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Group theory and spectroscopy

by

Giulio Racah

The Hebrew University, Jerusalem

Reprint of lectures delivered at the
Institute for Advanced Study, Princeton, in Spring 1951

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GENEVE

INTRODUCTORY NOTE

For many years now these lectures have been out of print. They have been repeatedly requested in the CERN Library but they were unobtainable. By a fortunate chance a copy reached us recently as part of the Pauli Memorial Collection at CERN.

Realising that these lectures would be greatly appreciated by physicists, we requested Professor Racah's permission to re-issue them in a photo-offset edition in our Report Series. He very kindly agreed and supplied us with a list of corrections. We wish to thank him in the name of all those who will welcome this reprint.

L. van Hove.

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These notes are based on a series of seminar lectures given during the 1951 Spring term at the Institute for Advanced Study. Owing to limitations of time only particular topics were considered, and there is no claim to completeness. Since it is intended to publish later a more complete treatment of the subject, comments about these notes as well as suggestions concerning the desirability of adding related topics will be appreciated and should be addressed to the author at the Hebrew University, Jerusalem, Israel.

As the limited supply of these notes makes it impossible to send more than one copy to any institution, it is requested that all copies be made available to departmental libraries.

The Institute for Advanced Study

GROUP THEORY AND SPECTROSCOPY

by

Giulio Racah
The Hebrew University, Jerusalem

Spring, 1951

Notes by Eugen Merzbacher and David Park

Group Theory and Spectroscopy

by

Giulio Racah

These lectures will treat the applications of group theory to problems of spectroscopy and nuclear structure. While developing the mathematical tools for this purpose, we shall occasionally forego the elaboration of a rigorous proof. In such cases, references will be quoted.

Lecture 1.

GENERAL NOTIONS ON CONTINUOUS GROUPS

§1. Continuous Groups and Infinitesimal Groups.

We start with a set of n variables x_0^i $(i = 1 \ldots n)$, which may be regarded as coordinates of a point in a certain space. Consider now the set of equations

$$x^i = f^i(x_0^1, \ldots, x_0^n; a^1, \ldots, a^r) \qquad (i = 1 \ldots n), \qquad (1)$$

in which the a^ρ appear as a set of r independent parameters. Omitting indices, we shall write this and similar relations in the form

$$x = f(x_0; a) \qquad \text{or} \qquad x = S_a x_0. \qquad (1')$$

These equations define a set S of transformations, depending on the parameters a, which map the ~~vector~~ point x_0 onto x. We shall assume that the f^i have all the required derivatives, and that the f^i depend essentially on the parameters, i.e. that no two transformations with different parameters are the same for all values of x_0, so that r is the smallest number of parameters needed to specify the transformations completely and uniquely.

The set of transformations f is said to form a group if it obeys the following two conditions:

i) The result of performing successively any two transformations of the set is another transformation belonging to this set. Formally, if $x = f(x_0;a)$ and $x' = f(x;b)$ then there exists a set of parameters c^ρ ~~such that~~

$$c^\rho = \varphi^\rho(a;b) \qquad (2)$$

such that

$$x' = f(x;b) = f(f(x_0,a);b) = f(x_0;c) = f(x_0;\varphi(a;b)). \qquad (3)$$

ii) Corresponding to every transformation there exists a unique inverse, which also belongs to the set: Given equation (1) there exists a set of ~~parameter~~ parameters $\bar{a}$ such that $x_0 = f(x;\bar{a})$.

The uniqueness of $\bar{a}$ is guaranteed if the Jacobian of the transformation does not vanish:

$$\left| \frac{\partial f}{\partial x_0} \right| \neq 0 \qquad (4)$$

Transforming x_0 onto x and then inversely x back to x_0, we obtain according to i) a transformation which belongs to the group and is characterized by the set of parameters a_0. Since the transformation depends on the parameters in an essential way, the a_0 so constructed cannot depend on the particular value of the parameters from which we started. The transformation $f(x,a_0)$ is called the identity.

Since it imposes no restriction, we shall take:

$$a_0^\rho = 0 \qquad (\rho = 1\ldots r).$$

r is called the order of the group. (Note that this usage is different from that found in the theory of finite groups.)

We also remind the reader of the following definitions:

A mapping of one group onto another is said to be homomorphic or a homomorphism if it preserves the operation of group multiplication. We call such a mapping an isomorphism if, in addition, the correspondence between elements of the two groups is one-to-one. Since the combination law of the transformations (1) is given in terms of the parameters a, there can be transformations corresponding to different values of n which are homomorphic or even isomorphic.

A group of linear transformations which is homomorphic with a given group is called a representation of this group.

The fundamental idea of Sophus Lie's theory of continuous groups is to consider not the whole of a group, but that part of it which lies near the identity, consisting of the so-called infinitesimal transformations. Thus, instead of the finite displacement of a point under a transformation, we consider the application of successive infinitesimal displacements - we think of a generalized velocity field describing the motion of a point from its original position x_0 to its final position x.

We have now two equivalent expressions for x:

$$a) \quad x = f(x_0;a) \qquad \text{or b)} \quad x = f(x;0). \tag{5}$$

Corresponding to these we can represent in either of two ways a transformation, as a result of which the new components of x differ infinitesimally from the old ones - by differentiation of (5a) or by introducing a parameter of infinitesimal size in (5b):

$$x + dx = f(x_0; a + da) \qquad \text{or} \qquad x + dx = f(x, \delta a)$$

or (employing the summation convention)

$$dx = \frac{\partial f(x_0, a)}{\partial a^\sigma}\, da^\sigma \qquad \text{or} \qquad dx = \left(\frac{\partial f(x,a)}{\partial a^\sigma}\right)_{a=0} \delta a^\sigma . \tag{6}$$

The last may be written

$$dx^i = u^i_\sigma(x)\, \delta a^\sigma , \qquad u^i_\sigma(x) = \left(\frac{\partial f^i(x,a)}{\partial a^\sigma}\right)_{a=0} \tag{7}$$

which defines the "velocity field" $u^i_\sigma(x)$ mentioned above. In the notation of (2) we may write

$$a + da = \varphi(a; \delta a).$$

Since it follows from (2) and (5) that $\varphi(a;0) = a$, we have

$$a + da = a + \left(\frac{\partial \varphi(a;b)}{\partial b^\tau}\right)_{b=0} \delta a^\tau .$$

Thus, da is a linear combination of δa:

$$da^\rho = \mu^\rho_\tau(a)\, \delta a^\tau , \qquad \mu^\rho_\tau(a) = \left(\frac{\partial \varphi^\rho(a,b)}{\partial b^\tau}\right)_{b=0} . \tag{8}$$

Solving for δa, we get

$$\delta a^\sigma = \lambda^{\cdot\sigma}_{\rho}(a)\, da^\rho \tag{8'}$$

where

$$\lambda \mu = 1 , \quad \text{i.e.,} \quad \lambda^\sigma_\rho \mu^\rho_\tau = \delta^\sigma_\tau . \tag{9}$$

From (6), (7) and (8') we get the first fundamental formula:

$$\frac{\partial x^i}{\partial a^\rho} = u^i_\tau(x) \; \lambda^\tau_\rho(a). \qquad (A)$$

