2, C_2, G_2). In contrast to the root diagrams, the weight diagrams are directly of physical interest.

An explicit method to calculate all the weights in terms of the highest weight and the simple roots is given by the next theorem. We have learnt earlier that the roots equal differences of weights.

* I have introduced a small change of notation in the labelling of representations. Dynkin and Behrends et al. label irreducible representations with numbers $a_1, a_2, \dots, a_\ell$ where a_i are (non-negative) integers. I have used for labelling the numbers $j_1, j_2, \dots, j_\ell$ where the j's are (non-negative) integers or half-integers. The new notation possibly brings out still more the fact that a general Lie group of rank ℓ is a simple "generalization" of O_3 and has ℓ distinct "angular momenta" $j_1, j_2, \dots, j_\ell$ rather than just one (j_1).

Let $\underline{\Lambda}$ and W be the highest weight and the set of all weights respectively of a given irreducible representation.

An element $\underline{m} \in W$ is said to belong to the layer $\Delta^{(k)}$ if it can be obtained by subtracting K simple roots from $\underline{\Lambda}$. Clearly $\Delta^{(0)}$ consists only of $\underline{\Lambda}$, and

$$W = \Delta^{(0)} \cup \Delta^{(1)} \cup \Delta^{(2)} \dots\dots$$

Note that all the layers are disjointed.

Theorem 13

Every element $\underline{m}^{(k)} \in \Delta^{(k)}$ can be expressed as

$$\underline{m}^{(k)} = \underline{m}^{(k-1)} - \underline{r}(\alpha),$$

where

$$\underline{m}^{(k-1)} \in \Delta^{(k-1)}$$

and

$$\underline{r}(\alpha) \in \pi.$$

However, if $\underline{m}^{(k-1)}$ belongs to $\Delta^{(k-1)}$ and $\underline{r}(\alpha)$ is an arbitrary simple root, the difference $\underline{m}^{(k-1)} - \underline{r}(\alpha) \in \Delta^{(k)}$ if and only if the following condition is satisfied:

$$2\,\underline{m}^{(k-1)} \cdot \underline{r}(\alpha)/\underline{r}(\alpha)\cdot \underline{r}(\alpha) + Q > 0,$$

where the number Q is defined by the requirements:

$$\underline{m}^{(k-1)} + q\,\underline{r}(\alpha) \quad \in W \quad \text{for } q \leqslant Q,$$
$$m^{(k-1)} + q\,\underline{r}(\alpha) \quad \bar{\in} W \quad \text{for } q = Q+1.$$

Example:

Perhaps the best way to show that the theorem is actually quite harmless and simple in practice is to construct the weights for a specific case. Consider the group $A_2 \approx SU_3$ for which $\ell = 2$. The Dynkin diagram is O—O. The π-space is two-dimensional; and if we call the roots α and β, the diagram tells us that their lengths are equal ($|\alpha|^2 = |\beta|^2$) and the angle between them is 120° so that

$$\underline{\alpha}\cdot\underline{\beta}/\underline{\alpha}\cdot\underline{\alpha} = -\tfrac{1}{2}.$$

Consider now the regular representation $(\frac{1}{2}, \frac{1}{2})$. The dimensionality in this case is d = 8, so that the representation could accommodate 8 particles. The "components" of the highest weight $\underline{\Lambda}$ (ie) j_α, j_β are given by

$$j_\alpha = \underline{\Lambda}\cdot\underline{\alpha}/\underline{\alpha}\cdot\underline{\alpha} = \tfrac{1}{2}, \qquad (11.1)$$

$$j_\beta = \underline{\Lambda}\cdot\underline{\beta}/\underline{\beta}\cdot\underline{\beta} = \tfrac{1}{2}. \qquad (11.2)$$

Noticing that $\underline{\alpha}$ and $\underline{\beta}$ do not form an orthogonal basis, we find from (11.1) and (11.2) that

$$\underline{\Lambda} = \underline{\alpha} + \underline{\beta}.$$

Now using Theorem 13, if we are given an arbitrary weight M and we wish to know whether $\underline{M}-\underline{\alpha}$ is a possible weight or not, we proceed as follows:

Write the series $\underline{M}$, $\underline{M}+\underline{\alpha}$, $\underline{M}+2\underline{\alpha}$, ... $\underline{M}+(Q+1)\underline{\alpha}$ where Q is an integer. The series terminates for a Q defined by the requirement that while $\underline{M}$, $\underline{M}+\underline{\alpha}$, ... $\underline{M}+Q\underline{\alpha}$ are weights, $\underline{M}+(Q+1)\underline{\alpha}$ is not a weight. Now compute the number,

$$Q + M_\alpha \quad \text{where } M_\alpha = 2\,\underline{M}\cdot\underline{\alpha}/\underline{\alpha}\cdot\underline{\alpha}.$$

If $M_\alpha + Q > 0$, then $\underline{M}-\underline{\alpha}$ is a weight; otherwise it is not. In starting this procedure the crucial point to remember is that $\underline{\Lambda} + \underline{\alpha}$ where $\underline{\alpha}$ is a simple root is never a possible weight.

Consider now the case when $\underline{M} = \underline{\Lambda}$. Since $\underline{\Lambda} + \underline{\alpha}$ is not a weight, Q = 0. Since

$$\Lambda_\alpha = \underline{\Lambda}\cdot\underline{\alpha}\,/\,\underline{\alpha}\cdot\underline{\alpha} = j_\alpha > 0, \tag{11.3}$$

we see from (11.3) that $\underline{\Lambda}-\underline{\alpha}$ is indeed a weight. Likewise, since $j_\beta > 0$, $\underline{\Lambda}-\underline{\beta}$ is also a weight.

We can now start with $(\underline{\Lambda}-\underline{\alpha})$ and test if $(\underline{\Lambda}-\underline{\alpha})-\underline{\alpha}$ and $(\underline{\Lambda}-\underline{\alpha})-\underline{\beta}$ are possible weights or not. It is easy to see that $\underline{\Lambda}-2\underline{\alpha}$ is not a weight, but $\underline{\Lambda}-\underline{\alpha}-\underline{\beta}$ is. Proceeding in this fashion, we find that all possible weights are given by the diagram shown in Fig. 3.

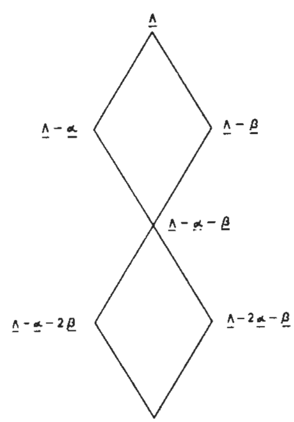


Fig. 3

Notice that the weight $\underline{\Lambda}-\underline{\alpha}-\underline{\beta}$ is of multiplicity two. The diagram does not further fan out, and we obtain a totality of eight weights. Writing $\Lambda = \underline{\alpha}+\underline{\beta}$, we have the following system of weights:

$$\underline{\alpha}+\underline{\beta},\ \underline{\alpha},\ \underline{\beta},\ 0,\ 0,\ -\underline{\beta},\ -\underline{\alpha},\ -(\underline{\alpha}+\underline{\beta}). \tag{11.4}$$

The multiplicities are spindle-shaped: they increase, come to a maximum and decrease again. (The weight zero has multiplicity two.) This is a general result which will not be discussed further.

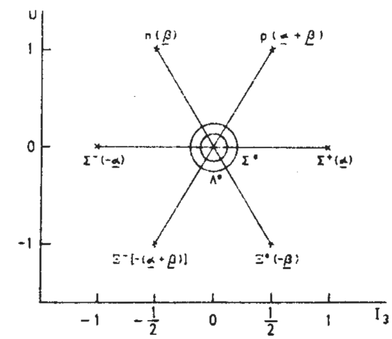


Fig. 4

Euclidean diagrams

Fig.4 gives the Euclidean diagram of these weights. The two rings in the centre indicate the two zero weights. A tentative identification of the stable baryons with the appropriate weights has also been made in the figure, provided we identify

$$m_1 = I_3,$$

$$m_2 = (2/\sqrt{3})U,$$

where $\underline{m} = \binom{m_1}{m_2}$ in a Euclidean basis.

For illustrative purposes, here are some more weight diagrams corresponding to the representations [4] shown in Fig. 5.

Before concluding this section we state one important theorem and make one final remark.

Theorem 14

For the adjoint representation, the root vectors and the non-zero weight vectors coincide. The weight zero occurs with a multiplicity equal to the rank of the group.

An illustration of this theorem is given by the weight diagram of the $(\frac{1}{2}, \frac{1}{2})$ representation of SU_3 computed earlier in this section. Because of

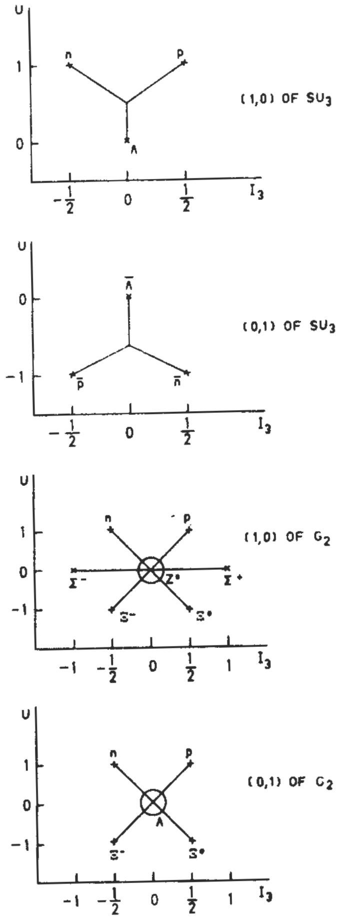


Fig. 5

this rather remarkable property clearly the adjoint representation has a greater claim to attention than any other.

Remark

In O_3, the eigenvalues of J_3 (the weights) are non-degenerate for any given representation and hence suffice to label the representation. For general Lie groups, except for the highest weight, all others may possess multiplicities of > 1 (compare the weight (0, 0) for SU_3 which has multiplicity 2). If the multiplicity is > 1 we need additional operators all commuting with each other and with the $\underline{H}$'s, whose eigenvalues will enable us to re-

move the degeneracy and label uniquely the eigenvectors of the $\underline{H}$'s, belonging to the same given weight. (A Casimir operator which has the same eigenvalue for all vectors of a given representation is clearly useless for this purpose.) The number of extra operators needed can be shown to equal $(N-\ell)/2-\ell = (N-3\ell)/2$. For O_3, $N = 3$, $\ell = 1$ so that no extra operator is needed to characterize all the eigenkets of J_3 in a representation specified (uniquely) by the highest weight j. For SU_3, however, $N = 8$, $\ell = 2$ so that we need one more operator besides I_3 and U to label uniquely the eigenkets of I_3 and U. It is not hard to show that in this case such an operator is given by $\underline{I}^2$. For C_2, $(N-3\ell)/2 = 2$. Thus, even additional to $\underline{I}^2$ (and U and I_3), one more quantum number is needed to form a complete set of commuting observables. For G_2, $(N-3\ell)/2 = 4$.

12. REDUCIBLE REPRESENTATIONS

Let us take stock of the situation. For a physicist working in symmetry problems, the information necessary for progress is the following:

(i) Classification of irreducible representation for a group of rank ℓ. We possess a complete solution of this problem.

(ii) The eigenvalues of the commuting operators $H_1, \ldots, H_\ell$. This is the same problem as the problem of determination of weights. Again we possess a complete solution of this.

(iii) Determination of the extra $(N-3\ell)/2$ operators to enable a unique labelling of the eigenkets of $H_1, \ldots, H_\ell$. For groups like A_2, B_2, C_2, D_2 we know how to construct such operators but a general systematic procedure apparently is not known.

(iv) The reduction of a reducible representation into the direct sum of irreducible representations. There are two parts of this problem: first, finding out which irreducible representations make their appearance in this direct sum; second, to find the Clebsch-Gordon coefficients. Theorem 15 will give the procedure for solving the first problem. The second problem will be dealt with by Ruegg and Goldberg in their lectures for some special (fortunately for the physicist, extremely important) cases. No general solution however exists.

First, some obvious definitions:

Kronecker products

If R_1, R_2, R_3 are three linear spaces of dimensions m, n and mn respectively, we shall say R_3 is the Kronecker product of R_1 and R_2 ($R_3 = R_1 \times R_2$) provided to every vector $|\xi_1\rangle \in R_1$, $|\xi_2\rangle \in R_2$, there corresponds a vector $|\xi_3\rangle \in R_3$ (notation $|\xi_3\rangle = |\xi_1\rangle \times |\xi_2\rangle$) such that:

(i) The operation $|\xi_1\rangle \times |\xi_2\rangle$ is linear in each argument;

(ii) R_3 is spanned by vectors of the form $|\xi_1\rangle \times |\xi_2\rangle$.

If ϕ_1 and ϕ_2 are linear representations of a Lie algebra operating in R_1 and R_2, the representation ϕ_3 defined in $R_1 \times R_2$ by the formula,

$$\phi_3 \{|\xi_1\rangle \times |\xi_2\rangle\} = \{\phi_1 |\xi_1\rangle\} \times |\xi_2\rangle + |\xi_1\rangle \times \{\phi_2 |\xi_2\rangle\},$$

is called the Kronecker product of ϕ_1 and ϕ_2 and will be denoted as

$$\phi_3 = \phi_1 \times \phi_2 .$$

Theorem 15

(i) Addition of weights

If Δ_{ϕ_1} is the weight space of ϕ_1 and Δ_{ϕ_2} is the weight space of the representation ϕ_2, then $\Delta_{\phi_3} = \Delta_{\phi_1} + \Delta_{\phi_2}$.

(ii) If $\underline{\Lambda}_1$ and $\underline{\Lambda}_2$ are the greatest weights of ϕ_1 and ϕ_2, the greatest weight of ϕ_3 is $\underline{\Lambda}_1 + \underline{\Lambda}_2$.

This theorem is an obvious generalization of the addition theorem for angular momenta in O_3 which we consider in detail. If j_1 and j_2 are the highest weights of two irreducible representations $\phi(j_1)$ and $\phi(j_2)$, the (reducible) product representation has the highest weight $j_1 + j_2$. Also the totality of its weights is given by

Weight →	j_1+j_2,	j_1+j_2-1,	j_1+j_2-2,	...,	$-j_1-j_2$
multiplicity →	1,	2,	3,	...,	1

The multiplicities are easily deduced. For example, j_1+j_2-1 arises in two ways: either as the sum $j_1+(j_2-1)$ or equally as the sum of the weights $(j_1-1)+j_2$. The usual procedure to find the irreducible representations contained in $\phi(j_1)\times\phi(j_2)$ can be stated thus: Take away from the totality of weights those which belong to the representation $\phi(j_1+j_2)$. Among the remaining weights occurs the weight j_1+j_2-1 with unit multiplicity. Clearly this must be the highest weight of the representation $\phi(j_1+j_2-1)$ which therefore must also be contained in $\phi(j_1)\times\phi(j_2)$. Taking away all the weights belonging to $\phi(j_1+j_2-1)$, we next identify the occurrence of $\phi(j_1+j_2-2)$ in the direct sum from the fact that the highest weight left is (j_1+j_2-2). This procedure is continued till we reach $\phi(|j_1-j_2|)$. At this stage all weights are exhausted, leading to the inference that

$$\phi(j_1)\times\phi(j_2) = \phi(j_1+j_2) + \phi(j_1+j_2-1) + \ldots + \phi(|j_1-j_2|).$$

The procedure is obviously completely general. Its only drawback is that in order to apply it we need to know all the weights. A simpler version has been developed by Racah, Speiser and Ruegg where, if $j_1 \geqslant j_2$, one adds all weights belonging to the representation $\phi(j_2)$ (i.e. $j_2, j_2-1, \ldots, -j_2$) to the highest weight j_1 of $\phi(j_1)$. For O_3, the resulting weights are clearly the highest weights of the irreducible representations contained in $\phi(j_1)\times\phi(j_2)$. For the more general cases this sum may lead to a certain number of negative weights which certainly cannot qualify as highest weights. These then have to be excluded, and the procedure for this is explained in Ruegg's lecture.

Cartan composition

If ϕ_1 and ϕ_2 are two irreducible representations, the Kronecker product $\phi_1 \times \phi_2$ is in general a reducible representation. Consider its greatest component, $\overline{\phi_1 \times \phi_2}$. This is an irreducible representation with the highest weight $\underline{\Lambda}_1 + \underline{\Lambda}_2$. The operation of Kronecker multiplication of two irreducible representations followed by the operation of isolating the greatest component lead to the formation of a new irreducible representation $(\overline{\phi_1 \times \phi_2})$ and is called the cartan composition of irreducible representations.

Those irreducible representations of an algebra which cannot be obtained from other irreducible representations are called basic representations by Cartan. These representations are characterized by the fact that their highest weights cannot be split into the sums of two elements that are themselves highest weights. Clearly a representation ϕ is basic if, and only if, all the labelling numbers $j_1, j_2, \ldots, j_\ell$ are zero except one which equals $\frac{1}{2}$. Thus every simple algebra of rank ℓ has ℓ basic representations.

One can go further and show that all basic representations themselves can be constituted from a few so-called elementary representations by Kronecker multiplications followed by an antisymmetrization procedure which is somewhat familiar in ordinary tensor theory and will not be described here in detail. For A_ℓ and B_ℓ there are just two elementary representations. C_ℓ has one elementary representation and D_ℓ has three. One of the elementary representations ϕ of A_ℓ is realized as the group $SL(\ell+1)$ of all matrices of order $\ell+1$ with determinant + 1, the other being given by

$$\phi' = -\{\phi_1\}^T.$$

For B_ℓ, one of the elementary representations is obtained by considering the group $O(2\ell+1)$ of all unimodular orthogonal transformations of the $(2\ell+1)$ dimensional space, while the second elementary representation is the so-called spinor representation. The realization of the group C_ℓ in the form of the group $Sp(2n)$ of the symplectic matrices of order 2ℓ gives its elementary representation, while for D_ℓ ($\ell \geqslant 5$) one elementary representation is given by the group of unimodular orthogonal matrices of order 2ℓ and in addition there are two distinct spinor representations. For the elementary representations of the exceptional groups reference may be made to Dynkin.

This brief description of the results in representation theory does not even touch the practical problem of reduction of representation in the manner the physicist wants it solved. For this we must fall back on our amateur methods, multiplying matrices, symmetrizing and antisymmetrizing tensor indices, though perhaps somewhat emboldened by the knowledge that this is also the entire, and when I say entire - I mean entire, stock-in-trade of the professional group theorist.

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Note on Unitary Symmetry in Strong Interactions*)

Susumu OKUBO

Department of Physics, University of Tokyo, Tokyo
and
Department of Physics, University of Rochester
Rochester, N.Y., U.S.A

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Assuming invariance of theory under three-dimensional unitary group, various consequences have been investigated. Both Sakata's and Gell-Mann's scheme can be treated in the same fashion and in a simpler way. Mass formula for particles belonging to the same irreducible representation has been derived and compared with experiments.

§ 1. Introduction

The purpose of this note is to investigate consequences of the three-dimensional unitary group (denoted as U_3 hereafter), which is a certain generalization of the usual isotopic space group. Though many authors[1),2),3)] have examined this problem, our procedure is simpler and some new results have been obtained. Also, we can treat different schemes of U_3 such as Sakata's[1),2)] or Gell-Mann's[3)] on the same footing by our method.

First of all, we shall give some motivations for introducing U_3. All known interactions obey certain symmetries, i.e. they are subject to the corresponding transformation groups. We can classify all known groups appearing in the studies of elementary particles into the following three categories.

(I) *Space-group*
  (i) Lorentz group (ii) Charge conjugation
(II) *Isotopic-groups*
  (i) Isotopic spin rotation $R_3^{(I)}$
  (ii) Baryon gauge transformation $R_2^{(B)}$
  (iii) Charge gauge transformation $R_2^{(Q)}$
  (iv) Strangeness gauge transformation $R_2^{(S)}$
  (v) Leptonic gauge transformation $R_2^{(L)}$
(III) *Gauge-transformation of the 2nd kind*
  (i) Electro-magnetic field
  (ii) Yang-Mills field

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In this list, we have included the charge conjugation into the space-group, because of the TCP theorem. These three groups of transformations are correlated with each other in some degree, but here we do not go into details. Furthermore, we restrict ourselves only in the study of the iso-space groups (II), in this paper. Moreover, we do not take account of leptons also, though they might be treated on the same footing.[4] Then, the groups (II) consist of 4 groups. However, by virtue of the Nakano-Nishijima-Gell-Mann formula, we have one following relation:

$$Q=I_3+1/2\cdot(N+S). \tag{1}$$

Thus, only 3 out of the 4 groups are independent. So, the known strong interactions have to be invariant under the following group G:

$$G=R_3^{(I)}\times R_1^{(B)}\times R_1^{(Q)}.$$

Now, for the moment, let us suppose that the nature obeys some higher symmetry than this. Then, the invariant group U of this higher symmetry must include G as a sub-group. One of them including G is U_3, which is relatively uncomplicated. This is one motivation for adopting U_3. Besides, we may note that the 3-dimension is the minimum dimension for non-trivial representation of the group G. This may be taken as another motivation for U_3.[5]

In the next section, we shall give the classification of particles belonging to a given irreducible representation by means of restricting U_3 into U_2 (two-dimensional unitary group). In § 3 we shall give applications of U_3. Furthermore, the following mass formula will be proved:

$$M=a+b\cdot S+c\cdot[I(I+1)-1/4\cdot S^2]. \tag{2}$$

This relation holds for particles belonging to a given irreducible representation of U_3, and S and I stand for the strangeness and isospin of particles contained in the representation, respectively. This formula has been proved in the lowest order perturbation violating U_3-symmetry of the type $\bar{\Lambda}\Lambda$, but in any orders for the strong U_3-invariant interactions. The proof of Eq. (2) will be given in the Appendix. As an application of Eq. (2), we note that if N, Λ, Σ and Ξ belong to an irreducible representation as in the Gell-Mann scheme, we have

$$1/2\cdot[M_N+M_\Xi]=3/4\cdot M_\Lambda+1/4\cdot M_\Sigma,$$

which is satisfied in good accuracy. Another application of our formula Eq. (2) is that the mass of a neutral-isoscalar meson π_0' would be given by

$$M(\pi_0')=4/3\cdot M(K)-1/3\cdot M(\pi)\simeq 600\text{ Mev},$$

where π_0' is the meson belonging to the same representation as π, K and $\bar{K}$ mesons. Similarly, we should have

$$M(K^*)=3/4\cdot M(\omega)+1/4\cdot M(\rho)$$

where ρ, ω and K^* are bosons representing resonant states of $(\pi\text{-}\pi)$, $(\pi\text{-}\pi\text{-}\pi)$ and $(\pi\text{-}K)$ system, respectively. We note that this relation is satisfied within an error of 12%.

§ 2. Classification of particles in U_3

The three-dimensional unitary group U_3 is defined by the following transformation on a vector $\phi_\mu(\mu=1,2,3)$:

$$\phi_\mu\to\sum_{\lambda=1,2,3}a_\mu^\lambda\phi_\lambda\quad(\mu=1,2,3) \tag{3}$$

where a_μ^λ satisfies

$$\sum_{\mu=1,2,3}(a_\mu^\lambda)^*a_\mu^\nu=\delta_\lambda^\nu\quad(\nu,\lambda=1,2,3). \tag{4}$$

In the Sakata model,[6] we identify ϕ_1, ϕ_2 and ϕ_3 with the proton, the neutron and the Λ, respectively. However, this is not the only way. We shall assume that ϕ_1, and ϕ_2 form an isotopic doublet and ϕ_3 an isotopic singlet. As for other quantum numbers, we can assign according to the following cases:

(a) ϕ_1, ϕ_2 and ϕ_3 have the baryon number $N=1$. ϕ_1 and ϕ_2 have the strangeness quantum number $S=0$, ϕ_3 has the strangeness $S=-1$.

(b) We do not assign any baryon numbers to ϕ_1, ϕ_2 and ϕ_3, but assign $Y=0$ for ϕ_1 and ϕ_2, and $Y=-1$ for ϕ_3 where Y stands for the hypercharge $Y=N+S$.

(c) We do not assign any baryon numbers to ϕ_1, ϕ_2 and ϕ_3, but assign a new quantum number $Z=N+3\cdot S$ as $Z=1$ for ϕ_1 and ϕ_2, and $Z=-2$ for ϕ_3.

The first assignment (a) corresponds to the usual Sakata model, and the second one (b) is practically the same as the Gell-Mann scheme,[3] and so we refer to it as "Gell-Mann scheme" for simplicity,[*] though not exactly. The third scheme is actually convenient if we consider the unitary-unimodular group of 3 dimensions instead of U_3, and so refer to it as "the unitary-unimodular scheme". We may give possible schemes other than (a), (b) and (c), but it will not be so fruitful.

First, let us consider the case (a) (referred to as "Sakata scheme" hereafter). In this scheme, consider a special transformation:

$$\phi_1\to\varepsilon_1\phi_1,\qquad\phi_2\to\varepsilon_2\phi_2,\qquad\phi_3\to\varepsilon_3\phi_3$$

$$|\varepsilon_\mu|=1\quad(\mu=1,2,3). \tag{5}$$

This is a special transformation of Eqs. (3) and (4). Then, a component of every tensor $T_{\beta_1\cdots\beta_m}^{\alpha_1\cdots\alpha_n}$ would transform as

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*) Note added in proof: Exactly the same scheme has been proposed by Y. Yamaguchi in 1960, so that we should call it as Yamaguchi-Gell-Mann scheme hereafter. Y. Yamaguchi: private communication.

$$T \longrightarrow \varepsilon_1^{\alpha}\, \varepsilon_2^{\beta}\, \varepsilon_3^{\gamma}\, T.$$

In our case, the baryon number N and the strangeness S is obviously given by

$$N = \alpha + \beta + \gamma$$

$$S = -\gamma. \tag{6}$$

Now, all irreducible tensor representation of U_3 are characterized by three integers f_1, f_2 and f_3 satisfying a condition $f_1 \geq f_2 \geq f_3$. We shall denote it as $U_3(f_1, f_2, f_3)$, hereafter. The dimension of the representation is given[7] by

$$D = 1/2 \cdot (f_1 - f_2 + 1)(f_1 - f_3 + 2)(f_2 - f_3 + 1). \tag{7}$$

Also, comparing the character of $U_3(f_1, f_2, f_3)$ with Eq. (6), we find that the baryon number N of this representation is

$$N = f_1 + f_2 + f_3. \tag{8}$$

Now, to specify sub-quantum numbers S and the isospin I in $U_3(f_1, f_2, f_3)$, we fix the direction of the 3rd component ϕ_3. So, we restrict ourselves within the two-dimensional unitary group U_2, whose irreducible representations are specified by two integers f_1', and f_2' satisfying $f_1' \geq f_2'$ and will be referred to as $U_2(f_1', f_2')$. Then, the branching rule[8] for this decomposition tells us that U_3 can be decomposed according as

$$U_3(f_1, f_2, f_3) \to \sum_{(f_1', f_2')} U_2(f_1', f_2'), \tag{9}$$

where we sum over all possible integer pairs (f_1', f_2') satisfying the following conditions:

$$f_1 \geq f_1' \geq f_2 \geq f_2' \geq f_3. \tag{10}$$

The decomposition Eq. (9) is an analogue of the well-known decomposition of R_4 into R_3 (R_n being the n-dimensional rotation group).

$$R_4(l, l') \to \sum_{L=|l-l'|}^{l+l'} R_3(L).$$

Now, two-dimensional unitary group is a product of two-dimensional unitary-unimodular group (which we can identify as the usual isotopic rotation group) and a gauge group, which defines the nucleon charge. Then, the isospin I is immediately given by

$$I = 1/2 \cdot (f_1' - f_2') \tag{11}$$

and also, comparing the character of $U_2(f_1', f_2')$ with Eq. (6), we get

$$S = (f_1' + f_2') - (f_1 + f_2 + f_3). \tag{12}$$

In this way, we could specify sub-quantum numbers S and I. Furthermore, we note[9] that two representations $U_3(f_1, f_2, f_3)$ and $U_3(-f_3, -f_2, -f_1)$ are contragradient to each other, i.e. they are charge-conjugate of each other in our case.

This remark does not apply to the cases (b) and (c), since the nucleon number is not defined in these cases.

In order to explain our procedure, consider various cases:

(i) $(f_1, f_2, f_3) = (1, 0, 0)$

This is a 3-dimensional representation by Eq. (7) and the decomposition Eqs. (9) and (10) tells us two choices $(f_1', f_2') = (1, 0)$ or $(0, 0)$. By Eqs. (8), (11) and (12), $N=1$ and the former belongs to $(I=1/2, S=0)$, and the latter to $(I=0, S=-1)$. So the natural identification would be the triplet (p, n, Λ).

(ii) $(f_1, f_2, f_3) = (1, 0, -1)$

By Eqs. (7) and (8), this is a boson representation with 8 components. Also, by the remark given after Eq. (12), it must be self-conjugate, i.e. it must contain a particle and its anti-particle together. Now, the decomposition Eqs. (9) and (10) gives us the choice $(f_1', f_2') = (1, 0)$, $(0, -1)$, $(1, -1)$ and $(0, 0)$, and by Eqs. (11) and (12) they have $(I=1/2, S=1)$, $(I=1/2, S=-1)$, $(I=1, S=0)$, and $(I=0, S=0)$, respectively. By the remark given in the beginning, the first two must be charge conjugate of each other and the last two must be self-conjugate under charge conjugation operation. Natural identification would be (K_+, K_0), $(\overline{K}_+, \overline{K}_0)$, (π_+, π_0, π_-) and π_0', where the last one is a new pseudoscalar boson. We may identify the newly found states K^*, $\overline{K}^*$, ρ and ω mesons under the same category.

(iii) $(f_1, f_2, f_3) = (2, 0, -1)$

This is a fermion state with 15 components by Eqs. (7) and (8), and they contain the following particles by Eqs. (10), (11) and (12).

$$(I=1/2, S=-2),\ (I=1, S=-1),\ (I=0, S=-1),$$

$$(I=1/2, S=0),\quad (I=1, S=+1),\ (I=3/2, S=0).$$

We might identify the first four as Ξ, Σ, Λ and N, respectively, but then we have two other unwanted particles. This interpretation is originally due to Yamaguchi,[2] but as we will see in a later section this identification seems to give small masses for $(I=1, S=1)$ and $(I=3/2, S=0)$ particles so as to make them stable, and so it would be more natural to adopt the case (i) as representing Λ and N. Furthermore, if we take the viewpoint (ii) for bosons, then $(I=1/2, S=-2)$ has to be identified still as Ξ particles. This is because the transition $\Xi \to \Lambda + \overline{K}$ must be possible and therefore Ξ (and also Σ since $\Sigma \to \Lambda + \pi$) has to be in a product representation $U_3(1, 0, 0) \times U_3(1, 0, -1)$. However,[10] we have

$$U_3(1, 0, 0) \times U_3(1, 0, -1) = U_3(2, 0, -1) + U_3(1, 1, -1) + U_3(1, 0, 0)$$

but $U_3(1, 1, -1)$ and $U_3(1, 0, 0)$ do not contain a particle with $(I=1/2, S=-2)$. As for Σ, the same argument shows that it must belong either to $U_3(2, 0, -1)$ or to $U_3(1, 1, -1)$. Ikeda et al.[1] identify $(I=3/2, S=0)$ in $U_3(2, 0, -1)$ as

N^* (the first π-N scattering resonance), then the spin of Ξ has to be 3/2, since N^* has the space-spin 3/2. Similarly, $(I=1, S=-1)$ and $(I=0, S=-1)$ states in $U_3(2, 0, -1)$ may be interpreted as Y_1^*(π-Λ scattering resonance) and Y_0^* (π-Σ scattering resonance), respectively. Then, they must have spin 3/2 also. In this case, we have to assign $U_3(1, 1, -1)$ for Σ.

(iv) $(f_1, f_2, f_3)=(1, 1, -1)$

This is a fermion state with six components. We have $(I=1/2, S=0)$, $(I=0, S=+1)$ and $(I=1, S=-1)$, and the last one may be interpreted as Σ. However, we have a new state with $(I=0, S=+1)$, so, we should observe a resonance for the reaction $K_+ + n$ scattering, which has not so far been found experimentally.

Up to now, we have investigated the case (a), i.e. the Sakata-scheme. Now, let us consider the case (b). In this case, we cannot assign any baryon numbers to ϕ_μ, so that Eq. (8) has no meaning as to indicate the baryon number. Eq. (11) is unchanged as before, but in Eq. (12), S has to be replaced by Y, so that in our scheme (b), we have

$$I=1/2\cdot(f_1'-f_2')$$
$$Y=(f_1'+f_2')-(f_1+f_2+f_3). \tag{13}$$

In this case, the representation $(1, 0, -1)$ gives four states; $(I=1/2, Y=1)$, $(I=1/2, Y=-1)$, $(I=1, Y=0)$ and $(I=0, Y=0)$. As for bosons, our assignment is unchanged, since S and Y are the same for bosons. So, we can assign $(\pi, K, \bar{K}, \pi_0')$ and $(\rho, K^*, \bar{K}^*, \omega)$ to $U_3(1, 0, -1)$. A new phenomenon is that we can also assign $(N, \Xi, \Sigma, \Lambda)$ to $U_3(1, 0, -1)$ since the nucleon number is no longer defined and the corresponding quantum numbers Y and I can be given correctly. This is exactly the same as in Gell-Mann's scheme, though the starting points are quite different. As we shall see in the next section, our scheme is essentially the same as Gell-Mann's as for all practical purposes, and so we can call our scheme (b) as Gell-Mann's. We may note the following decomposition:[10]

$$U_3(1, 0, -1)\times U_3(1, 0, -1)=2U_3(1, 0, -1)+U_3(0, 0, 0)+U_3(2, 0, -2)$$
$$+U_3(2, -1, -1)+U_3(1, 1, -2)$$

so that Y_1^*, Y_0^* and N^* in the Gell-Mann scheme have to be included in one of the right-hand side, since they decay into one-boson and one-fermion state. This will be treated in a forthcoming paper.

Finally, we may study the consequence of our scheme (c). This was given, since it is more natural when we think of the unitary-unimodular group of 3-dimension (we refer to it as $SL(3)$) rather than U_3. In $SL(3)$, there is no distinction between covariant and contravariant tensors. This is because a constant totally anti-symmetric tensor $\epsilon^{\lambda\mu\nu}$ is invariant under $SL(3)$, so that ϕ^λ behaves like $\epsilon^{\lambda\mu\nu}T_{\mu\nu}$ where $T_{\mu\nu}$ is a tensor. More generally, we have that the

representation (f_1, f_2, f_3), which we have written[11] as $U_3(f_1, f_2, f_3)$ up to now, is the same representation as (f_1+e, f_2+e, f_3+e) where e is an arbitrary integer. Then, obviously Eqs. (12) or (13) is not invariant under $SL(3)$, since it is not invariant under $f_\mu\to f_\mu+e(\mu=1, 2, 3)$ and $f_\mu'\to f_\mu'+e(\mu=1, 2)$. Invariant quantum numbers under $SL(3)$ under our decomposition Eq. (9) are given by

$$Z=3(f_1'+f_2')-2(f_1+f_2+f_3),$$
$$I=1/2(f_1'-f_2') \tag{14}$$

where $Z=N+3\cdot S$. We omit the details for these derivations. In this case, we can repeat the same procedures as before, but it gives almost the same results as in the case (a), so we will not go too far. Here we may note also that if we give up additivity of quantum numbers, we may assign $Z=3\cdot Y+N(N-1)$ for Eq. (14). In this case, we can assign $(1, 0, -1)$ both for bosons and fermion, and we have the same result as Gell-Mann's again. We shall not consider our case (c) any longer in this paper, and restrict ourselves only in discussions of the cases (a) and (b).

§ 3. Tensor representation and applications

First, let us consider the Sakata scheme (a), and we take the representations $U_3(1, 0, 0)$ and $U_3(1, 0, -1)$ for (Λ, n, p) and $(\pi, \pi_0', K, \bar{K})$ systems, respectively. Then, p, n and Λ can be represented by a vector ϕ_μ.

$$\phi_1=p,\ \phi_2=n,\ \phi_3=\Lambda \tag{15}$$

and $(\pi, \pi_0', K, \bar{K})$ can be represented by a traceless tensor f_ν^μ, so that $f_\mu^\mu=0$. The identification is

$$\pi_+=f_1^2,\ \pi_-=f_2^1,\ \pi_0=\frac{1}{\sqrt{2}}(f_1^1-f_2^2),\ \pi_0'=-\frac{3}{\sqrt{6}}f_3^3,$$
$$K_+=f_1^3,\ K_0=f_2^3,\ \bar{K}_-=f_3^1,\ \bar{K}_0=f_3^2 \tag{16}$$

and also $(\rho, \omega, K^*, \bar{K}^*)$ can be represented by a traceless tensor F_ν^μ exactly in the same fashion as Eq. (16) by replacing $\pi\to\rho$, $\pi_0'\to\omega$, $K\to K^*$, $\bar{K}\to\bar{K}^*$. Actually, F_ν^μ has a vector suffix due to space-spin, but we omit it for simplicity.

The invariant interactions among baryon-boson and among boson-boson would be given by

$$H_1=ig\,\bar{\phi}_\mu\gamma_5\phi_\nu f_\nu^\mu \tag{17}$$
$$H_2=ig\,F_\nu^\mu\cdot(f_\lambda^\nu\cdot\partial f_\mu^\lambda-\partial f_\lambda^\nu\cdot f_\mu^\lambda) \tag{18}$$

where the repeated indices mean summations over 1, 2 and 3. In Eq. (17), we note that $\bar{\phi}_\mu$ behaves as a contra-variant vector ϕ^μ. Using the representations Eqs. (15) and (16), these Hamiltonians can be written as

$$H_1=ig\frac{1}{\sqrt{2}}\bar{N}\gamma_5(\tau\cdot\pi)N+ig\bar{N}\gamma_5\Lambda K+ig\bar{\Lambda}\gamma_5N\bar{K}$$

$$+ig\frac{1}{\sqrt{6}}(\bar{N}\gamma_5 N-2\bar{\Lambda}\gamma_5\Lambda)\pi_0', \qquad (17)'$$

$$H_2=\frac{ig}{\sqrt{2}}\boldsymbol{\rho}(\bar{K}\boldsymbol{\tau}\partial K-\partial\bar{K}\boldsymbol{\tau}K)+\sqrt{2}\cdot g\cdot\boldsymbol{\rho}(\boldsymbol{\pi}\times\partial\boldsymbol{\pi})$$

$$+\frac{ig}{\sqrt{2}}\bar{K}^*\boldsymbol{\tau}[K\partial\boldsymbol{\pi}-(\partial K)\boldsymbol{\pi}]+\frac{3}{\sqrt{6}}ig\bar{K}^*[K\partial\pi_0'-\partial K\pi_0']$$

$$+\frac{ig}{\sqrt{2}}[\boldsymbol{\pi}(\partial\bar{K})-\partial\boldsymbol{\pi}\bar{K}]\cdot\boldsymbol{\tau}K^*+\frac{3}{\sqrt{6}}ig[\pi_0'\partial\bar{K}-\partial\pi_0'\bar{K}]K^*$$

$$+\frac{3}{\sqrt{6}}ig\omega[\bar{K}\partial K-\partial\bar{K}K]. \qquad (18)'$$

We note that Eq. (18)′ agrees with that given by Gell-Mann.[3)]

Now, let us consider the Gell-Mann scheme (b). Here, as for bosons, Eq. (16) is unchanged. For baryons, we introduce two traceless tensors N_ν^μ and M_ν^μ (so that $M_\mu^\mu=N_\mu^\mu=0$) as representing

$$\Sigma_+=N_1^2,\ \Sigma_-=N_2^1,\ \Sigma_0=\frac{1}{\sqrt{2}}(N_1^1-N_2^2),\ \Lambda=-\frac{3}{\sqrt{6}}N_3^3, \qquad (19a)$$

$$p=N_1^3,\ n=N_2^3,\ \Xi_-=N_3^1,\ \Xi_0=N_3^2,$$

$$\bar{\Sigma}_-=M_1^2,\ \bar{\Sigma}_+=M_2^1,\ \bar{\Sigma}_0=\frac{1}{\sqrt{2}}(M_1^1-M_2^2),\ \bar{\Lambda}=-\frac{3}{\sqrt{6}}M_3^3, \qquad (19b)$$

$$\bar{\Xi}_-=M_1^3,\ \bar{\Xi}_0=M_2^3,\ \bar{p}=M_3^1,\ \bar{n}=M_3^2.$$

Then, we have two invariant forms for baryon-boson interactions.

$$H_3=ig\,M_\nu^\mu\gamma_5 N_\lambda^\nu f_\mu^\lambda, \qquad (20a)$$

$$H_4=ig\,M_\nu^\mu\gamma_5 f_\lambda^\nu N_\mu^\lambda. \qquad (20b)$$

Explicit calculation gives

$$H_3=\frac{ig}{\sqrt{2}}\bar{N}\gamma_5(\boldsymbol{\tau}\cdot\boldsymbol{\pi})N+\frac{g}{\sqrt{2}}(\bar{\boldsymbol{\Sigma}}\gamma_5\times\boldsymbol{\Sigma})\boldsymbol{\pi}+\frac{g}{\sqrt{6}}[i\bar{\boldsymbol{\Sigma}}\boldsymbol{\pi}\gamma_5\Lambda+\text{c.c.}]$$

$$+\frac{g}{\sqrt{6}}[\bar{\Lambda}\gamma_5\Xi\tau_2K+\text{c.c.}]-\frac{\sqrt{6}}{3}g[i\bar{N}\gamma_5K\Lambda+\text{c.c.}]$$

$$-\frac{g}{\sqrt{2}}[\bar{K}\boldsymbol{\tau}\tau_2\bar{\Xi}\gamma_5\boldsymbol{\Sigma}+\text{c.c.}]$$

$$-\frac{ig}{\sqrt{6}}\pi_0'[2(\bar{\Xi}\gamma_5\Xi)+\bar{\Lambda}\gamma_5\Lambda-\bar{\boldsymbol{\Sigma}}\gamma_5\boldsymbol{\Sigma}-\bar{N}\gamma_5N], \qquad (21a)$$

$$H_4=\frac{g}{\sqrt{6}}[i\bar{\Lambda}\gamma_5\boldsymbol{\Sigma}\boldsymbol{\pi}+\text{c.c.}]-\frac{g}{\sqrt{2}}(\bar{\boldsymbol{\Sigma}}\gamma_5\times\boldsymbol{\Sigma})\boldsymbol{\pi}$$

$$-\frac{ig}{\sqrt{2}}\bar{\Xi}(\boldsymbol{\tau}\cdot\boldsymbol{\pi})\gamma_5\Xi$$

$$+\frac{g}{\sqrt{6}}[i\bar{N}\gamma_5K\Lambda+\text{c.c.}]$$

$$-\frac{2g}{\sqrt{6}}[\bar{\Lambda}\gamma_5\Xi\tau_2K+\text{c.c.}]$$

$$+\frac{g}{\sqrt{2}}[i\bar{N}\gamma_5\boldsymbol{\tau}K\boldsymbol{\Sigma}+\text{c.c.}]$$

$$+\frac{ig}{\sqrt{6}}\pi_0'[\bar{\boldsymbol{\Sigma}}\gamma_5\boldsymbol{\Sigma}+\bar{\Xi}\gamma_5\Xi-\bar{\Lambda}\gamma_5\Lambda-2\bar{N}\gamma_5N] \qquad (21b)$$

where we have put

$$N=\begin{pmatrix}p\\n\end{pmatrix},\quad \boldsymbol{\Sigma}=\begin{pmatrix}\Sigma_1\\\Sigma_2\\\Sigma_3\end{pmatrix},\quad \Xi=\begin{pmatrix}-\Xi_0\\\Xi_-\end{pmatrix},\quad K=\begin{pmatrix}K_+\\K_0\end{pmatrix},\quad \boldsymbol{\pi}=\begin{pmatrix}\pi_1\\\pi_2\\\pi_3\end{pmatrix},$$

and Eqs. (21a) and (21b) are connected with L_D and L_F of Gell-Mann[3)] by

$$H_3=\frac{1}{2\sqrt{2}}[L_D+L_F],$$

$$H_4=\frac{1}{2\sqrt{2}}[L_D-L_F],$$

when we take the same coupling constants.

As applications of our formalism, we may think of the boson-baryon scattering in the case of the Sakata scheme. In this case, we can form the following invariants of which the S-matrix element is a linear combination:

$$T_\nu^\mu f_\lambda^\nu \tilde{f}_\mu^\lambda,\ T_\nu^\mu \tilde{f}_\lambda^\nu f_\mu^\lambda,\ T_\mu^\mu f_\beta^\alpha \tilde{f}_\alpha^\beta$$

where we have put $T_\nu^\mu=\bar{\psi}_\mu\psi_\nu$, and f, and $\tilde{f}$ represent for incoming and outgoing bosons. From this, we can prove the following identities among total cross-sections.

$$\sigma(\pi_++p)=\sigma(K_++p),\quad \sigma(K_-+n)=\sigma(\pi_++\Lambda),$$

$$\sigma(\pi_-+p)=\sigma(K_-+p)=\sigma(K_++\Lambda),\ \text{etc.}$$

$$\sigma(\pi_0'+p)=1/3\cdot\sigma(\pi_0+p)+2/3\cdot\sigma(K_0+p).$$

These have been derived also by Hara and Singh.[18)] They are also investigating similar identities in the case of Gell-Mann scheme. We can get similar identities among magnetic moments of baryons. In the case of Sakata-scheme, let us assume that the electromagnetic current j_μ has a transformation property as T_1^1 component of a tensor T_ν^μ. This can be taken, since the usual current $ie\bar{p}\gamma_\mu p$ has such form. Then, the method mentioned in the above immediately gives

$\mu(\Lambda)=\mu(n)$ and also we can prove that K_0 and $\bar{K}_0$ have no electromagnetic structures. This is because we can prove $\langle K_0|j_\mu|K_0\rangle=-\langle \bar{K}_0|j_\mu|\bar{K}_0\rangle$ similarly, but j_μ changes its sign under charge conjugation, and therefore $\langle K_0|j_\mu|K_0\rangle$ has to be identically zero.

In the case of Gell-Mann scheme (b), we can give some relations among magnetic moments of baryons. By the same reason as in the above, let us assume that the electromagnetic current j_μ behaves as T_1^1 of a tensor T_ν^μ, with respect to U_3. We have to take the expectation value of j_μ, i.e. T_1^1. From invariance, we have

$$\langle T_\nu^\mu\rangle = aM_\lambda^\mu N_\nu^\lambda + bM_\nu^\lambda N_\lambda^\mu + c\cdot\delta_\nu^\mu\cdot(M_\beta^\alpha N_\alpha^\beta)$$

where M and N represent baryons as in Eq. (19) and we have omitted spinor indices. By putting $\mu=\nu=1$, and comparing with Eq. (19), we have $\mu(p)=a+c$, $\mu(n)=c$, etc. Then, we have the following relations:

$$\begin{aligned}
&\mu(p)=\mu(\Sigma_-),\\
&\mu(\Xi_0)=\mu(n),\\
&\mu(\Xi_-)=\mu(\Sigma_-),\\
&\mu(\Lambda)=1/6\cdot[\mu(p)+\mu(\Sigma_-)+4\mu(n)],\\
&\mu(\Sigma_0)=1/2\cdot[\mu(\Sigma_+)+\mu(\Sigma_-)].
\end{aligned}\tag{22}$$

Furthermore, if we demand that T_ν^μ is traceless, i.e. $T_\mu^\mu=0$, then we should have $a+b+3c=0$ and then this condition gives one more relation:

$$\mu(\Lambda)=(1/2)\mu(n).\tag{23}$$

Relations Eqs. (22) and (23) have been given also by Coleman and Glaschow[13] by somewhat more direct method. We note that they used $T_\beta^\alpha=M_\lambda^\alpha N_\beta^\lambda-M_\beta^\lambda N_\lambda^\alpha$, so that obviously $T_\alpha^\alpha=0$ is satisfied. From our derivation, however, it is clear that the explicit form for T_1^1 is unnecessary.

We can give other applications of our method for the weak leptonic decays of bosons and fermions. In case of the strangeness-violating leptonic decays, the interaction Hamiltonian would be given by

$$H_1=G\,\mathfrak{J}_\mu[\bar{\nu}\gamma_\mu(1+\gamma_5)e+\bar{\nu}\gamma_\mu(1+\gamma_5)\mu]+\text{c.c.}\tag{24}$$

where $\mathfrak{J}_\mu$ is the strangeness-violating current. Let us consider the case of Gell-Mann scheme, and assume that $\mathfrak{J}_\mu$ has the transformation property as T_1^3 component of a tensor T_ν^μ, so that it has the same character as K_+. Then, we may construct two tensors $M_\lambda^3N_1^\lambda$ and $M_1^\lambda N_\lambda^3$ out of M and N, and it would be natural to take

$$\begin{aligned}
\mathfrak{J}_\mu&=aM_\lambda^3N_1^\lambda+bM_1^\lambda N_\lambda^3 \qquad (25)\\
&=a\left[\frac{1}{\sqrt{6}}(\bar{\Xi}_-\cdot\Lambda)+(\bar{\Xi}_0\cdot\Sigma_++\frac{1}{\sqrt{2}}\bar{\Xi}_-\cdot\Sigma_0)-\frac{\sqrt{6}}{3}(\bar{\Lambda}\cdot p)\right]
\end{aligned}$$

$$+b\left[\frac{1}{\sqrt{6}}(\bar{\Lambda}\cdot p)+(\bar{\Sigma}_-\cdot n+\frac{1}{\sqrt{2}}\bar{\Sigma}_0\cdot p)-\frac{\sqrt{6}}{3}(\bar{\Xi}_-\cdot\Lambda)\right]$$

where we omitted γ-matrices. Of course, this behaves as a component of an isotopic spinor[14] in the usual isospin assignment.

§ 4. Applications of mass formula

If there are no interactions violating U_3 symmetry, all particles belonging to the same irreducible representation have to have the same mass, the same spin and parity. So we should have the same mass for pion and kaon, which is not true. We must therefore have some interactions violating U_3. According to Yamaguchi,[2] we may suppose that such interactions may be moderately strong, as compared with the very strong U_3-conserving interactions. Our purpose in this note is to investigate the result of mass-splitting among particles in a given irreducible representation due to this moderately strong U_3-violating interaction. In the Appendix, we shall prove that the mass splitting is given by the following formula.*)

$$M=a+b\cdot S+c\cdot[1/4\cdot S^2-I(I+1)].\tag{26}$$

Eq. (26) has been proved in the lowest order perturbation for such U_3-violating interaction with the transformation property T_3^3 of a tensor T_ν^μ but in any orders for U_3-conserving very strong interactions. In Eq. (26), a, b and c are constants which do not depend upon such sub-quantum numbers as the strangeness S and isospin I, but may depend upon the nature of the interaction and upon the irreducible representation to be considered. Eq. (26) may be rewritten as

$$M=a'+b'Y+c'[1/4\cdot Y^2-I(I+1)]\tag{27}$$

if we use the hypercharge $Y=N+S$ instead of S. Formula Eqs. (26) or (27) holds for both the Sakata and the Gell-Mann scheme. For the details, the reader may consult the Appendix.

Now, in this section, we shall investigate the result of Eqs. (26) or (27). First, let us consider boson system (π, π_0', K and $\bar{K}$). An application of (26) or (27) immediately gives that we have a relation

$$M(K)=1/2\cdot[M(K)+M(\bar{K})]=3/4\cdot M(\pi_0')+1/4\cdot M(\pi).\tag{28}$$

From this, we can calculate the mass of π_0' with $M(\pi_0')\simeq 600$ Mev. It is interesting to note that a similar value has been predicted by other methods.[15] The same formula as Eqs. (28) holds for the (ω, ρ, K^*, $\bar{K}^*$) system.

$$M(K^*)=1/2\cdot[M(K^*)+M(\bar{K}^*)]=3/4\cdot M(\omega)+1/4\cdot M(\rho).\tag{29}$$

*) A similar formula has already been suggested by R. P. Feynman at Gatlinburg Conference held in 1958.

The calculated value for $M(K^*)$ by using $M(\omega)$ and $M(\rho)$ is 780 Mev, compared to the experimental value 885 Mev. This relation Eq. (29) holds as long as $(\rho, \omega, K^*, \bar{K}^*)$ belongs to the same irreducible representation. Previously we have assigned $(1, 0, -1)$ for these, but another possibility is that these may belong to 27-dimensional representation $(2, 0, -2)$ instead of the 8-dimensional $U_3(1, 0, -1)$ representation. Then, the method of § 2 tells us that we have 5 more states $(I=1, S=\pm 2)$, $(I=3/2, S=\pm 1)$ and $(I=2, S=0)$ in addition to $(\rho, \omega, K^*, \bar{K}^*)$. Then, we can use our formula Eq. (26) and we can calculate the mass of these states in terms of $M(\rho)$ and $M(\omega)$, to get

$$M(I=1, S=\pm 2) \simeq 770 \text{ Mev},$$

$$M(I=3/2, S=\pm 1) \simeq 720 \text{ Mev}, \quad M(I=2, S=0) \simeq 700 \text{ Mev}.$$

However, we do not observe $I=3/2$ resonance for the K-π system, and so this value for $M(I=3/2, S=\pm 1)$ contradicts the experiment. Accordingly, it seems that our assignment of $(1, 0, -1)$ for $(\rho, \omega, K^*, \bar{K}^*)$ is more reasonable than that of $(2, 0, -2)$. The above argument equally applies both to the Sakata and the Gell-Mann schemes.

As for baryons, let us first consider the Gell-Mann scheme; then $(\Lambda, \Sigma, N, \Xi)$ belongs to $U_3(1, 0, -1)$ representation. Then, by using Eq. (27), we have a relation

$$1/2 [M(N) + M(\Xi)] = 3/4 \cdot M(\Lambda) + 1/4 \cdot M(\Sigma) \tag{30}$$

which is satisfied with good accuracy.

In the case of Sakata scheme, we do not have such relation unless we include $(N, \Xi, \Lambda, \Sigma)$ in $U_3(2, 0, -1)$ representation as we mentioned in § 2. Then, we have Eq. (30) still. However, $U_3(2, 0, -1)$ representation contains two other states with $(I=3/2, S=0)$ and $(I=0, S=+1)$. We can calculate the masses of these particles by Eq. (26) and by using the experimental masses of N, Λ, and Σ. Then, we get

$$M(I=3/2, S=0) \simeq 1050 \text{ Mev} \quad (< M(N) + M(\pi)),$$

$$M(I=0, S=+1) \simeq 770 \text{ Mev} \quad (< M(N))$$

which seems to have too small masses not to be detected experimentally. Thus, this assignment originally due to Yamaguchi would not be so good. Therefore, we take the view that $U_3(2, 0, -1)$ represents Ξ, N^*, Y_0^*, Y_1^*, etc., as has been mentioned in § 2. In this case, we have the following relations:

$$M(Y_1^*) = 1/2 \cdot [M(\Xi) + M(N^*)],$$

$$M(I=1/2, S=0) = 1/2 \cdot [M(Y_0^*) + M(I=1, S=+1)], \tag{31}$$

$$M(I=1, S=+1) = M(Y_1^*) + 2[M(Y_0^*) - M(\Xi)].$$

The first relation gives us $M(Y_1^*) \simeq 1280$ Mev by using the experimental values

for $M(\Xi)$ and $M(N^*)$ and it should be compared to the experimental value of $M(Y_1^*) \simeq 1385$ Mev. Similarly, the last two equations give us

$$M(I=1, S=+1) \simeq 1560 \text{ Mev},$$

$$M(I=1/2, S=0) \simeq 1480 \text{ Mev}$$

where we have used the experimental masses for Y_0^* and Y_1^*. Consequently, we may identify the $(I=1/2, S=0)$ state as the 2nd pion-nucleon resonance, if it corresponds to the $p_{3/2}$ resonance instead of the usual $d_{3/2}$ resonance. As for $(I=1, S=+1)$, resonance for $K_+ + n$ or $K_+ + p$ scattering has not been discovered yet, and this gives a trouble to this scheme.

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Appendix

Derivation of Mass Formula

Here, we shall prove the mass formula Eq. (26).

Let us consider infinitesimal U_3 transformation. Then, the infinitesimal generator A_ν^μ of U_3 satisfies the Lie equation:

$$[A_\beta^\alpha, A_\nu^\mu] = \delta_\beta^\mu \cdot A_\nu^\alpha - \delta_\nu^\alpha \cdot A_\beta^\mu. \tag{A·1}$$

This relation holds actually for general linear transformation of arbitrary dimension. The unitary restriction gives

$$(A_\beta^\alpha)^\dagger = A_\alpha^\beta \tag{A·2}$$

where $Q^\dagger$ means the hermitian conjugate of Q. For comparison's sake, our A_ν^μ is related to Ikeda et al.[11]'s $X_{\mu\nu}$ by

$$A_\nu^\mu = -1/2 \cdot [(1+i) X_{\nu\mu} + (1-i) X_{\mu\nu}],$$

$$X_{\mu\nu} = -1/2 \cdot [(1+i) A_\nu^\mu + (1-i) A_\mu^\nu]. \tag{A·3}$$

However, their notation $X_{\mu\nu}$ makes the mixed tensor character of A_ν^μ obscure.

For an arbitrary mixed tensor T_ν^μ, the commutation relation is given by

$$[A_\beta^\alpha, T_\nu^\mu] = \delta_\beta^\mu \cdot T_\nu^\alpha - \delta_\nu^\alpha \cdot T_\beta^\mu. \tag{A·4}$$

Comparing this with Eq. (A·1), we see that A_ν^μ has the property of a mixed tensor.

Generalized Casimir operators of our Lie algebra can be given by

$$M_1 = A_\mu^\mu = \langle A\rangle,$$
$$M_2 = A_\mu^\nu \cdot A_\nu^\mu = \langle A\cdot A\rangle, \qquad (A\cdot 5)$$
$$M_3 = A_\mu^\nu \cdot A_\nu^\lambda \cdot A_\lambda^\mu = \langle A\cdot A\cdot A\rangle$$

where the repeated indices mean summation over 1, 2 and 3, and we used the notations $\langle Q\rangle$ and defined product tensor $Q\cdot R$ of two tensor Q_μ^ν and R_μ^ν by

$$\langle Q\rangle = Q_\mu^\mu,$$
$$(Q\cdot R)_\mu^\nu = Q_\mu^\lambda \cdot R_\lambda^\nu. \qquad (A\cdot 6)$$

It is easy to see that M_1, M_2 and M_3 commute with all A_μ^ν and therefore they commute with each other. Thus, they are constants in a given irreducible representation. Again, we will give a relation between our M_i and N, M, M' of Ikeda et al.[1]

$$N = -M_1,$$
$$M = 1/2\cdot M_2,$$
$$M' = -1/2\cdot M_3 + 3/4\cdot M_2 - 1/4\cdot (M_1)^2,$$

and so the relation between eigenvalues of M_i and f_1, f_2, f_3 of $U_3(f_1, f_2, f_3)$ is given[1] by

$$M_1 = -(f_1 - f_2 - f_3),$$
$$M_2 = (f_1^2 - f_2^2 + f_3^2) + 2(f_1 - f_3),$$
$$M_3 = -(f_1^3 - f_2^3 + f_3^3) + [-3/2\cdot f_1^2 + 3/2\cdot f_2^2 + 9/2\cdot f_3^2] \qquad (A\cdot 7)$$
$$-1/2\cdot (f_1 + f_2 + f_3)^2 + (2f_1 + 2f_2 - 4f_3).$$

Note that $M_4 = \langle A\cdot A\cdot A\cdot A\rangle$, etc., are unnecessary. They are given as functions of M_1, M_2 and M_3 as will be seen shortly.

Now, we will prove the following theorem.

[Theorem I]

In any irreducible representations of U_3, any mixed tensors T_μ^ν can be regarded as a linear combination:

$$T_\mu^\nu = a\cdot \delta_\mu^\nu + b\cdot A_\mu^\nu + c(A\cdot A)_\mu^\nu. \qquad (A\cdot 8)$$

Eq. (A·8) means that it holds good when we take matrix elements of both sides in a given irreducible representation. Constants a, b and c are independent of tensor suffices μ and ν and of sub-quantum numbers S and I of the representation, but may depend upon f_1, f_2 and f_3 and upon the nature of the tensor T_μ^ν. Eq. (A·8) is an analogue of the so-called vector algebra in R_3, i.e.

$$\langle J, m|V_\mu|J, m'\rangle = \langle J||V||J\rangle\langle J, m|J_\mu|J, m'\rangle$$

where $V_\mu (\mu = 1, 2, 3)$ is a vector in R_3, and J_μ means the angular momentum operator in R_3.

Before proving our theorem, we will show that this equation will give the desired mass formula Eq. (26).

First, let us consider the case of Sakata scheme. In that case, the nucleon number N, the strangeness quantum number S, and the isotopic spin operator $\boldsymbol{I}$ are defined[1] by

$$N = -\langle A\rangle,$$
$$S = A_3^3,$$
$$I_+ = (I_1 + iI_2) = -A_1^2,\ I_- = (I_1 - iI_2) = -A_2^1, \qquad (A\cdot 9)$$
$$I_3 = 1/2(A_2^2 - A_1^1).$$

Now, let us suppose that the mass-splitting interaction is given by T_3^3 which has the same property as $\bar{A}\cdot A$ in the case of Sakata model. Then, the mass splitting is given by diagonal matrix element of T_3^3.

$$\Delta M = \langle i|T_3^3|i\rangle.$$

Then, noting (A9) and

$$(A\cdot A)_3^3 = 1/2\cdot\langle A\cdot A\rangle + 1/2\cdot S^2 + 1/2(3\cdot S - \langle A\rangle) - (\boldsymbol{I})^2 - 1/4\cdot (S - \langle A\rangle)^2,$$

we find that our theorem I (Eq. (A·8)) gives the desired mass formula Eq. (26).

In the case of Gell-Mann scheme, we have only to replace S by Y, hence we get Eq. (27). In this case, N is simply a parameter to distinguish representations.

Now, let us prove our theorem Eq. (A·8). First, we will show the following lemma.

[Lemma I]

In the three-dimensional space, suppose that a tensor $S_{\mu\nu}^{\alpha\beta}$ is anti-symmetric with respect to exchanges of α and β and of μ and ν and furthermore $S_{\mu\nu}^{\mu\beta} = 0$, i.e. traceless; then $S_{\mu\nu}^{\alpha\beta}$ is identically zero. Schematically, this means that $S_{\mu\nu}^{\alpha\beta} = -S_{\mu\nu}^{\beta\alpha} = -S_{\nu\mu}^{\alpha\beta}$ and $S_{\mu\nu}^{\mu\beta} = 0 \rightarrow S_{\mu\nu}^{\alpha\beta} \equiv 0$.

[Proof]

Let us consider a tensor

$$T_{\mu\nu\lambda}^{\alpha\beta\gamma} = S_{\mu\nu}^{\alpha\beta}\cdot\delta_\lambda^\gamma - S_{\mu\nu}^{\alpha\gamma}\cdot\delta_\lambda^\beta - S_{\mu\nu}^{\gamma\beta}\cdot\delta_\lambda^\alpha$$
$$+ S_{\lambda\nu}^{\gamma\beta}\cdot\delta_\mu^\alpha - S_{\lambda\nu}^{\gamma\alpha}\cdot\delta_\mu^\beta - S_{\lambda\nu}^{\alpha\beta}\cdot\delta_\mu^\gamma.$$

Then, $T_{\mu\nu\lambda}^{\alpha\beta\gamma}$ is totally anti-symmetric for any two exchanges of α, β and γ and satisfies traceless condition $T_{\alpha\nu\lambda}^{\alpha\beta\gamma} = 0$. However, such tensor must be identically zero in the three-dimensional space, since only non-zero independent component must be $T_{\mu\nu\lambda}^{123}$ and by traceless-condition, this has to be identically zero, (for example, consider the case $\mu = 1$). Thus we have $T_{\mu\nu\lambda}^{\alpha\beta\gamma} \equiv 0$. Then, by putting $\gamma = \nu$ and summing over ν, we find

$$T^{\alpha\beta\nu}_{\mu\nu\lambda}=S^{\alpha\beta}_{\mu\lambda}-S^{\alpha\beta}_{\lambda\mu}=2S^{\alpha\beta}_{\mu\lambda}=0. \quad (Q.E.D.)$$

Our lemma I is not surprising at all, since such tensor $S^{\alpha\beta}_{\mu\nu}$ must be an irreducible representation in U_n but such type of irreducible representation is not possible in U_3. (However, it is possible in $U_n (n\geq 4)$ and has signature (1, 1, −1, −1) in U_4.)

[Lemma II]

In U_3, for any two arbitrary tensors M_ν^μ and N_ν^μ, we have the following identities:

$$(M_\nu^\mu\cdot N_\beta^\alpha+M_\beta^\alpha\cdot N_\nu^\mu)-(M_\nu^\alpha\cdot N_\beta^\mu+M_\beta^\mu\cdot N_\nu^\alpha)$$
$$=\delta_\nu^\mu[\langle M\rangle N_\beta^\alpha+M_\beta^\alpha\langle N\rangle-(M\cdot N)_\beta^\alpha-M_\beta^\lambda\cdot N_\lambda^\alpha]$$
$$-\delta_\nu^\alpha[\langle M\rangle N_\beta^\mu+M_\beta^\mu\langle N\rangle-(M\cdot N)_\beta^\mu-M_\beta^\lambda\cdot N_\lambda^\mu]$$
$$-\delta_\beta^\mu[\langle M\rangle\cdot N_\nu^\alpha+M_\nu^\alpha\cdot\langle N\rangle-(M\cdot N)_\nu^\alpha-M_\nu^\lambda\cdot N_\lambda^\alpha]$$
$$+\delta_\beta^\alpha[\langle M\rangle\cdot N_\nu^\mu+M_\nu^\mu\cdot\langle N\rangle-(M\cdot N)_\nu^\mu-M_\nu^\lambda\cdot N_\lambda^\mu]$$
$$-(\delta_\nu^\mu\cdot\delta_\beta^\alpha-\delta_\nu^\alpha\cdot\delta_\beta^\mu)\cdot[\langle M\rangle\cdot\langle N\rangle-\langle M\cdot N\rangle].$$

[Proof]

Define a tensor $Q^{\alpha\beta}_{\mu\nu}$ by

$$Q^{\alpha\beta}_{\mu\nu}=(M_\mu^\alpha\cdot N_\nu^\beta-M_\mu^\beta\cdot N_\nu^\alpha)-(M_\nu^\alpha\cdot N_\mu^\beta-M_\nu^\beta\cdot N_\mu^\alpha)\cdot$$

Then, $Q^{\alpha\beta}_{\mu\nu}$ is anti-symmetric for exchanges of α and β and of μ and ν. Furtheremore, construct a new tensor $S^{\alpha\beta}_{\mu\nu}$ by

$$S^{\alpha\beta}_{\mu\nu}=Q^{\alpha\beta}_{\mu\nu}-(\delta_\mu^\alpha\cdot Q^{\lambda\beta}_{\lambda\nu}+\delta_\mu^\beta\cdot Q^{\alpha\lambda}_{\lambda\nu}+\delta_\nu^\alpha\cdot Q^{\lambda\beta}_{\mu\lambda}+\delta_\nu^\beta\cdot Q^{\alpha\lambda}_{\mu\lambda})$$
$$+1/2\cdot(\delta_\mu^\alpha\cdot\delta_\nu^\beta-\delta_\nu^\alpha\cdot\delta_\mu^\beta)Q^{\lambda\theta}_{\lambda\theta}.$$

We can see that $S^{\alpha\beta}_{\mu\nu}$ satisfies the conditions of lemma I, and must be identically zero. This gives the desired identity. (Q.E.D.)

[Theorem II]

In U_3, for any tensor T_ν^μ and for infinitesimal operator A_ν^μ, which satisfy the commutation relations Eqs. (A·1) and (A·4), we have the following identity.

$$2\cdot[(A\cdot T\cdot A)_\nu^\mu+(T\cdot A\cdot A)_\nu^\mu+(A\cdot A\cdot T)_\nu^\mu]-(2\langle A\rangle+9)\cdot[(A\cdot T)_\nu^\mu+(T\cdot A)_\nu^\mu]$$
$$-2\cdot\langle T\rangle(A\cdot A)_\nu^\mu+[6\langle A\rangle+12+(\langle A\rangle)^2]T_\nu^\mu$$
$$-1/2\cdot[\langle A\cdot A\rangle T_\nu^\mu+T_\nu^\mu\cdot\langle A\cdot A\rangle]+[6\langle T\rangle+2\langle A\rangle\langle T\rangle-2\langle A\cdot T\rangle]A_\nu^\mu$$
$$+\delta_\nu^\mu\cdot(-\langle T\rangle\cdot[(\langle A\rangle)^2-\langle A\cdot A\rangle+4\langle A\rangle+4]+(2\langle A\rangle+6)\langle A\cdot T\rangle$$
$$-2\langle A\cdot A\cdot T\rangle)=0.$$

Note that $[\langle A\rangle, T_\nu^\mu]=0$, $[\langle T\rangle, A_\nu^\mu]=0$ but $[\langle A\cdot A\rangle, T_\nu^\mu]\neq 0$.