If u is to represent the velocity field of a transformation (1), equation (A) must be completely integrable, i.e. it must be capable of admitting solutions with n arbitrary constants x_0. The integrability condition $\frac{\partial^2 x^i}{\partial a^\sigma \partial a^\rho} = \frac{\partial^2 x^i}{\partial a^\rho \partial a^\sigma}$ becomes

$$\left(u^j_\chi \frac{\partial u^i_\nu}{\partial x^j} - u^j_\nu \frac{\partial u^i_\chi}{\partial x^j} \right) \lambda^\kappa_\rho \lambda^\nu_\sigma + u^i_\nu \left(\frac{\partial \lambda^\nu_\sigma}{\partial a^\rho} - \frac{\partial \lambda^\nu_\rho}{\partial a^\sigma} \right) = 0 ,$$

and, using (9), this gives

$$u^j_\chi \frac{\partial u^i_\nu}{\partial x^j} - u^j_\nu \frac{\partial u^i_\chi}{\partial x^j} = c_{\chi\nu}^{\;\;\tau}(a) \; u^i_\tau \qquad (10)$$

where

$$c_{\chi\nu}^{\;\;\tau}(a) = \left(\frac{\partial \lambda^\tau_\rho}{\partial a^\sigma} - \frac{\partial \lambda^\tau_\sigma}{\partial a^\rho} \right) \mu^\rho_\chi \, \mu^\sigma_\nu . \qquad (11)$$

Since u is independent of a, differentiation of (10) by a^ρ gives

$$\frac{\partial c_{\chi\nu}^{\;\;\tau}(a)}{\partial a^\rho} \; u^i_\tau = C .$$

But the a's have been assumed essential, so that by (7) the u's are linearly independent, hence the c's are independent of a. Equation (10) is

$$u^j_\chi \frac{\partial u^i_\nu}{\partial x^j} - u^j_\nu \frac{\partial u^i_\chi}{\partial x^j} = c_{\chi\nu}^{\;\;\tau} \; u^i_\tau \qquad (P_1)$$

and from (11)

$$\frac{\partial \lambda_\rho^\tau}{\partial a^\sigma} - \frac{\partial \lambda_\sigma^\tau}{\partial a^\rho} = c_{\chi\nu}{}^\tau \lambda_\rho^\chi \lambda_\sigma^\nu . \qquad (B_2)$$

(B_1) is a necessary condition on the velocity field if the latter is to generate a group, and (B_2) is a corresponding restriction on the manner in which the a's combine.

An infinitesimal transformation on the x induces on any function F(x) a variation

$$dF(x) = \frac{\partial F}{\partial x^i} dx^i = \delta a^\sigma u_\sigma^i \frac{\partial F}{\partial x^i} \equiv \delta a^\sigma X_\sigma F \qquad (12)$$

where

$$X_\sigma = u_\sigma^i (x) \frac{\partial}{\partial x^i} \qquad (13)$$

(12) shows that every infinitesimal transformation of F(x) is generated by a linear combination of the operators X which are called the infinitesimal operators of the group S. From (B_1) it follows that they satisfy the relation

$$X_\rho X_\sigma - X_\sigma X_\rho \equiv [X_\rho X_\sigma] = c_{\rho\sigma}{}^\tau X_\tau . \qquad (14)$$

Evidently

$$c_{\rho\sigma}{}^\tau = - c_{\sigma\rho}{}^\tau . \qquad (C_1)$$

Substituting (14) into the Jacobi identity

$$[[X_\rho X_\sigma] X_\tau] + [[X_\sigma X_\tau] X_\rho] + [[X_\tau X_\rho] X_\sigma] = 0$$

we get

$$c_{\rho\sigma}^{\ \mu} c_{\mu\tau}^{\ \nu} + c_{\sigma\tau}^{\ \mu} c_{\mu\rho}^{\ \nu} + c_{\tau\rho}^{\ \mu} c_{\mu\sigma}^{\ \nu} = 0 . \qquad (C_2)$$

We have shown that equations (C) are implied if the f^i form a group. That the converse to this statement holds is the content of the three fundamental theorems of Lie, which we shall not prove. They state that

I. If there exist $f^i = x^i$ satisfying (A) then they form a group.

II. If there exist u's satisfying (B_1), then there exist λ's, determined within isomorphism, which satisfy (B_2), so that equation (A) is integrable.

III. For every set of c's satisfying (C), there exist u's satisfying (B_1).

We shall write an infinitesimal transformation of the group S in the form $S_a = 1 + \delta a^\sigma X_\sigma$, where δa^σ is an infinitesimal quantity defined to be of the first order. If we combine two such transformations, we get

$$S_a S_b = (1 + \delta a^\rho X_\rho)(1 + \delta b^\sigma X_\sigma) = 1 + \delta a^\rho X_\rho + \delta b^\sigma X_\sigma ,$$

where the first non-vanishing infinitesimal terms have been retained. Thus, to the operation of multiplication in S corresponds addition in the infinitesimal group of S. If the first order quantities vanish, we have to consider quantities of higher order. But the second theorem of Lie implies that in this connection we need never go beyond the second order of infinitesimals - i.e. we have only to worry about <u>commutators</u>, which are expressions of the form $S_a S_b S_a^{-1} S_b^{-1}$, and to ask that the corresponding infinitesimal operator of the second order, $\delta a^\rho \delta b^\sigma [X_\rho X_\sigma]$ be contained in the linear manifold of infinitesimal operators.

§2. Parameter Groups and Adjoint Groups.

On comparing (8) with (7), we see that a close formal analogy exists between the functions μ and u. In fact, the $\varphi(a;b)$ of (2) which connect the parameters according to the composition law of the group may themselves be considered as defining a group in the same way as does (1'):

$$a'^{\rho} = \varphi^{\rho}(a;b)$$

This relation can be regarded as a mapping of the a onto the a' according to a transformation whose parameter is b. We shall prove that these transformations form a group P_1, which is isomorphic with S and is called the first parameter group. Indeed, if $a' = \varphi(a;b)$ and $a'' = \varphi(a';c)$ then

$$a'' = \varphi(\varphi(a;b);c) = \varphi(a;\varphi(b;c))$$

where the last equality follows from the associative property of the transformations f^i. We thus see that the law of composition is the same for the first parameter group and the original group of transformations f^i.

The analogous group of transformations on the argument b of $\varphi(a;b)$ is called the second parameter group P_2. P_2 is anti-isomorphic with S; that is, it is isomorphic when the factors are taken in the reverse order. But since $(xy)^{-1} = y^{-1}x^{-1}$ and since a group contains x^{-1} if it contains x, the two are in fact isomorphic. Let a (or c) be a transformation belonging to the first (or second) parameter group. Let the operation of P_1 transform b into $b' = \varphi(b,a)$ and let the operation of P_2 transform b' into $b'' = \varphi(c,b')$. Then it is clear that

$$b'' = \varphi(c,b') = \varphi(c,\varphi(b,a)) = \varphi(\varphi(c,b),a),$$

hence every element of P_1 commutes with every element of P_2.

The μ's are the velocity field of P_1 (see equation (8)), and they define the infinitesimal operator

$$A_\tau = \mu^\rho_\tau(a) \frac{\partial}{\partial a^\rho} \quad . \tag{15}$$

Correspondingly, in P_2

$$B_\tau = \bar{\mu}^\rho_\tau(b) \frac{\partial}{\partial b^\rho} \tag{16}$$

Another operation which it is useful to consider is conjugation: Given an element S_a of S, to every element S_b of the group there corresponds an element $S_{b'} = S_a S_b S_a^{-1}$. The operation $b \longrightarrow b'$ is a faithful mapping of the group onto itself which depends on S_a and which is called conjugation of S by S_a. Consider now the set of conjugations obtained by letting S_a run through all the elements of S. These conjugations themselves constitute a group of transformations, **homomorphic to S,** , but not in general isomorphic with S. It is easily seen that isomorphism between S and the group of conjugations holds if and only if the identity is the only element of S which commutes with all elements of S.