[Theorem III]

$$6(A\cdot A\cdot A)_\nu^\mu-[6\langle A\rangle+18]\cdot(A\cdot A)_\nu^\mu+[3\cdot(\langle A\rangle)^2-3\cdot\langle A\cdot A\rangle+$$

$$+12\cdot\langle A\rangle+12]\cdot A_\nu^\mu$$
$$-[(\langle A\rangle)^3+4(\langle A\rangle)^2+4\langle A\rangle-3\langle A\rangle\cdot\langle A\cdot A\rangle+2\langle A\cdot A\cdot A\rangle$$
$$-6\langle A\cdot A\rangle]\delta_\nu^\mu=0.$$

Theorem III can be obtained from theorem II by putting $T=A$. From this, we see that $(A\cdot A\cdot A\cdot A)_\nu^\mu$ can be expressed as a linear combination of δ_ν^μ, A_ν^μ, $(A\cdot A)_\nu^\mu$ and $(A\cdot A\cdot A)_\nu^\mu$, and so $\langle A\cdot A\cdot A\cdot A\rangle$ is a function of $\langle A\rangle$, $\langle A\cdot A\rangle$ and $\langle A\cdot A\cdot A\rangle$. So are $\langle A^n\rangle$ $(n\geq 4)$, as has already been mentioned.

To prove theorem II, we put $M_\nu^\mu=N_\nu^\mu=A_\nu^\mu$ in lemma II, and multiply T_α^ν from the left, and using commutation relations Eqs. (A·1) and (A·4), we find our theorem II, when we change the indices suitably. We may give another direct proof of theorem II as follows. Any tensor $Q^{\alpha\beta\gamma\theta}_{\pi\lambda\nu\mu}$ which is anti-symmetric with respect to any exchanges of two variables among α, β, γ and θ must be identically zero in U_3. Therefore, we have

$$\sum_P(-1)^P T_\pi^\alpha\cdot A_\lambda^\beta\cdot A_\nu^\gamma\cdot\delta_\mu^\theta=0$$

where P means permutations among α, β, γ and θ. Then putting $\pi=\beta$, $\gamma=\lambda$, $\theta=\nu$ and taking traces, we find our theorem II again after somewhat long calculations.

Now, we shall prove our theorem I, Eq. (A·8). Using the commutation relations

$$[M_2, T_\nu^\mu]=3(A\cdot A\cdot T)_\nu^\mu-3(T\cdot A\cdot A)_\nu^\mu-3[M_3, T_\nu^\mu],$$
$$[M_3, S_\nu^\mu]=2(A\cdot S)_\nu^\mu-2(S\cdot A)_\nu^\mu,$$

we can rewrite theorem II as follows.

$$3(T\cdot A\cdot A)_\nu^\mu-(T\cdot A)_\nu^\mu\cdot(2\langle A\rangle+9)+T_\nu^\mu\cdot[1/2\cdot(\langle A\rangle)^2$$
$$-1/2\cdot\langle A\cdot A\rangle+3\langle A\rangle+6]$$
$$=-1/2\cdot[M_2, (TA)_\nu^\mu-(\langle A\rangle+3)T_\nu^\mu]-1/3[M_3, T_\nu^\mu]$$
$$+(A\cdot A)_\nu^\mu\cdot\langle T\rangle-A_\nu^\mu\cdot[(\langle A\rangle+3)\cdot\langle T\rangle-\langle T\cdot A\rangle]$$
$$-\delta_\nu^\mu([\langle A\rangle+3]\langle T\cdot A\rangle-\langle T\cdot A\cdot A\rangle-1/2\cdot\langle T\rangle\cdot[(\langle A\rangle)^2-\langle A\cdot A\rangle+4\langle A\rangle+4]). \tag{A·9}$$

Now, in a given irreducible representation, M_2 and M_3 are constants, so the matrix elements $\langle\alpha|[M_2, Q]|\beta\rangle=0$ and $\langle\alpha|[M_3, Q]|\beta\rangle=0$, hence we can omit the first and second terms in the right-hand side of Eq. (A·9) in our case. Thus, we have

$$3(T\cdot A\cdot A)_\nu^\mu-(T\cdot A)_\nu^\mu(2\langle A\rangle+9)+T_\nu^\mu\cdot[1/2\cdot(\langle A\rangle)^2-1/2\cdot\langle A\cdot A\rangle+3\langle A\rangle+6]$$
$$=(A\cdot A)_\nu^\mu\cdot\langle T\rangle-A_\nu^\mu\cdot[(\langle A\rangle+3)\langle T\rangle-\langle T\cdot A\rangle]$$
$$-\delta_\nu^\mu([\langle A\rangle+3]\langle T\cdot A\rangle-\langle T\cdot A\cdot A\rangle-$$

$$-1\cdot 2\cdot\langle T\rangle\cdot[(\langle A\rangle)^2-\langle A\cdot A\rangle+4\langle A\rangle+4]). \qquad (A\cdot 10)$$

Eq. (A·10) is true when we take any matrix elements in a given irreducible representation. Now, T_ν^μ is arbitrary, as long as it satisfies the commutation relation Eq. (A·4), and so we can replace T by $T\cdot A$ and $T\cdot A\cdot A$ in Eq. (A·10). For quantities like $T\cdot A\cdot A\cdot A$ or $T\cdot(A\cdot A\cdot A\cdot A)$, we use our theorem III and we can reduce them to a linear combination of T, $T\cdot A$ and $T\cdot A\cdot A$. Then, Eq. (A·10) gives three equations of the form

$$a_{1i}(T\cdot A\cdot A)_\nu^\mu+a_{2i}(T\cdot A)_\nu^\mu+a_{3i}(T)_\nu^\mu$$
$$=b_{1i}(A\cdot A)_\nu^\mu+b_{2i}(A)_\nu^\mu+b_{3i}\cdot\ .\ . \qquad (i=1,2,3) \qquad (A\cdot 10)$$

We can give an explicit form for a_{ij} and b_{ij}, but as it is a little complicated, here we simply remark that a_{ij} are functions of only $\langle A\rangle$, $\langle A\cdot A\rangle$ and $\langle A\cdot A\cdot A\rangle$, i.e. a_{ij} depend only upon f_1, f_2 and f_3 by Eq. (A·7), b_{ij} depend upon f_1, f_2 and f_3, and also upon $\langle T\rangle$, $\langle T\cdot A\rangle$ and $\langle T\cdot A\cdot A\rangle$, which are constants in the irreducible representation which we are considering. We can solve Eq. (A·10), since the determinant $\det(a_{ij})$ is, in general, not identically zero; thus we get

$$T_\nu^\mu=a\cdot\delta_\nu^\mu+b\cdot A_\nu^\mu+c(A\cdot A)_\nu^\mu$$

and two other equations for $(T\cdot A)_\nu^\mu$ and $(T\cdot A\cdot A)_\nu^\mu$. This is the desired formula theorem I.

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Note on Unitary Symmetry in Strong Interaction. II

——*Excited States of Baryons*——

Susumu OKUBO

Department of Physics, University of Tokyo, Tokyo
and
Department of Physics, University of Rochester
Rochester, N.Y., U.S.A.

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A classification of the baryon isobars has been investigated on basis of the unitary symmetry model which has been developed in a previous paper under the same title.

The purpose of this note is to investigate problems of baryon isobars from the viewpoint of the unitary symmetry model.[1] In this model, the mass differences among mesons and among baryons are neglected. As the result, one may wonder if such a model can be applicable to the study of the baryon isobars which appear in the meson-baryon scattering, where these mass differences are certainly not negligible. It is almost probable that our model will present a very poor approximation for this problem if compared quantitatively. However, it might be possible that many of qualitative features could be roughly explained by our model. It is due to this hope that this work has been undertaken. So all results given in this paper should not be taken in its face value, but only in a qualitative sense. In this paper, we shall concern ourselves with the case of studies of Yamaguchi-Gell-Mann scheme,[2] since the case of the Sakata scheme has been treated already[1),3)] and would not produce any new results. We may note that our results here could be applied also for study of meson-meson resonances or for baryon-baryon scattering resonances, with small changes.

As has been noted in the previous paper,[1),2)] the baryon octet (N, Ξ, Σ, Λ) and the meson octet (K, $\bar{K}$, π, π_0') belong to irreducible representations $U_3(1,0,-1)$ of the 3-dimensional unitary group U_3, and they are represented by two traceless tensors N_ν^μ and f_ν^μ, respectively, as follows:

$$\pi_+=f_1^2,\ \pi_-=f_2^1,\ \pi_0=\frac{1}{\sqrt{2}}(f_1^1-f_2^2),\ \pi_0'=-\frac{3}{\sqrt{6}}f_3^3,$$
$$K_+=f_1^3,\ K_0=f_2^3,\ \bar{K}_+=f_3^1,\ \bar{K}_0=f_3^2, \qquad (1)$$
$$\Sigma_+=N_1^2,\ \Sigma_-=N_2^1,\ \Sigma_0=\frac{1}{\sqrt{2}}(N_1^1-N_2^2),\ \Lambda=-\frac{3}{\sqrt{6}}N_3^3.$$

$$p=N_1^3,\ n=N_2^3,\ \Xi_-=N_3^1,\ \Xi_0=N_3^2.$$

We may note that the same representation Eq. (1) has been given by many others[4] in matrix notations. Now, as has been remarked in (I), the baryon isobars N^*, N^{**}, Y_0^* and Y_1^* have to belong to some of the following irreducible representations in the right-hand side of the next equation.

$$U_3(1,0,-1)\times U_3(1,0,-1)=2U_3(1,0,-1)+U_3(0,0,0)+U_3(2,0,-2)$$
$$+U_3(2,-1,-1)+U_3(1,1,-2)\ . \qquad (2)$$

The same is also true for meson-meson scattering isobars or for baryon-baryon scattering resonances, since both the mesons and the baryons belong to the same irreducible representations $U_3(1,0,-1)$, and therefore their scattering states to the product representation $U_3(1,0,-1)\times U_3(1,0,-1)$. Thus all results given in this paper can be immediately translated from our baryon isobar case into the meson-meson and baryon-baryon scattering cases, but here we study only in the case of meson-baryon scattering problem. Below, we list a classification of particles contained in each of these irreducible representations. This can be easily done by applying the technique developed previously.[1]

(*a*) $U_3(1,0,-1)$

$(I=1/2,\ Y=1)$, $(I=1/2,\ Y=-1)$, $(I=1,\ Y=0)$, $(I=0,\ Y=0)$.

(*b*) $U_3(0,0,0)$

$(I=0,\ Y=0)$.

(*c*) $U_3(2,0,-2)$

$(I=2,\ Y=0)$, $(I=3/2,\ Y=+1)$, $(I=3/2,\ Y=-1)$,

$(I=1,\ Y=2)$, $(I=1,\ Y=-2)$, $(I=1,\ Y=0)$,

$(I=1/2,\ Y=1)$, $(I=1/2,\ Y=-1)$, $(I=0,\ Y=0)$.

(*d*) $U_3(2,-1,-1)$

$(I=3/2,\ Y=1)$, $(I=1,\ Y=0)$, $(I=1/2,\ Y=-1)$ $(I=0,\ Y=-2)$.

(*e*) $U_3(1,1,-2)$

$(I=3/2,\ Y=-1)$, $(I=1,\ Y=0)$, $(I=1/2,\ Y=+1)$ $(I=0,\ Y=+2)$,

where Y stands for hypercharge, so that $Y=S+1$ in terms of the strangeness S in the present case. First of all, we note that a particle with $I=1$ and $Y=0$ is contained in all representations except in $U_3(0,0,0)$. Thus, we cannot identify the representation to which Y_1^* belongs. We shall investigate all of these in turn.

Case (a): $U_3(1,0,-1)$

If Y_1^* belongs to this representation, we have to identify other three particles

in this representation. Obviously, we can identify the particle with $(I=0,\ Y=0)$ as Y_0^*, and the one with $(I=1/2,\ Y=1)$ as the second pion-nucleon resonance N^{**}, while the one with $(I=1/2,\ Y=-1)$ can be considered as an excited state of Ξ. Now, the second resonance N^{**} of the pion-nucleon system is to be considered likely to have the character of a $d_{3/2}$ resonance.[5] Accordingly, we have to assign the same $d_{3/2}$ resonances for all Y_1^*, Y_0^* and Ξ^* in this case. This is not so bad, because the spin[6] of Y_1^* appears to be 3/2. However, we should remark that it is unnecessary to identify the $(I=1/2,\ Y=1)$ state as N^{**}. As has been stated in the beginning, our approximation is quite poor, and as the result the state with $(I=1/2,\ Y=1)$ might disappear when we take account of the mass differences among meson octet and among the baryon octet. The above statement is meant to indicate the following: "When we neglect these mass differences, the state with $(I=1/2,\ Y=1)$ then certainly exists because of U_3 symmetry. Now, we have to change the masses of the pion and the kaon and of the nucleon and the Ξ-particle from the common values. We may suppose that we can take such a procedure continuously with respect to these masses. Then, in course of these operations, the state with $(I=1/2,\ Y=1)$ may cease to represent a resonance state." If such thing could ever happen, then we cannot say anything about the spin of Y_1^* and Y_0^*. But we do not adopt such a view here.

The irreducible representation $U_3(1,0,-1)$ can be characterized by a traceless tensor T_ν^μ whose identifications with the real isobar states can be expressed exactly in the same way as Eq. (1). Let us consider the decay of these isobars into one baryon and one meson states. We can form the following two invariant expressions for these processes:

$$S_1=M_\nu^\mu f_\lambda^\nu T_\mu^\lambda\,,$$
$$S_2=M_\nu^\mu f_\mu^\lambda T_\lambda^\nu, \qquad (3)$$

where we have put $M_\nu^\mu=(N_\mu^\nu)^\dagger$ for creation operators of baryons. This occurrence of two independent forms corresponds to the double appearance of $U_3(1,0,-1)$ representation in the product $U_3(1,0,-1)\times U_3(1,0,-1)$ as we can see from Eq. (2), and thus the same situation does not happen to other representations in the right-hand side of Eq. (2). At any rate, we cannot determine the branching ratio of $Y_1^*\to\Sigma+\pi$ against $Y_1^*\to\Lambda+\pi$ in our case, unless we make some additional assumptions. One tempting hypothesis is to assume the invariance of our theory under the transpose operation; i.e. we assume the invariance under interchanges of lower and upper suffixes. By this operation, a tensor F_ν^μ is changed into F_μ^ν, so that $S_1\leftrightarrow S_2$ in Eq. (3) and we have the following from Eq. (1).

$$\pi_+\leftrightarrow\pi_-,\ \pi_0\leftrightarrow\pi_0,\ K_+\leftrightarrow\overline{K}_+,\ K_0\leftrightarrow\overline{K}_0,\ \pi_0'\leftrightarrow\pi_0',$$
$$\Sigma_+\leftrightarrow\Sigma_-,\ \Sigma_0\leftrightarrow\Sigma_0,\ p\leftrightarrow\Xi_-,\ n\leftrightarrow\Xi_0,\ \Lambda\leftrightarrow\Lambda. \qquad (4)$$

We may note that similar transformations have already been proposed by many authors.[7] Then, we can compute the kinematical weights for the various processes, since the only invariant expression is now S_1+S_2 instead of an arbitrary linear combination of S_1 and S_2 of Eq. (3). We list our results obtained in this fashion in the following tables. In Table I the relative weights have their origin in numerical coefficients due to generalized Clebsch-Gordon coefficients. If we could neglect the mass differences among baryons, then the widths for these processes are proportional to the relative weights. However, we should take account of the baryon mass differences at least for the calculation of the phase-volume. Thus, for evaluations of relative widths, we should multiply to these weights the d-wave phase volume which is given by

$$\frac{1}{M^2}\cdot k^5 \tag{5}$$

where M is the mass of the mother isobar, and k is the magnitude of the spatial momentum of the meson in the rest system of the isobar.

Table I. Relative weights and widths for decays in case (a).

type of process	relative weight	relative width
$(N^{**})_+ \to \left\{ \begin{array}{l} n+\pi_+ \\ p+\pi_0 \end{array} \right.$	1	1
$(Y_1^*)_+ \to \Lambda+\pi_+$	4/9	0.014
$(Y_1^*)_+ \to \Sigma_{+,0}+\pi_{0,+}$	0	0
$(Y_0^*)_0 \to \Sigma_{\pm,0}+\pi_{\mp,0}$	4/3	0.008
$(\Xi)_-^* \to \Xi_{0,-}+\pi_{-,0}$	1	?

One interesting aspect is that Y_1^* does not decay into a pion and a Σ, in agreement with experiment. However, this is not characteristic only of the present scheme, since the representation $U_3(2, 0, -2)$ also forbids $Y_1^* \to \Sigma+\pi$. Actually, it is a natural consequence of the invariance of theory under the transpose operation Eq. (4), as has been shown by Sakurai.[7] As we shall see shortly, the representation $U_3(2, 0, -2)$ is also invariant under this operation.

Now, we will investigate the case (c), since the case (b) is quite trivial.

Case (*c*) : $U_3(2, 0, -2)$

This is a 27-dimensional representation, which is characterized by a tensor $T^{\mu\nu}_{\alpha\beta}$ having the following properties:

$$T^{\mu\nu}_{\alpha\beta}=T^{\nu\mu}_{\alpha\beta}=T^{\mu\nu}_{\beta\alpha},\quad T^{\mu\nu}_{\mu\beta}=0. \tag{6}$$

We can form a base of the unitary representation $U_3(2, 0, -2)$ from this $T^{\mu\nu}_{\alpha\beta}$, which is given by

(i) $(I=2,\ Y=0)$

$$T_{11}^{22},\ T_{12}^{22}-T_{11}^{12},\ \frac{1}{\sqrt{6}}(T_{11}^{11}+T_{22}^{22}-4T_{12}^{12}),\ T_{12}^{11}-T_{22}^{12},\ T_{22}^{11}$$

(ii) $(I=3/2,\ Y=1)$

$$\sqrt{2}\,T_{11}^{23},\ \sqrt{\frac{2}{3}}(2T_{12}^{23}-T_{11}^{13}),\ -\sqrt{\frac{2}{3}}(2T_{12}^{13}-T_{22}^{23}),\ -\sqrt{2}T_{22}^{13}.$$

(iii) $(I=3/2,\ Y=-1)$

$$\sqrt{2}\,T_{13}^{22},\ -\sqrt{\frac{2}{3}}(2T_{13}^{12}-T_{23}^{22}),\ -\sqrt{\frac{2}{3}}(2T_{23}^{12}-T_{13}^{11}),\ \sqrt{2}\,T_{23}^{11}.$$

(iv) $(I=1,\ Y=2)$

$$T_{11}^{33},\ \sqrt{2}\,T_{12}^{33},\ T_{22}^{33}.$$

(v) $(I=1,\ Y=-2)$

$$T_{33}^{22},\ -\sqrt{2}\,T_{33}^{12},\ T_{33}^{11}.$$

(vi) $(I=1,\ Y=0)$

$$\sqrt{5}\,T_{31}^{32},\ \sqrt{\frac{5}{2}}(T_{23}^{23}-T_{13}^{13}),\ -\sqrt{5}\,T_{23}^{13}.$$

(vii) $(I=1/2,\ Y=1)$

$$\sqrt{\frac{10}{3}}T_{31}^{33},\ \sqrt{\frac{10}{3}}T_{32}^{33}.$$

(viii) $(I=1/2,\ Y=-1)$

$$\sqrt{\frac{10}{3}}\,T_{33}^{23},\ -\sqrt{\frac{10}{3}}T_{33}^{13}.$$

(ix) $(I=0,\ Y=0)$

$$\sqrt{\frac{10}{3}}T_{33}^{33}.$$

In the table listed in the above, all terms in a given sub-classification as (I, Y) have the same transformation properties as spherical harmonics $Y_M^{(I)}$ $(M=I, I-1, \cdots, -I)$ in the decreasing order from the left to the right. The relative numerical coefficients belonging to different sub-classifications with different (I, Y) have been determined from a requirement that

$$\sum_{\mu,\nu,\alpha,\beta}(T^{\mu\nu}_{\alpha\beta})^*T^{\mu\nu}_{\alpha\beta}=\sum_{A=1}^{27}(X_A)^*X_A \tag{7}$$

where $X_A (A=1, \cdots, 27)$ represents each term listed in the above. The condition Eq. (7) shows that these 27 X_A's form the desired unitary base of our representation $U_3(2, 0, -2)$. Thus, we can identify each X_A with each isobar states appearing in $U_3(2, 0, -2)$ as in Eq. (1).

In this case, we have an undesired isobar with $(I=1, Y=2)$, which could be detected in kaon-nucleon scattering but so far not found. However, we may suppose again that such state would not appear if we take the mass differences among mesons and among baryons. We may identify $(I=1, Y=0)$, $(I=0, Y=0)$ and $(I=3/2, Y=1)$ with Y_1^*, Y_0^* and N^*, respectively, where N^* represents for the first pion-nucleon resonance. Then, all these have to be resonances in the $p_{3/2}$ states, since the last one is known to be so. Then, the state with $(I=1/2, Y=1)$ in our representation must be a resonance in $p_{3/2}$ state also and thus this state is difficult[5] to be identified with N^{**}, though we cannot completely rule out a possibility of the $p_{3/2}$ resonance for N^{**} at the moment. The possible existence of other states in $U_3(2, 0, -2)$ does not lead to any disagreement with the experimental data.

Now, let us consider the decay matrix element of isobars into mesons and baryons. In this case, there is only one invariant form under U_3.

$$S=M_\alpha^{\ \mu} f_\beta^{\ \nu} T_{\mu\nu}^{\alpha\beta}. \tag{7'}$$

We may note that Eq. (7') is invariant under the transpose operation as has been mentioned already. We can reduce this in terms of X_A and of meson and baryon components by using the table listed in the above and by Eq. (1).

Table II. Relative weights and widths for decays in the case (c).

type of the decay	relative weight	relative width
$(N^*)_{++}\to p+\pi_+$	1	1
$(Y_1^*)_+\to \Lambda+\pi_+$	3/5	0.36
$(Y_1^*)_+\to \Sigma_{+,0}+\pi_{0,+}$	0	0
$(Y_0^*)\to \Sigma_{\pm,0}+\pi_{\mp 0}$	1/20	0.01
$(N^{**})_+\to \begin{cases} p+\pi_0 \\ n+\pi_+ \end{cases}$	1/10	0.45
$(Y_2^*)_{++}\to \Sigma_+ +\pi_+$	2	?
$(Z)_{++}\to p+K$	2	?

Then, we can compute the kinematical weight factors for the decay as before. For the calculation of the relative widths in the above table, we have multiplied the p-wave phase volume factor:

$$k^3/M^2. \tag{8}$$

Again, Y_1^* does not decay into $\Sigma+\pi$, because of the transpose invariance of $U_3(2, 0, -2)$ as has been mentioned already. In the table, Y_2^* means the state with $(I=2, Y=0)$, and Z represents the state with $(I=1, Y=2)$.

We should note that appearance of Y_2^*, Y_1^* and Y_0^* could be easily understood[8] in terms of the static p-wave pion-hyperon interactions, if we assume that $f_{\Lambda\Sigma}\gg f_{\Sigma\Sigma}$. Indeed, this is the case if we take the D-type interaction[1),2)] in Gell-Mann's notation, which is also invariant under the transpose transformation.

Case (*d*): $U_3(2, -1, -1)$

This is a 10-dimensional representation, and can be specified by a tensor

$F_{\alpha\beta}^{\mu\nu}$ having the following properties:

$$F_{\alpha\beta}^{\mu\nu}=F_{\beta\alpha}^{\mu\nu}=-F_{\alpha\beta}^{\nu\mu},\ F_{\mu\beta}^{\mu\nu}=0. \tag{9}$$

The unitary base $X_A (A=1, \cdots, 10)$ of $U_3(2, -1, -1)$ can be formed from $F_{\alpha\beta}^{\mu\nu}$ in the same way as in the previous case, giving that

(i) $(I=3/2, Y=1)$

$$F_{11}^{23},\ \sqrt{3}\ F_{12}^{23}\ (=-\sqrt{3}\ F_{11}^{13}),\ -\sqrt{3}\cdot F_{12}^{13}(=\sqrt{3}\ F_{22}^{23}),\ -F_{22}^{13}.$$

(ii) $(I=1, Y=0)$.

$$\sqrt{3}\ F_{13}^{23}(=\sqrt{3}\ F_{11}^{12}),\ \sqrt{6}\ F_{12}^{12}(=-\sqrt{6}\ F_{13}^{13}=\sqrt{6}\ F_{23}^{23}),$$
$$-\sqrt{3}\ F_{23}^{13}(=\sqrt{3}\ F_{22}^{12}).$$

(iii) $(I=1/2, Y=-1)$

$$\sqrt{3}\ F_{13}^{12}(=\sqrt{3}\ F_{33}^{23}),\ -\sqrt{3}\ F_{33}^{13}(=\sqrt{3}\ F_{23}^{12}).$$

(iv) $(I=0, Y=-2)$

$$F_{33}^{12}.$$

It is interesting to note that we have a particle with the strangeness -3. The decay matrix element is again unique and has the same form as Eq. (7') when we replace $T_{\alpha\beta}^{\mu\nu}$ by $F_{\alpha\beta}^{\mu\nu}$. Then, once again we can compute the weights and the relative widths. Now, we have the decay $Y_1^*\to\pi+\Sigma$ in this case.

Table III. Relative weights and widths for decays in the case (d).

type of decay	relative weight	relative width
$(N^*)_{++}\to p+\pi_+$	1	1
$(Y_1^*)_+\to \Lambda+\pi_+$	1/2	0.30
$(Y_1^*)_+\to \Sigma_{0,+}+\pi_{+,0}$	1/3	0.043

Case (*e*): $U_3(1, 1, -2)$

This is the contragradient representation of $U_3(2, -1, -1)$; i.e. the one which can be obtained from $U_3(2, -1, -1)$ by the transpose operation. Thus, it is specified by a tensor $G_{\alpha\beta}^{\mu\nu}$ satisfying the following conditions.

$$G_{\alpha\beta}^{\mu\nu}=G_{\alpha\beta}^{\nu\mu}=-G_{\beta\alpha}^{\mu\nu},\ G_{\mu\beta}^{\mu\nu}=0. \tag{10}$$

Similarly, we can construct the unitary base by

(i) $(I=3/2, Y=-1)$

$$G_{13}^{22},\ -\sqrt{3}\ G_{13}^{12}(=\sqrt{3}\ G_{23}^{22}),\ -\sqrt{3}\ G_{23}^{12}(=\sqrt{3}\ G_{13}^{11}),\ G_{23}^{11}.$$

(ii) $(I=1, Y=0)$

$$-\sqrt{3}\ G_{13}^{23}(=\sqrt{3}G_{12}^{22}),\ -\sqrt{6}\ G_{12}^{12}(=\sqrt{6}\ G_{13}^{13}=-\sqrt{6}\ G_{23}^{23}),$$
$$\sqrt{3}\ G_{23}^{13}(=\sqrt{3}\ G_{12}^{11}).$$

(iii) $(I=1/2,\ Y=1)$

$$\sqrt{3}\ G_{13}^{33}(\equiv-\sqrt{3}\ G_{12}^{33}),\ \sqrt{3}\ G_{11}^{13}(\equiv\sqrt{3}G_{23}^{33}).$$

(iv) $(I=1/2,\ Y=+2)$

$$G_{12}^{33}.$$

We may identify $(I=1,\ Y=0)$ and $(I=1/2,\ Y=1)$ with Y_1^* and N^{**}, respectively, and we can compute the widths in a similar fashion.

Table IV. Relative weights and widths for decays in the case (e).

type of decay	relative weight	relative width
$(N^{**})_+\to N+\pi$	1	1
$(Y_1^*)_+\to\Sigma_{+,0}+\pi_{0,+}$	2/3	1.9×10^{-3}
$(Y_1^*)_+\to\Lambda+\pi_+$	1	3.2×10^{-3}

Finally, we shall give an application of the mass formula, which has been derived in (I). For particles belonging to the same irreducible representation, we have a relation among masses of these particles. It is given by

$$M=a+b\cdot Y+c\cdot[1/4\ Y^2-I(I+1)] \tag{11}$$

where a, b and c are some constants. This relation has been proved in the lowest order perturbation of a certain type of interactions causing the mass-differences, but in all orders of the U_3-conserving interactions. As has been stated in the beginning, this would not be a good approximation for the meson-baryon scattering problem, where the mass differences between the pion and the kaon is quite important. Thus, we should not expect that our results to be given in the below have some quantitative meanings. At any rate, Eq. (11) has three unknown constants, a, b and c. Thus, we have six relations among masses of particles contained in $U_3(2, 0, -2)$. If we use the experimental masses of Y_1^*, Y_0^* and N^*, then the masses of six other particles in $U_3(2, 0, -2)$ can be computed in terms of these three masses. In this way, we have

$$M(I=2,\ Y=0)\simeq1345\text{ Mev},$$
$$M(I=3/2,\ Y=-1)\simeq1505\text{ Mev},$$
$$M(I=1,\ Y=2)\simeq1125\text{ Mev},$$
$$M(I=1,\ Y=-2)\simeq1665\text{ Mev},$$
$$M(I=1/2,\ Y=1)\simeq1265\text{ Mev},$$
$$M(I=1/2,\ Y=-1)\simeq1535\text{ Mev}. \tag{12}$$

A serious trouble is that the mass of the particle with $(I=1,\ Y=2)$ is so low that it is stable against the decay into a nucleon and a kaon. However, this difficulty may not be so serious, since such state may disappear as remarked

already. We may note that we have a similar trouble in the case of the Sakata scheme.[1] It is also interesting to compare Eqs. (11) and (12) to those obtained in the case of the global symmetry model,[9] and to those of the Sakata scheme.[1),10)]

We have made a group-theoretical classification of isobar states. As has been mentioned in the beginning, almost all of the results given in this paper are also immediately applicable to the study of the meson-meson resonances or of the baryon-baryon scatterings, with small modifications. However, we would not go into details for these cases. From our analysis on baryon isobars, it seems to be difficult to identify the best irreducible representation for these at the moment. One interesting problem is to determine the parity of the resonances so as to enable us to distinguish whether the resonances are of the $p_{3/2}$ or $d_{3/2}$ character.

Acknowledgement

The author would like to express his thanks to Professor H. Miyazawa and other members of the University of Tokyo for their hospitality.

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SPIN AND UNITARY SPIN INDEPENDENCE OF STRONG INTERACTIONS*

F. Gürsey† and L. A. Radicati‡
Brookhaven National Laboratory, Upton, New York
(Received 15 July 1964)

The purpose of this Letter is twofold. We want first to point out that the group SU(4) introduced by Wigner[1] to classify nuclear states can be extended to the relativistic domain and it is, therefore, relevant for particle physics. We will next show that when strangeness is taken into account the group SU(4) becomes enlarged to[2] SU(6) which contains, as a subgroup, $SU(3)\otimes[SU(2)]_q$. $[SU(2)]_q$ is the unitary subgroup (little group) of the Lorentz group that leaves invariant the momentum four-vector q.

The group we consider here embodies SU(3) and the ordinary spin in the same way as Wigner's SU(4) embodies isotopic spin and ordinary spin. Preliminary results on the classification of particles based on SU(6) seem encouraging enough to motivate a study of this group.[3]

We begin by discussing the first point. Let us assume that the ρ, ω, and π mesons are coupled to the nuclear field through a symmetrical Lagrangian of the form

$$L_{NM} = g\{\bar\psi\gamma_\mu\psi\omega_\mu + \bar\psi\gamma_\mu\tau^a\psi\rho_\mu^{\ a} + i\bar\psi\gamma_5\gamma_\mu\tau^a\psi\varphi_\mu^{\ a}\}, \quad (1)$$

where a denotes the isotopic spin index. Let us further impose the subsidiary conditions

$$\partial_\mu\omega_\mu = 0, \quad \partial_\mu\rho_\mu^{\ a} = 0,$$

$$\partial_\lambda\varphi_\mu^{\ a} - \partial_\mu\varphi_\lambda^{\ a} = 0, \quad (2)$$

which insure that $\omega_\mu, \rho_\mu^{\ a}, \varphi_\mu^{\ a}$ describe, respectively, particles with $(J=1^-, T=0)$, $(J=1^-, T=1)$, and $(J=0^-, T=1)$. The pion field π^a is related to the axial vector field $\varphi_\mu^{\ a}$ through $\varphi_\lambda^{\ a} = (1/\mu)\partial_\lambda\pi^a$, μ being the mass common to all mesons.

The conditions (2) are compatible with the equations of motion only if L includes, besides L_{NM} [Eq. (2)], additional terms such that the mesons (ρ, ω, π) are coupled to conserved currents. The ω and ρ are coupled to the conserved baryon and isotopic-spin currents, respectively, while the pion is coupled to a conserved axial-vector current.

It can now be shown[4] that L is invariant under a group[5] $\mathcal{G}_4$ which induces for each momentum q of the mesons a unitary unimodular transformation among the 15 degenerate states ω, ρ, and π. In counting the multiplicity we include, for a given momentum, the spin states just as for Wigner's supermultiplets. Under this transformation the nucleon $(S=\frac{1}{2}, T=\frac{1}{2})$ transforms like the four-dimensional representation of the group.

In the nonrelativistic limit, L_{NM} gives rise to a potential which describes spin- and isospin-independent exchange forces (Majorana forces) between nucleons. This potential is, therefore, invariant under Wigner's group SU(4). If now a purely spin-dependent perturbation is introduced, ω and ρ remain degenerate whereas the pion splits from them within the supermultiplet. We note that ω, ρ, and π are associated with the adjoint representation of SU(4). When this representation is reduced under the subgroup $SU(2)\otimes[SU(2)]_q$ it splits into states with $(J=1^-, T=0)$, $(J=1^-, T=1)$, and $(J=0^-, T=1)$.

These considerations are readily extended to include strange particles. In this case the SU(2) isotopic-spin group is replaced by SU(3) so that $\mathcal{G}_4$ goes over into a group $\mathcal{G}_6$ whose little group is $[SU(6)]_q$ which admits $SU(3)\otimes[SU(2)]_q$ as a subgroup.

The representation of SU(6) can be characterized by five integers $(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5)$ where the λ_i's are functions of the five Casimir operators. Table I shows some of the representations of SU(6) together with their SU(3) and spin structure. The symbols (m, n) in the third column refer to the SU(3) and spin multiplicity, respectively.

The lowest nontrivial representation (10000) has six dimensions. It represents a fundamental SU(3) triplet with (ordinary) spin $\frac{1}{2}$. Its SU(3) $\otimes$SU(2) content is (3, 2). The conjugate representation (00001) describes the antiparticle and its content is (3*, 2).

A Lagrangian similar to (1) can be written which couples invariantly the fundamental triplet to mesons corresponding to the 35-dimensional adjoint representation. When a spin-dependent perturbation is introduced the 35 states split into a pseudoscalar octet and a degenerate vector nonet[6] with negative parity. These can be identified with the observed (π, K, η) and $(\rho, \omega, K^*, \varphi)$ multiplets. SU(6) provides, therefore, a natural explanation of the degeneracy of the vector octet and the vector singlet in the nonet. All other meson-meson resonances must belong to self-conjugate representations of SU(6). Possible candidates are (00000) with even or odd parity, (10001) with even parity, (11011) with even or odd parity, etc.

The baryon octet and the $J=\frac{3}{2}^+$ decuplet can be grouped as a 56-dimensional representation obtained from the symmetrical combination of three fundamental triplets. The reduction of the direct product of $\underline{6}\otimes\underline{6}\otimes\underline{6}$ gives rise to three representations with 20, 70, and 56 dimensions. The fact that the ground state of the three-body configuration is symmetrical (56-dimensional representation) in the spin and unitary-spin variables implies that the two-body forces between them are repulsive. This seems to exclude a scheme based on only three fundamental quarks[7] whereas it is consistent with model II discussed in Appendix IV of reference 6. The connection of higher representations with possible baryon resonances is discussed by Pais.[3]

Table I. Some representations of SU(6) and their unitary spin and spin content.

Labeling $(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5)$	Dimensions $D(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5)$	Unitary spin and spin multiplicities (n, m)
(00000)	1	(1, 1)
(10000)	6	(3, 2)
(00001)	6*	(3*, 2)
(01000)	15	(3*, 3), (6, 1)
(00100)	20	(8, 2), (1, 4)
(20000)	21	(6, 3), (3*, 1)
(10001)	35	(8, 3), (8, 1), (1, 3)
(30000)	56	(10, 4), (8, 2)
(11000)	70	(10, 2), (8, 4), (8, 2), (1, 2)

The splitting between the $J=0^-$ octet and the $J=1^-$ nonet suggests that the mass operator contains a spin-dependent term which can only be a function of $J(J+1)$. A simple mass formula for an SU(6) supermultiplet is the mass squared[6] formula

$$\mu^2 = \mu_0^{\ 2} + \alpha J(J+1) + \gamma[T(T+1) - \tfrac{1}{4}Y^2]$$

for mesons and

$$M = M_0 + aJ(J+1) + bY + c[T(T+1) - \tfrac{1}{4}Y^2]$$

for baryons.

These are by no means the most general mass formulas that can be written on the basis of a broken SU(6) symmetry. The mass formula problem is further discussed by Pais.[3]

The interaction Lagrangian with conserved currents is generated from the free Lagrangian through a gauge transformation[4] associated with the group $\mathcal{G}_6$.[8] As in the case of the electromagnetic interaction this implies parity conservation for the strong interactions invariant under $\mathcal{G}_6$. Hence all the states of an SU(6) supermultiplet must have the same parity. Our scheme is, therefore, different from others that have been discussed recently[2,9,10]; in particular it does not predict 0^+ and 1^+ mesons degenerate with the existing 0^- and 1^- mesons. The degenerate states associated with the meson states for given momentum $\vec{q}$ and given SU(6) quantum numbers are simply the states corresponding to the opposite momentum and the same SU(6) quantum numbers.

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†On leave from the Middle East Technical University, Ankara, Turkey.

‡On leave from Scuola Normale Superiore, Pisa, Italy.

[1] E. P. Wigner, Phys. Rev. 51, 105 (1937). For recent evidence on the validity of the supermultiplet model,

see P. Franzini and L. A. Radicati, Phys. Letters 6, 322 (1963).

[2]The group SU(6) has been suggested in a somewhat different context by M. Gell-Mann, to be published. Gell-Mann's point of view is, however, different from the one discussed here, being based on the algebra of the conserved and quasiconserved currents.

[3]For a more detailed analysis of the applications, see A. Pais, following Letter [Phys. Rev. Letters 13, (1964)].

[4]F. Gürsey and L. A. Radicati, to be published.

[5]The group $\mathfrak{G}_4$ is noncompact and may be regarded as an extension of the Lorentz group by means of the isotopic spin group. The generators of $\mathfrak{G}_4$ are the covariant spin operators, the isotopic spin operators and their products. The little group of $\mathfrak{G}_4$ for fixed momentum q is $[\mathrm{SU}(4)]_q$.

[6]F. Gürsey, T. D. Lee, and M. Nauenberg, Phys. Rev. 135, B467 (1964).

[7]M. Gell-Mann, Phys. Letters 8, 214 (1964).

[8]It is clear that the fundamental triplets will be coupled to the mesons through F-type coupling only. Since the baryons do not belong to the lowest representation of SU(6), the gauge operators generate a larger algebra which produces F-type couplings with the vector mesons and F- and D-type couplings with the pseudoscalar mesons.

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IMPLICATIONS OF SPIN-UNITARY SPIN INDEPENDENCE*

A. Pais†
Brookhaven National Laboratory, Upton, New York
(Received 15 July 1964)

It is the purpose of this note to discuss further the possibility[1] that a broken $[\mathrm{SU}(6)]_{\tilde{q}}$ is a useful symmetry in strong interactions.

To introduce some questions which arise, consider Wigner's nuclear SU(4)-multiplet theory.[2] Representations of this group label multinucleon states in a given nuclear l shell. This is useful largely because spin-orbit coupling can be neglected to a good approximation for low-lying states. Spin-orbit forces will lead to some recoupling and accordingly the classification under SU(4) gets less good for higher excitations, as emphasized by Wigner.

Likewise for SU(6). Call $(M)_{\tilde{q}}$ and $(B)_{\tilde{q}}$ the respective meson and baryon representations. For M-B scattering one must reduce out $\{(B)\otimes(M)\}_\alpha$ where α represents the orbital variables. After taking out the center of mass, one can choose $\alpha=(k,l,l_z)$, l = orbital angular momentum. For each partial wave there may be recoupling between l and the (B,M) spins. Where this is unimportant, we can just reduce out $(B)\otimes(M)$.

This leads to a maximum possible spin for the baryon resonances, namely $\frac{5}{2}$ with the proposed choice of representations.[1] Higher spins are a sure sign of (l,s) coupling. In the region where this starts to happen (it appears[3] to be ~2 BeV), the assignment of resonances to "new" SU(6) multiplets becomes considerably more complicated.

In view of this complexity, it may be asked whether it is necessary to put (8, 2) and (10, 4) in 56, as proposed,[1] because the breakdown SU(6) → factorized [SU(3)⊗SU(2)] (first stage) → broken SU(3) (second stage) has a first stage of which the scale is not known beforehand. However, the choice 56 becomes more suggestive through mass considerations. The success of the Gell-Mann–Okubo formula as an effective first-order perturbation leads one to try the assumption that SU(6) → broken SU(3) is additive in the first- and second-stage breakdowns with coefficients that depend on the (five) Casimir operators C_i of SU(6) only. This is achieved by $M=M_0+a_0(C_i)F_8+b_0(C_i)\times d_{8jk}F_jF_k$, or[4]

$$M=M_0+a(C_i)Y+b(C_i)[I(I+1)-\tfrac{1}{4}Y^2-\tfrac{1}{3}F^2] \qquad (1)$$

$(F^2=F_iF_i)$. M_0 is the central mass of an SU(3) multiplet,

$$M_0=M_{00}(C_i)+m(C_i,F^2,d_{ijk}F_iF_jF_k,J(J+1)). \qquad (2)$$

M_{00} is the central mass of the SU(6) multiplet. We shall see shortly that the dependence of the SU(6)-breaking term m on both spin and unitary-spin invariants is essential, and the same is true for the C_i dependence of the quantities a,b, etc.

Application of Eq. (1) to the meson 35 yields (using the quadratic mass relation) $\rho^2-\pi^2=K^{*2}-K^2$, known[5] to be true within the ρ-mass accuracy. Equation (1) as a linear mass formula gives for the 56 a calculated (10, 4) equidistance ≃ 130 MeV, derived from the (8, 2), close enough to the experimental value ≈145 MeV to make the choice 56 quite attractive.[6] The first-stage split between (10, 4) and (8, 2) is ≈235 MeV, comparable in magnitude to a, the F-type octet split.

There is an important new aspect to the (effective) $\bar{B}BM$ coupling in this theory. It follows from

$$\underline{56}\otimes\underline{56}^* = \underline{1} + \underline{35} + \underline{405} + \underline{2695} \quad (3)$$

that this coupling is, in fact, unique, because of the single occurrence of 35. Hence, the F/D ratio is determined by SU(6). The fact that both F and D must occur in this coupling was noted by Gürsey and Radicati.[7] Hence, there is no R invariance (unless one "doubles" the theory which is unattractive).

Let us next consider a few consequences based on the additional assumption that the (spin, F-spin) multiplets need not be strongly recoupled to l. As (10, 4) decays into baryon and meson (where energetically possible) one should at least know whether 56 is in 35⊗56. It is, as

$$\{\underline{35}\otimes\underline{56}\}_\alpha = \underline{1134} + \underline{700} + \underline{70} + \underline{56}. \quad (4)$$

For the decay of the (10, 4) the label α now specifically refers to $l = 1$. [For the one-particle states on the right-hand side of Eq. (4) we may imagine to be in their rest frame.] Equation (3) also indicates which other SU(6) representations are possible candidates for resonances which can decay into (octet + meson) or (decuplet + meson).

It is natural to consider next the other "small" representation, 70, of Eq. (3) with content

$$\underline{70} = (1, 2) + (8, 4) + (10, 2) + (8, 2).$$

It is tempting to fill (1, 2) with $Y_0^*(1405)$. For this to work, one needs spin$(Y_0^*) = \frac{1}{2}$. It is furthermore desirable for Y_0^* to have odd parity, in order that it can be the resonant state sought for in the interpretation of (K^-, p) data.[8] This would fix the parity of the other terms in Eq. (3) to be negative. Thus, the incomplete γ octet[9] becomes a possible candidate for $(8, 4)^-$ in $\underline{70}^-$. There would then be harmony between the spin-parity of this last multiplet and the desirable properties of $Y_0^*(1405)$.

Concerning the status of the γ octet, for both $Y_0^*(1520)$ and $N^{**}(1512)$ the evidence for $\frac{3}{2}^-$ is good.[10] The assignment $\frac{3}{2}^-$ to $Y_1^{**}(1660)$ seems dubious.[11] However, according to Willis[12] this possibility cannot be excluded. In connection with the SU(3) mass formula this assignment for Y_1^{**} would imply a $\Xi^*(1600)$ with $\frac{3}{2}^-$. If this at all exists,[13] its production seems to be at most ~1-2% of $\Xi^*(1530)$. [This would mean a first-stage split (8, 4)-(1, 2) of ≈185 MeV, comparable to the one for (10, 4)-(8, 2).] It seems that a $\frac{3}{2}^-$ octet could well be there, even though not all the correct ingredients may be at hand as yet.

If these assignments within the representation $\underline{70}^-$ are correct, there is a prediction of the existence of an $\frac{1}{2}^-$ octet and decuplet. In the spirit of Eq. (1), one may anticipate that there should be octet-decuplet relations also within the 70. If this is so and if we assume, to give an example, that the γ octet is fixed by the masses 1512, 1520, 1660, and (1600?) then the equidistance in $(10, 2)^-$ should be ~-60 MeV, i.e., it is a 10 with its "Ω" as lowest state. This, in turn, would imply a sum rule for $(8, 2)^-$, namely, $\Sigma^* \sim \Xi^* + 60$ MeV. These assignments to 70 can only possibly work if the first-stage split m of Eq. (2) depends on unitary spin as well as on spin. The simplest possibility of a dependence of m on F is $\alpha(C_i)F^2$ with $\alpha(C_i) > 0$ which would give equidistant central masses for the sequence (1, 2), (8, 2), and (10, 2) with (1, 2) lowest. Note that a and b in Eq. (1) are generally different for the 56 and the 70, due to their C_i dependence.[14]

The content of 1134 and 700 is, of course, very complex. In particular one (1, 2) and one (1, 4) are herein contained. The assignment of Y_0^* to 70 is therefore not unambiguous; one must hope that some simplicity prevails.

Finally, note that the small baryon representation[1] 20 is a baryon-two-meson state (for example, in 70⊗35). One can also discuss two-meson states, using $\underline{35}\otimes\underline{35} = \underline{1} + \underline{35} + \underline{35} + \underline{189} + \underline{280} + \underline{280}^* + \underline{405}$.

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†Permanent address: Rockefeller Institute, New York, New York.

[1]F. Gürsey and L. Radicati, preceding Letter [Phys. Rev. Letters 13, 173 (1964)]. Notations used here are the same as in this Letter.

[2]E. Wigner, Phys. Rev. 51, 105 (1937).

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limit is amusing in that it closely resembles the old strong-coupling treatment for the (3,3) resonance which has, in fact, SU(4) characteristics in its algebra [W. Pauli and S. Dancoff, Phys. Rev. 62, 85 (1942)].

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[8]R. H. Dalitz, Ann. Rev. Nucl. Sci. 13, 338 (1963).

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[12]W. J. Willis, private communication.

[13]P. L. Connolly et al., Proceedings of the Sienna International Conference on Elementary Particles (Società Italiana di Fisica, Bologna, Italy, 1963), Vol. I, p. 125.

[14]One should also consider the (l,s) coupling as a "third stage" which may lead to recurrences of SU(6) multiplets with higher J values. This coupling is not the same as the spin-orbit coupling of T. Kycia and K. Riley, Phys. Rev. Letters 10, 266 (1963) (K-R). An effective (l,s) coupling in the present meaning may be responsible for the $\Delta l = 2$ recurrences noted by K-R. If this picture makes sense, then the K-R mass plot indicates that the third stage is (once again) linearly independent of the first one and that both 56 and 70 recur with J raised by 2.

SPIN AND UNITARY SPIN INDEPENDENCE OF STRONG INTERACTIONS*

F. Gürsey,† A. Pais,‡ and L. A. Radicati§
Brookhaven National Laboratory, Upton, New York
(Received 5 August 1964)

In this note we pursue further the consequences of the assumption that strong interactions are spin and unitary spin (F-spin) independent.[1,2] In particular we discuss the meson-baryon vertex and some subgroups of SU(6).

Within a representation of SU(6), states of given four-momentum may be partially labeled by the eigenvalues of five commuting elements in the Lie algebra,[3] for which we may take (S_i = spin, $F_\lambda = F$ spin) S_3, F_3, F_8, S_3F_3, S_3F_8.

In this approach, a Fourier component $P_B{}^A(q)$ of the pseudoscalar octet ($A, B = 1, 2, 3$ are the SU(3)-tensor indices, q is the momentum) is united with a Fourier component $V_B{}^A(k, q)$ of the vector nonet ($\langle S_3\rangle \equiv k = -1, 0, 1$ is a polarization index) in the representation $\underline{35}$ of SU(6), described by a 6×6 matrix $M(q)$ given by

$$M(q) = \sigma_\mu M_\mu(q), \tag{1}$$

$$M_\mu = (iq_\mu/|q|)(F_A{}^B - \tfrac{1}{3}\delta_A{}^B F_C{}^C)P_B{}^A(q) + \textstyle\sum_k n_\mu(k,q)F_A{}^B V_B{}^A(k,q), \tag{2}$$

with $\mu = 1, \cdots, 4$. σ_μ are the Pauli spin matrices ($\mu = 1, 2, 3$) and the unit matrix ($\mu = 4$). $q_\mu/|q|$ and $n_\mu(k, q)$ form an orthogonal tetrad ($|q|^2 = q_0^2 - \vec{q}^2$). In particular, we have (see below)

$$V_3{}^3(k,q) = \varphi(k,q), \quad V_1{}^1(k,q) + V_2{}^2(k,q) = 2^{1/2}\omega(k,q) \tag{3}$$

where φ and ω stand for the corresponding vector mesons.

A matrix element of $M(q)$ is written as $M_\alpha{}^\beta(q)$, $\alpha, \beta = 1, \cdots, 6$, and we have[4] $M_\beta{}^\beta(q) = 0$. With this notation, the $\underline{56}$ representation of SU(6) which unites[1,2] the baryon octet (b) and decuplet (d) is written as $B^{\alpha\beta\gamma}(q)$; the anti-particles are $\overline{B}_{\alpha\beta\gamma}(q)$. In both cases there is total symmetry in α, β, and γ.

We turn to the ($\overline{B}, B, M$) vertex in the pure SU(6) limit, where all B's have mass M_{00} and all M's have μ_{00}. This vertex will, for example, contain the minimal coupling of protons to ρ^0 which we normalize to $g\bar{p}\gamma_\mu p\rho_\mu{}^0$. The general ($\bar{b}, b, V$) vertex also contains $\sigma_{\mu\nu}q_\nu$ coupling, which we leave aside for the moment. At low energies this minimal part of the vertex will then contain only the s-wave ($\bar{b}bV$) and the p-wave ($\bar{b}bP$) coupling. The minimal vertex is unique[3] and has the form (for given Fourier components of the fields)

$$J_\gamma{}^\delta(-q)M_\delta{}^\gamma(q) = 6g\overline{B}_{\alpha\beta\gamma}(p)B^{\alpha\beta\delta}(p')M_\delta{}^\gamma(q), \quad q = p - p'. \tag{4}$$

$J_\gamma{}^\delta$ is the baryon part of the strong current. The full current, which also contains M terms, can be decomposed into an axial current octet $(\mathfrak{a}_\mu)_A{}^B$ and a vector current nonet $(\mathfrak{v}_\mu)_A{}^B$ by the method of Eq. (2). We have

$$\partial_\mu(\mathfrak{v}_\mu)_A{}^B = 0; \quad \partial_\mu(\mathfrak{a}_\nu)_A{}^B - \partial_\nu(\mathfrak{a}_\mu)_A{}^B = 0. \tag{5}$$

By simultaneous reduction in spin and F-spin we can decompose (4) into b, d and P, V. We state some results for this minimal vertex.

(1) ($\bar{b}bV$). Its strength is normalized as noted above. Hence, from the fact that ρ is coupled to the conserved isospin current, g is determined by the rate for $\rho \to 2\pi$. Thus[5]

$$\frac{g^2}{4\pi} \simeq \tfrac{1}{2}. \tag{6}$$

The coupling is pure F, as $\mathfrak{v}$ is conserved [see Eq. (5)].

(2) ($\bar{b}bP$). As noted before[1] this is a mixture of D and F. We call their ratio $(D/F)_A$ and find

$$(D/F)_A = \tfrac{3}{2}. \tag{7}$$

This ratio will reappear in the weak decays if we assume that the same axial vector current is involved in weak interactions.

In order to define the total strength of this coupling we again go to low energies and consider the p-wave term $g_A p^\dagger \vec{\sigma} p \cdot \nabla\pi^0/\mu_{00}$. We find

$$g_A = 5g/3. \tag{8}$$

As g is not renormalized, the same is true for g_A. This comes about because $\mathfrak{v}$ and $\mathfrak{a}$ currents can transform into each other in the SU(6) limit. In order to go from g_A to the pseudoscalar constant g_{PS}, we use the central mass values of SU(6). In this way we get

$$\frac{g_{PS}{}^2}{4\pi} = \frac{25}{9}\left(\frac{2M_{00}}{\mu_{00}}\right)^2 \frac{g^2}{4\pi}. \tag{9}$$

Using Eq. (6) and mean masses $M_{00} \simeq 1100$ MeV, $\mu_{00} \simeq 700$ MeV, we get $g_{PS}^2/4\pi \simeq 12.5$. We do not attach significance to the precise value, but believe that the estimate is fair and the result encouraging.

(3) $(\bar{d}bP)$. This d-decay vertex is also contained in Eq. (4). Its strength is related to the width Γ_{33} of N_{33}^* by the following formula:

$$\Gamma_{33} = \frac{12}{25}\frac{g_{PS}^2}{4\pi}\frac{k^3}{m_{33}^2}\left[\frac{m_N m_{33}}{M_{00}^2}\right]. \qquad (10)$$

The true masses m_N and m_{33} for nucleon and N_{33}^* enter through the usual device of using the true phase space. g_{PS} is defined by Eq. (9). With the factor in square brackets $\simeq 1$, we get $\Gamma_{33} \sim 60$ MeV. This is of the right order, but these d widths cannot be too precise in the symmetry limit, as we know from[6] the properties of $Y_1^* \to \Sigma + \pi$.

(4) $(\bar{d}dP)$ and $(\bar{d}dV)$. The p-wave transition $N^{*++}(S_3 = \frac{3}{2}) \to N^{*++}(\frac{3}{2}) + \pi^0$ and the corresponding S-wave transition with a ρ^0 each have strength $3g$. Thus there is strong direct d-P and d-V interaction. It would be interesting to know whether this could explain to some extent the different value for $(D/F)_A$ found here as compared to other estimates.[7]

Remarks.–(i) The above considerations can be readily extended to include induced terms. Since in the low-energy limit the vertex is SU(6)-invariant for each partial wave, the static limit is obtained by taking the s-wave contribution of the minimal vector meson coupling together with the contribution of the induced pseudoscalar meson coupling. For the p wave the induced vector meson Pauli term completes the minimal pseudoscalar term.

(ii) For each partial wave the four-point function for B-M scattering contains only three independent amplitudes. Likewise, B-B scattering can be expressed in terms of four independent amplitudes. This implies a large number of selection rules.

We now turn to the discussion of an important subgroup of SU(6), which we denote by $W(Y) \otimes \mathrm{SU}(4)(T) \otimes \mathrm{SU}(2)(X)$. To define this subgroup we follow the usual procedure to study the algebra associated with the fundamental (6-dimensional) representation. Let

$$\lambda_\pm = \tfrac{1}{2}(1 \pm \xi), \quad \xi = (4/\sqrt{3})F_8 + \tfrac{1}{3}. \qquad (11)$$

Thus $\xi^2 = 1$, $\lambda_\pm^2 = \lambda_\pm$, $\lambda_+\lambda_- = \lambda_-\lambda_+ = 0$. $W(Y)$ has the elements λ_+, λ_-. SU(4)(T) is generated by $\lambda_+ S_i$, F_i, and $\lambda_+ S_i F_k$ $(i, k = 1, 2, 3)$, and SU(2)(X) by $X_i = \lambda_- S_i$.

In Table I we list for some of the representations of SU(6) those representations of SU(4)(T) $\otimes$ SU(2)(X) which correspond to a definite eigenvalue of $\lambda_+ F_8$. We recall that ω, π, and ρ form the adjoint 15-dimensional representation of SU(4)(T) while φ is a scalar under SU(4)(T). Conversely, the requirement that the physical φ and ω belong to definite representations of SU(4)(T) defines the mixing of the "unphysical" SU(3) singlet $\omega^{(0)}$ and the octet member $\varphi^{(0)}$, for these physical mesons. Equation (3) is in accordance with this choice.

N appears in a 20 representation, together with N^*. This differs from Wigner's assignment[8] for the nucleon which was also provisionally used earlier.[1] It is most probable that Wigner's theory appears as a valid approximation to the SU(6) model in the nonrelativistic limit.

Table I. SU(4) multiplets in SU(6).

Representations[a] and dimensions of SU(6) $(\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5)$, D_6	Representations[b] and dimensions of SU(4)(T) $(PP'P'')$, D_4	Representation of SU(2)(X) X	G'-parity $G' = G\xi$	Particles
(10000), 6	$(\frac{1}{2}\frac{1}{2}\frac{1}{2})$, 4	0		
	(000), 1	$\frac{1}{2}$		
(10001), 35	(110), 15	0	−1	ω, π
			+1	ρ
	(000), 1	1	+1	φ
(30000), 56	$(\frac{3}{2}\frac{1}{2}\frac{1}{2})$, 20	0		N, N^*
	(000), 1	$\frac{3}{2}$		Ω^-

[a] Defined as in reference 1.
[b] Defined as in reference 11.

In the fourth column of Table I, we list for the mesons ω, π, ρ, and φ the eigenvalues of the operator $G' = G\xi$, where G is the usual G parity. Just as G is convenient for dealing with (π, ρ, ω), so G' will be convenient for dealing simultaneously with (π, ρ, ω) and φ. For these particles G' coincides with Bronzan and Low's number A.[9] The $(N, \bar{N})$ system is closed under G', and so are each of the systems $(K, \bar{K})$, $(K^*, \bar{K}^*)$, $(N^*, \bar{N}^*)$, and $(\Omega, \bar{\Omega})$. The other particles involved (for example the η) are all definite mixtures of states even and odd under G'. The behavior of all particles under G' is therefore fully specified and hence G' provides selection rules in the SU(4) limit. For example, the reactions $\pi + N \to N + n\pi + \varphi$, $N + N \to N + N + n\pi + \varphi$, and $N + \bar{N} \to n\pi + \varphi$ should be suppressed compared to the corresponding ω reactions. On the other hand, $\varphi \to K + \bar{K}$ and $K^- + p \to \Lambda + n\pi + \varphi$ are "allowed"; that is, not SU(4)-inhibited relative to ω. All these results appear to be in qualitative agreement with experiment. It may be stressed that these consequences of the theory carry no restrictions on particles present in intermediate states.

The reduction of the two-meson product (110) $\times$(110) in SU(4)(T) is worth noting. It yields (in terms of dimensions) $\underline{1} + \underline{15} + \underline{15} + \underline{20} + \underline{45} + \underline{45}^* + \underline{84}$. The $\underline{20}$, which is characterized by the representation $(2, 0, 0)$, has (T, S) content $(1, 1) + (1, 5) + (5, 1) + (3, 3)$ and thus contains an isoscalar of spin 2 which could be identified with the f^0(1250 MeV) with positive parity. If this is correct, then the $\underline{20}^+$ would also contain an isotriplet of axial vector mesons. However, as was the case for higher baryon resonances,[2] one must be prepared for a possible nonuniqueness. Thus in the present[10] case, the $\underline{84}$ also contains a (1, 5).

Finally, we note that SU(6) invariance may also prove useful in the analysis of nuclear forces. In the static limit the mesons (ρ, ω, π) will still generate SU(4)(T)-invariant Majorana forces between nucleons. φ will not contribute in the limit of perfect symmetry, while a contribution to Wigner forces will arise from η exchange. While SU(4)(T) allows for an arbitrary mixture of Wigner versus Majorana forces, SU(6) invariance makes this mixture unique. It would be interesting to investigate the relationship with the Serber mixture[12] of nuclear forces.

One of us (F.G.) would like to thank Dr. R. Serber for many stimulating discussions. Details of this work will be published elsewhere.[13]

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†On leave from the Middle East Technical University, Ankara, Turkey.

‡Permanent address: Rockefeller Institute, New York, New York.

§On leave from Scuola Normale Superiore, Pisa, Italy.

[1]F. Gürsey and L. A. Radicati, Phys. Rev. Letters 13, 173 (1964).

[2]A. Pais, Phys. Rev. Letters 13, 175 (1964). In the fourth line from the end of this paper please read "$\underline{280} + \underline{280}^*$" for "$\underline{280} + \underline{280}$."

[3]The F_λ are normalized by the same convention as in M. Gell-Mann, Phys. Rev. 125, 1067 (1962), Eq. (4.16). For the S_i we have $[S_1, S_2] = iS_3$, cyclically.

[4]$M_\beta^{\ \beta}(q)$ is a separate representation of SU(6)(q) and describes a spinless unitary singlet.

[5]M. Gell-Mann, D. Sharp, and W. G. Wagner, Phys. Rev. Letters 8, 261 (1962).

[6]V. Gupta and V. Singh, to be published.

[7]R. Cutkosky, Ann. Phys. (N.Y.) 23, 415 (1963); S. Glashow and L. Rosenfeld, Phys. Rev. Letters 10, 192 (1963); A. Martin and K. Wali, Nuovo Cimento 31, 1324 (1964).

[8]E. P. Wigner, Phys. Rev. 51, 106 (1937).

[9]J. E. Bronzan and F. E. Low, Phys. Rev. Letters 12, 522 (1964).

[10]The complete (T, S) contents are as follows: $\underline{45}$ (and $\underline{45}^*$) = (1, 3) + (3, 1) + (3, 3) + (3, 5) + (5, 3); $\underline{84}$ = (1, 1) + (1, 5) + (5, 1) + 2×(3, 3) + (3, 5) + (5, 3) + (5, 5). The $\underline{84}$ representation provides a first instance of multiple occurrence of the same (T, S) submultiplet.

[11]E. Feenberg and E. P. Wigner, Rept. Progr. Phys. 8, 274 (1941).

[12]J. M. Blatt and V. F. Weisskopf, Theoretical Nuclear Physics (John Wiley & Sons, Inc., New York, 1952), p. 170.

[13]After completion of this work, one of us (F.G.) was informed by Dr. B. Sakita that he, independently, had the idea of extending Wigner's supermultiplet theory to elementary particles and putting the mesons in the adjoint representation of SU(6).

MASS FORMULAS IN THE SU(6) SYMMETRY SCHEME*

T. K. Kuo and Tsu Yao
Brookhaven National Laboratory, Upton, New York
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Recently it was proposed by Gürsey and Radicati[1] and Pais[2] that the SU(6) symmetry scheme incorporating spin and unitary spin may have important consequences in particle physics. They discussed, among other things, a possible mass formula and applied it to some low-dimensional representations. In this note[3] we propose that the SU(6) symmetry is broken analogously as in SU(3),[4,5] namely, the primary symmetry-breaking term in the Hamiltonian transforms like the $I=0, Y=0, J=0$ member of the $\underline{35}$ representation. The major result of this assumption is that in a given SU(6) representation, states with the same I, Y, and J belonging to different SU(3) multiplets are mixed in a definite way.

The 36 traceless operators $B_\mu{}^\nu$ of SU(6) are defined such that their representation in the six-dimensional vector space C_6 are given by

$$(B_\nu{}^\mu)_{ij}=\delta_{\mu j}\delta_{\nu i}-\tfrac{1}{6}\delta_{\mu\nu}\delta_{ij}, \quad (1)$$

$(\mu,\nu,i,j=1,2,\cdots,6)$. These operators satisfy the commutation relations

$$[B_\nu{}^\mu, B_\beta{}^\alpha]=\delta_\beta{}^\mu B_\nu{}^\alpha-\delta_\nu{}^\alpha B_\beta{}^\mu. \quad (2)$$

The symmetry-breaking term is proposed to be $T_3{}^3+T_6{}^6$, where

$$[B_\nu{}^\mu, T_\beta{}^\alpha]=\delta_\beta{}^\mu T_\nu{}^\alpha-\delta_\nu{}^\alpha T_\beta{}^\mu. \quad (3)$$

Note that the hypercharge operator Y is $-(B_3{}^3+B_6{}^6)$. It can be shown that[6]

$$T_\nu{}^\mu=a_0\delta_\nu{}^\mu+a_1B_\nu{}^\mu+a_2(B\cdot B)_\nu{}^\mu+a_3(B\cdot B\cdot B)_\nu{}^\mu$$
$$+a_4(B\cdot B\cdot B\cdot B)_\nu{}^\mu+a_5(B\cdot B\cdot B\cdot B\cdot B)_\nu{}^\mu, \quad (4)$$

where the a_i's are constants depending only on the five Casimir operators of the group.

For the few low-dimension representations discussed below, only the first three terms in Eq. (4) are needed. Therefore, for those SU(6) supermultiplets we can write down the following mass formula:

$$M=M_0+aY+b\{(B\cdot B)_{SU(4)}-2Q(Q+1)-\tfrac{1}{2}Y^2\}. \quad (5)$$

For mesons mass squared is to be used in Eq. (5). The symbol $(B\cdot B)_{SU(4)}$ denotes the quadratic Casimir operator of the SU(4) subgroup which is considered by Gürsey, Pais, and Radicati.[7] $\vec{Q}$ is an angular momentum vector with components

$$Q_3=\tfrac{1}{2}(B_3{}^3-B_6{}^6),$$
$$Q_+=B_3{}^6,$$
$$Q_-=B_6{}^3. \quad (6)$$

In the quark language,[8] $Q=\frac{1}{2}$ for the $S=\pm1$ quarks and $Q=0$ for the $S=0$ quarks. In Table I we shall give all the eigenvalues of $(B\cdot B)_{SU(4)}$ and Q of all the particles in the $\underline{20}$, $\underline{35}$, $\underline{56}$, and $\underline{70}$ representations.

Now let us discuss the $\underline{35}$ representation $[\underline{35}=(\underline{8},\underline{1})+(\underline{8},\underline{3})+(\underline{1},\underline{3})]$ which has as members the pseudoscalar-meson octet and the vector-meson nonet. Since

$$\underline{35}\otimes\underline{35}=\underline{1}\oplus\underline{35}\oplus\underline{35}\oplus\underline{189}\oplus\underline{280}\oplus\underline{280}^*\oplus\underline{405}, \quad (7)$$

the matrix element

$$\langle\underline{35}|T_\nu{}^\mu|\underline{35}\rangle$$
$$=a_0+a_1\langle\underline{35}|B_\nu{}^\mu|\underline{35}\rangle+a_2\langle\underline{35}|(B\cdot B)_\nu{}^\mu|\underline{35}\rangle. \quad (8)$$

Table I. SU(4) multiplets in SU(6).

Particles	No. of states	Representation of SU(4)	$(B\cdot B)_{\mathrm{SU}(4)}$	Q
		$\underline{35}$		
ρ, ω, π	15	$\underline{15}$	8	0
K^*, K	8	$\underline{4}$	4	$\frac{1}{2}$
$\bar{K}^*, \bar{K}$	8	$\underline{4}^*$	4	$\frac{1}{2}$
φ	3	$\underline{1}$	0	1
η	1	$\underline{1}$	0	0
		$\underline{56}$		
N^*, N	20	$\underline{20}$	16	0
Y_1^*, Σ, Λ	20	$\underline{10}$	9	$\frac{1}{2}$
Ξ^*, Ξ	12	$\underline{4}$	4	1
Ω	4	$\underline{1}$	1	$\frac{3}{2}$
		$\underline{70}$		
$N_{3/2}^{1/2}, N_{1/2}^{1/2}, N_{1/2}^{3/2}$	20	$\underline{20}$	10	0
$Y_1^{1/2}, Y_0^{1/2}, Y_0^{3/2}$	12	$\underline{6}$	5	$\frac{1}{2}$
$Y_1^{1/2}, Y_0^{1/2}, Y_1^{3/2}$	20	$\underline{10}$	9	$\frac{1}{2}$
$\Xi_{1/2}^{1/2}, \Xi_{1/2}^{3/2}$	12	$\underline{4}$	4	1
$\Xi_{1/2}^{1/2}$	4	$\underline{4}$	4	0
$\Omega_0^{1/2}$	2	$\underline{1}$	1	$\frac{1}{2}$
		$\underline{20}$		
$Y_0^{1/2}, Y_1^{1/2}, Y_0^{3/2}$	12	$\underline{6}$	5	$\frac{1}{2}$
$N_{1/2}^{1/2}$	4	$\underline{4}^*$	4	0
$\Xi_{1/2}^{1/2}$	4	$\underline{4}$	4	0

This simplification from Eq. (4) is the result that in Eq. (7) $\underline{35}$ occurs only twice in $\underline{35}\otimes\underline{35}$. The immediate consequence of Eq. (5) is as follows:

(1) For vector mesons we get the familiar result

$$m_\omega^2 = m_\rho^2,$$

$$m_\varphi^2 + m_\rho^2 = 2m_{K^*}^2. \tag{9}$$

(2) For pseudoscalar mesons we get the usual mass sum rule,[4,5]

$$m_K^2 = \tfrac{1}{4}(3m_\eta^2 + m_\pi^2). \tag{10}$$

(3) We also obtain the relation

$$m_{K^*}^2 - m_\rho^2 = m_K^2 - m_\pi^2, \tag{11}$$

which was noticed before.[2]

(4) In connection with Eq. (9), we also obtain from Eq. (8) the mixing of φ^0 and ω^0 unambiguously, such that the physical φ and ω are given by

$$\varphi = -(\tfrac{2}{3})^{1/2}\varphi^0 + (\tfrac{1}{3})^{1/2}\omega^0,$$

$$\omega = (\tfrac{1}{3})^{1/2}\varphi^0 + (\tfrac{2}{3})^{1/2}\omega^0. \tag{12}$$

(5) We further notice that the primary symmetry-breaking term $(T_3^3 + T_6^6)$ still leaves ρ and π degenerate (also K^* and K). This degeneracy can be lifted by a spin-dependent mass term which can only be a function of $J(J+1)$. We emphasize that the inclusion of this spin-dependent term will not affect the results in Eqs. (9)-(12). (See below for more details.)