If we regard the relation $x' = S_a x$ as a coordinate transformation, it is well known that the effect of operating with S_b on a function of x' is given in terms of x' by the operation of the conjugate of S_b by S_a acting on the same function of x:

$$S_{b'} x' = S_a S_b x = (S_b x)'$$

The conjugation gives the change in the parameters of an operation if this operation is considered in the new system of coordinates x'. The advantage of the group of conjugations over the parameter group is that the conjugate element $S_a S_e S_a^{-1}$ is infinitesimal if S_e is infinitesimal, irrespective of the magnitude of S_a.

If in the first system of coordinates S_e is expressed by $S_e = 1 + \varepsilon e^\rho X_\rho$, then, after the transformation with S_a the same transformation S_e will be expressed by $1 + \varepsilon e'^\rho X'_\rho$, and hence

$$e'^\rho X'_\rho = e^\rho X_\rho \tag{17}$$

The group of transformations $e^\rho \longrightarrow e'^\rho$ is called the adjoint group. We wish to determine its infinitesimal operators, produced by transformations S_a in the neighborhood of the identity. With $S_a = 1 + \delta a^\sigma X_\sigma$ and $S_\rho = 1 + \varepsilon X_\rho$ we have

$$S'_\rho = 1 + \varepsilon X'_\rho = S_a S_\rho S_a^{-1} = (S_a S_\rho S_a^{-1} S_\rho^{-1}) S_\rho = (1 + [\delta a^\sigma X_\sigma, \varepsilon X_\rho])\cdot(1 + \varepsilon X_\rho)$$

or

$$X'_\rho - X_\rho \equiv dX_\rho = \delta a^\sigma [X_\sigma, X_\rho] = c_{\sigma\rho}{}^\tau \, \delta a^\sigma X_\tau \quad ,$$

by (14). From (17), we have

$$de^\tau X_\tau = -e^\rho \, dX_\rho = e^\rho \, c_{\rho\sigma}{}^\tau \delta a^\sigma X_\tau$$

or

$$de^\tau = e^\rho \, c_{\rho\sigma}{}^\tau \delta a^\sigma . \tag{18}$$

If E_σ are the infinitesimal operators of the adjoint group, we find by comparison of (18) with (7) and (13) that

$$E_\sigma = e^\rho \, c_{\rho\sigma}{}^\tau \frac{\partial}{\partial e^\tau} . \tag{19}$$

§3. Subgroups, simple and semi-simple groups.

a) A group is Abelian if all its elements commute. It follows from the correspondence between commutators and square brackets that for an Abelian group all square brackets, and consequently all structure constants, vanish:

$$c_{\rho\sigma}^{\tau} = 0 \tag{20}$$

b) A subgroup of a group S is a subset of elements of S which satisfies the group postulates. Thus, if $X_1, X_2, \ldots, X_p$ are the infinitesimal operators of a subgroup, the structure constants of the group must satisfy the relations

$$c_{\rho\sigma}^{\tau} = 0 \quad (\rho, \sigma \leq p, \ \tau > p). \tag{21}$$

c) An invariant subgroup, H, of a group S is a subgroup of S which contains all the conjugates (images) of its elements. Thus, with S_n, it contains $S_x S_n S_x^{-1}$ for any S_x in S. If so, it also contains the commutator $S_x S_n S_x^{-1} S_n^{-1}$. Thus, the square bracket connecting an infinitesimal element of H with any infinitesimal element of S must belong to H. If $X_1, X_2, \ldots, X_p$ are the infinitesimal operators of an invariant subgroup of S, the structure constants of S must satisfy

$$c_{\rho\sigma}^{\tau} = 0 \quad (\rho \leq p, \ \tau > p) \tag{22}$$

d) A group is simple if it has no invariant subgroups besides the unit element.

e) A group is semi-simple if it has no Abelian invariant subgroups besides the unit element.

The distinction between groups which have Abelian invariant subgroups and those which do not have such subgroups is important, because Abelian subgroups, though apparently easiest to deal with, can actually be most troublesome from the point of view of representations, as the following example will show:

We consider the group of rectilinear motions in one dimension, in which the transformation $x' = x + a$ followed by $x'' = x' + b$ is equivalent to $x'' = x + a + b$. This group can be represented by square matrices of the second rank, in terms of which the composition law just given would read

$$\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix}$$

However, none of the matrices of this particular representation can be brought to diagonal form by a similarity transformation. This peculiar behavior is closely related to the Abelian property. Indeed, as we shall show later, semi-simple groups never exhibit it. Moreover, the physical applications in which we shall be interested will require the use only of semi-simple groups. We shall therefore from this point on restrict ourselves to the study of semi-simple groups. To this end, we must have a criterion for their identification.

Such a criterion can be formulated very simply in terms of a symmetrical tensor of the second rank which we construct from the $c_{\rho\sigma}^{\ \ \tau}$:

$$g_{\rho\sigma} = c_{\rho\lambda}^{\ \ \mu} c_{\sigma\mu}^{\ \ \lambda} . \qquad (23)$$

If the group is semi-simple, then necessarily,

$$\det |g_{\rho\tau}| \neq 0 \qquad (24)$$

For suppose it possesses an Abelian invariant subgroup, the indices of whose elements are denoted by $\bar{\rho}$, $\bar{\sigma}$, ... Then,

$$g_{\rho\bar{\sigma}} = c_{\rho\lambda}^{\ \ \mu} c_{\bar{\sigma}\mu}^{\ \ \lambda}$$

$$= c_{\rho\lambda}^{\ \ \mu} c_{\bar{\sigma}\mu}^{\ \ \bar{\lambda}} \qquad \text{by (22)}$$

$$= c_{\rho\bar{\lambda}}^{\ \ \bar{\mu}} c_{\bar{\sigma}\bar{\mu}}^{\ \ \bar{\lambda}} \qquad \text{by (22)}$$

$$= 0 \qquad \text{by (20)}$$

That the condition (24) is sufficient as well as necessary has been shown by Cartan.

We can use the tensor $g_{\mu\nu}$ to define a relation of orthogonality between contravariant vectors or to form new tensors by lowering of indices. As an example,

$$c_{\rho\sigma\lambda} = c_{\rho\sigma}^{\ \ \tau} g_{\tau\lambda} \qquad (25)$$

and this new tensor is totally antisymmetric, for by (23),

$$c_{\rho\sigma\lambda} = c_{\rho\sigma}^{\ \ \tau} c_{\tau\mu}^{\ \ \nu} c_{\lambda\nu}^{\ \ \mu}$$

$$= -c_{\sigma\mu}^{\ \ \tau} c_{\tau\rho}^{\ \ \nu} c_{\lambda\nu}^{\ \ \mu} - c_{\mu\rho}^{\ \ \tau} c_{\tau\sigma}^{\ \ \nu} c_{\lambda\nu}^{\ \ \mu} \qquad \text{by } (C_2)$$

$$= c_{\sigma\mu}^{\ \ \tau} c_{\rho\tau}^{\ \ \nu} c_{\lambda\nu}^{\ \ \mu} + c_{\mu\rho}^{\ \ \tau} c_{\tau\sigma}^{\ \ \nu} c_{\nu\lambda}^{\ \ \mu} \qquad \text{by } (C_1)$$

The last line has the desired property, since it is invariant under cyclic permutation of the indices and is, by construction, skew in ρ and σ .