We next come to a discussion of the $\underline{56}$ representation $[\underline{56} = (\underline{10}, \underline{4}) + (\underline{8}, \underline{2})]$ which has as members the baryon octet and decuplet. Since

$$\underline{35}\otimes\underline{56} = \underline{56}\oplus\underline{70}\oplus\underline{700}\oplus\underline{1134}, \tag{13}$$

$\underline{56}$ occurs only once, the matrix element

$$\langle\underline{56}|T_\nu^{\ \mu}|\underline{56}\rangle = a_0 + a_1\langle\underline{56}|B_\nu^{\ \mu}|\underline{56}\rangle. \tag{14}$$

Thus the mass formula for $\underline{56}$ reduces to the simple form

$$M = M_0 + a_1 Y. \tag{15}$$

(1) Now both the decuplet and the octet are equally spaced:

$$M_\Omega - M_{\Xi^*} = M_{\Xi^*} - M_{Y_1^*} = M_{Y_1^*} - M_{N^*}, \tag{16}$$

$$M_\Xi - M_\Sigma = M_\Sigma - M_N, \tag{17a}$$

$$M_\Lambda = M_\Sigma. \tag{17b}$$

(2) Furthermore,

$$M_{\Xi^*} - M_{Y_1^*} = M_\Xi - M_\Sigma. \tag{18}$$

(3) We still have the degeneracy between Ξ^* and Ξ, etc., which can be removed by a spin-dependent mass term as before.

(4) Now Λ and Σ are still degenerate. This degeneracy can be removed by adding a term of the form $\lambda[I(I+1)-\frac{1}{4}Y^2]$ to Eq. (5). Equations (17a) and (17b) are now combined to give the usual octet mass formula,

$$2(M_\Xi + M_N) = 3M_\Lambda + M_\Sigma. \tag{17c}$$

We note that Eqs. (10)-(12), (16), and (18) are not changed. [Equation (9) becomes[9] $m_\varphi^2 + \frac{1}{2}(m_\rho^2 + m_\omega^2) = 2m_{K^*}^2$.] The general mass formula can now be written as

$$M = M_0 + aY + b[(B\cdot B)_{SU(4)} - 2Q(Q+1) - \tfrac{1}{2}Y^2] + \mu J(J+1) + \lambda[I(I+1) - \tfrac{1}{4}Y^2]. \tag{19}$$

So far we have only reproduced some familiar results. Now we proceed to a discussion of the $\underline{70}$ representation [$\underline{70} = (\underline{8}, \underline{4}) + (\underline{10}, \underline{2}) + (\underline{8}, \underline{2}) + (\underline{1}, \underline{2})$]. Again we obtain an equation similar to Eq. (8), since

$$\underline{35} \otimes \underline{70} = \underline{20} \oplus \underline{56} \oplus \underline{70} \oplus \underline{70} \oplus \underline{540} \oplus \underline{560} \oplus \underline{1134}. \tag{20}$$

For the spin-$\frac{3}{2}$ baryon resonances we have the familiar octet mass formula.[2] In the case of the spin-$\frac{1}{2}$ resonances we again encounter the mixing problem just as in Eq. (12) where φ^0 and ω^0 get mixed by the symmetry-breaking term. Here the $I=0, Y=0$ members of $(\underline{8}, \underline{2})$ and $(\underline{1}, \underline{2})$ are mixed. Furthermore, the $I=\frac{1}{2}, Y=-1$ members of $(\underline{8}, \underline{2})$ and $(\underline{10}, \underline{2})$ are mixed. So are the $I=1, Y=0$ members of $(\underline{8}, \underline{2})$ and $(\underline{10}, \underline{2})$. The mixing angle is found to be $\theta = 45°$ in all three cases. From Eq. (19) there are six mass sum rules among the nine (in general) nondegenerate particles:

$$N_{3/2}' + 3\Xi_\pm' = \Omega' + 3\Sigma_\pm', \tag{21a}$$

$$\Omega' + N_{3/2}' = \Xi_\mp' + \Sigma_\pm', \tag{21b}$$

$$2(N_{1/2}' + \Xi_\pm') = 3\Lambda_\pm' + \Sigma_\mp', \tag{21c}$$

where $N_{3/2,\,1/2}'$ have $I = \frac{3}{2}, \frac{1}{2}$, respectively. The subscript + denotes the heavier, and − the lighter of the two particles with the same I and Y. We note that Eq. (21a) takes a form hitherto not discussed. So far very few spin-$\frac{1}{2}$ resonances have been positively identified in the experiments. It is hoped that Eqs. (21) may be helpful in finding spin-$\frac{1}{2}$ resonances in the future.

It is a pleasure to thank Professor F. Gürsey and Professor A. Pais for stimulating discussions.

Note added in proof. – The transformation properties of the terms $J(J+1)$ and $[I(I+1)-\frac{1}{4}Y^2]$, which are not considered in this Letter, have since been discussed by Bég and Singh.[10] In fact, their Eq. (22) reduces to our Eq. (19) for $b=f=0$.

*Work performed under the auspices of the U. S. Atomic Energy Commission.

[1] F. Gürsey and L. A. Radicati, Phys. Rev. Letters 13, 173 (1964).

[2] A. Pais, Phys. Rev. Letters 13, 175 (1964).

[3] Details of this Letter will be published elsewhere.

[4] M. Gell-Mann, Phys. Rev. 125, 1067 (1962).

[5] S. Okubo, Progr. Theoret. Phys. (Kyoto) 27, 949 (1962).

[6] This is a generalization of Eq. (A.8) in Okubo's paper.

[7] F. Gürsey, A. Pais, and L. A. Radicati, Phys. Rev. Letters 13, 299 (1964).

[8] M. Gell-Mann, Phys. Letters 8, 214 (1964).

[9] F. Gürsey, T. D. Lee, and M. Nauenberg, Phys. Rev. 135, B467 (1964).

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SPLITTING OF SPIN-UNITARY SPIN SUPERMULTIPLETS

Mirza A. Baqi Bég

and

Virendra Singh*
Institute for Advanced Study, Princeton, New Jersey
(Received 26 August 1964)

1. Recent work of Gürsey, Pais, and Radicati[1-3] appears to indicate that the ideas propounded by Wigner[4] 27 years ago may, with appropriate generalization to accommodate strangeness, find spectacular fulfillment in the domain of particle physics. In the Sakata model one immediately gets SU(6) in place of Wigner's SU(4); in the eightfold way a similar picture is easily constructed with the help of quarks.[5] In a relativistic theory the full invariance group, of course, is not SU(6); however, the classification of particle states with respect to SU(6) still seems permissible.[6]

The qualitative success of SU(6) classifications, in spite of the marked lack of degeneracy in the supermultiplets, prompts one to ask: What is the nature of the phenomenological interaction responsible for the breakdown? Unless one can pin down the transformation properties of this interaction the symmetry will be of little practical use. In this connection it should be noted that mass formulas were written down in references 1 and 2 on the basis of physical intuition. It is not clear, a priori, whether these formulas can be derived by starting with any number of SU(6) tensor operators.

The purpose of this note is to report some results that have emerged in a systematic study of the problems mentioned above. We consider all the mass formulas that can be derived by considering tensor operators transforming according to real representations of dimensionality less than 1000, which can contribute to the meson, baryon, and low-lying resonance spectra. These representations and their SU(3)⊗SU(2) content are[7]

$$\underline{35}=(\underline{1},\underline{3})\oplus(\underline{8},\underline{3})\oplus(\underline{8},\underline{1}), \tag{1}$$

$$\underline{189}=(\underline{1},\underline{1})\oplus(\underline{8},\underline{1})\oplus(\underline{27},\underline{1})\oplus2(\underline{8},\underline{3})\oplus(\underline{10},\underline{3})\oplus(\underline{10}^*,\underline{3})\oplus(\underline{1},\underline{5})\oplus(\underline{8},\underline{5}), \tag{2}$$

$$\underline{405}=(\underline{1},\underline{1})\oplus(\underline{8},\underline{1})\oplus(\underline{27},\underline{1})\oplus2(\underline{8},\underline{3})\oplus(\underline{10},\underline{3})\oplus(\underline{10}^*,\underline{3})\oplus(\underline{27},\underline{3})\oplus(\underline{1},\underline{5})\oplus(\underline{8},\underline{5})\oplus(\underline{27},\underline{5}). \tag{3}$$

The tensors we consider will, of course, be singlets under SU(2); under SU(3) we shall take the ones that either are singlet or transform like the $I=Y=0$ member of an octet.

Incidentally, the $\underline{35}$ representation has already been considered by Kuo and Yao.[8] The choice turns out to be rather inadequate since the spin degeneracy is not lifted at all and for baryons the isospin degeneracy is not lifted either.

Before we write down the mass formulas it is necessary to establish the requisite notation.

2. The first problem is to set up a scheme for labeling SU(6) states. Mathematically, such a scheme is afforded by the reduction chain

$$SU(6)\supset U(1)\otimes SU(5)\supset U(1)\otimes U(1)\otimes SU(4)\cdots. \tag{4}$$

We are unable, however, to find any physical meaning for the quantum numbers that emerge in this chain. We begin therefore by considering the physical chain (P chain)

$$SU(6)\supset SU(2)\otimes SU(3)\supset SU(2)\otimes U(1)\otimes SU(2), \tag{5}$$

which fails to furnish us with enough labels. We therefore supplement this chain with an unphysical chain (U chain)

$$SU(6)\supset U(1)\otimes SU(2)\otimes SU(4)\supset U(1)\otimes SU(2)\otimes SU(2)\otimes SU(2). \tag{6}$$

The subgroups in one chain do not generally commute with those in another and appropriate "recoupling" transformations are needed.

3. We denote by $A_\beta{}^\alpha$, $\alpha,\beta=1,2,\cdots,6$, the 35 infinitesimal generators of SU(6) satisfying the canonical commutation rules. From an inspection of the adjoint representation one can pick out the generators of the commuting SU(3) and SU(2), respectively. These are

$$SU(3):\ A_i^{\ j}+A_{i+3}^{\ \ j+3},\quad i,j=1,2,3; \tag{7}$$

$$SU(2)_J:\ J_+=\sum_{i=1}^{3}A_i^{\ i+3},$$

$$J_-=\sum_{i=1}^{3}A_{i+3}^{\ \ i},$$

$$J_3=\tfrac{1}{3}\sum_{i=1}^{3}(A_i{}^i-A_{i+3}{}^{i+3}). \qquad (8)$$

The subscript J implies ordinary spin. The isospin and hypercharge operators in SU(3) are, of course,

$$\mathrm{SU}(2)_I:\ I_+=A_1{}^2+A_4{}^5,$$
$$I_-=A_2{}^1+A_5{}^4,$$
$$I_3=\tfrac{1}{2}(A_1{}^1-A_2{}^2+A_4{}^4-A_5{}^5); \qquad (9)$$
$$\mathrm{U}(1):\ Y=-(A_3{}^3+A_6{}^6). \qquad (10)$$

Equations (7)-(10) complete the identification of operators in the P chain. For the U chain one has

$$\mathrm{SU}(4):\ A_i{}^j+\tfrac{1}{4}\delta_i{}^j(A_3{}^3+A_6{}^6),\ i,j=1,2,4,5; \qquad (11)$$
$$\mathrm{U}(1):\ Y; \qquad (12)$$
$$\mathrm{SU}(2)_S:\ S_+=A_3{}^6,$$
$$S_-=A_6{}^3,$$
$$S_3=\tfrac{1}{2}(A_3{}^3-A_6{}^6). \qquad (13)$$

We use the subscript S to make explicit the fact that in the defining representation $\vec{S}$ is the spin of the strangeness-bearing quark. The commuting subgroups of SU(4) are $\mathrm{SU}(2)_I$ and

$$\mathrm{SU}(2)_N:\ N_+=A_1{}^4+A_2{}^5,$$
$$N_-=A_4{}^1+A_5{}^2,$$
$$N_3=\tfrac{1}{2}(A_1{}^1-A_4{}^4+A_2{}^2-A_5{}^5). \qquad (14)$$

$\vec{N}$ is the spin of quarks with no strangeness. Note the relationship between the two chains and the identity

$$\vec{J}=\vec{N}+\vec{S}. \qquad (15)$$

We shall need only the quadratic Casimir operators of the various groups. For SU(6) our definition is

$$C_2^{(6)}=\tfrac{1}{2}\sum_{\lambda=1}^{6}(A\cdot A)_\lambda{}^\lambda=\tfrac{1}{2}\sum_{\lambda,\mu=1}^{6}\{A_\mu{}^\lambda,A_\lambda{}^\mu\}. \qquad (16)$$

Similarly, $C_2^{(3)}$, $C_2^{(4)}$, and $C_2^{(2)}(L)=L(L+1)$ ($L=I,J,S,N$).

4. We proceed to the construction of tensor operators. For $T^{(35)}$ a general construction is immediately available from Ginibre's theorem[9] and an analogous construction can be worked out for $T^{(189)}$ and $T^{(405)}$. We shall not quote these general constructions since we are interested only in representations α such that $\alpha^*\otimes\alpha$ contains $\underline{189}$ and $\underline{405}$ no more than once.

For $T^{(189)}$ $[T^{(405)}]$, we first extract the antisymmetric (symmetric) part of $\underline{6}\otimes\underline{6}=\underline{15}\oplus\underline{21}$ to obtain the basis tensors of the $\overline{15}$- $(\overline{21}$-) dimensional representation and at the same time identify the quantum numbers associated with each component. With this information in hand we can write down the five $I=J=Y=0$ states that occur in the reducible representation $\underline{15}^*\otimes\underline{15}=\underline{1}\oplus\underline{35}\oplus\underline{189}$ $(\underline{21}^*\otimes\underline{21}=\underline{1}\oplus\underline{35}\oplus\underline{405})$. The extraction of orthogonal linear combinations with prescribed transformation properties is then a straightforward task. A knowledge of the basis tensor leads immediately to the corresponding tensor operator.

All of these tensor operators can be expressed in terms of the Casimir operators of subgroups in either the P or the U chains. The mass operators[10] can then be read off from these expressions.

5. We indicate by $M_{(n)}{}^{(m)}$ the mass operator containing a symmetry-breaking term transforming like an SU(6) tensor of multiplicity n with an SU(3) component of multiplicity m and $I=Y=0$, and singlet under $\mathrm{SU}(2)_J$.

The five "irreducible" mass formulas are

$$M_{(35)}{}^{(8)}=a_1+b_1Y+c_1[2S(S+1)-C_2^{(4)}+\tfrac{1}{4}Y^2], \qquad (17)$$
$$M_{(189)}{}^{(1)}=a_2+b_2[2J(J+1)-C_2^{(3)}], \qquad (18)$$
$$M_{(189)}{}^{(8)}=a_3+b_3\{[2J(J+1)-C_2^{(3)}]+3[2I(I+1)-\tfrac{1}{2}Y^2-2N(N+1)+2S(S+1)]-\tfrac{3}{4}[2S(S+1)-C_2^{(4)}+\tfrac{1}{4}Y^2]\}, \qquad (19)$$
$$M_{(405)}{}^{(1)}=a_4+b_4[2J(J+1)+C_2^{(3)}], \qquad (20)$$
$$M_{(405)}{}^{(8)}=a_5+b_5\{[2J(J+1)+C_2^{(3)}]+(21/8)[2S(S+1)-C_2^{(4)}+\tfrac{1}{4}Y^2]+3[2I(I+1)-\tfrac{1}{2}Y^2+2N(N+1)-2S(S+1)]\}, \qquad (21)$$

where the coefficients depend only on the Casimir operators of SU(6).

If the symmetry-breaking term in the actual mass operator contains contributions from all the five tensors listed above, the mass operator[10] is

$$M=a+bC_2^{(3)}+cJ(J+1)+dY+e[2S(S+1)-C_2^{(4)}+\tfrac{1}{4}Y^2]+f[N(N+1)-S(S+1)]+g[I(I+1)-\tfrac{1}{4}Y^2]. \qquad (22)$$

We proceed to examine the consequences of Eq. (22) for the 56- and 35-dimensional representations.

6. In the 56-dimensional representation there exist the following identities:

$$2J(J+1)-C_2^{(3)}=-\tfrac{9}{2}, \qquad (23)$$

$$2S(S+1)-C_2^{(4)}+\tfrac{1}{4}Y^2=-8Y-15/2, \qquad (24)$$

$$I(I+1)-\tfrac{1}{4}Y^2-N(N+1)+S(S+1)=-Y+\tfrac{3}{4}. \qquad (25)$$

Equation (22) therefore collapses into

$$M=M_0+M_1J(J+1)+M_2Y+M_3[I(I+1)-\tfrac{1}{4}Y^2], \qquad (26)$$

a result conjectured by Gürsey and Radicati.[1] Mass relationships based on this formula are satisfied to great accuracy. One now has an explanation for the empirically known fact that the mass formula for the baryon octet can be used with the same coefficients for the resonance decuplet in broken SU(3).

7. For the 35-dimensional representation, Eq. (22) must be used with care since there is no analog of the identity (23) and hence the mass operator is not a priori diagonal in either the P or the U chains. Only the ω and φ states, however, are affected.

Let ω_U, φ_U be eigenstates of operators in the U chain, and ω_P, φ_P of those of the P chain. They are related through the equations

$$\omega_U=(\tfrac{1}{3})^{1/2}\omega_P+(\tfrac{2}{3})^{1/2}\varphi_P, \qquad (27)$$

$$\varphi_U=-(\tfrac{2}{3})^{1/2}\omega_P+(\tfrac{1}{3})^{1/2}\varphi_P. \qquad (28)$$

Since the bulk of the mass operator is diagonal in the U chain, it is convenient to start with ω_U and φ_U as the basis and subsequently carry out the diagonalization of the mass matrix. The new eigenvectors are the physical ω and φ.

By a straightforward evaluation of the quantum numbers that occur in Eq. (22) (see Table I), we can write down the squares of meson masses[11] in terms of a, b, c, d, e, f and obtain sum rules by elimination. For pseudoscalar mesons one recovers the usual sum rule [meson label ≡ (meson mass)^2]

$$4K-\pi=3\eta. \qquad (29)$$

No other sum rules are possible since our original formula was much too general.

If we drop the contribution of $M_{(189)}^{(8)}$, we get the constraint $f=g$. One extra sum rule is now obtained, to wit,

$$\omega\varphi=\tfrac{1}{3}(\pi+K^*-K)(3K^*-\rho+K-\pi)$$
$$-\tfrac{1}{9}(4K^*-\rho)(5K^*-\rho+\pi-K-2\omega-2\varphi). \qquad (30)$$

Table I. Quantum numbers of mesons and baryons. Center dots mean "not an eigenstate."

Particle	I	N	S	J	$C_2^{(3)}$	$C_2^{(4)}$
π	1	0	0	0	6	8
ρ	1	1	0	1	6	8
ω_U	0	1	0	1	···	8
ω_P	0	···	···	1	6	···
η	0	0	0	0	6	0
K	1/2	1/2	1/2	0	6	15/4
K^*	1/2	1/2	1/2	1	6	15/4
$\bar{K}$	1/2	1/2	1/2	0	6	15/4
$\bar{K}^*$	1/2	1/2	1/2	1	6	15/4
φ_U	0	0	1	1	···	0
φ_P	0	···	···	1	0	···
N	1/2	1/2	0	1/2	6	63/4
N^*	3/2	3/2	0	3/2	12	63/4
Σ	1	1	1/2	1/2	6	9
Λ	0	0	1/2	1/2	6	9
Y_1^*	1	1	1/2	3/2	12	9
Ξ	1/2	1/2	1	1/2	6	15/4
Ξ^*	1/2	1/2	1	3/2	12	15/4
Ω	0	0	3/2	3/2	12	0

With the present mass values Eq. (30) appears to be obeyed quite well. We are thus led to conjecture that the 189-octet contribution is indeed absent. It is important to state, however, that no further contributions can be dropped without running into serious contradiction with physical reality.

8. The mass operator,[10] we are led to propose, is therefore

$$M=a+bC_2^{(3)}+c\,J(J+1)+dY$$
$$+e[2S(S+1)-C_2^{(4)}+\tfrac{1}{4}Y^2]$$
$$+f[I(I+1)-\tfrac{1}{4}Y^2+N(N+1)-S(S+1)]. \qquad (31)$$

Applications of this formula to the 70-dimensional representation will be the subject of a forthcoming communication.

If one uses Eq. (31) to define the meson central mass and Eq. (26) to define the baryon central mass, one obtains ~610 MeV and ~970 MeV, respectively. Equation (9) of reference 3 now gives $g_{ps}^2/4\pi \sim 13$, a gratifying result.

We are deeply indebted to Professor A. Pais for his interest and enthusiastic encouragement. Many conversations with Professor F. J. Dyson are gratefully acknowledged. One of us (V.S.) wishes to thank Professor J. R. Oppenheimer for hospitality at the Institute for Advanced Study.

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*On leave from (and address after 1 September 1964)

Tata Institute for Fundamental Research, Bombay, India.

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[7]We are aware that representations are not, in general, determined by their dimensionality. However, there is no ambiguity in the representations considered in this paper.

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[10]Note that these mass operators commute with all the Casimir operators of SU(6) and thus cannot reproduce the off-diagonal elements of the symmetry-breaking interaction. The dependence on state labels, in a given SU(6) representation, is, however, correctly reproduced (Wigner-Eckart theorem).

[11]We have followed the canonical practice of using masses for fermions and $(\text{masses})^2$ for bosons.

SPLITTING OF THE 70-PLET OF SU(6)

Mirza A. Baqi Bég
The Rockefeller Institute, New York, New York

and

Virendra Singh*
Institute for Advanced Study, Princeton, New Jersey
(Received 18 September 1964)

1. In a previous note,[1] hereafter called I, we proposed an expression for the mass operator responsible for lifting the degeneracies of spin-unitary spin supermultiplets [Eq. (31)-I]. The purpose of the present note is to apply this expression to the 70-dimensional representation of SU(6).

The importance of the 70-dimensional representation has already been underlined by Pais.[2] Since

$$\underline{35}\times\underline{56}=\underline{56}+\underline{70}+\underline{700}+\underline{1134}, \quad (1)$$

it follows that $\underline{70}$ is the natural candidate for accommodating the higher meson-baryon resonances. Furthermore, since the $SU(3)\otimes SU(2)$ content is

$$\underline{70}=(\underline{1},\underline{2})+(\underline{8},\underline{2})+(\underline{10},\underline{2})+(\underline{8},\underline{4}), \quad (2)$$

we may assume that partial occupancy of the $\underline{70}$ representation has already been established through the so-called γ octet[2] $(\frac{3}{2})^-$. Recent experiments appear to indicate that some $(\frac{1}{2})^-$ states may also be at hand.[3] With six masses at one's disposal, our formulas can predict the masses of all the other occupants of $\underline{70}$ and also provide a consistency check on the input. Our discussion of the $\underline{70}$ representation thus appears to be of immediate physical interest.

The question of numerical predictions cannot be properly treated without a critical analysis of the available experimental input. Such an analysis, however, is outside the proper province of this note.[4]

2. The first problem at hand is the construction of basis tensors of the 70-dimensional representation and identification of the I, Y, J values associated with each component. The simplest procedure is to start with the reducible representation $\underline{15}\otimes\underline{6}=\underline{20}\oplus\underline{70}$ and remove the completely antisymmetric part. Alternatively one can start with $\underline{21}\otimes\underline{6}=\underline{56}\oplus\underline{70}$ and remove the completely symmetric part.

3. The $J=\frac{3}{2}$ tensors are pure under SU(3). For the $J=\frac{1}{2}$ tensors the SU(3) reduction is accomplished by separating out the part completely symmetric in SU(3) indices (decuplet), the part completely antisymmetric (singlet), and the part with "mixed symmetry" (octet). By taking appropriate linear combinations one can then write down all the eigenstates of the P chain. If SU(3) representations are labeled by the usual (p, q), then

$$C_2^{(3)}=\tfrac{2}{3}(p^2+q^2+pq+3p+3q). \tag{3}$$

4. Next one takes linear combinations of states in the P chain in order to obtain eigenstates of N and S. States which share the same values of Y and S can be combined into one or more sets, each set providing a basis for an irreducible representation of SU(4). These representations can be reduced with respect to $SU(2)_I\otimes SU(2)_N$; this reduction, in fact, provides a powerful check on the SU(4) assignments and the Wigner numbers[5] characterizing SU(4) representations. If the Wigner numbers are denoted by (Q, Q', Q''), then

$$C_2^{(4)}=Q^2+4Q+Q'^2+2Q'+Q''^2. \tag{4}$$

5. Our notation for particle states which are eigenstates of operators in the P chain is as follows[6]:

$$(\underline{1},\underline{2}):\ \Lambda_P', \tag{5}$$

$$(\underline{8},\underline{2}):\ \tilde{N}, \tilde{\Sigma}_P, \tilde{\Lambda}_P, \tilde{\Xi}_P; \tag{6}$$

$$(\underline{10},\underline{2}):\ \tilde{N}^*, \tilde{Y}_P^*, \tilde{\Xi}_P^*, \tilde{\Omega}; \tag{7}$$

$$(\underline{8},\underline{4}):\ N_\gamma, \Sigma_\gamma, \Lambda_\gamma, \Xi_\gamma. \tag{8}$$

For the eigenstates in the U chain, we have[6]

$$(1,0;\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2}):\ N_\gamma, \tilde{N}, \tilde{N}^*; \tag{9}$$

$$(0,\tfrac{1}{2};1,1,1):\ \Sigma_\gamma \text{ or } \tilde{\Sigma}_U, \Lambda_U'; \tag{10}$$

$$(0,\tfrac{1}{2};1,0,0):\ \Lambda_\gamma \text{ or } \tilde{\Lambda}_U, \tilde{Y}_U^*; \tag{11}$$

$$(-1,0;\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}):\ \tilde{\Xi}_U; \tag{12}$$

$$(-1,1;\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}):\ \tilde{\Xi}_U^* \text{ or } \Xi_\gamma; \tag{13}$$

$$(-2,\tfrac{1}{2};0,0,0):\ \tilde{\Omega}. \tag{14}$$

Here the numbers in parentheses are $(Y, S; Q, Q', Q'')$ and "A or B" implies that A and B are states distinguished by J spin but totally identical with respect to $U(1)\otimes SU(2)_S\otimes SU(4)$.

Recoupling formulas, relating states in the two chains, are all of the form

$$\begin{pmatrix} U_1 \\ U_2 \end{pmatrix}=\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix}, \tag{15}$$

where $U_1=\Lambda_U'$, $\tilde{\Sigma}_U$, and $\tilde{\Xi}_U$ correspond, respectively, to $U_2=\tilde{\Lambda}_U$, $\tilde{Y}_U^*$, and $\tilde{\Xi}_U^*$. Similarly, we obtain the correspondences between P_1 and P_2 from the foregoing by changing the subscripts.

6. We have tabulated, in Table I, the quantum numbers associated with the states listed above. In order to use this table it is convenient, as in I, to start with the U chain as the basis and diagonalize the mass operator. One obtains in this way the masses of the real particles as well as the corresponding eigenvectors.

7. We obtain the following seven "sum" rules,[7] four additive and three multiplicative, connecting the 13 masses which occur in the 70-dimensional representation of SU(6) (notation: particle label ≡ particle mass):

$$3\Lambda_\gamma+\Sigma_\gamma=2(N_\gamma+\Xi_\gamma), \tag{16}$$

$$4(\tilde{Y}_R^*+\tilde{\Sigma}_R)-2(\tilde{N}^*+\tilde{N}+\tilde{\Xi}_R^*+\tilde{\Xi}_R)=6(\tilde{N}^*-\tilde{N})-3(\tilde{Y}_R^*+\tilde{\Sigma}_R-\tilde{\Lambda}_R-\Lambda_R'), \tag{17}$$

$$2(\tilde{\Omega}-\tilde{N}^*)=3(\tilde{\Xi}_R^*+\tilde{\Xi}_R-\tilde{Y}_R^*-\tilde{\Sigma}_R), \tag{18}$$

$$2(\tilde{\Omega}-\tilde{N}^*)=3(\Sigma_\gamma+\Lambda_\gamma)-6N_\gamma, \tag{19}$$

SU(6) AND ELECTROMAGNETIC INTERACTIONS

M. A. B. Bég
The Rockefeller Institute, New York, New York

and

B. W. Lee*
Institute for Advanced Study, Princeton, New Jersey

and

A. Pais
The Rockefeller Institute, New York, New York
(Received 23 September 1964)

1. The purpose of this note is to discuss some properties of the electromagnetic vertex of baryons under the assumption that the effective electromagnetic current associated with the strongly interacting particles transforms according to the adjoint representation of the group[1-3] SU(6). In particular we show that, in the limit where SU(6) is broken by electromagnetism only, all of the following quantities can be expressed uniquely in terms of the proton magnetic moment $\mu(p)$: (a) the magnetic moments of all baryon octet members, (b) those of the spin-$\frac{3}{2}$ decuplet, (c) all allowed transition moments between octet and decuplet. We recall[1,2] that the octet and the decuplet are united in the 56-dimensional representation of SU(6) and that $\underline{56}^*\otimes\underline{56}$ contains $\underline{35}$ only once. All our results about baryons stem from this single occurrence of $\underline{35}$.

In a pure SU(3) treatment it is customary to define the charge operator Q as follows:

$$Q = (F_3 + F_8/\sqrt{3}). \tag{1}$$

The magnetic moment operator is

$$\vec{M} = \mu_0 Q' \vec{J}. \tag{2}$$

$\vec{J}$ is the appropriate spin matrix ($\vec{J} = \vec{\sigma}/2$ for spin $\frac{1}{2}$), μ_0 is a scale factor. Q' is an operator with the same SU(3) transformation properties as Q. The quantity commonly called the magnetic moment is the matrix element of M_3 between states of highest J_3. The (diagonal) magnetic moments within any given SU(3) multiplet are given by[4]

$$\mu = bQ + c[U(U+1) - \tfrac{1}{4}Q^2 - \tfrac{1}{6}C_2^{(3)}], \tag{3}$$

U being the usual U spin.[5] The embedding of SU(3)⊗SU(2) in SU(6) removes the D/F arbitrariness reflected in Eq. (3) and gives the unique relations mentioned earlier. We next state our results:

(a) Baryon octet. – We find

$$\mu_8 = \tfrac{2}{3}[(\tfrac{1}{2}Q+1)^2 - U(U+1)]\mu(p). \tag{4}$$

Beyond the SU(3) relations, first tabulated by Coleman and Glashow,[6] Eq. (4) gives the additional SU(6) relation

$$\beta = \mu(n)/\mu(p) = -\tfrac{2}{3}, \tag{5}$$

in remarkable agreement with the experimental ratio ≈ -0.684.

(b) The spin-$\frac{3}{2}$ decuplet. – As $U = 1 - Q/2$, SU(3) predicts that $\mu_{10} = \text{const} \times Q$. More specifically we find from SU(6) that

$$\mu_{10} = Q\mu(p). \tag{6}$$

Thus, for example, $\mu(\Omega) = -\mu(p)$.

(c) Decuplet-octet transitions. – We denote the amplitude of the M_1 transitions by $\langle n'J'M'|\mu|nJM\rangle$, where n' and n are particle labels. In this notation $\langle p\frac{1}{2}\frac{1}{2}|\mu|p\frac{1}{2}\frac{1}{2}\rangle \equiv \mu(p)$. For the decuplet-octet transitions $J = \frac{3}{2}$, $J' = \frac{1}{2}$, and it is sufficient to quote the results for $M = \frac{1}{2}$ and $M' = \frac{1}{2}$. Amplitudes for other M and M' can be obtained by elementary SU(2) rotations. In the following it is therefore understood that $J = \frac{3}{2}$, $M = \frac{1}{2}$, $J' = M' = \frac{1}{2}$, and the explicit dependence need not be exhibited.

Note that SU(3) alone gives the following relationships for transitions allowed by conservation of charge and hypercharge[7]:

$$\langle p|\mu|N_+^*\rangle = -\langle\Sigma_+|\mu|Y_+^*\rangle = \langle n|\mu|N_0^*\rangle = 2\langle\Sigma_0|\mu|Y_0^*\rangle$$
$$= \tfrac{1}{2}\sqrt{3}\langle\Lambda|\mu|Y_0^*\rangle = \langle\Xi_0|\mu|\Xi_0^*\rangle, \tag{7}$$

$$\langle\Sigma_-|\mu|Y_-^*\rangle = \langle\Xi_-|\mu|\Xi_-^*\rangle = 0. \tag{8}$$

SU(6) now gives the additional relation

$$\langle p|\mu|N_+^*\rangle = \tfrac{2}{3}\sqrt{2}\mu(p). \tag{9}$$

This is in qualitative agreement with the estimates of Gourdin and Salin[8] who obtain $\langle p|\mu|N_+^*\rangle \cong 1.6\times(2\sqrt{2}/3)\mu(p)$ from a study of $\gamma + p \to \pi + N$ near the 33 resonance.

2. Derivations. – For given momentum $\vec{q}$, the states of the 56-dimensional representation of $SU(6)_{\vec{q}}$ are described by the completely symmetric tensor $B^{\alpha\beta\gamma}(\vec{q})$, $\alpha,\beta,\gamma = 1, 2, \cdots, 6$. This tensor is reducible under the group $SU(3)\otimes SU(2)_{\vec{q}}$, the explicit reduction[9] in the rest frame ($\vec{q} = 0$) being

$$B^{\alpha\beta\gamma}(0) = B^{\alpha\beta\gamma} = \chi^{(ijk)}d^{(ABC)}$$
$$+\frac{1}{3\sqrt{2}}[(2\epsilon^{ij}\chi^k + \epsilon^{jk}\chi^i)\epsilon^{ABD}b_D{}^C$$
$$+(\epsilon^{ij}\chi^k + 2\epsilon^{jk}\chi^i)\epsilon^{BCD}b_D{}^A], \tag{10}$$

$i,j,k = 1,2$; $A,B,C,D = 1,2,3$. Here ϵ^{ij} and ϵ^{ABC} are the Levi-Civita symbols in two and three dimensions, respectively. χ^i is a (normalized) Pauli spinor. The $\chi^{(ijk)}$ are the spin-$\frac{3}{2}$ wave functions.[10] $b_B{}^A$ is the usual baryon octet tensor,[11] $d^{(ABC)}$ is the SU(3)-decuplet tensor.[11]

Our assumption is that the charge operator transforms like an (8, 1) member of a 35 representation, and the magnetic moment operator transforms like an (8, 3) member of a 35 representation[12] (we do not assume that the same 35 representation appears in both cases). Under these assumptions, the effective, low-frequency limit of the electromagnetic vertex of the baryons may be written as[13]

$$3B_{\alpha\beta\gamma}B^{\alpha\beta\delta}[e\varphi\delta_l{}^k + \mu(p)\cdot i(\vec{\sigma}\cdot\vec{q}\times\vec{\epsilon})_l{}^k]Q_D{}^C;$$
$$\gamma = (k, C),\ \ \delta = (l, D), \tag{11}$$

where φ is an electrostatic potential and $\vec{\epsilon}$ a polarization vector $\perp\vec{q}$. Expanding the coefficient of φ in terms of particle states we get the respective charges of the particles, while the magnetic term yields the results quoted in Eqs. (4)-(9).[14]

3. Remarks. – (a) A more general definition of Q has been proposed[15] which would lead to the addition on the right-hand side of Eq. (3) of a constant (independent of Q, U, and $C_2^{(3)}$). The inclusion of such a term would diminish the pre-

dictive power of SU(6) and would in particular render β arbitrary [see Eq. (5)].

(b) It has been noted[3] that the subgroup $SU(4)(T)$ of SU(6) gives an arbitrary mixture of Wigner versus Majorana forces between nucleons, while this mixture is unique for SU(6). This statement has an electromagnetic analog, namely, the isoscalar vs isovector ratio is fixed in SU(6) but arbitrary in $SU(4)(T)$, so that $SU(4)(T)$ does not make any predictions for β. However, if one assumes that the effective electromagnetic current transforms according to the adjoint representation of $SU(4)(T)$, one obtains[16] $\beta = -1$.

(c) It has been noted in reference 3 and independently by Sakita[17] that SU(6) relates the structure of Pauli-type vector-meson terms to that of the p-wave pseudoscalar term. Sakita has studied an assignment where the baryon octet is contained in the 20-dimensional representation of SU(6). β is unique also for this choice and we find $\beta = -2$ in this case. This may serve as a further indication that the $\underline{56}$ representation is preferable.

(d) All our results can also be obtained by the method of vector addition of magnetic moments,[18] by regarding the baryons as composite structures built up out of spin-$\frac{1}{2}$ quarks[19] with composite wave functions dictated by SU(6). This method can of course be applied to other SU(6) representations as well. In this way one easily shows that SU(6) yields a new relation for the $\underline{35}$ meson representation, namely $\mu(\rho_+) = 3\langle\pi_+, 0, 0|\mu|\rho_+, 1, 0\rangle$. We hasten to add that this remark is not meant to shed light on the existence of quarks.

4. Finally, we discuss some implications of our results from the point of view of a local Lagrangian field theory. It should be stressed that the conclusions obtained so far have come from an analysis of an effective vertex[14] under the assumption that this vertex has prescribed SU(6) properties. Likewise the results found in reference 3 referred exclusively to an SU(6)-invariant effective strong-interaction vertex. However, in the present electromagnetic case we are in the unique position to be able to compare a specific numerical prediction of the SU(6) theory with an equally specific answer of local field theory. Loosely speaking, the situation is the following: According to Eq. (5), $\beta = -\frac{3}{2}$. This comfortable value for β is a pure number, independent of any coupling constants. In field theory we have been accustomed for many years to say, "In the limit where the strong interactions are 'turned off,' we should have $\mu(n) = 0$, $\mu(p) = 1$, hence $\beta = 0$; or, conversely, the 'anomalous' magnetic moments of nucleons come about by 'turning on' the strong interactions." Thus we arrive at a paradox which comforts while it mocks: We cannot assume both that the SU(6) group is valid and that local field theory with minimal electromagnetic interactions applies to nucleons.

We shall next attempt to state this imcompatibility in more precise terms. Let us consider the following set of assumptions: (I) Strong and electromagnetic effects are derivable from a Lagrangian $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}(g) + \mathcal{L}(e)$. Here $\mathcal{L}_0$ is the free Lagrangian, $\mathcal{L}(g)$ symbolizes all strong-interaction terms, and $\mathcal{L}(e)$ stands for the electromagnetic terms. $\mathcal{L}_0$, $\mathcal{L}(g)$, and $\mathcal{L}(e)$ contain explicitly the local nucleon fields. (II) $\mathcal{L}(e)$ is minimal, that is, it contains no derivatives of the electromagnetic potentials, while also the SU(3) trace of the charge operator shall vanish ($Q = F_3 + F_8/\sqrt{3}$). (III) $\mathcal{L}_0 + \mathcal{L}(g)$ is invariant under a group which contains SU(6) as a subgroup. As SU(6) is a linear group, this means in particular that $\mathcal{L}_0$ and $\mathcal{L}(g)$ are separately SU(6)-invariant. Furthermore, $\mathcal{L}(e)$ shall have the definite SU(6) properties assumed above for the effective electromagnetic vertex. (IV) It is possible to calculate in such a theory the magnetic moment of the neutron, which we denote by $\mu_n(e,g)$, and likewise for other particles. Moreover, $\mu_n(e,0)$ exists and is identical with the neutron magnetic moment calculated from $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}(e)$; likewise for the proton. We conclude that the assumptions (I) to (IV) are incompatible.

We are now faced with two connected questions. First, one should prove this statement in a direct fashion rather than having recourse to the numerical result for β. Second, if one believes (as we do) that the results obtained with the SU(6) assumptions are not a series of numerical coincidences, one will have to revise some of the assumptions (I) to (IV) and the question is which ones. These questions will be studied further.

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[9]To each fixed α corresponds a fixed pair of labels (i,A). For $\alpha=1,\cdots,6$ these respective pairs are (1,1), (1,2), (1,3), (2,1), (2,2), and (2,3). In Eq. (11) we use the correspondence $\alpha=(i,A)$; $\beta=(j,B)$, $\gamma=(k,C)$. Note that $\epsilon^{ABD}b_D{}^C+\epsilon^{BCD}b_D{}^A+\epsilon^{CAD}b_D{}^B=0$, $\epsilon^{ij}\chi^k+\epsilon^{jk}\chi^i+\epsilon^{ki}\chi^j=0$.

[10]$\chi^{(ijk)}$ is totally symmetric in i,j, and k. Normalizations are $\|\chi^{(111)}\|=1$, $\|\chi^{(112)}\|=\frac{1}{3}$, etc.

[11]We use a normalization such that $b_3{}^1=p$. Furthermore, $d^{(111)}=N_{++}^*$, $d^{(112)}=N_+^*/\sqrt{3}$, etc. For fixed α,β,γ, the norm of $B^{\alpha\beta\gamma}$ is equal to 1 if $\alpha=\beta=\gamma$; $\frac{1}{3}$ if $\alpha=\beta\neq\gamma$; $\frac{1}{6}$ if $\alpha\neq\beta\neq\gamma$.

[12]Note that in terms of SU(6) generators the charge and magnetic moment operators are given by: $Q=e\times(A_1{}^1+A_4{}^4)$; $M_+=\mu_0[\frac{2}{3}T_1{}^4-\frac{1}{3}T_2{}^5-\frac{1}{3}T_3{}^6]$; $M_-=\mu_0[\frac{2}{3}T_4{}^1-\frac{1}{3}T_5{}^2-\frac{1}{3}T_6{}^3]$; $M_3=\mu_0[\frac{1}{3}(T_1{}^1-T_4{}^4)-\frac{1}{6}(T_2{}^2-T_5{}^5)-\frac{1}{6}(T_3{}^3-T_6{}^6)]$, where $T_\alpha{}^\beta$ is an irreducible tensor operator transforming as the generator $A_\alpha{}^\beta$ (see Bég and Singh, reference 4).

[13]The normalization factors are so chosen that for the proton Eq. (11) reduces to $ep^\dagger p\varphi+\mu(p)p^\dagger\vec{\sigma}p\cdot\vec{H}$.

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A SCHEMATIC MODEL OF BARYONS AND MESONS *

M. GELL-MANN
California Institute of Technology, Pasadena, California

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If we assume that the strong interactions of baryons and mesons are correctly described in terms of the broken "eightfold way" [1-3], we are tempted to look for some fundamental explanation of the situation. A highly promised approach is the purely dynamical "bootstrap" model for all the strongly interacting particles within which one may try to derive isotopic spin and strangeness conservation and broken eightfold symmetry from self-consistency alone [4]. Of course, with only strong interactions, the orientation of the asymmetry in the unitary space cannot be specified; one hopes that in some way the selection of specific components of the F-spin by electromagnetism and the weak interactions determines the choice of isotopic spin and hypercharge directions.

Even if we consider the scattering amplitudes of strongly interacting particles on the mass shell only and treat the matrix elements of the weak, electromagnetic, and gravitational interactions by means of dispersion theory, there are still meaningful and important questions regarding the algebraic properties of these interactions that have so far been discussed only by abstracting the properties from a formal field theory model based on fundamental entities [3] from which the baryons and mesons are built up.

If these entities were octets, we might expect the underlying symmetry group to be SU(8) instead of SU(3); it is therefore tempting to try to use unitary triplets as fundamental objects. A unitary triplet t consists of an isotopic singlet s of electric charge z (in units of e) and an isotopic doublet (u, d) with charges z+1 and z respectively. The anti-triplet $\bar{t}$ has, of course, the opposite signs of the charges. Complete symmetry among the members of the triplet gives the exact eightfold way, while a mass difference, for example, between the isotopic doublet and singlet gives the first-order violation.

For any value of z and of triplet spin, we can construct baryon octets from a basic neutral baryon singlet b by taking combinations $(b t \bar{t})$, $(b t t \bar{t} \bar{t})$, etc. **. From $(b t \bar{t})$, we get the representations **1** and **8**, while from $(b t t \bar{t} \bar{t})$ we get **1**, **8**, **10**, $\overline{\mathbf{10}}$, and **27**. In a similar way, meson singlets and octets can be made out of $(t \bar{t})$, $(t t \bar{t} \bar{t})$, etc. The quantum number $n_t - n_{\bar{t}}$ would be zero for all known baryons and mesons. The most interesting example of such a model is one in which the triplet has spin $\frac{1}{2}$ and $z = -1$, so that the four particles d^-, s^-, u^0 and b^0 exhibit a parallel with the leptons.

A simpler and more elegant scheme can be constructed if we allow non-integral values for the charges. We can dispense entirely with the basic baryon b if we assign to the triplet t the following properties: spin $\frac{1}{2}$, $z = -\frac{1}{3}$, and baryon number $\frac{1}{3}$. We then refer to the members $u^{\frac{2}{3}}$, $d^{-\frac{1}{3}}$, and $s^{-\frac{1}{3}}$ of the triplet as "quarks" [6] q and the members of the anti-triplet as anti-quarks $\bar{q}$. Baryons can now be constructed from quarks by using the combinations $(q q q)$, $(q q q q \bar{q})$, etc., while mesons are made out of $(q \bar{q})$, $(q q \bar{q} \bar{q})$, etc. It is assuming that the lowest baryon configuration $(q q q)$ gives just the representations **1**, **8**, and **10** that have been observed, while the lowest meson configuration $(q \bar{q})$ similarly gives just **1** and **8**.

A formal mathematical model based on field theory can be built up for the quarks exactly as for p, n, Λ in the old Sakata model, for example [3] with all strong interactions ascribed to a neutral vector meson field interacting symmetrically with the three particles. Within such a framework, the electromagnetic current (in units of e) is just

$$i\{\tfrac{2}{3}\bar{u}\gamma_\alpha u - \tfrac{1}{3}\bar{d}\gamma_\alpha d - \tfrac{1}{3}\bar{s}\gamma_\alpha s\}$$

or $\mathscr{F}_{3\alpha} + \mathscr{F}_{8\alpha}/\sqrt{3}$ in the notation of ref. [3]. For the weak current, we can take over from the Sakata model the form suggested by Gell-Mann and Lévy [7], namely $i\,\bar{p}\gamma_\alpha(1+\gamma_5)(n\cos\theta + \Lambda\sin\theta)$, which gives in the quark scheme the expression ***

$$i\,\bar{u}\gamma_\alpha(1+\gamma_5)(d\cos\theta + s\sin\theta)$$

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** This is similar to the treatment in ref. [1]. See also ref. [5].

*** The parallel with $i\,\bar{\nu}_e\gamma_\alpha(1+\gamma_5)e$ and $i\,\bar{\nu}_\mu\gamma_\alpha(1+\gamma_5)\mu$ is obvious. Likewise, in the model with d^-, s^-, u^0, and b^0 discussed above, we would take the weak current to be $i(\bar{b}^0\cos\theta + \bar{u}^0\sin\theta)\gamma_\alpha(1+\gamma_5)s^- + i(\bar{u}^0\cos\theta - \bar{b}^0\sin\theta)\gamma_\alpha(1+\gamma_5)d^-$. The part with $\Delta(n_t - n_{\bar{t}}) = 0$ is just $i\,\bar{u}^0\gamma_\alpha(1+\gamma_5)(d^-\cos\theta + s^-\sin\theta)$.

or, in the notation of ref. 3),

$$[\mathcal{F}_{1\alpha} + \mathcal{F}^5_{1\alpha} + i(\mathcal{F}_{2\alpha} + \mathcal{F}^5_{2\alpha})] \cos\theta + [\mathcal{F}_{4\alpha} + \mathcal{F}^5_{4\alpha} + i(\mathcal{F}_{5\alpha} + \mathcal{F}^5_{5\alpha})] \sin\theta \, .$$

We thus obtain all the features of Cabibbo's picture [8] of the weak current, namely the rules $|\Delta I| = 1$, $\Delta Y = 0$ and $|\Delta I| = \frac{1}{2}$, $\Delta Y/\Delta Q = +1$, the conserved $\Delta Y = 0$ current with coefficient $\cos\theta$, the vector current in general as a component of the current of the F-spin, and the axial vector current transforming under SU(3) as the same component of another octet. Furthermore, we have [3] the equal-time commutation rules for the fourth components of the currents:

$$[\mathcal{F}_{j4}(x) \pm \mathcal{F}^5_{j4}(x),\ \mathcal{F}_{k4}(x') \pm \mathcal{F}^5_{k4}(x')] = -2 f_{jkl} [\mathcal{F}_{l4}(x) \pm \mathcal{F}^5_{l4}(x)]\, \delta(x-x') \, ,$$

$$[\mathcal{F}_{j4}(x) \pm \mathcal{F}^5_{j4}(x),\ \mathcal{F}_{k4}(x') \mp \mathcal{F}^5_{k4}(x')] = 0 \, ,$$

$i = 1, \ldots 8$, yielding the group SU(3) × SU(3). We can also look at the behaviour of the energy density $\theta_{44}(x)$ (in the gravitational interaction) under equal-time commutation with the operators $\mathcal{F}_{j4}(x') \pm \mathcal{F}_{j4}{}^5(x')$. That part which is non-invariant under the group will transform like particular representations of SU(3) × SU(3), for example like $(3, \bar{3})$ and $(\bar{3}, 3)$ if it comes just from the masses of the quarks.

All these relations can now be abstracted from the field theory model and used in a dispersion theory treatment. The scattering amplitudes for strongly interacting particles on the mass shell are assumed known; there is then a system of linear dispersion relations for the matrix elements of the weak currents (and also the electromagnetic and gravitational interactions) to lowest order in these interactions. These dispersion relations, unsubtracted and supplemented by the non-linear commutation rules abstracted from the field theory, may be powerful enough to determine all the matrix elements of the weak currents, including the effective strengths of the axial vector current matrix elements compared with those of the vector current.

It is fun to speculate about the way quarks would behave if they were physical particles of finite mass (instead of purely mathematical entities as they would be in the limit of infinite mass). Since charge and baryon number are exactly conserved, one of the quarks (presumably $u^{\frac{2}{3}}$ or $d^{-\frac{1}{3}}$) would be absolutely stable *, while the other member of the doublet would go into the first member very slowly by β-decay or K-capture. The isotopic singlet quark would presumably decay into the doublet by weak interactions, much as Λ goes into N. Ordinary matter near the earth's surface would be contaminated by stable quarks as a result of high energy cosmic ray events throughout the earth's history, but the contamination is estimated to be so small that it would never have been detected. A search for stable quarks of charge $-\frac{1}{3}$ or $+\frac{2}{3}$ and/or stable di-quarks of charge $-\frac{2}{3}$ or $+\frac{1}{3}$ or $+\frac{4}{3}$ at the highest energy accelerators would help to reassure us of the non-existence of real quarks.

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* There is the alternative possibility that the quarks are unstable under decay into baryon plus anti-di-quark or anti-baryon plus quadri-quark. In any case, some particle of fractional charge would have to be absolutely stable.

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THE SYMMETRY GROUP OF VECTOR AND AXIAL VECTOR CURRENTS*

MURRAY GELL-MANN
California Institute of Technology, Pasadena, California

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Abstract

We review, modify slightly, generalize, and attempt to apply a theory proposed earlier of a higher broken symmetry than the eightfold way. The integrals of the time components of the vector and axial vector current octets are assumed to generate, under equal time commutation, the algebra of SU(3) × SU(3). *The energy density of the strong interactions is assumed to consist of a piece invariant under the algebra, a piece that violates conservation of the axial vector currents only and belongs to the representation* (3, 3*) *and* (3*, 3), *and a piece that violates the eightfold way and probably belongs to* (1, 8) *and* (8, 1). *Assuming the algebraic structure is exactly correct, there is still the question of whether one can assign particles approximately to super-supermultiplets. The pseudoscalar meson octet, together with a pseudoscalar singlet, a scalar octet, and a scalar singlet, may belong to* (3, 3*) *and* (3*, 3). *The vector meson octet, together with an axial vector octet, may belong to* (1, 8) *and* (8, 1). *The baryon octet with* $J = 1/2^+$, *together with a singlet with* $J = 1/2^-$, *may belong to* (3, 3*) *and* (3*, 3), *as suggested before. Several crude coupling patterns and mass rules emerge, to zeroth or first order in the symmetry violations. Some are roughly in agreement with experiment, but certain predictions, like that of the existence of a scalar octet, have not been verified. Whether or not they are useful as an approximate symmetry, the equal time commutation rules fix the scale of the weak interaction matrix elements. Further rules of this kind are found to hold in certain Lagrangian field theory models and may be true in reality. In particular, we encounter an algebraic system based on* SU(6) *that relates quantities with different kinds of behavior under Lorentz transformations.*

I. Introduction

THE "eightfold way" theory of a broken higher symmetry for strong interactions was proposed [1, 2] at a time when the value of a badly violated symmetry was unclear for two reasons:

(1) It was not obvious what real significance could be assigned to the algebraic properties of the higher symmetry.

(2) It was not known whether the particle spectrum would show unmistakable evidence of the higher symmetry.

A solution was offered to the first problem when we pointed out [3] that the weak vector currents with $|\Delta I| = 1/2$, $\Delta Y/\Delta Q = +1$, and $|\Delta I| = 1$, $\Delta Y = 0$ generate an algebraic system through the equal-time commutation relations of their time-components and that this algebra is preserved even though the conservation of the strangeness-changing currents is violated. We assumed that the algebra in question is that of SU(3); no matter how badly the eightfold way is broken, the vector current octet is then the current of the F-spin. (This result was a simple generalization of the conserved vector current hypothesis, that the $\Delta Y = 0$ vector

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current is the current of the I-spin.) We went on to consider the equal-time commutation relations of the vector currents with the energy density, which interacts with gravitation. The assumption of the broken eightfold way is that the energy density is the sum of two pieces, one of which is invariant under the F-spin, and the other of which transforms like one component (by definition, the eighth) of an octet:

$$\mathcal{H} = \mathcal{H}_{\text{inv}} - c\,\phi_8. \tag{1}$$

The second problem arose when we attempted to arrange the isotopic multiplets of strongly interacting particles in supermultiplets that correspond, in order c^0, to irreducible representations of SU(3). We then derived, to zeroth and first order in c, coupling patterns and mass rules that were to be compared with experiment. We had no guarantee, however, even if our algebraic system was the right one, that these rules would be sufficiently well obeyed to show traces of the higher symmetry and the manner in which it is violated. Fortunately, ample evidence is now available to support the eightfold way symmetry, together with the pattern of violation in equation (1). In fact, the mass rules derived to first order in c are surprisingly accurate.

The success of the broken eightfold way, despite the large violations of symmetry involved, suggests that it may be worthwhile to study in detail the still higher symmetry associated with the axial vector currents. We proposed [3] that the axial vector currents with $|\Delta I| = 1/2$, $\Delta Y/\Delta Q = +1$, and $|\Delta I| = 1$, $\Delta Y = 0$ belong to an octet with respect to F-spin and that the time-components of the vector and axial vector octets together generate, under equal time commutation, the algebra SU(3) × SU(3). Moreover, we suggested that the term $\mathcal{H}_{\text{inv}}$ in the energy density in equation (1) consists of two parts

$$\mathcal{H}_{\text{inv}} = \overline{\mathcal{H}} - \lambda\, u_0, \tag{2}$$

where $\overline{\mathcal{H}}$ is invariant under the full algebra and leaves both vector and axial vector currents conserved, while the term $-\lambda u_0$, transforming like a particular pair of representations of SU(3) × SU(3), violates conservation of the axial vector currents, while still commuting with the F-spin. These algebraic statements do not, of course, depend for their proposed validity on the smallness of the parameter λ.

We did not take altogether seriously, in ref. 3, the idea that these algebraic relations might be applied, like those of the eightfold way, by trying to assign particles to irreducible representations and finding rules to zeroth and first order in λ, to be compared with experiment. The success of the broken eightfold way, however, now makes it less ridiculous to see whether we can find traces of this even more badly broken symmetry. We would try to group the strongly interacting particle supermultiplets into super-supermultiplets, usually including particles of both parities, and to check by experiment very crude relations derived to low order in λ. In Section IV we describe the most plausible scheme of this kind, and note that an octet of scalar mesons is required for its success. Since no such octet has been clearly established at this time, we must reserve judgment on whether the approximation of small λ is of any use in describing the strong interactions.

It was mentioned in ref. 3 that an algebra is generated by the time-components of the vector and axial vector currents together with the symmetry-breaking term u_0 in the energy density. In Section VII we follow up this idea and discuss the possibility that the "extended algebra" may be rather small; we show that in a special model it is the algebra of SU(6) and suggest that such may be the case in reality. It is interesting that the extended algebra ties together quantities with different Lorentz transformation properties, such as the scalar u_0 and the four-vector currents.

The algebra [presumably SU(3) × SU(3)] of the vector and axial vector currents and the extended algebra [possibly SU(6)] can be used, in the form of equal time commutation relations, to supplement dispersion relations in the calculation of weak current matrix elements. It may also be true that more and more information about the strongly interacting particles can be expressed in algebraic language by repeated use of the notion of equal-time commutation relations.

II. Review of the Theory

We treat the strong interactions exactly, and the electromagnetic, weak, and gravitational interactions in lowest order. Even though we discuss the scattering amplitudes for strongly interacting particles on the mass shell only (by the method of dispersion relations or "S-matrix theory") we must still acknowledge that in lowest order the matrix elements of the weak and electromagnetic interactions can be determined for arbitrary mementum transfer by measurement and by analytic continuation of the measurable amplitudes. Presumably the same is true in principle of gravitation. Thus we may deal with electromagnetic and weak current operators and a stress-energy-momentum tensor operator $\theta_{\alpha\beta}$, all functions of a space-time variable x, with the matrix elements for any momentum transfer k given by Fourier transform and with time derivatives given by commutation with the space integral of the energy density $\mathscr{H} = -\theta_{44}$.

Incidentally, we notice in this way that the "S-matrix theory" of strong interactions, with electromagnetism, weak interactions, and gravitation treated as small purturbations, is just a branch of abstract field theory, since the current operators and $\theta_{\alpha\beta}$ are all field operators.

The weak current may be broken up, according to quantum numbers conserved by the strong interactions, first into a vector and an axial vector part, and then into pieces characterized by different values of $|\Delta I|$ and $\Delta Y/\Delta Q$. We restrict our attention here to the familiar terms with $|\Delta I| = 1$, $\Delta Y = 0$, and $|\Delta I| = 1/2$, $\Delta Y/\Delta Q = +1$. If there are others, they may lead to bigger algebras than we have here, but need not invalidate our conclusions.

The integrals of the time components of all these currents generate some minimal algebraic system under equal time commutation. For those currents that are conserved, the corresponding integrals (like the electric charge) are constant operators; the others vary with time. But the structure constants of the algebra remain unchanged under all conditions and correspond to a law of nature that specifies the minimal algebra of the vector and axial vector currents that we are studying.

In ref. 3, we made three assumptions that determine the algebra: (a) The vector weak current, like the electromagnetic current, is a component of the F-spin current $\mathscr{F}_{i\alpha}(x)$, where $i = 1\ldots 8$ and α is a Lorentz index. We have, then,

$$F_i(t) = -i\int \mathscr{F}_{i4}\, d^3x, \tag{3}$$

$$[F_i(t),\, F_j(t)] = i f_{ijk} F_k(t). \tag{4}$$

Of course, F_1, F_2, and F_3 are conserved by the strong interactions and are just the components of the isotopic spin; thus the conserved vector current hypothesis is included here. The components F_4, F_5, F_6, and F_7 actually vary with time. (b) The axial vector weak current is the same component of another current $\mathscr{F}_{i\alpha}^5(x)$ that transforms like an octet with respect to F-spin. We have

$$F_i^5(t) = -i\int \mathscr{F}_{i4}^5\, d^3x, \tag{5}$$

$$[F_i(t),\, F_j^5(t)] = i f_{ijk} F_k^5(t). \tag{6}$$

(c) The commutation rules of the operators $F_i^5(t)$ close the algebraic system by giving

$$[F_i^5(t),\, F_j^5(t)] = i f_{ijk} F_k(t). \tag{7}$$

We now define

$$2F_i^{\pm}(t) \equiv F_i(t) \pm F_i^5(t) \tag{8}$$

and notice that F_i^+ and F_i^- are two commuting F-spins, so that we are really dealing with the algebra of SU(3) × SU(3). The two sets of operators, which we may think of as "left-handed" and "right-handed" F-spins respectively, are connected by parity:

$$PF_i^{\pm}P^{-1} = F_i^{\mp}. \tag{9}$$

The total F-spin itself is, according to equation (8), just the sum of the left- and right-handed parts:

$$F_i = F_i^+ + F_i^-. \tag{10}$$

We will be concerned with irreducible representations of the system consisting of F_i^+, F_i^-, and P. We indicate [3] the behavior with respect to (F_i^+, F_j^-) by a pair of representations, such as $(\underline{3}, \underline{3}^*)$, $(\underline{8}, \underline{8})$, etc. Since parity interchanges F_i^+ and F_i^-, an irreducible representation with respect to parity and the two F-spins will have such forms as $(\underline{8}, \underline{8})$ or $(\underline{3}, \underline{3}^*)$ and $(\underline{3}^*, \underline{3})$ or $(\underline{1}, \underline{8})$ and $(\underline{8}, \underline{1})$.

If a representation of F_i^+, F_i^-, and P is to contain a component invariant under F_i, it must have the form $(\underline{1}, \underline{1})$ or $(\underline{3}, \underline{3}^*)$ and $(\underline{3}^*, \underline{3})$ or $(\underline{8}, \underline{8})$, etc., so that the product of the two indicated representations will contain $\underline{1}$.

The simplest choice, then, for the term λu_0 in equation (2), which violates the conservation of F_i^+ and F_i^- separately while conserving F_i, is to have it belong to $(\underline{3}, \underline{3}^*)$ and $(\underline{3}^*, \underline{3})$, as proposed in ref. 3. The operator u_0 thus belongs to a set of nine scalar and nine pseudoscalar quantities, in each case forming an octet and a singlet with respect to total F-spin. The scalar octet is labeled $u_1 \dots u_8$ and the singlet u_0, while the pseudoscalar octet and singlet are labeled $v_1 \dots v_8$ and v_0 respectively.

To specify the transformation properties of the u's and v's, we introduce [3] a generalization of the symbols f_{ijk} and d_{ijk} to the case $i = 0, 1, \dots 8$ instead of $i = 1 \dots 8$. To the 3×3 matrices λ_i $(i = 1 \dots 8)$ we adjoin the matrix $\lambda_0 = (2/3)^{1/2}\, 1$ and obtain the rules

$$[\lambda_i, \lambda_j] = 2 i f_{ijk} \lambda_k, \tag{11}$$

$$\{\lambda_i, \lambda_j\} = 2 d_{ijk} \lambda_k, \tag{12}$$

$$\mathrm{Tr}\, \lambda_i \lambda_j = 2 \delta_{ij}, \tag{13}$$

where i, j, and k run from 0 to 8. Here, f_{ijk} vanishes when any index is zero, and d_{ijk} equals $(2/3)^{1/2} \delta_{ij}$ when k is zero, etc.

We then obtain, for the transformation properties of the u_i and v_i, the results [3]

$$\begin{aligned} [F_i, u_j] &= i f_{ijk} u_k, \\ [F_i, v_j] &= i f_{ijk} v_k, \\ [F_i^5, u_j] &= -i d_{ijk} v_k, \\ [F_i^5, v_j] &= i d_{ijk} u_k. \end{aligned} \tag{14}$$

We note that equations (4), (6), and (7) indicate the representation to which the currents $\mathcal{F}_{i\alpha}(x)$ and $\mathcal{F}_{i\alpha}^5(x)$ belong, namely $(\underline{1}, \underline{8})$ and $(\underline{8}, \underline{1})$, with $i = 1 \dots 8$. At equal times, we have

$$\begin{aligned} [F_i, \mathcal{F}_{j\alpha}] &= i f_{ijk} \mathcal{F}_{k\alpha}, \\ [F_i, \mathcal{F}_{j\alpha}^5] &= i f_{ijk} \mathcal{F}_{k\alpha}^5, \\ [F_i^5, \mathcal{F}_{j\alpha}] &= i f_{ijk} \mathcal{F}_{k\alpha}^5, \\ [F_i^5, \mathcal{F}_{k\alpha}^5] &= i f_{ijk} \mathcal{F}_{k\alpha}. \end{aligned} \tag{15}$$

The essential physics of the theory is contained in the equations written so far and is taken over directly from ref. 3. Two more points need to be added, however, which are modifications of the corresponding points in the earlier article. One of these concerns the component of $\mathcal{F}_{i\alpha}^+$ utilized for the weak current, and is discussed in the next section. The other point is connected with the transformation properties under SU(3) × SU(3) of the term in the energy density that violates the eightfold way, namely ϕ_8 in equation (1). The simplest possibilities for a unitary octet are, of course, $[(\underline{3}, \underline{3}^*)$ and $(\underline{3}^*, \underline{3})]$ and $[(\underline{1}, \underline{8})$ and $(\underline{8}, \underline{1})]$. It now appears that the latter may be more nearly satisfactory than the former, as we shall see in the next section. We had previously assumed not only that ϕ_8 transformed as $[(\underline{3}, \underline{3}^*)$ and $(\underline{3}^*, \underline{3})]$, i.e., like u_8, but also that ϕ_8 was equal to u_8.

III. Universality of the Weak Interactions

We know that the electromagnetic current of the strongly interacting particles (or "hadrons" to use Okun's expression) is given by the formula

$$j_\alpha = e\left(\mathcal{F}_{3\alpha} + \frac{1}{\sqrt{3}}\mathcal{F}_{8\alpha}\right). \tag{16}$$

(A constant term may have to be added if there exist hadrons corresponding to certain kinds of spinor representations of SU(3), as discussed in Section VI.)

What about the hadron weak current coupled to leptons? It must be a linear combination of $\mathcal{F}_{1\alpha}^{+} + i\mathcal{F}_{2\alpha}^{+}$ with $\Delta Y = 0$ and $\mathcal{F}_{4\alpha}^{+} + i\mathcal{F}_{5\alpha}^{+}$ with $\Delta Y/\Delta Q = +1$, if we stick to the assumptions (a) and (b) of Section II. The choice of the linear combination is motivated in part by the requirement of universality of the weak interactions: the algebraic properties of the total weak current should be the same for leptons and for hadrons [4, 5].

We write the effective weak interaction in the local approximation (or at least the part coming from a product of charged currents) as

$$\frac{G}{\sqrt{2}} J_\alpha^{+} J_\alpha, \tag{17}$$

where $J_\alpha = J_\alpha(\text{leptons}) + J_\alpha(\text{hadrons})$. The situation is then the following, as described in ref. 4.

For the now obsolete case of one neutrino for electron and muon, J_α(leptons) has the form

$$\bar{\nu}\gamma_\alpha(1+\gamma_5)e + \bar{\nu}\gamma_\alpha(1+\gamma_5)\mu = 2\sqrt{2}\,\bar{\nu}\gamma_\alpha\frac{(1+\gamma_5)}{2}\frac{(e+\mu)}{\sqrt{2}}.$$

The "weak charge" $-i\int d^3x\, J_4(\text{leptons})$ evidently may be written in the form $2\sqrt{2}\,(K_1 + iK_2)$, where K_1, K_2, and $-i[K_1, K_2]$ have the commutation rules of an angular momentum or an isotopic spin. The weak charge and its hermitian conjugate, for leptons, generate the algebra of SU(2), with $\frac{1+\gamma_5}{2}\frac{e+\mu}{\sqrt{2}}$ and $\frac{1+\gamma_5}{2}\nu$ appearing as the lower and upper components of a spinor.

With distinct neutrinos for electron and muon, as in the real situation, J_α(leptons) becomes

$$\bar{\nu}_e\gamma_\alpha(1+\gamma_5)e + \bar{\nu}_\mu\gamma_\alpha(1+\gamma_5)\mu = 2\bar{\nu}_e\gamma_\alpha\frac{1+\gamma_5}{2}e + 2\bar{\nu}_\mu\gamma_\alpha\frac{1+\gamma_5}{2}\mu. \tag{18}$$

This time the leptonic weak charge has the form $2(K_1 + iK_2)$, where again K_1 and K_2 are the first two components of an angular momentum and the algebra of SU(2) is generated. Now $\frac{1+\gamma_5}{2}e$ and $\frac{1+\gamma_5}{2}\nu_e$ form a spinor and so do $\frac{1+\gamma_5}{2}\mu$ and $\frac{1+\gamma_5}{2}\nu_\mu$.