If the group S is semi-simple, then Cartan's criterion (24) implies that we can form from $g_{\rho\sigma}$ the reciprocal tensor $g^{\rho\sigma}$ which can be used to raise indices and define orthogonality between covariant vectors.

As an example of the foregoing we consider the group of rigid motions in three dimensions, consisting of rotations and translations. The infinitesimal rotations are generated by operators L_j ($j = 1,2,3$) satisfying

$$[L_1 L_2] = i\, L_3 \quad \text{etc.} \tag{26}$$

and the infinitesimal displacements by operators $P_1 = L_4$, $P_2 = L_5$, $P_3 = L_6$ which commute among themselves but which satisfy

$$[L_1 L_5] = i\, L_6 \quad \text{etc.}$$

so that the only non-vanishing structure constants are

$$c_{12}^{\ \ 3} = c_{23}^{\ \ 1} = c_{15}^{\ \ 6} = c_{26}^{\ \ 4} = c_{34}^{\ \ 5} = c_{31}^{\ \ 2} = c_{61}^{\ \ 5} = c_{42}^{\ \ 6} = c_{53}^{\ \ 4} = i$$

plus a corresponding list given by (C_1). For the $g_{\rho\sigma}$ we find

$$g_{11} = g_{22} = g_{33} = 4\,, \quad g_{44} = g_{55} = g_{66} = 0, \quad g_{\rho\sigma} = 0 \quad (\rho \neq \sigma).$$

The determinant det $g_{\rho\sigma}$ vanishes, as required by Cartan's criterion, since the translations form an Abelian invariant subgroup.

If we consider only the group of three-dimensional rotations, defined by (26), we find that it is simple and that the metric tensor is

$$g_{\rho\sigma} = 2\delta_{\rho\sigma} \tag{27}$$

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Lecture 2.

CLASSIFICATION OF THE SEMI-SIMPLE GROUPS.

§1. The Standard Form of the Infinitesimal Group.

In order to obtain a standard coordinate system for the set of infinitesimal operators of a semi-simple group we consider an eigenvalue problem of the form

$$[A\ X] = \rho\ X \tag{28}$$

where A is a fixed arbitrary infinitesimal operator $A = a^\mu X_\mu$ while $X = x^\nu X_\nu$ is an eigenvector corresponding to the eigenvalue ρ. Using (14) we can write (28) explicitly as

$$a^\mu x^\nu c_{\mu\nu}^{\ \ \tau} X_\tau = \rho\, x^\tau X_\tau .$$

Since the infinitesimal operators are linearly independent it follows that

$$(a^\mu c_{\mu\nu}^{\ \ \tau} - \rho\, \delta_\nu^\tau) x^\nu = 0 . \tag{29}$$

From (29) we get the secular equation

$$\det(a^\mu c_{\mu\nu}^{\ \ \tau} - \rho\, \delta_\nu^\tau) = 0 \tag{30}$$

If there exist r linearly independent eigenvectors, they can be used as a basis for a coordinate system in the r-dimensional space. However, generally, r linearly independent eigenvectors may not exist if the secular equation has degenerate roots. Usually, in physical problems, conditions like hermiticity or symmetry of the matrix insure the existence of r linearly independent eigenvectors. But for semi-simple infinitesimal groups Cartan has shown that if A is chosen so that the secular equation (30) has the maximum number of different roots, then only $\rho = 0$ is degenerate; and that if ℓ be the multiplicity of this root, there are corresponding to this root ℓ linearly independent eigenvectors $H_1, \ldots, H_\ell$ which commute with each other. ℓ is called the rank of the semi-simple group. (Since A commutes with itself, the rank of a semi-simple group is at least one.)

We shall use Latin indices $1, \ldots \ell$ for the coordinates in the subspace of dimension ℓ, spanned by the H_i, while Greek indices $\alpha \ldots \nu$ will be employed for the $r-\ell$ dimensional subspace which is spanned by the eigenvectors $E_\alpha \ldots E_\nu$ corresponding to the non-vanishing distinct roots $\alpha \ldots \nu$. For the latter indices the summation convention will be suspended. The three indices ρ, σ, τ will be used to refer to the whole r-dimensional space.

The basic vectors H_i and E_α are defined by the relations

$$[A\, H_i] = 0 \qquad (i = 1 \ldots \ell) \tag{31}$$

$$[A\, E_\alpha] = \alpha E_\alpha \tag{32}$$

Further, since A is an eigenvector of (28) with eigenvalue zero, it can be written in the form:

$$A = \lambda^i H_i \,. \tag{33}$$

We shall now discuss the commutators of H's and E's, in order to obtain information about the $c_{\rho\sigma}^{\ \ \tau}$. First, from Cartan's theorem, we have

$$[H_i\, H_k] = 0 \qquad \text{or} \qquad c_{ik}^{\ \ \tau} = 0 \,. \tag{34}$$

Second, we consider $[H_i E_\alpha]$. To do this we write

$$[A[H_i E_\alpha]] + [H_i[E_\alpha A]] + [E_\alpha[A\, H_i]] = 0 \,.$$

By (31) and (32) this is

$$[A[H_i E_\alpha]] = \alpha [H_i E_\alpha]. \qquad (35)$$

Thus $[H_i E_\alpha]$ is an eigenvector of (28) belonging to $\rho = \alpha$, and since these eigenvectors are not degenerate, we must have

$$[H_i E_\alpha] = \alpha_i \, E_\alpha, \quad \text{or} \quad c_{i\alpha}^{\ \tau} = \alpha_i \delta_\alpha^\tau . \qquad (36)$$

From (32), (33) and (36) follows that

$$\alpha = \lambda^i \alpha_i \qquad (37)$$

From here on the letter α or the term "root" will be used to denote either the form (37) or the vector with covariant components α_i in the ℓ-dimensional space.

Finally, to find $[E_\alpha E_\beta]$, we form

$$[A[E_\alpha E_\beta]] + [E_\alpha [E_\beta A]] + [E_\beta [A \, E_\alpha]] = 0 .$$

By (32), this is

$$[A[E_\alpha E_\beta]] = (\alpha + \beta) \, [E_\alpha E_\beta]. \qquad (38)$$

Hence $[E_\alpha E_\beta]$ belongs as eigenvector to the root $\alpha + \beta$ if $\alpha + \beta$ is a root, and vanishes if $\alpha + \beta$ is not a root. If $\alpha + \beta$ is a non-vanishing root, we shall write

$$[E_\alpha E_\beta] = N_{\alpha\beta} \, E_{\alpha+\beta} \quad \text{or} \quad c_{\alpha\beta}^{\ \alpha+\beta} = N_{\alpha\beta} . \qquad (39)$$

If $\beta = -\alpha$ then evidently we have

$$[E_\alpha E_{-\alpha}] = c_{\alpha-\alpha}^{\ \ i} \, H_i , \qquad (40)$$

and for the rest,

$$c_{\alpha\beta}^{\ \ \tau} = 0. \qquad (\tau \neq \alpha + \beta) \qquad (41)$$