Let us now demand universality for the weak interactions. In the one-neutrino case, we would require that the weak charge for hadrons have the form $2\sqrt{2}\,(K_1 + iK_2)$ and in the two-neutrino case that it have the form $2(K_1 + iK_2)$, where K_1 and K_2 are the first two components of an angular momentum. Writing the weak charge for hadrons in the general form

$$A(F_1^{+} + iF_2^{+})\cos\theta + A(F_4^{+} + iF_5^{+})\sin\theta,$$

we may verify that we have $A(K_1 + iK_2)$, where

$$K_1 = F_1^+ \cos\theta + F_4^+ \sin\theta,$$

$$K_2 = F_2^+ \cos\theta + F_5^+ \sin\theta,$$

$$K_3 = F_3^+ \cos^2\theta + \left(\frac{\sqrt{3}}{2} F_8^+ + \frac{1}{2} F_3^+\right) \sin^2\theta - F_6^+ \sin\theta \cos\theta.$$

Now from the approximate equality of vector coupling constants in the decay of the muon and the decay of the nucleus 0^{14}, we know that $A \cos\theta$ is around 2.

Clearly, then, universality in the one-neutrino case gives us $A = 2\sqrt{2}$, $\theta = 45°$ or

$$J_\alpha = 2(\mathcal{F}_{1\alpha}^+ + i\mathcal{F}^{\ +} + \mathcal{F}_{4\alpha}^+ + i\mathcal{F}_{5\alpha}^+)$$

as in ref. 3. However, for the actual case of two neutrinos, we must take $A = 2$ with θ small and have

$$J_\alpha = 2\cos\theta(\mathcal{F}_{1\alpha}^+ + i\mathcal{F}_{2\alpha}^+) + 2\sin\theta(\mathcal{F}_{4\alpha}^+ + i\mathcal{F}_{5\alpha}^+). \tag{19}$$

In a recent paper, Cabibbo [6] has combined our assumptions (a) and (b) quoted in Section II with the choice (18) of current components suitable for universality in the two-neutrino case and has shown that such a theory is in reasonable agreement with present information on leptonic decays of hadrons, with $\theta \approx 0.26$. We may therefore adopt equation (18) with some confidence, provided experimental leptonic and hadronic weak interactions exhibit no further complications.

The weak charge in general thus has the form $2(K_1 + iK_2)$. Moreover, the electric charge in units of e has the general form $K_3 + K_0$, where K_0 commutes with K_1, K_2, and K_3. The weak and electric charge operators, for both leptons and hadrons, thus generate the algebra of U(1) × SU(2). It is only when we take these charges for hadrons and break them up according to the quantum numbers conserved by the strong interactions that we get the group SU(3) × SU(3).

IV. Crude Results for Small λ

We now attempt to make use of the broken symmetry model for rough predictions about the strongly interacting particles. For the most part we shall put $c = 0$ and forget about violations of the eightfold way, concentrating on axial vector current conservation and *its* violation. In the limit $\lambda \to 0$, where all the axial vector currents are conserved, if the axial vector β-decay coupling constant is not to vanish, we must have either vanishing baryon masses or vanishing pseudoscalar meson masses. We choose vanishing baryon masses, and thus the point of view expressed here differs from that of many authors [7]. Under these conditions, the "renormalization constant" for the axial vector current becomes unity in the limit $\lambda \to 0$. The β-decay interaction in the zero-momentum transfer case is thus completely fixed, in the limit $\lambda \to 0$, by the symmetry pattern.

Since we are going to try assigning dominant representations to the particles, let us begin with the eight baryons having $J = 1/2^+$. For convenience we describe them, in the limit $\lambda \to 0$, by "fields" $\psi_1 \ldots \psi_8$. If they belonged to $(\underline{1}, \underline{8})$ and $(\underline{8}, \underline{1})$, then ψ_i and $\gamma_5\psi_i$ would transform under F_i and F_i^5 in the same way that $\mathcal{F}_{i\alpha}$ and $\mathcal{F}_{i\alpha}^5$ respectively transform in equation (15). In the zero momentum transfer case, then, in the limit $\lambda \to 0$, we would have the following pattern for the weak current $\mathcal{F}_{i\alpha}^+$:

$$-i f_{ijk} \overline{\psi}_j \gamma_\alpha (1 + \gamma_5) \psi_k.$$

Both the vector and axial vector currents would be coupled through F rather than D. This seems to be far from the truth [6].

Instead, we try the other baryon representation suggested in ref. 3, namely $(\underline{3}, \underline{3}^*)$ and $(\underline{3}^*, \underline{3})$. We then have to add a ninth particle, described by ψ_0. Under F_i and F_i^5, ψ_i and $\gamma_5\psi_i$ transform like u_i and iv_i in equation (14). The coupling pattern as $\lambda \to 0$, for any momentum transfer, is then

$$-i f_{ijk} \overline{\psi}_j \gamma_\alpha \psi_k + d_{ijk} \overline{\psi}_j \gamma_\alpha \gamma_5 \psi_k \tag{20}$$

with $i, j, k = 0, 1, \ldots 8$. For the baryon octet, then, the vector current is coupled through F and the axial vector current through D, with equal coefficients. This situation resembles the experimental one [6], the admixture of F in the axial vector pattern being of the order of 30%. There is a single form factor for the whole expression (20) in the limit $\lambda \longrightarrow 0$, and the anomalous magnetic and induced pseudoscalar form factors vanish. Since the vector coupling is through F, the neutron has no electrical interaction in the limit.

The interpretation of the ninth baryon depends on whether the mass associated with ψ_0 is positive or negative. A negative mass would lead us to treat $\gamma_5 \psi_0$ as the appropriate operator, so that the new particle with positive mass would have $J = 1/2^-$. Now in first order in λ we can compute the ratio of octet and singlet masses, using the transformation property ($\underline{3}$, $\underline{3}^*$) and ($\underline{3}^*$, $\underline{3}$) of the mass term u_0. The result is that the mass associated with the singlet is minus twice that of the octet. Thus $\gamma_5 \psi_0$ should describe, to first order in λ, a baryon with $J = 1/2^-$ and twice the average mass of the baryon octet with $J = 1/2^+$. If the extra baryon is identified with $\Lambda(1405)$, the spin and parity assignments may well be right but 1405 MeV is a far cry from twice the average mass of the baryon octet. We should perhaps not expect better agreement, however, with an approximation that treats baryon masses in first order.

The pseudoscalar octet should be assigned to the same representation as the divergence of the axial vector current, so that the Goldberger-Treiman relation can have some validity even to lowest order in λ. As explained in ref. 3, we have

$$\int d^3x\, \partial_\alpha \mathscr{F}_{i\alpha}^5 = \dot{F}_i^5 = i\left[\int \mathscr{H} d^3x,\, F_i^5\right] = \lambda\, i\left[F_i^5,\, \int u_0\, d^3x\right]$$
$$= \lambda d_{i0k} \int v_k\, d^3x = \sqrt{\frac{2}{3}}\, \lambda \int v_i\, d^3x \qquad (21)$$

neglecting c. So the divergence of the current, in lowest order, belongs to ($\underline{3}$, $\underline{3}^*$) and ($\underline{3}^*$, $\underline{3}$) and we make the same assignment for the pseudoscalar octet, along with a pseudoscalar singlet, a scalar octet, and a scalar singlet. For convenience, we describe these by "fields" π_i and σ_i, $i = 0, \ldots 8$, transforming like v_i and u_i respectively.

To order λ^0, these eighteen mesons all have a common mass. In order λ, they split according to the following pattern: the scalar singlet has a squared mass equal to $\overline{\mu^2} - 2\Delta$, the pseudoscalar octet has $\overline{\mu^2} - \Delta$, the scalar octet has $\overline{\mu^2} + \Delta$, and the pseudoscalar singlet has $\overline{\mu^2} + 2\Delta$. It is possible that the scalar singlet may be identified with a low-lying $J = 0^+$, $I = 0$ mesonic state decaying quickly into 2π; such states have been reported at various masses and at least one of them may exist. We then expect a scalar octet lying higher than the pseudoscalar one and a pseudoscalar singlet lying still higher.

The matrix elements of the weak current between one of these mesons and another follow the same pattern, in the limit $\lambda \longrightarrow 0$, as for the baryons in equation (20), since these mesons and the nine baryons belong to the same representation:

$$-if_{ijk}(\pi_j \partial_\alpha \pi_k + \sigma_j \partial_\alpha \sigma_k) + d_{ijk}(\sigma_j \partial_\alpha \pi_k - \pi_j \partial_\alpha \sigma_k). \qquad (22)$$

In the limit, there is a common form factor for the whole of (22).

If the scalar singlet meson represented by σ_0 (let us call it σ) really has a lower mass than K^+, then the K_{e4} decay $K^+ \longrightarrow \pi^+ + \pi^- + e^+ + \nu$ should be dominated by the chain $K^+ \longrightarrow \sigma + e^+ + \nu$, $\sigma \longrightarrow \pi^+ + \pi^-$. Treating σ as stable and using the zeroth order pattern (22), we may compute the ratio of the rates of $K^+ \longrightarrow \sigma + e^+ + \nu$ and $K^+ \longrightarrow \pi^0 + e^+ + \nu$:

$$\frac{K^+ \longrightarrow \sigma + e^+ + \nu}{K^+ \longrightarrow \pi^0 + e^+ + \nu} = \frac{8}{3}\frac{f(m_\sigma/m_K)}{f(m_\pi/m_K)}, \qquad (23)$$

where

$$f(\beta) = 1 - 8\beta^2 + 24\beta^4 \ln \beta^{-1} + 8\beta^6 - \beta^8. \qquad (24)$$

Roughly, then, the relative rate of $K^+ \longrightarrow \pi^+ + \pi^- + e^+ + \nu$ should be given by 2/3 of the expression (23) and the mass of the $\pi^+\pi^-$ system should be clustered around m_σ. The experimental data [8] still do not permit any firm conclusions, except that the ratio calculated in (23) must be a few times 10^{-3} if the decay through σ takes place. By contrast, if the σ mass is as low as 310 MeV, our rough formula gives about 0.21, in complete disagreement with observation. With a mass near 400 MeV, agreement is possible.

The symmetry, in the limit $\lambda \longrightarrow 0$, permits trilinear couplings of the scalar and pseudoscalar mesons to one another and to the nine baryons. In each case the allowed coupling pattern is formed with the symbol d'_{ijk}, which equals d_{ijk} except when one index is zero and the other two equal but not zero; the value of d' is then $-\sqrt{1/6}$. The effective couplings are:

$$2g\, d'_{ijk}\bar{\psi}_i(\sigma_j + i\pi_j\gamma_5)\psi_k, \tag{25}$$

$$(h/6)d'_{ijk}(\sigma_i + i\pi_i)(\sigma_j + i\pi_j)(\sigma_k + i\pi_k) + \text{hermitian conjugate}. \tag{26}$$

The coupling of the pseudoscalar octet to the baryon octet through D is, of course, in reasonable agreement with experiment, as we might have expected from the Goldberger-Treiman relation and the axial vector current coupling through D.

Other predictions, however, are not in good agreement with the present experimental situation. No scalar octet has been found. The nuclear forces arising from the scalar particles should be gigantic, according to (25), and evidence for such forces is not convincing at the present time. Also the scalar coupling of the ninth baryon $\gamma_5\psi_0$ to the pseudoscalar octet and the baryon octet should also be very strong, according to (25), and lead to a width of several BeV (!) for the ninth baryon if we take the prediction literally. An estimate in the next section of h in equation (26) indicates large widths also for the scalar mesons if they are significantly above the thresholds for decay into two pseudoscalar mesons.

We may thus adopt several different attitudes:

(1) The SU(3) × SU(3) algebraic system, assuming it is correct, does not provide a useful approximate symmetry.
(2) Higher order effects in λ, for example as indicated by the Goldberger-Treiman relation (see next section), may reduce some of the coupling constants.
(3) The scalar mesons will turn up with large couplings, and the ninth baryon will turn up as a very vague bump with a huge width.
(4) Something is wrong with our choices of representations.

At present, it is not easy to choose among these possibilities. They are discussed further in Section V. Meanwhile, we return to the assignment of representations.

The vector meson situation is complicated by the $\phi-\omega$ mixing and will not be fully treated here, but we should expect a vector meson octet that dominates the vector form factors to transform like the currents, i.e., according to $(\underline{1}, \underline{8})$ and $(\underline{8}, \underline{1})$. Thus there should be an axial vector octet nearby. In fact, the splitting between the two octets is of second order in λ, since we cannot make $(\underline{3}, \underline{3}^*)$ and $(\underline{3}^*, \underline{3})$ out of $[(\underline{1}, \underline{8})$ and $(\underline{8}, \underline{1})]$ times itself.

In conclusion, we mention the violation of the eightfold way by the term in ϕ_8. If ϕ_8 were to belong to $(\underline{3}, \underline{3}^*)$ and $(\underline{3}^*, \underline{3})$, then the splitting of the baryon octet with $J = 1/2^+$ would go mainly with D, since to order $\lambda^0 c^1$ the only allowed coupling is analogous to (25). In fact, the splitting is mostly F. We may therefore consider the possibility that ϕ_8 belongs to $(\underline{1}, \underline{8})$ and $(\underline{8}, \underline{1})$. The baryon octet is then split only in order $\lambda^1 c^1$ and both F and D come in.

The scalar and pseudoscalar octets are split open in order $\lambda^0 c^1$ by a term ϕ_8 transforming like $(\underline{1}, \underline{8})$ and $(\underline{8}, \underline{1})$. Moreover, in that order, the spacing is the same for both octets.

V. The Goldberger-Treiman Relation

The Goldberger-Treiman relation states in essence that certain matrix elements of the divergence $\partial_\alpha \mathcal{F}^5_{i\alpha}$ of the axial vector current components obey unsubtracted dispersion relations in the invariant momentum squared carried by the current and that these dispersion relations are dominated by the intermediate state with one pseudoscalar meson [3, 9].

The pion decay amplitude is described by the quantity f_π, which is defined, as in ref. 3, by the formula

$$<0|\partial_\alpha \mathscr{F}_{i\alpha}^5|\pi> = \frac{m_\pi^2}{2f_\pi}\Phi_i, \qquad (i = 1, 2, 3) \tag{27}$$

where Φ_i is the wave function of the pion. The width for the decay $\pi^+ \longrightarrow \mu^+ + \nu_\mu$ is then

$$\Gamma_\pi = G^2 \cos^2\theta\, m_\pi\, m_\mu^2 (1 - m_\mu^2/m_\pi^2)^2 \left(\frac{f_\pi^2}{4\pi}\right)^{-1} (64\pi^2)^{-1}, \tag{28}$$

which differs by the factor $\cos^2\theta$ from the corresponding expression in ref. 3. Likewise, we have

$$<0|\partial_\alpha \mathscr{F}_{i\alpha}^5|K> = \frac{m_K^2}{2f_K}\Psi_i, \qquad (i = 4, 5, 6, 7) \tag{29}$$

where Ψ_i is the wave function of the kaon. The width for the decay $K^+ \longrightarrow \mu^+ + \nu_\mu$ is

$$\Gamma_K = G^2 \sin^2\theta\, m_K\, m_\mu^2 (1 - m_\mu^2/m_K^2)^2 \left(\frac{f_K^2}{4\pi}\right)^{-1} (64\pi^2)^{-1}, \tag{30}$$

where in ref. 3 we would not have had the factor $\sin^2\theta$. Cabibbo [6] has pointed out that with $\theta \approx 0.26$ we get $f_\pi \approx f_K$. (The attempt [3] to make $f_\pi/f_K \approx m_\pi/m_K$ is thus unnecessary.)

The Goldberger-Treiman relation for neutron β decay now states that

$$2m_N\left(\frac{-G_A}{G\cos\theta}\right) \approx g_{NN\pi} f_\pi^{-1}, \tag{31}$$

where $(-G_A/G\cos\theta)$ is the axial vector "renormalization factor" for the nucleon. The matrix element between neutron and proton of the divergence of the axial vector current has been approximately expressed as the product of the pion-nucleon coupling constant $g_{NN\pi}$ and the pion decay constant f_π^{-1}. Experimentally, equation (31) is satisfied with an error of around 10%.

If we now generalize to the baryon octet and the pseudoscalar meson octet, in the approximation of the eightfold way (which includes $f_\pi = f_K = f$), we have in general a part of g that goes with the D coupling and a part with the F coupling; the same is true of $(-G_A/G\cos\theta)$. Not only should the relation (31) hold, then, but also the F/D ratios should be the same for the meson-baryon coupling and for the axial vector current. Cabibbo's value of 0.30/0.95 for the F/D ratio for the current agrees well with all estimates of F/D for the meson coupling.

If scalar mesons exist, the Goldberger-Treiman relation should apply to the β-decay matrix elements between scalar and pseudoscalar meson states, in relation to the strong coupling constants for the scalar-pseudoscalar-pseudoscalar ($\sigma\,\pi\,\pi$) vertices.

Now let us examine what happens to the Goldberger-Treiman relation for the $NN\pi$ and $\sigma\pi\pi$ cases when we have approximate conservation of both vector and axial vector currents, i.e., $c = 0$ and $\lambda \longrightarrow 0$, assuming our assignments of both baryons and mesons to $(\underline{3}, \underline{3}^*)$ and $(\underline{3}^*, \underline{3})$ are correct. In the limit $\lambda \longrightarrow 0$, the matrix elements $<0|u_i|\sigma_i>$ and $<0|v_i|\pi_i>$ are all non-zero and equal. The quantity f^{-1}, proportional to $<0|\partial_\alpha F_{i\alpha}^5|\pi_i>$, is evidently of order λ. Likewise, the mass of the nucleon is of order λ. The renormalization $(-G_A/G\cos\theta)$ approaches unity, with the octet pattern becoming pure D. The coupling constant $g_{NN\pi}$ remains finite and the octet coupling pattern here too becomes pure D. Evidently, then, the Goldberger-Treiman relation (27) can hold approximately in the limit $\lambda \longrightarrow 0$ with both sides of order λ. We have, then, in the double approximation of $\lambda \longrightarrow 0$ and exact validity of the Goldberger-Treiman relation,

$$2m_N \approx g f^{-1}, \tag{32}$$

where g multiplies the whole baryon-baryon-meson coupling pattern, as in (25). The analogous equation for the $\sigma\pi\pi$ vertices is

$$2\Delta \approx h f^{-1}, \tag{33}$$

where 2Δ is the difference in mass squared between the scalar and pseudoscalar octets in order λ and h multiplies the whole trilinear meson coupling pattern, as in (26). This double approximation is equivalent to saying that the violation of SU(3) × SU(3) mass degeneracy is accomplished formally by the displacement

$$\sigma_0 \longrightarrow \sigma_0 - \sqrt{\frac{3}{8}}\, f^{-1} \tag{34}$$

in effective couplings such as (25) and (26).

Now if we consider large violations of symmetry, so that higher order effects in λ are important (and even effects involving c, which violate the eightfold way), we may suppose that the Goldberger-Treiman approximation is still good. For example, as we mentioned above, the actual value of $(-G_A/G \cos\theta)$ is around 1.25, with about 0.95 going with the D pattern and about 0.30 with the F pattern, in contrast with the value 1 and the pure D coupling that we would have in the limit $\lambda \longrightarrow 0$. Likewise, the baryon-baryon-meson coupling departs from the pure D coupling that is acquired in the limit $\lambda \longrightarrow 0$, but these two departures seem to follow the Goldberger-Treiman relation in that the F/D ratio is similar in the two cases.

We may look, for example, at the decay of the ninth baryon into $\Sigma + \pi$. In order λ, the mass difference between the two baryons is the same as m_N and in order 1 the coupling constant is the same as $g_{NN\pi}$, with the appropriate ratio of d'_{ijk} coefficients. If, now, in higher order in λ and c the mass difference becomes quite different, we might expect the coupling constant to change in proportion, keeping the Goldberger-Treiman equation approximately valid. Instead of having the effective coupling constant g for the decay equal to $2 m_N f_\pi$, as in (32), we would have approximately $2(m - m_\Sigma) f_\pi r$, where m is the real mass of the ninth baryon and r is the renormalization factor for the axial vector current matrix element between Σ and the ninth baryon. If m is 1405 MeV, for instance, then the coupling constant g in question is reduced by the factor $r(m - m_\Sigma) m_N^{-1} \approx 0.23 r$ and the width of the ninth baryon is then

$$\Gamma \approx 4 \frac{[2 f_\pi (m - m_\Sigma) r]^2}{4\pi} k \frac{m_\Sigma + E}{2m},$$

where k is the decay momentum. The width comes out about 200 MeV(r^2) instead of about 3 BeV(r^2); the actual width of $\Lambda(1405)$ is around 60 MeV.

Similar corrections should be applied to the various coupling constants h for the various $(\sigma\,\pi\,\pi)$ vertices; instead of the completely symmetrical formula (33) we can use similar expressions in which the actual differences of mass squared are inserted in place of the first order pattern based on the single quantity Δ. For a scalar K particle of mass μ, the decay into $K + \pi$ would be regulated by a value of h approximately equal to $f_\pi(\mu^2 - m_K^2) r$, where r is the renormalization factor for the axial vector current matrix element between the scalar K particle and K itself. The decay width for this case is then

$$\Gamma \approx \frac{3}{2} \frac{[f_\pi(\mu^2 - m_K^2) r]^2}{4\pi} \frac{k}{\mu^2},$$

where k is the decay momentum. For $\mu = 725$ MeV, for example, Γ comes out around 80 MeV(r^2).

None of this is of much use, of course, if the scalar octet does not exist. If it is not found, we will have to abandon the idea of using the group SU(3) × SU(3) of the vector and axial vector currents as an approximate symmetry of the strong interactions.

We should mention one intermediate possibility, which involves a different assignment of representation to the pseudoscalar octet, namely $(\underline{1}, \underline{8})$ and $(\underline{8}, \underline{1})$. No scalar or pseudoscalar unitary singlet would be predicted. The Goldberger-Treiman relation could not in this case be approximately valid for small λ, since the two sides would be of different order in λ, with the pseudoscalar octet not transforming like v_i under SU(3) × SU(3). Conceivably, however, for a particular value of λ, not particularly small, we could have the relation. The coupling of baryons to the pseudoscalar and scalar mesons would be forbidden as $\lambda \longrightarrow 0$ and the pseudoscalar octet would acquire its coupling through the violation of the symmetry. The scalar octet would have the opposite properties under charge conjugation to the scalar octet hitherto discussed. Thus in the limit $c \longrightarrow 0$ of the eightfold way, this scalar octet could have no Yukawa coupling to the baryon octet and no coupling to two mesons of the pseudoscalar octet. When the violation of the eightfold way is turned on, the non-strange members of the scalar octet would still lack these couplings, but the strange members could have them by violating the eightfold way. Such an "abnormal" scalar octet would have very different experimental properties from a normal one, particularly for the $Y = 0$ members, and would be readily identifiable as such.

VI. Triplets, Real and Mathematical

So far, we have concerned ourselves with the assumption that the group of the vector and axial vector octets of currents is SU(3) × SU(3), with the transformation properties under the group of the terms λu_0 and $c\phi_8$ in the energy density, and with the possibility that in a crude approximation λ as well as c might be treated as small. We may, however, go further and ask whether there are additional algebraic relations among the quantities we have introduced. In order to obtain such relations that we may conjecture to be true, we use the method of *abstraction* from a Lagrangian field theory model. In other words, we construct a mathematical theory of the strongly interacting particles, which may or may not have anything to do with reality, find suitable algebraic relations that hold in the model, postulate their validity, and then throw away the model. We may compare this process to a method sometimes employed in French cuisine: a piece of pheasant meat is cooked between two slices of veal, which are then discarded [10].

In ref. 3, the Sakata model was employed in this way. However, certain adjustments had to be made to get to the eightfold way. Instead, we may employ the quark model [11], which gives the eightfold way directly; there are also other models [11], based on a fundamental triplet and a fundamental singlet, that are perfectly compatible with the eightfold way.

Any such field-theoretic model must contain some basic set of entities, with non-zero baryon number, out of which the strongly interacting particles can be made. If this set is a unitary octet, the theory is very clumsy; it is hard to arrange any coupling that will reduce the symmetry from SU(8) to SU(3) without introducing [1, 2] in addition a Yang-Mills octet of fundamental vector mesons. Thus the only reasonably attractive models are based on unitary triplets and perhaps singlets.

If we adopt such a viewpoint, we should say that the correct dynamical description of the strongly interacting particles requires either the bootstrap theory or else a theory based on a fundamental triplet. In neither case do the familiar neutron and proton play any basic role.

It is, of course, a striking fact that no unitary triplets have so far been identified among the strongly interacting particles; however, they may turn up. Their appearance may, of course, be consistent with either the bootstrap theory or a theory with a fundamental triplet. Their non-appearance could certainly be consistent with the bootstrap idea, and also possibly with a theory containing a fundamental triplet which is hidden, i.e., has effectively infinite mass.

Thus, without prejudice to the independent questions of whether the bootstrap idea is right and whether real triplets will be discovered, we may use a mathematical field theory model containing a triplet in order to abstract algebraic relations.

If we want to use just a triplet and no singlet, we must have quarks, with baryon number 1/3 and electric charges −1/3, −1/3, and +2/3. Such particles presumably are not real but we may use then in our field theory model anyway. Since the quark model is mathematically the simplest, we shall in fact employ it in the next section, as in ref. 11, for our process of abstraction.

If we consider a model with a basic triplet t and a singlet b, then we are free to take for these particles integral electric charges and baryon number equal to one; say we do so. The singlet must be neutral, and we can then form the known supermultiplets from (b), $(b\ t\ \bar{t})$, $(b\ \bar{b}\ b\ \bar{t}\ t\ \bar{t}\ t)$, etc. The triplet can have electric charges $q, q, q+1$ or $-q, -q, -q-1$, where q is any integer. In the former case, the electric charge Q in units of e is given by the relation

$$Q = (q + 1/3)(n_t - n_{\bar{t}}) + F_3 + \frac{F_8}{\sqrt{3}},$$

where $n_t - n_{\bar{t}}$ is the number of triplets minus the number of antitriplets. In the latter case we have the relation

$$Q = -(q + 1/3)(n_t - n_{\bar{t}}) + F_3 + \frac{F_8}{\sqrt{3}}.$$

With integral charges and baryon number one, there is no reason to require any member of a real triplet to be absolutely stable; we may permit them all to decay into ordinary baryons. However, the question arises whether such decays take place by weak interactions or by moderately strong interactions that violate SU(3) but not isotopic spin conservation. If the latter, we want the violation of SU(3) to transform like $\underline{3}$ and $\underline{3}^*$, so that in second order it gives the familiar octet behavior of the violation. But $I = 0$, $Q = 0$ occurs in $\underline{3}$ and $\underline{3}^*$ only when $q = 0$. Thus if the decay of triplets into baryons is to be attributed to moderately strong interactions that give rise, in second order, to the octet violation of SU(3), we must have $q = 0$. This is the model used by Maki [12] and by Tarjanne and Teplitz [13].

With other values of q, we presumably have the triplet decaying into ordinary baryons by the weak interaction. As pointed out previously [11, 14], the most interesting case of this kind is the one with $q = -1$, where the four members of the basic triplet and singlet present a perfect analogy with the known leptons.

Absolute stability of one member of a triplet is of course, a possibility for any value of q.

These kinds of triplets with integral charges are, of course, more likely to correspond to real particles than the quarks, and we may also use them in field theory models to abstract algebraic relations, obtaining essentially the same ones as for the quarks in the next section.

VII. Further Algebraic Relations

We start with the simple Lagrangian model of quarks discussed in ref. 11. There is a triplet t of fermion fields corresponding to three spin 1/2 quarks: the isotopic doublet u and d, with charges 2/3 and $-1/3$ respectively, and the isotopic singlet s, with charge $-1/3$. A neutral vector meson field B_α is introduced, too. The Lagrangian is simply

$$-\bar{t}\gamma_\alpha \partial_\alpha t - \mathscr{L}_B - iF B_\alpha \bar{t}\gamma_\alpha t$$

as $\lambda \longrightarrow 0$ and $c \longrightarrow 0$, where $\mathscr{L}_B$ is the free Lagrangian for the field B and

$$\bar{t}\gamma_\alpha t = \bar{u}\gamma_\alpha u + \bar{d}\gamma_\alpha d + \bar{s}\gamma_\alpha s.$$

Now we may add to the Lagrangian a quark mass term

$$\lambda u_0 = m_0(\bar{u}u + \bar{d}d + \bar{s}s) = m_0 \bar{t}t.$$

The energy density acquires a term that is just the negative of this. In the model we may put

$$u_i = \bar{t}\frac{\lambda_i}{2}t, \qquad v_i = -i\bar{t}\gamma_5\frac{\lambda_i}{2}t, \qquad i = 0, 1, \ldots 8,$$

and $\lambda = \sqrt{6}\, m_0$. Likewise, we have, in the model,

$$\mathscr{F}_{i\alpha} = i\bar{t}\frac{\lambda_i}{2}\gamma_\alpha t, \qquad \mathscr{F}_{i\alpha}^5 = i\bar{t}\frac{\lambda_i}{2}\gamma_\alpha\gamma_5 t.$$

For the moment, we forget the term $c\phi_8$ that breaks the eightfold way.

The non-singularity of the model enables us to generalize [3, 11] the commutation relations (14) and (15) and equation (20) to the local relations

$$[\mathscr{F}_{i4}(\underline{x}, t), u_j(\underline{x}', t)] = -f_{ijk}u_k(\underline{x}, t)\delta(\underline{x} - \underline{x}'), \text{ etc.} \tag{35}$$

$$[\mathscr{F}_{i4}(\underline{x}, t), \mathscr{F}_{j4}(\underline{x}', t)] = -f_{ijk}\mathscr{F}_{k4}(\underline{x}, t)\delta(\underline{x} - \underline{x}'), \text{ etc.,} \tag{36}$$

$$\partial_\alpha \mathscr{F}_{i\alpha}^5 = \sqrt{\frac{2}{3}}\lambda v_i + \mathscr{O}(c). \tag{37}$$

We have proposed that these relations be abstracted from the model and postulated as true. In an exact calculation of the matrix elements of the $\mathcal{F}_{i\alpha}$, u_i, and v_i by means of linear homogeneous dispersion relations without subtractions, the nonlinear relations (35) and (36) supply the scale factors that determine such things as the axial vector current renormalization.

In the quark model, the term $c\phi_8$ in the Lagrangian could be put in as a mass difference between singlet and doublet quarks, but ϕ_8 would then be the same as u_8 and would transform like $(\underline{3}, \underline{3}^*)$ and $(\underline{3}^*, \underline{3})$. If we want ϕ_8 to belong to $(\underline{1}, \underline{8})$ and $(\underline{8}, \underline{1})$, we could put it into the model as a coupling of the meson B_α to the current $\mathcal{F}_{8\alpha}$. Such a term is reminiscent of Ne'eman's "Fifth Interaction" [15] or of Sakurai's use [16] of $\phi-\omega$ mixing as a dynamical mechanism for violating the eightfold way.

We mentioned in ref. 3 that in the model there are further commutation relations, besides (35) and (36), which we might or might not take seriously, namely the commutation relations of the u's and v's. It is interesting that when these operators are commuted, in the model, they bring back the operators $\mathcal{F}_{i4}$ and $\mathcal{F}_{i4}^5$, along with a new operator, the helicity charge density, which we may call $\mathcal{F}_{04}^5$. The algebra of the u_i, v_i, $\mathcal{F}_{i4}$, and $\mathcal{F}_{i4}^5$ then closes; we have 18 u's and v's, 8 $\mathcal{F}_{i4}$'s, and 9 $\mathcal{F}_{i4}^5$'s, corresponding to the 35 generators of the algebra SU(6). In the model, the new current $\mathcal{F}_{0\alpha}^5$ is just $(i/2)\,\bar{t}\,\lambda_0\,\gamma_\alpha\,t$.

Evidently, we can look upon all 35 operators as generating infinitesimal unitary transformations among the three left-handed quarks and the three right-handed quarks. This algebraic system connects scalars and pseudoscalars with four-vectors and four-pseudovectors and thus represents a new stage in the generalization of symmetry. Whereas F-spin connects only systems of the same parity and behavior under proper Lorentz transformations, the group SU(3) × SU(3) of the vector and axial vector currents connects systems which may have different parity but must still have the same behavior under the proper Lorentz group, and SU(6) now connects systems with different parity and/or different space-time behavior.

Of course, it is not clear, even in the model, that SU(6) is of any use as an approximate symmetry. If it were, it would arrange particles of various spins and parities in super-super-supermultiplets. However, it does appear to be true that a huge number of special algebraic properties can be abstracted from a field theory model. The situation is reminiscent of the growth of dispersion relations from an obscure equation for forward scattering of light to a huge set of relations among all scattering amplitudes, nearly sufficient to determine the whole S-matrix. Conceivably, the study of algebraic relations will undergo a comparable transformation.

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GROUP U(6)⊗U(6) GENERATED BY CURRENT COMPONENTS*

R. P. Feynman, M. Gell-Mann, and G. Zweig†
California Institute of Technology, Pasadena, California
(Received 2 November 1964)

It has been suggested[1-3] that the equal-time commutation rules of the time components of the vector and axial-vector current octets ($\mathfrak{F}_{i\alpha}$ and $\mathfrak{F}_{i\alpha}{}^5$, respectively) are the same as if these currents had the simple form $\mathcal{G}_{i\alpha}$ and $\mathcal{G}_{i\alpha}{}^5$, defined as follows:

$$\mathcal{G}_{i\alpha}=\tfrac{1}{2}i\bar{q}\lambda_i\gamma_\alpha q,$$

$$\mathcal{G}_{i\alpha}{}^5=\tfrac{1}{2}i\bar{q}\lambda_i\gamma_\alpha\gamma_5 q, \qquad (1)$$

where q is an SU(3) triplet with spin $\frac{1}{2}$—for example, the quarks[2] or aces.[4] Here the matrices λ_i ($i=1,\cdots,8$) are the SU(3) analogs of the Pauli matrices, as defined in reference 1. The operators

$$F_i(t)=-i\int d^3x\,\mathfrak{F}_{i4},$$

$$F_i{}^5(t)=-i\int d^3x\,\mathfrak{F}_{i4}{}^5, \qquad (2)$$

then generate at equal times the algebra of SU(3)⊗SU(3), which may be a very approximate symmetry of the strong interactions,[1,3] while the F_i generate a subalgebra corresponding to SU(3), which is a fairly good symmetry of the strong interactions.