We shall now show that if α is a root, then $-\alpha$ is also a root. This is done by forming the tensor $g_{\alpha\tau}$. The restrictions (36), (40) and (41), when applied to (23), give

$$g_{\alpha\tau} = c_{\alpha i}^{\ \ \alpha} c_{\tau\alpha}^{\ \ i} + \sum_{\beta \neq -\alpha} c_{\alpha\beta}^{\ \ \alpha+\beta} c_{\tau\,\alpha+\beta}^{\ \ \beta} + c_{\alpha-\alpha}^{\ \ i} c_{\tau i}^{\ \ -\alpha} \qquad (42)$$

But by (36) and (41), each term on the right of (42) exists only when $\tau = -\alpha$, so that

$$g_{\alpha\tau} = 0 \qquad (\tau \neq -\alpha). \qquad (43)$$

Thus, if $-\alpha$ is not a root, Cartan's criterion (24) for the semi-simple groups is violated. By a suitable normalization of E_α we may set

$$g_{\alpha-\alpha} = 1 , \qquad (44)$$

and we can order our basis so that the tensor $g_{\rho\sigma}$ is written in the form

$$g_{\rho\sigma} = \left(\begin{array}{c|c} g_{ik} & 0 \\ \hline 0 & \begin{matrix} 01 & & \\ 10 & & \\ & 01 & \\ & 10 & \\ 0 & & \ddots \end{matrix} \end{array} \right) \qquad (45)$$

Since det $g_{\rho\sigma}$ is the product of the elementary determinants, it follows from (24) that

$$\det g_{ik} \neq 0 . \qquad (46)$$

Further,

$$g_{ik} = \sum_{\alpha} c_{i\alpha}^{\ \ \alpha} c_{k\alpha}^{\ \ \alpha} = \sum_{\alpha} \alpha_i \, \alpha_k. \qquad (47)$$

It may be noted that the g_{ik} defined by (47) has a non-vanishing determinant only if the vectors α span the entire ℓ-dimensional space. g_{ik} will be used as the metric tensor for this space.

Using the inverse tensor we can now establish the following useful identity:

$$
\begin{aligned}
c_{\alpha-\alpha}{}^{i} &= g^{ik}\, c_{\alpha-\alpha\, k} \\
&= g^{ik}\, c_{k\,\alpha-\alpha} \text{ by antisymmetry in subscripts,} \\
&= g^{ik}\, c_{k\alpha}{}^{\alpha} \quad \text{by (44)} \\
&= g^{ik}\, \alpha_k = \alpha^i \quad \text{by (36),}
\end{aligned}
\tag{48}
$$

so that (40) can be written as

$$[E_\alpha E_{-\alpha}] = \alpha^i H_i\ , \tag{49}$$

where the α^i are the contravariant components of the vector α. Collecting (34), (36), (39) and (49) we have for the standard forms of the commutation relations

$$
\begin{aligned}
&[H_i\, H_k] = 0 \\
&[H_i E_\alpha] = \alpha_i E_\alpha \\
&[E_\alpha\, E_\beta] = N_{\alpha\beta}\; E_{\alpha+\beta} \text{ when } \alpha+\beta \text{ is a non-vanishing root} \\
&[E_\alpha\, E_{-\alpha}] = \alpha^i\, H_i\ .
\end{aligned}
\tag{50}
$$

As an example of the foregoing, we can take the operators of rotation in three dimensions, generated by L_1, L_2, L_3 such that

$$[L_1\, L_2] = i\, L_3\ , \quad \text{etc.} \tag{51a}$$

If we take A equal to L_3, the two relations

$$[L_3, L_1 \pm i L_2] = \pm (L_1 \pm i L_2) \tag{51b}$$

show that $L_1 \pm i L_2$ are eigenvectors corresponding to $\rho = \pm 1$. Use of the normalization condition (44) yields

$$H_1 = L_3, \quad E_1 = \frac{L_1 + i L_2}{\sqrt{2}}, \quad E_{-1} = \frac{L_1 - i L_2}{\sqrt{2}} . \tag{51c}$$

§2. Properties of the Roots.

We shall now prove the following

Theorem: If α and β are roots, then $\frac{2(\alpha\beta)}{(\alpha\alpha)}$ is an integer and $\beta - \frac{2(\alpha\beta)}{(\alpha\alpha)}\alpha$ is also a root.*

This theorem is to hold for arbitrary α and β, but we shall start by restricting β to be some root, γ, such that $\alpha + \gamma$ is not a root. According to (50) we can generate a set of operators

$$[E_{-\alpha}\ E_\gamma] = N_{-\alpha\gamma} E_{\gamma-\alpha} \equiv E'_{\gamma-\alpha}$$

$$[E_{-\alpha}\ E'_{\gamma-\alpha}] = E'_{\gamma-2\alpha}$$

$$\cdots\cdots\cdots\cdots$$

$$[E_{-\alpha} E'_{\gamma-j\alpha}] = E'_{\gamma-(j+1)\alpha} \tag{52}$$

where the primes indicate that, for the moment, we are not interested in the normalization of the E_β. Since there is only a finite number of E_β, this process must eventually stop after, say, g steps. Thus

* We use the notation $(\alpha\beta)$ for the scalar product $\alpha_i \beta^i$.

$$[E_{-\alpha}E'_{\gamma-g\alpha}] = E'_{\gamma-(g+1)\alpha} = 0 \qquad (53)$$

According to (39), $E_{\gamma-j\alpha}$ may be obtained again by an equation of the form

$$[E_{\alpha}E'_{\gamma-(j+1)\alpha}] = \mu_{j+1}E'_{\gamma-j\alpha} \qquad (54)$$

In order to evaluate the coefficients μ_{j+1} we eliminate $E'_{\gamma-(j+1)\alpha}$ from (52) and (54), thus finding

$$\mu_{j+1}E'_{\gamma-j\alpha} = -[E'_{\gamma-j\alpha}[E_{\alpha}E_{-\alpha}]] - [E_{-\alpha}[E'_{\gamma-j\alpha}E_{\alpha}]]$$

by Jacobi's identity,

$$= -[E'_{\gamma-j\alpha}, \alpha^k H_k] + \mu_j[E_{-\alpha}E'_{\gamma-(j-1)\alpha}],$$

by (50) and (54).

The use of (50) and (52) gives at once a recurrence relation for the μ_j:

$$\mu_{j+1} = \mu_j + (\alpha\gamma) - j(\alpha\alpha). \qquad (55)$$

This relation holds only for $j \geq 1$, as μ_0 is not defined by (54); however, the preceding argument shows that (55) can be extended to hold also for $j = 0$ if we define

$$\mu_0 = 0 . \qquad (56)$$

From (55) and (56) we obtain immediately

$$\mu_j = j(\alpha\gamma) - \frac{j(j-1)}{2}(\alpha\alpha). \qquad (57)$$

It follows from (53) and (54) that $\mu_{g+1} = 0$, whence we have

$$(\alpha\gamma) = \tfrac{1}{2}g(\alpha\alpha), \qquad (58)$$

where g is, by definition, a non-negative integer. Introducing (58) into (57) we get

$$\mu_j = \frac{j(g-j+1)}{2}(\alpha\alpha). \qquad (59)$$

If $(\alpha\alpha)$ were zero for some root α, this root would, according to (58), be orthogonal to every root. But, as the roots span the entire ℓ-dimensional space, this would contradict (46). Hence we can write

$$g = \frac{2(\alpha\gamma)}{(\alpha\alpha)}, \qquad (60)$$

and we have proved that if α and γ are roots and $\alpha + \gamma$ is not a root, then there exists a string of roots,

$$\gamma, \gamma-\alpha, \ldots, \gamma - \frac{2(\alpha\gamma)}{(\alpha\alpha)}\alpha \equiv \gamma - g\alpha, \qquad (61)$$

which is invariant under reflection with respect to the hyperplane through the origin perpendicular to the vector α. To return to the Theorem which is to be proved, we note that for any root β, there exists some integer $j \geq 0$, such that $\beta + j\alpha$ is a root but $\beta + (j+1)\alpha$ is not. We can now set $\beta + j\alpha = \gamma$ in the above discussion, so that the string (61) can be written

$$\beta + j\alpha, \beta + (j-1)\alpha, \ldots, \beta, \ldots, \beta - k\alpha \qquad (62)$$

$$(j + k = g),$$

and as

$$2(\alpha\beta) = 2(\alpha\gamma) - 2j(\alpha\alpha) = (g - 2j)(\alpha\alpha),$$

$\frac{2(\alpha\beta)}{(\alpha\alpha)}$ is an integer, and $\beta - \frac{2(\alpha\beta)}{(\alpha\alpha)}\alpha$ is contained in the string (62).