We now propose to extend these considerations to the space components of the currents as well. First we define[1-3] a ninth λ matrix $\lambda_0\equiv(\frac{2}{3})^{1/2}1$ and a corresponding ninth pair of currents $\mathfrak{F}_{0\alpha}$ and $\mathfrak{F}_{0\alpha}{}^5$ (where $\mathfrak{F}_{0\alpha}$ would be $\sqrt{6}$ times the baryon current in a true quark or ace theory). We then assume that the equal-time commutation relations of all the 72 components of the $\mathfrak{F}_{i\alpha}$ and $\mathfrak{F}_{i\alpha}{}^5$ ($i=0,\cdots,8$) are the same as those of the $\mathcal{G}_{i\alpha}$ and $\mathcal{G}_{i\alpha}{}^5$, at least as far as terms proportional to the spatial δ function are concerned. (There are also, in general, terms[5] involving gradients of the δ function, which vanish on space integration and which we ignore here.) The system of $\mathcal{G}_{i\alpha}$ and $\mathcal{G}_{i\alpha}{}^5$ is closed under equal-time commutation, and the space integrals $\int\mathcal{G}_{i\alpha}d^3x$ and $\int\mathcal{G}_{i\alpha}{}^5d^3x$ generate the algebra of U(6)⊗U(6). Our assumption thus implies that $\int\mathfrak{F}_{i\alpha}d^3x$ and $\int\mathfrak{F}_{i\alpha}{}^5d^3x$ also generate the algebra of U(6) ⊗U(6). We assume further that this algebra is a very approximate symmetry of the strong interactions.

We now exhibit some of the structure of the algebra by looking at the $\mathcal{G}_{i\alpha}$ and $\mathcal{G}_{i\alpha}{}^5$. We note that the space integrals of the densities

$$\bar{q}\lambda_i\gamma_4 q=q^\dagger\lambda_i q \quad (i=0,\cdots,8)$$

and

$$i\bar{q}\lambda_i\gamma_n\gamma_5 q=q^\dagger\lambda_i\sigma_n q \quad (n=1,2,3)$$

generate the subalgebra corresponding to U(6); the same is then true of the corresponding components of the $\mathfrak{F}$'s. We may refer to the algebra of the space integrals of these $\mathfrak{F}$ components as the A spin, with generators A_r ($r=0,1,\cdots,35$). Now the space integrals of the densities $q^\dagger\lambda_i\frac{1}{2}(1+\gamma_5)q$ and $q^\dagger\lambda_i\sigma_n\frac{1}{2}(1+\gamma_5)q$ also generate a group U(6), and so do the corresponding terms with $\frac{1}{2}(1-\gamma_5)$. The corresponding integrals of $\mathfrak{F}$ components thus give a left-handed A spin A_r^+ and a right-handed A spin A_r^-, respectively, with

$$A_r = A_r^+ + A_r^- \quad (r = 0, 1, \cdots, 35). \tag{3}$$

Those 36 components of $\mathfrak{F}_{i\alpha}$ and $\mathfrak{F}_{i\alpha}^5$ (out of a total of 72) that are the densities of the A_r do not go just into themselves under Lorentz transformations, but yield instead the complete system of 72 components of the $\mathfrak{F}_{i\alpha}$ and $\mathfrak{F}_{i\alpha}^5$, which form the densities of A_r^+ and A_r^-

We have assumed above that the A^+ and A^- spins are separately very approximate symmetries of the strong interactions. We may now add the further assumption that the total A spin is a good symmetry, nearly as good as the subset that constitutes the F spin. This approximate conservation of A spin is then our way of describing the success achieved by the SU(6) symmetry of Gürsey and Radicati,[6] Sakita,[7] and Zweig,[8] treated further in a series of recent Letters.[9-14] In reference 10, our interpretation of the symmetry is hinted at, but otherwise it is described in different language, which does not make clear the physical identification of the symmetry operators with integrals of components of the vector and axial-vector currents occurring in the weak and electromagnetic interactions. Also, the Lorentz-complete system, obeying the commutation rules of U(6)⊗ U(6), is not given.

In a relativistic situation, where a state like ρ exists part of the time as 2π, part of the time as $N+\bar{N}$, part of the time as $\Delta+\bar{\Delta}$, etc., with a different set of channel spins in each case, it is evidently not sufficiently specific to talk of "spin independence" of strong interactions. In contrast, our statement in terms of the approximate conservation of the Gamow-Teller operator $\int\mathfrak{F}_{in}{}^5 d^3x$ $(n=1,2,3)$ does have a definite meaning.

One set of consequences of our approach is that the Gamow-Teller matrix elements within an SU(6) supermultiplet can be exactly computed in the limit of SU(6) symmetry. We adopt the assignments of the $J^\pi=\frac{1}{2}^+$ baryon octet and $J^\pi=\frac{3}{2}^+$ baryon decimet to the SU(6) representation $\underline{56}$, and the assignment of the vector-meson octet and singlet and the pseudoscalar octet to the representation $\underline{35}$; these assignments have explained at least six well-known facts.[15] The axial-vector strength, within the baryon octet, comes out to be $1(D)+\frac{2}{3}(F)$; for the nucleon, this gives $(-G_A/G_V)=5/3$, as indicated in reference 10, to be compared with an observed value more like 1.2. The agreement is fair, as is the agreement of the D/F ratio with the results on leptonic hyperon decays. The matrix elements of the Gamow-Teller operator between octet and decimet are also exactly specified in the limit of SU(6) symmetry and can be checked by neutrino experiments.

Let us now go on to discuss the badly broken symmetry U(6)⊗ U(6), which bears about the same relation to U(6) symmetry as the U(3) ⊗ U(3) symmetry generated by the time components of vector and axial-vector currents[1,3] bears to the eightfold way. On the way from the full U(6)⊗ U(6) down to U(3), we could pass through U(6) or through U(3)⊗ U(3) symmetry as an intermediate stage; these are alternatives in somewhat the same way as are L-S and j-j coupling in atomic physics. It seems that the operators of U(6), all of which have nonrelativistic limits, form a much better symmetry system than those of U(3)⊗ U(3); hence, the useful procedure is to go from U(6) ⊗ U(6) to U(6), and then to U(3) and U(2). [Actually U(6) is not much worse than U(3).]

The baryons are presumed to have zero mass in the limit of U(6)⊗ U(6) symmetry, as in the limit of U(3)⊗ U(3) symmetry.[1,3] The perturbation that reduces the symmetry of U(6) is assumed to transform like $(\underline{6},\underline{6}^*)$ and $(\underline{6}^*,\underline{6})$ under (A^+,A^-), and like $\underline{1}$ under A. Thus it transforms like a common quark mass term $\bar{q}q$, which takes a left-handed q going like $(\underline{6},\underline{1})$ into a right-handed q going like $(\underline{1},\underline{6})$, and vice versa.[16] The $J^\pi=\frac{1}{2}^+$ octet and $J^\pi=\frac{3}{2}^+$ decimet belonging to $\underline{56}$ can be placed either in $(\underline{1},\underline{56})$ and $(\underline{56},\underline{1})$, or in $(\underline{6},\underline{21})$ and $(\underline{21},\underline{6})$, if we restrict ourselves to representations that transform like $3q$. The latter is very attractive, because it splits into a $\underline{56}$ and a $\underline{70}$, where the masses to first order in the perturbation are in the ratio 1:−2; as in reference 3, we must interpret negative mass as positive mass with negative parity, and so we are led to a $\underline{56}$ with unit mass and a $\underline{70}$ with opposite parity and roughly twice the mass. The $\underline{70}$ contains a $\frac{3}{2}^-$ octet, a $\frac{1}{2}^-$ singlet, a $\frac{1}{2}^-$ octet, and a $\frac{1}{2}^-$ decimet. Thus the prediction of reference 3 that the $\frac{1}{2}^+$ octet is accompanied by a $\frac{1}{2}^-$ singlet of roughly twice the mass is contained in our present result. The $\frac{3}{2}^-$ octet has probably been seen [including $N(1512)$], but the $\frac{1}{2}^-$ octet and decimet have not so far been identified.

In the limit of U(6)⊗U(6) symmetry, the vector and pseudoscalar mesons of the 35 can be put into either of two pairs of representations that transform like $q+\bar{q}$. The mesons could go like (35, 1) and (1, 35), or else like (6, 6*) and (6*, 6). If they belong to the adjoint representation pair (1, 35) and (35, 1), as the current components do, then the usual 35 is accompanied by another 35, consisting of a normal axial-vector octet and singlet and an abnormal scalar octet. [Here, "normal" means that the $Y=0$, $I=0$ member of an axial vector, scalar, or pseudoscalar SU(3) multiplet is even under charge conjugation; "abnormal" means it is odd.] If the mesons belong to (6, 6*) and (6*, 6), then the usual 35 is accompanied by a 1 (a normal pseudoscalar singlet), another 1 (a normal scalar singlet), and a 35 consisting of an abnormal axial-vector octet and singlet and a normal scalar octet. In either case, the perturbation that reduces U(6)⊗U(6) to U(6) does not split the mesons into U(6) multiplets in first order; in second order, they are split. The assignment to (6, 6*) and (6*, 6) is appealing because the pseudoscalar singlet could be identified with $\eta(960)$, the scalar octet may include $\kappa(725)$, and the abnormal axial octet may include the meson at about 1220 MeV with $I=1$ that decays into $\pi+\omega$.

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[16]At the end of reference 3, it is suggested that perhaps the perturbation in the energy density that reduces U(3)⊗U(3) symmetry to U(3) symmetry generates, together with the algebra of U(3)⊗U(3), a small algebra, which could be that of U(6). Such a use of U(6) is not the same as the use we are discussing in this Letter. However, we might consider the analogous possibility that the perturbation in the energy density that reduces U(6)⊗U(6) to U(6) generates, together with the algebra of U(6)⊗U(6), the algebra of U(12), corresponding to all unitary transformations on the four Dirac components and the three unitary-spin components of a quark field. Even if this is true, of course, U(12) need not be a useful symmetry of strong interactions.

INTRINSICALLY BROKEN U(6)⊗U(6) SYMMETRY FOR STRONG INTERACTIONS

K. Bardakci,* J. M. Cornwall,* P. G. O. Freund,*† and B. W. Lee‡
The Institute for Advanced Study, Princeton, New Jersey
(Received 4 November 1964)

With an ever increasing internal-symmetry group for hadrons, the possibility of combining internal symmetry and space-time symmetry has become all the more appealing. Recent attempts[1,2] in this direction have, however, been confined to essentially nonrelativistic situations where the spin degrees of freedom can be regarded as internal. The group that has emerged from these investigations as a likely global group is SU(6). Following up this line of thought, we wish to extend these results to relativistic quantum field theory. We shall produce a chain of symmetries culminating in $W_6 \equiv$ U(6)⊗U(6) that arises naturally in this case. In contradistinction to symmetries previously considered in physics, the largest members of this chain are intrinsically broken. In other words, there does not exist a total Lagrangian that possesses W_6 symmetry, since the kinetic energy and mass terms will automatically break it. The symmetry will show up only in the interaction term and will consequently make sense in terms of a strong-coupling limit.

Let us consider a triplet of spin-$\frac{1}{2}$ fermions (quarks),[3] and let $\psi^i(x)$, $i=1,2,3$, be the corresponding Dirac fields.[4] Introducing their left- and right-handed Weyl components R and L,

$$\begin{pmatrix}L\\0\end{pmatrix}=\frac{(1+\gamma_5)}{2}\psi,\quad \begin{pmatrix}0\\R\end{pmatrix}=\frac{(1-\gamma_5)}{2}\psi, \tag{1}$$

we consider the following 72-parameter group of transformations:

$$L\to(1+ia_{i\mu}\sigma^\mu\lambda_i)L,$$

$$R\to(1+ia_{i\mu}^{\ *}\sigma^\mu\lambda_i)R, \tag{2}$$

where $\sigma^\mu=(1,\vec{\sigma})$; λ_i, $i=0,1,\cdots,8$, are the 3×3 Hermitean matrices defined by Gell-Mann[5] [$\lambda_0=(\frac{2}{3})^{1/2}1$], and $a_{i\mu}$ are complex parameters. The generators of the transformations (2) can be written as

$$\lambda_i,\ \sigma_{\mu\nu}\lambda_i,\ i\gamma_5\lambda_i, \tag{3}$$

in the space of Dirac spinors. The matrices in (3) are reducible; the irreducible components are

$$\sigma_\mu\lambda_i,\ \pm i\sigma_\mu\lambda_i. \tag{4}$$

The generators with upper (lower) sign act on L (R). This makes clear that the transformation group (2) is the complexification of U(6)–the full linear group in six dimensions GL(6). Finite-dimensional representations of GL(6) of physical interest can be classified by a method similar to the "unitary trick" of Weyl for the Lorentz group.[6] We form out of the set (3) the generators

$$\tfrac{1}{2}(1\pm\gamma_5)\Sigma_\mu\lambda_i,\quad \Sigma_\mu=(1,\sigma_{23},\sigma_{31},\sigma_{12}), \tag{5}$$

which span the group $W_6 \equiv U(6)\otimes U(6)$. The generators in (5) induce on R and L the infinitesimal transformations

$$R\rightarrow(1+i\alpha_\mu{}^i\sigma^\mu\lambda_i)R,$$

$$L\rightarrow(1-i\beta_\mu{}^i\sigma^\mu\lambda_i)L,$$

where $\alpha_\mu{}^i$ and $\beta_\mu{}^i$ are real.

A parity-conserving, Lorentz-invariant four-Fermion interaction which is invariant under GL(6) is[7]

$$\mathcal{L}_I=g\{\tfrac{1}{2}\bar\psi\sigma_{\mu\nu}\lambda_i\psi\bar\psi\sigma^{\mu\nu}\lambda_i\psi+\bar\psi\lambda_i\psi\bar\psi\lambda_i\psi+\bar\psi\gamma_5\lambda_i\psi\bar\psi\gamma_5\lambda_i\psi\}$$

$$=2g\delta_{\mu\nu}\{R^+\sigma_\mu\lambda_iLR^+\sigma_\nu\lambda_iL$$

$$+L^+\sigma_\mu\lambda_iRL^+\sigma_\nu\lambda_iR\}. \tag{6}$$

The equal-time commutation relations satisfied by the field ψ are not in general invariant under GL(6), and therefore it cannot be an invariance of the underlying Hilbert space. We consider it rather to be a dynamical symmetry possessed by the interaction Lagrangian and physical matrix elements in some approximation.

Another interaction Lagrangian invariant under GL(6) is

$$\mathcal{L}_I'=g'\{\bar\psi\gamma_\mu\lambda_i\psi\bar\psi\gamma^\mu\lambda_i\psi-\bar\psi\gamma_\mu\gamma_5\lambda_i\psi\bar\psi\gamma^\mu\gamma_5\lambda_i\psi\}. \tag{7}$$

There is a U(6) subgroup of GL(6) obtained by taking the $a_{i\mu}$ real. This is the group considered in reference 1. Both interaction Lagrangians (6) and (7) are invariant under this subgroup, which leaves the canonical commutation relations invariant.

The complete Lagrangian is the sum of kinetic energy, mass, and interaction terms:

$$\mathcal{L}=\mathcal{L}_K+\mathcal{L}_M+\mathcal{L}_I,$$

$$\mathcal{L}_K=i\bar\psi\gamma\cdot\partial\psi,$$

$$\mathcal{L}_M=-m\bar\psi\psi.$$

Since the four-momentum has four components, while GL(6) [or, for that matter, any of its SU(6) subgroups] has no four-dimensional representation, it is clear that there is no possibility of making $\mathcal{L}_K$ GL(6)-invariant (a similar argument applies to $\mathcal{L}_M$). Thus the only term of $\mathcal{L}$ that is GL(6) invariant is $\mathcal{L}_I$. If in some strong-coupling sense $\mathcal{L}_I \gg \mathcal{L}_K+\mathcal{L}_M$, then it is fair to claim that $\mathcal{L}$ exhibits <u>intrinsically broken</u> GL(6) <u>invariance</u>. Before we analyze in more detail this intrinsic symmetry breaking, let us give the representations to which the quarks and their bilinear covariants belong. The quarks form the representation[8] (1, 6) ⊗(6, 1) of W_6, and the bilinear covariants $\bar\psi\gamma_\mu(1+\gamma_5)\lambda_i\psi$ and $\bar\psi\gamma_\mu(1-\gamma_5)\lambda_i\psi$ form, respectively, the representations (35, 1)⊕(1, 1) and (1, 35)⊕(1, 1). Finally, $\bar\psi\sigma_{\mu\nu}\lambda_i\psi\oplus\bar\psi\lambda_i\psi\oplus\bar\psi\gamma_5\lambda_i\psi$ together form (6*, 6)⊕(6, 6*). The particle contents of all these representations are given in Table I. It is important to remember that irreducible representations of W_6 do not, in general, have definite Lorentz-transformation properties. In order to build up objects with well-defined transformation properties under the Lorentz group, one must consider reducible representations of W_6. There is some latitude in choosing the representations

Table I. Spin–parity–unitary-spin content of representations of W_6.

Representation of W_6	Spin, parity, unitary-spin dimensionality (J^P, N_3)
(35, 1)⊕(1, 35)⊕(1, 1)⊕(1, 1)	$(1^+,8)\oplus(1^-,8)\oplus(1^+,1)\oplus(1^-,1)\oplus(0^+,8)\oplus(0^-,8)\oplus(0^+,1)\oplus(0^-,1)$
(6*, 6)⊕(6, 6*)	$(1^+,8)\oplus(1^-,8)\oplus(1^+,1)\oplus(1^-,1)\oplus(0^+,8)\oplus(0^-,8)\oplus(0^+,1)\oplus(0^-,1)$
(6, 1)⊕(1, 6)	$(\tfrac{1}{2}^+,3)$
(56, 1)⊕(1, 56)	$(\tfrac{1}{2}^+,8)\oplus(\tfrac{3}{2}^+,10)$

in which one places the baryons. One obvious candidate is (56, 1)⊕(1, 56).

We now return to the intrinsic symmetry breaking, coming from the kinetic energy and mass terms in $\mathcal{L}$. Their transformation properties (as members of incomplete W_6 multiplets) are easily found to be (35, 1) ⊕(1, 35) and (6*, 6)⊕(6, 6*). We are treating the kinetic energy and mass terms as perturbations on an otherwise symmetric Lagrangian. Such a procedure is nonconventional (i.e., not describable in terms of the usual Feynman diagrams), but if this feature is ignored, one can proceed formally with group-theoretical arguments. It is simplest to think of the symmetry-breaking terms in the language of spurions, with the spurions possessing the requisite W_6 transformation properties. Spurions can only contribute in pairs to self-mass terms, with the pairs necessarily possessing the quantum numbers of the vacuum. These spurion pairs are of particular interest when classified according to the U(6) subgroup of W_6 mentioned above ($a_{i\mu}$ real). For the breakdown of this U(6) symmetry our spurion-pair mechanism implies that to lowest order the symmetry-breaking terms in the mass formulas transform like members of the 35-, 189-, and 405-dimensional representation, as assumed by Bég and Singh.[9]

Because of this subgroup most of the nonrelativistic results[10] based on U(6) can be obtained from W_6. However, W_6 predicts a super-supermultiplet structure on top of U(6), most characteristically the axial-vector and scalar mesons listed in Table I. Correspondingly, more general mass formulas can be derived on the basis of W_6 symmetry.

It is of interest to find the "would-be-conserved" currents of our model, and to calculate their (nonvanishing) divergences.[11] For example, corresponding to the parameter a_{i2} there is a current

$$j_\mu{}^{i2}=\frac{\delta\mathcal{L}}{\delta a_{i2,\mu}}=\frac{i}{2}\bar\psi[\gamma_\mu,\sigma_{31}]_+\left(\frac{1+\gamma_5}{2}\right)\lambda^i\psi,$$

and its divergence

$$\partial_\mu j_\mu{}^{i2}=\frac{\delta\mathcal{L}}{\delta a_{i2}}=\bar\psi(\gamma_3\partial_1-\gamma_1\partial_3)\left(\frac{1+\gamma_5}{2}\right)\lambda^i\psi$$

$$+m\bar\psi\sigma_{31}\gamma_5\lambda^i\psi.$$

We then ascribe in the sense of a Goldberger-Treiman argument particles to these nonvanishing divergences of currents. In this way we may introduce a nonet each of vector, axial-vector, and pseudoscalar (the last arising from the noninvariant mass term) mesons, that form together an incomplete W_6 multiplet. The coupling of these particles to other physical states may be viewed as an effect of W_6 breakdown.

In conclusion we wish to point out that W_6 seems to be the natural group of hadrons.[12] One crucial test of its approximate validity would be the experimental discovery of the 1^+ and 0^+ mesons it predicts.

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†On leave of absence from the Enrico Fermi Institute for Nuclear Studies and the Department of Physics of the University of Chicago, Chicago, Illinois.

‡Alfred P. Sloan Foundation Fellow on leave from the Department of Physics, The University of Pennsylvania, Philadelphia, Pennsylvania.

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[4]$\psi(x)$ is a 12-component spinor.

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[6]The finite dimensional representations of GL(6) so generated are not unitary. However, under the decomposition GL(6)→$\mathcal{L}'$⊗SU(3), where $\mathcal{L}'$ is a group of Dirac γ matrices isomorphic with the Lorentz group, the representations decompose into sum of direct products made up of nonunitary finite-dimensional representations of $\mathcal{L}'$ and unitary representations of SU(3).

[7]We use four-Fermion interactions only as a guide in our search for symmetries.

[8]We label the representations of W_6 by the dimensionalities of the representations of its two commuting U(6) factors.

[9]M. A. B. Bég and V. Singh, Phys. Rev. Letters

13, 418 (1964). Note that the kinetic-energy term will not break the SU(3) symmetry; however, one can use the mass term to accomplish this by, e.g., making the third quark of different mass.

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[11]See, e.g., M. Gell-Mann and M. Lévy, Nuovo Cimento 16, 705 (1960).

[12]There is a larger contender for the title of symmetry group of hadrons, a 144-parameter group, which we do not discuss here.

About the strong-coupling limit on which our above discussions are based, we would still like to make the following remark: One possible way of handling this problem is to study the behavior of vertex functions in the limit $g\to\infty$. We have done such a study in the context of chain diagrams and found that for $g\to\infty$ the vertex functions exhibit W_6 symmetry.

PROBLEM OF COMBINING INTERACTION SYMMETRIES AND RELATIVISTIC INVARIANCE

W. D. McGlinn
Argonne National Laboratory, Argonne, Illinois
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Recently there has been discussion[1,2] concerning the possibility of combining interaction symmetries (for instance SU(3) for strong interactions) and relativistic invariance in a nontrivial way. One of the motivations is the possibility of obtaining exact mass formulas[3] for particles belonging to the same representations of the interaction group. In this paper the impossibility of such combinations, under a certain restrictive condition, is pointed out.

Consider an interaction symmetry defined by a semisimple Lie group I. When combined with Lorentz invariance, the usual assumption is that the group T that describes the full symmetry is $T=I\times L$, where L is the inhomogeneous Lorentz group. This, of course, leads to the conclusion that particles belonging to the same irreducible representation of I have the same mass since the irreducible representations of T are products of those of I and L.

It is clear that if particles belonging to the same representation of I are to have different masses, then the generators of I (denoted by I_i) do not in general commute with the translation operators, that is, in general

$$[I_i, p_\mu] \neq 0. \qquad (1)$$

The problem then is to find a Lie group T that has as generators those of L and I and for which inequality (1), in general, holds. A further restriction to be imposed on T is that the generators of I commute with the generators of the homogeneous Lorentz group. This of course implies that if one applies both a homogeneous Lorentz transformation and an interaction-symmetry transformation on a state, the transformed state is independent of the order in which the two transformations are applied. It further implies that the quantum numbers associated with the interaction symmetry do not change when one performs a homogeneous Lorentz transformation. With this restriction it is shown that indeed $[I_i, p_\mu]=0$ and thus $T=I\times L$.

The generators of the Lorentz group satisfy the familiar commutation relations

$$[M_{\mu\nu}, M_{\lambda\sigma}] = i[g_{\mu\sigma}M_{\lambda\nu} + g_{\mu\lambda}M_{\nu\sigma} + g_{\nu\sigma}M_{\mu\lambda} + g_{\lambda\nu}M_{\sigma\mu}], \qquad (2)$$

$$[M_{\mu\nu}, p_\lambda] = i[p_\nu g_{\mu\lambda} - p_\mu g_{\nu\lambda}], \qquad (3)$$

$$[p_\mu, p_\nu] = 0. \qquad (4)$$

We will denote the generators of the group I by J_i, $i=1,\cdots,n$, where n is the dimension of the group, and the generators of the Lorentz group by J_i, $n+1 \le i \le n+10$. In particular,

$$p_i = J_{n+i} \text{ for } 1 \le i \le 4,$$

$$M_{12} = J_{n+5},\; M_{13} = J_{n+6},\; M_{14} = J_{n+7},$$

$$M_{23} = J_{n+8},\; M_{24} = J_{n+9},\; M_{34} = J_{n+10}.$$

The full symmetry group T is determined by the commutation relations

$$[J_j, J_k] = C_{jk}^{\;i} J_i, \qquad (5)$$

where the structure constants must satisfy

$$C_{jk}^{\;i} = -C_{kj}^{\;i}, \qquad (6)$$

$$C_{is}^{\;p} C_{jk}^{\;s} + C_{ks}^{\;p} C_{ij}^{\;s} + C_{js}^{\;p} C_{ki}^{\;s} = 0. \qquad (7)$$

The restriction imposed immediately after Eq. (1) implies

$$C_{jk}^{\;i} = 0 \text{ for } j \le n \text{ and } k > n+4. \qquad (8)$$

Consider Eq. (7) for $i \le n$, $j > n+4$, $n+1 \le k \le n+4$, $p \le n$. The restriction on T implies $C_{js}^{\;p} = C_{ij}^{\;s} = 0$ and thus Eq. (7) reduces to

$$C_{is}^{\;p} C_{jk}^{\;s} = 0.$$

From Eq. (3) there is at most one value of s in this sum for a given j and k for which $C_{jk}^{\;s} \neq 0$ and for a given value of s, $n+1 \le s \le n+4$, a pair of values j and k exist for which $C_{jk}^{\;s} \neq 0$. Thus

$$C_{is}^{\;p} = 0 \text{ for } i \le n,\; n+1 \le s \le n+4,\; p \le n. \qquad (10)$$

Now consider Eq. (7) for $n+1 \le p \le n+4$, $i \le n$, $j > n+4$, and $n+1 \le k \le n+4$. For this case $C_{ij}^{\;s} = 0$ and Eq. (7) reduces to

$$C_{is}^{\;p} C_{jk}^{\;s} + C_{js}^{\;p} C_{ki}^{\;s} = 0. \qquad (11)$$

In Eq. (11) both s indexes need only range between $n+1$ and $n+4$. Consider the particular case $j=n+5$ and $k=n+1$. With the aid of Eq. (3), Eq. (1) can be reduced to

$$C_{i,n+2}^{\;p} C_{jk}^{\;n+2} + C_{j,n+1}^{\;p} C_{k,i}^{\;n+1} + C_{j,n+2}^{\;p} C_{ki}^{\;n+2} = 0,$$

where

$$C_{j,n+1}^{\;p} = 0 \text{ unless } p = n+2,$$

$$C_{j,n+2}^{\;p} = 0 \text{ unless } p = n+1.$$

This implies $C_{i,n+2}^{\;p} = 0$ for all i unless $p=n+1$ or $p=n+2$. Letting $p=n+2$, one obtains

$$C_{ik}^{\;k} = C_{i,n+2}^{\;n+2}.$$

If one considers Eq. (7) for the case $j=n+8$ and $k=n+3$ one concludes $C_{i,n+2}^{\;n+1}=0$. Continuing in a similar way we find

$$C_{ik}^{\;j} = C_i \delta_{jk},\quad i \le n,\; n+1 \le j,\; k \le n+4. \qquad (12)$$

Now consider Eq. (7) with $n+1 \le p$, $k \le n+4$, and i, $j \le n$. From the previous discussion for this case we can write

$$C_{jk}^{\;s} = \delta_{sk} C_j,\quad C_{js}^{\;p} = \delta_{sp} C_j,$$

$$C_{ks}^{\;p} = -\delta_{pk} C_s,\quad C_{is}^{\;p} = \delta_{ps} C_i,$$

$$C_{ki}^{\;s} = -\delta_{sk} C_i.$$

Inserting these relations in Eq. (7) leads to the conclusion that

$$C_s C_{ij}^{\;s} = 0 \text{ for } i, j \le n. \qquad (13)$$

If one introduces a standard coordinate system[4] of the group I, it is easy to see that Eq. (13) implies $C_s \equiv 0$. Thus

$$C_{ij}^{\;k} = 0 \text{ for } i \le n,\; n+1 \le j,\; k \le n+4. \qquad (14)$$

Now consider Eq. (7) with i, $j < n$, $n+1 \le k \le n+4$, $p > n+4$. From the original assumption and the previous discussion, it follows that $C_{js}^{\;p} = C_{is}^{\;p} = 0$ for this case. Thus Eq. (7) reduces to

$$C_{ks}^{\;p} C_{ij}^{\;s} = 0 \qquad (15)$$

Since this must be true for all i, $j \le n$, using again the standard coordinate system of the group I, one sees that

$$C_{ks}^{\;p} \equiv 0 \text{ for } p > n+4,\; n+1 \le k \le n+4,\; s \le n. \qquad (16)$$

Combining Eqs. (8), (10), (13), and (14) leads to

$$C_{kj}^{\;i} = 0 \text{ for } k \le n,\; j \ge n+1, \text{ all } i: \qquad (17$$

that is, $T=I\times L$. Thus if one demands that the i teraction symmetry commute with the homogeneous Lorentz transformation and requires the existence of a Lie group T whose generators are those of the interaction-symmetry group and the Lorentz group, then it follows that $T=I\times L$. Thi applies in particular to the group SU(3).

In conclusion, if one wishes to combine such a interaction symmetry with Lorentz invariance to form a larger group that will give mass splitting one must accept not only lack of commutation of the symmetry-interaction generators with the Lorentz translation generators but also their lack of commutation with the homogeneous Loren generators. It is felt that this will, in general, lead to interpretation difficulties.

It should be noted that Eqs. (13) and (15) are true even if the interaction-symmetry group is not semisimple; for some such groups one can still deduce Eqs. (14), (16), and (18).

The author wishes to thank Dr. M. Hamermesh for valuable discussions.

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ON THE ALGEBRA OF SU_6

Abdus Salam

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ON THE ALGEBRA OF SU_6

by
Abdus Salam
International Centre for Theoretical Physics, Trieste

The remarkable success of SU_6 ideas[1] in elementary particle physics makes it imperative to look for its relativistic basis. Consider the free Dirac Lagrangian $\mathcal{L} = \bar{\psi}(\not{p} - m)\psi$ for a single particle. $\mathcal{L}$ is invariant for the Pauli-Lubanski transformation

$$\psi' = (1 + i E_\mu \omega_\mu)\psi \tag{1}$$

where

$$\omega_\mu = \frac{1}{4} \epsilon_{\mu\nu\rho\kappa} \sigma_{\nu\rho} p_\kappa$$

Since $p_\mu \omega_\mu \equiv 0$, there are three independent generators with the [2] commutation relation

$$[\omega_\mu, \omega_\nu] = i \epsilon_{\mu\nu\rho\kappa} p_\rho \omega_\kappa \tag{2}$$

The generators give rise to an SU_2-like (in general non-compact) structure which satisfies for the spin 1/2 case the anti-commutation relation:

$$\{\omega_\mu, \omega_\nu\} = -\frac{1}{4}\left(\gamma_5[\gamma_\mu, \not{p}], \gamma_5[\gamma_\nu, \not{p}]\right) \tag{3}$$

$$= 2(p_\mu p_\nu - p^2 g_{\mu\nu})$$

Consider now the case when ψ is a three-component Sakata-like entity (representing quarks). It is possible to extend (1) to the general (SU_6) transformation:

$$\psi' = (1 + i E^i T^i + i E_\mu^\alpha T^\alpha \omega_\mu)\psi \tag{4}$$

Here $T^\alpha \begin{pmatrix} \alpha = 0, \ldots, 8 \\ i = 1, \ldots, 8 \end{pmatrix}$ are the usual U_3 generators with $T^0 = 1$ and from (2),

$$[T^\alpha \omega_\mu, T^\beta \omega_\nu] = \frac{1}{2}\{\omega_\mu, \omega_\nu\}[T^\alpha, T^\beta] + \frac{1}{2}[\omega_\mu, \omega_\nu]\{T^\alpha, T^\beta\}$$

$$= i(p_\mu p_\nu - p^2 g_{\mu\nu}) c_{ijk} T^k + \frac{i}{2} E_{\mu\nu\rho\kappa} p_\rho \omega_\kappa \left(\frac{1}{3}\delta_{ij} T^0 + d_{ijk} T^k\right) \qquad (5)$$

$$[T^i \omega_\mu, T^j] = \frac{i}{2} \omega_\mu c_{ijk} T^k$$

The adjoint representation-densities are given by $\bar{\psi} \gamma_\mu \omega_\nu T^\alpha \psi$ and $\bar{\psi} \gamma_\mu T^i \psi$ which satisfy as usual,

$$\bar{\psi} \not{p} \omega_\nu T^\alpha \psi = \bar{\psi} \not{p} T^i \psi = 0 \qquad (6)$$

One may now generalize the case of SU_6 above to the more general case [3] $(SU_6)_L \times (SU_6)_R$; i.e., start with the fields $\psi_{L,R} = \frac{1}{2}(1 \pm \gamma_5)\psi$ Clearly $m\bar{\psi}\psi$ term is not invariant for the full group (though the invariance is unaffected for the pure ω_μ transformations). There are altogether now 70 generators $\bar{\psi}_{L,R} \gamma_\mu \omega_\nu T^\alpha \psi_{L,R}$, $\bar{\psi}_{L,R} \gamma_\mu T^i \psi_{L,R}$. The conservation equations (6) however need modifying; thus:

$$\bar{\psi} \not{p} \omega_\nu \gamma_5 T^\alpha \psi \neq 0 = (2m \bar{\psi} \omega_\nu \gamma_5 T^\alpha \psi)$$

$$\bar{\psi} \not{p} \gamma_5 T^i \psi \neq 0 = (2m \bar{\psi} \gamma_5 T^i \psi).$$

From this point of view the 0^-, 1^- 35-fold (represented by the field operators $\bar{\psi} \omega_\nu \gamma_5 T^\alpha \psi$ and $\bar{\psi} \gamma_5 T^i \psi$) is a remnant of the broken $(SU_6)_L \times (SU_6)_R$ symmetry.

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$$\int d^3x \; \bar{\psi} \gamma_4 \omega_\nu \psi.$$

In checking the C.R. (2), (5) and (6) care is needed in writing anti-commutators like $\{\bar{\psi}(x), \not{p}\psi(y)\}\delta(x_0 - y_0)$.

[3] This is analogous to study of the extended Algebras $(SU_3)_L \times (SU_3)_R$ by A. Salam and J. C. Ward (Il Nuovo Cimento, 19, 167 (1961)), M. Gell-Mann (Phys. Rev. 125, 1067 (1962)) and Y. Nambu and P. Freund (Phys. Rev. Letters, 12, 714 (1964)).