In (39) we introduced a set of coefficients $N_{\alpha\beta}$, but we have yet to see whether some of them may not vanish. This we can do with the aid of the Theorem just proved. Assuming that with α and β, $\alpha+\beta$ is a root we evaluate

$$[E_{-\alpha}E_{\alpha+\beta}] = [E_{-\alpha}E_{\gamma-(j-1)\alpha}] = N_{-\alpha\,\alpha+\beta}E_{\gamma-j\alpha}.$$

With this we form

$$N_{-\alpha\,\alpha+\beta}[E_{\alpha}E_{\gamma-j\alpha}] = N_{\alpha\beta}N_{-\alpha\,\alpha+\beta}E_{\gamma-(j-1)\alpha}$$
$$= \mu_j E_{\gamma-(j-1)\alpha}, \qquad \text{by (54).}$$

Equations (59) and (62) now tell us that

$$N_{\alpha\beta}N_{-\alpha\,\alpha+\beta} = \mu_j = \frac{j(k+1)}{2}(\alpha\alpha), \tag{63}$$

and from this it is evident that $N_{\alpha\beta} \neq 0$ if $\alpha+\beta$ is a root and therefore $j \geq 1$.

It follows from this that if α is a root, 2α cannot be one, since E_α commutes with itself. From this, $k\alpha$ cannot be a root for any positive integer k, since if it were, it would determine a string which would contain 2α as an element. Hence, any string containing zero has only three elements α, 0, $-\alpha$.

We take now ℓ linearly independent roots $\alpha^{(1)}, \ldots, \alpha^{(\ell)}$ as the basis of a new coordinate system in the ℓ-dimensional space, and express all other root vectors as linear combinations:

$$\beta = \sum_{k=1}^{\ell} b_k\,\alpha^{(k)}. \tag{64}$$

Multiplying (64) by $\alpha^{(i)}$ and dividing by $(\alpha^{(i)}\alpha^{(i)})$, which was shown to be different from zero, we get

$$\frac{(\beta\alpha^{(i)})}{(\alpha^{(i)}\alpha^{(i)})} = \sum_{k=1}^{\ell} b_k \frac{(\alpha^{(k)}\alpha^{(i)})}{(\alpha^{(i)}\alpha^{(i)})} .$$

Using the fundamental relation (60) we deduce readily that the new covariant components b_k must be real, rational, and even, by a change of scale, integral numbers.

This shows that for a suitable choice of the H_i the α_i are real, and this implies that g_{ik} is a positive definite matrix, since for any(real) x^i

$$g_{ik}\, x^i x^k = \sum_{\alpha} (\alpha x)^2 \geq 0 . \qquad (65)$$

Hence the ℓ-dimensional space has an ordinary Euclidean metric.

§3. The Vector Diagrams.

The graphical representation of the root vectors is called a vector diagram. Schouten derived restrictions on these diagrams from which all simple Lie groups can be found. The complete classification (already found algebraically by Cartan) was obtained using this method by van der Waerden, who showed also that to every vector diagram corresponds only one infinitesimal Lie group. Since the roots belong to a lattice which is invariant under a group of reflections, Coxeter's construction of all finite groups generated by reflection leads to a third method of classifying the simple groups. We shall here sketch the method of Schouten and van der Waerden.

Suppose we have two roots, α and β, and let φ be the angle between them. We saw in the preceding section that

$$(\alpha\beta) = \tfrac{1}{2}\, m(\alpha\alpha) = \tfrac{1}{2}\, n(\beta\beta), \tag{66}$$

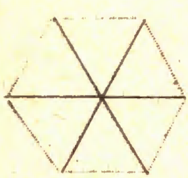
where m and n are integers. From this we get

$$\cos^2\varphi = \frac{(\alpha\beta)^2}{(\alpha\alpha)(\beta\beta)} = \frac{mn}{4}, \tag{67}$$

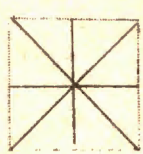
and from this we see that φ can have only the values 0°, 30°, 45°, 60°, and 90°. From (66) we deduce that the ratios of the lengths of the two vectors are $\sqrt{3}$ for 30°, $\sqrt{2}$ for 45°, 1 for 60°, and undetermined for 90°. For 0° we know already that $\alpha = \beta$.

We want to construct every possible vector diagram which satisfies these conditions and those obtained in §2. As Cartan has shown that every semi-simple group is a direct product of simple groups, we shall be interested only in the diagrams of simple groups. We shall therefore not consider diagrams which can be split into mutually orthogonal parts, since evidently every such part corresponds to an invariant subgroup.

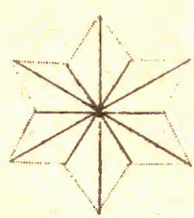
It is easy to see that the only possible two-dimensional diagrams are the ones drawn below. They are labelled by the letters which are traditional from Cartan's thesis; the numerical subscript denotes the rank of the group.



A_2



B_2



G_2

We shall now generalize these diagrams to ℓ dimensions. In what follows, we shall denote by e_i a set of mutually orthogonal unit vectors.

A_ℓ. The diagram A_2, above, may conveniently be regarded as consisting of all vectors of the form $e_i - e_k (i,k=1,2,3)$. Generalizing to ℓ dimensions, A_ℓ is formed from $\ell+1$ unit vectors e_i by forming the $\ell(\ell+1)$ differences $e_i - e_k$. These will lie in the plane $\sum_{i=1}^{\ell+1} x^i = 0$. There are $\ell(\ell+1)$ vectors, and adding to this the rank, which is the multiplicity of the root zero, we see that the group is of order $(\ell+1)^2-1$.

B_ℓ. We can generalize B_2 in ℓ dimensions by constructing B_ℓ out of all the vectors $\pm e_i$ and $\pm e_i \pm e_k$ $(i,k=1\ldots\ell)$. There are $2\ell^2$ vectors, and the order of the group is $\ell(2\ell+1)$.

C_ℓ. Another possible generalization of B_2, however, is to construct all vectors of the form $\pm 2e_i$ and $\pm e_i \pm e_k (i,k=1\ldots\ell)$. For $\ell=2$, C_2 differs from B_2 only by rotation through 45°. For $\ell > 2$, these diagrams are different from the B_ℓ. C_ℓ has the same order as B_ℓ.

D_ℓ. For $\ell > 2$, the diagram consisting of vectors $\pm e_i \pm e_k$ $(i,k= 1\ldots\ell)$ represents a simple group, which we shall call D_ℓ. There are $2\ell(\ell-1)$ vectors, and the group is of order $\ell(2\ell-1)$. For $\ell=2$, this construction gives only two orthogonal pairs of vectors and is therefore not simple. It may be noted that by a rotation of the axes given by

$$
\begin{aligned}
e'_1 &= \tfrac{1}{2}(e_1 + e_2 - e_3 - e_4) \\
e'_2 &= \tfrac{1}{2}(e_1 - e_2 + e_3 - e_4) \\
e'_3 &= \tfrac{1}{2}(e_1 - e_2 - e_3 + e_4) \\
e'_4 &= \tfrac{1}{2}(e_1 + e_2 + e_3 + e_4)
\end{aligned}
\qquad (68)
$$

the vector diagram A_3 may be brought into coincidence with D_3.

van der Waerden has shown that apart from these four classes of simple diagrams there are only five possible simple diagrams. One of them is G_2; the others are the following:

F_4. This diagram consists of the vectors of B_4 plus 16 more vectors $\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)$. There are 48 vectors and the group is of order 52.

E_6 consists of the vectors of A_5, the vectors $\pm\sqrt{2}\, e_7$, and all the vectors

$$\tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6) \pm \frac{e_7}{\sqrt{2}} ,$$

where in the first fraction we take three signs positive and three negative. There are 72 vectors and the group is of order 78.

E_7 consists of the vectors of A_7 and all the vectors

$$\tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 \pm e_7 \pm e_8),$$

where we take four signs positive and four negative. There are 126 vectors, and the group is of order 133.

E_8 consists of the vectors of D_8 and all the vectors

$$\tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 \pm e_7 \pm e_8),$$

with each sign occurring an even number of times. There are 240 vectors, and the group is of order 248.

The simplest realizations of the groups characterized by the vector diagrams A_ℓ, B_ℓ, C_ℓ, D_ℓ, are the classical groups, i.e. the special linear (unimodular), the orthogonal and the symplectic (complex) groups.

For the full linear group in $\ell+1$ dimensions we may choose the infinitesimal operators

$$X_{ik} = x^i \frac{\partial}{\partial x^k} , \qquad (i,k=1\ldots \ell+1) \tag{69}$$

with the commutation relations

$$[X_{ik}X_{mn}] = \delta_{km}X_{in} - \delta_{in}X_{mk} . \tag{70}$$

But the full linear group is not a semi-simple group; the operator $\sum_j X_{jj}$ commutes with every operator of the group, and the Abelian subgroup generated by this operator (i.e. the subgroup of the dilatations) is an invariant subgroup.

In order to have a semi-simple group we have to restrict ourselves to the unimodular subgroup (or 'special' linear group) in $\ell+1$ dimensions. Then the X_{ii} are no longer infinitesimal operators of the subgroup but should be replaced by

$$X'_{ii} = X_{ii} - \frac{1}{\ell+1} \sum_j X_{jj} , \tag{69'}$$

a change which does not affect the commutation relations (70). These operators correspond to the diagram A_ℓ if we make the identification

$$X'_{ii} \equiv H_i , \qquad X_{ik} \equiv E_{(e_i - e_k)} . \tag{71}$$

Although we have $\ell+1$ operators H_i, only ℓ of them are linearly independent, owing to the relation

$$\sum_{i=1}^{\ell+1} H_i = 0 . \tag{72}$$

For the orthogonal group in $2\ell+1$ dimensions, which leaves the quadratic form

$$\sum_{k=-\ell}^{\ell} x^k x^{-k} = x_o^2 + 2\sum_{k=1}^{\ell} x^k x^{-k}$$

invariant, we may choose the infinitesimal operators

$$X_{ik} = -X_{ki} = x^{i}\frac{\partial}{\partial x^{-k}} - x^{k}\frac{\partial}{\partial x^{-i}}, \qquad (i,k = 0, \pm 1, \ldots, \pm \ell), \qquad (73)$$

with the commutation relations

$$[X_{ik}X_{mn}] = \delta_{k+m}X_{in} - \delta_{k+n}X_{im} - \delta_{i+m}X_{kn} + \delta_{i+n}X_{km}, \qquad (74)$$

where δ_q is one if $q=0$, and zero otherwise. These operators correspond to the diagram B_ℓ if we identify

$$X_{i-i} = H_i, \; X_{\pm i \pm k} = E_{(\pm e_i \pm e_k)}, \; X_{0 \pm k} = E_{(\pm e_k)} \qquad (i,k > 0). \qquad (75)$$

For the symplectic group in 2ℓ dimensions, which leaves invariant the anti-symmetric bilinear form

$$\sum_{k=1}^{\ell} (x^{k}y^{-k} - x^{-k}y^{k}),$$

we may choose the infinitesimal operators

$$X_{ik} = X_{ki} = \varepsilon^{i}x^{i}\frac{\partial}{\partial x^{-k}} + \varepsilon^{k}x^{k}\frac{\partial}{\partial x^{-i}}, \qquad (i,k = \pm 1, \ldots, \pm \ell), \qquad (76)$$

with the commutation relations

$$[X_{ik}X_{mn}] = \varepsilon^{m}\delta_{k+m}X_{in} + \varepsilon^{n}\delta_{k+n}X_{im} + \varepsilon^{m}\delta_{i+m}X_{kn} + \varepsilon^{n}\delta_{i+n}X_{km}, \qquad (77)$$

where ε^{q} is +1 if q is positive and -1 if q is negative. These operators correspond to the diagram C_ℓ if we make the identification

$$X_{i-i} = H_i, \qquad X_{\pm i \pm k} = E_{(\pm e_i \pm e_k)} \qquad (i,k > 0). \qquad (78)$$

For the orthogonal group in 2ℓ dimensions, which leaves the quadratic form $\sum_{k=1}^{\ell} x^k x_{\cdot}^{-k}$ invariant, we may choose the same infinitesimal operators as in B_ℓ, with the same commutation relations, except that now $i,k \neq 0$. These operators correspond to the diagram D_ℓ if we make the same identification as in B_ℓ.

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Lectures 3 and 4

THE REPRESENTATIONS OF THE SEMI-SIMPLE GROUPS.

§1. Representations and Weights.

A group of linear transformations of a vector space R which is homomorphic to a given group is called a representation of this group. The dimension, N, of R is called the degree of the representation. If s and t are two elements of the group, and U(s) and U(t) the corresponding matrices of the representation, then U(s)U(t) = U(st). Two representations U(s) and V(s) are called equivalent if there is a constant matrix A such that

$$A\, U(s)\, A^{-1} = V(s)$$

for every element s.

A representation is reducible if it leaves a subspace R_1 of R invariant. If this is the case, the matrices of the representation can be given the form

$$\begin{pmatrix} A_1 & B \\ 0 & A_2 \end{pmatrix} \qquad (79)$$

where A_1 is a matrix whose dimensions equal that of R_1. If the representation leaves invariant two subspaces R_1 and R_2 such that $R_1 + R_2 = R$, then the representation can be written as

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \qquad (80)$$

We say in this case that the representation is fully reducible, or decomposable.

A Lie group is determined by the r infinitesimal operators and their commutation relations. Similarly, a representation of a Lie group is determined if we have r matrices, D_ρ, which satisfy the equation

$$D_\rho D_\sigma - D_\sigma D_\rho = [D_\rho D_\sigma] = c_{\rho\sigma}^{\tau} D_\tau . \qquad (81)$$

In particular, we may ask for a standard representation with matrices H_i and E_α which satisfy the relations (50). These same letters, which denoted infinitesimal operators in the previous work, will in this lecture consistently be used for the corresponding matrices.

Let u be a vector in the space R such that

$$H_i u = m_i u \qquad (i = 1 \ldots \ell). \qquad (82)$$

Thus, u is a simultaneous eigenvector of the ℓ matrices H_i. The set of eigenvalues $m_1, \ldots, m_\ell$ are the covariant components of a vector in the ℓ-dimensional space. We shall call this vector the weight of u; from now on the ℓ-dimensional space will be called the weight space. Evidently, u is also the eigenvector of the matrix $\lambda^i H_i$ corresponding to the eigenvalue

$$(\lambda m) = \lambda^i m_i . \qquad (83)$$

A weight will be called simple if to it belongs only one eigenvector.

The existence and various properties of the weights will now be proved.

A. Every representation has at least one weight.

Proof: H_1 has at least one eigenvalue, say m_1; let R_1 be the subspace of R spanned by the eigenvectors of H_1 belonging to m_1. Since

$H_1H_2u = H_2H_1u = m_1H_2u$, it follows that $H_2R_1 = R_1$. H_2 has at least one eigenvector in its invariant subspace R_1. Continuing the process, which is possible because every matrix has at least one eigenvector in every invariant subspace, we arrive at the subspace R_ℓ which consists of the simultaneous eigenvectors of $H_1 \ldots H_\ell$ corresponding to the weight $m=(m_1 \ldots m_\ell)$.

B. A vector u of weight m which is a linear combination of vectors u_k of weights $m^{(k)}$, all different from m, must vanish.

Proof: We form the matrix $\prod_k \lambda^i(H_i - m_i^{(k)})$ and let it operate on the equation $u = \sum_k u_k$. Since all H commute, each factor annihilates a term in the sum. Since the λ^i are arbitrary, the left hand side is also zero only if u vanishes.

C. From B it follows that vectors with different weights are linearly independent, so that there are at most N different weights.

D. If u is a vector of weight m, then $H_i u$ and $E_\alpha u$ have definite weights, m and $m+\alpha$ respectively.

Proof: For $H_i u$ this is an immediate consequence of (50). For $E_\alpha u$ we have

$$H_i E_\alpha u = [H_i E_\alpha] u + E_\alpha H_i u = (\alpha_i + m_i) E_\alpha u . \qquad (84)$$

E. If the representation is irreducible the H_i may simultaneously be expressed in diagonal form.

Proof: Starting with a vector u having a definite weight, we consider the space R_1 spanned by all possible products

$$\ldots E_\gamma E_\beta E_\alpha u \,, \qquad (85).$$

each of which, according to D, has a definite weight. Evidently, $E_\rho R_1 = R_1$. Thus, since the representation is assumed irreducible, R_1 coincides with R, and the vectors (85) span R. If we select from them as a basis N linearly independent vectors, each is an eigenvector of all H_i, which thus have been diagonalized.

F. For any weight m and root α, $\frac{2(m\alpha)}{(\alpha\alpha)}$ is an integer and $m - \frac{2(m\alpha)}{(\alpha\alpha)}\alpha$ is a weight.

Proof: The proof is analogous to the proof of the Theorem of §2, lecture 2, except that the weights are, in general, not simple, while in the previous case Cartan's theorem enabled us to assume that all non - vanishing roots are simple. We shall point out only the differences in the proofs.

We start out from a vector u_0 of weight m such that $m+\alpha$ is not a weight, and form the series of vectors

$$u_1 = E_{-\alpha} u_0 \,, \quad u_2 = E_{-\alpha} u_1, \ \ldots \,.$$

The relation

$$E_\alpha u_{j+1} = \mu_{j+1} u_j \,, \qquad (86)$$

which, because of the possible multiplicity of weights, is not as evident as its counterpart (54), may be proved by induction. Assume (86) to be true for a certain j-1; then

$$E_\alpha u_{j+1} = E_\alpha E_{-\alpha} u_j = [E_\alpha E_{-\alpha}]u_j + E_{-\alpha} E_\alpha u_j = \alpha^i H_i u_j + \mu_j E_{-\alpha} u_{j-1}$$
$$= [(\alpha m) - j(\alpha\alpha)]u_j + \mu_j u_j.$$

Hence (86) is true for j if it is true for j-1, and we have

$$\mu_{j+1} = (\alpha m) - j(\alpha\alpha) + \mu_j\,, \tag{87}$$

corresponding to (55). But $\mu_0 = 0$, by D, since $m+\alpha$ is not a root, and therefore (86) holds with $j+1 = 0$ and $\mu_0 = 0$. The rest of the proof parallels that of the analogous theorem for the roots.

G. By projecting the space R moduli $u_0,\ldots,u_g$ in a space of $N-(g+1)$ dimensions and by repeating the same considerations as in F, it may be proved that m and $m - \frac{2(\alpha\, m)}{(\alpha\alpha)}\alpha$ have the same multiplicity. All possible weights belong to a lattice which is invariant under the group S generated by the reflections with respect to the hyperplanes through the origin perpendicular to the roots. Weights which can be obtained from one another by operations of S are called <u>equivalent</u> and have the same multiplicity.

In the group A_ℓ we have $(\alpha\alpha) = |e_i - e_k|^2 = 2$; hence a weight

$$m = m_1e_1 + m_2e_2 + \ldots\; m_{\ell+1}e_{\ell+1} \tag{88}$$

must satisfy the condition that $2(m\cdot(e_i - e_k))/2 = m_i - m_k$ be an integer and, in addition, that

$$\sum_{i=1}^{\ell+1} m_i = 0, \tag{88'}$$

which follows from (72). Therefore the m_i are fractions with denominator $\ell+1$ which differ by integers. According to F and G, a weight equivalent to m is

$$m - (m_i - m_k)(e_i - e_k) = m_1e_1 + \ldots + m_ke_i + \ldots + m_ie_k + \ldots\; m_{\ell+1}e_{\ell+1},$$

hence the group S is the group of permutations of the components of m.

In B_ℓ we have, in addition to the condition that $m_i - m_k$ be an integer, the further condition that $2(m \cdot e_i)$ be an integer. Therefore the components of any weight are either all integers or all half-integers. The group S is the group of permutations of the components with any number of changes of sign.

In C_ℓ the additional condition is that $2(m \cdot 2e_i)/4$ be an integer, and therefore all components are integers. The group S is the same as in B_ℓ.

In D_ℓ we find that both $m_i - m_k$ and $m_i + m_k$ are integers. Therefore the weights are the same as in B_ℓ, but the group S is only the group of permutations of the components with an even number of changes of sign.

§2. <u>The Classification of the Irreducible Representations.</u>

We shall introduce a convention according to which the weights of the representations can be ordered. A weight $(m_1 \ldots m_\ell)$ is said to be positive if the first non-vanishing component is positive. One weight is said to be <u>higher</u> than another if the difference between them is positive. A weight is called <u>dominant</u> if it is higher than its equivalents.

<u>Theorem 1.</u> If a representation is irreducible, its highest weight is simple.

Proof: Assume that the vector u_o belongs to the highest weight, $m^{(o)}$. According to D of §1 it is sufficient to prove that every vector of the form

$$\ldots E_\delta E_\gamma E_\beta E_\alpha u_o \tag{89}$$

which is of weight $m^{(o)}$ can be written as ku_o, where k is a constant. We shall show, in addition, that k depends only on the series $\alpha, \beta, \gamma, \delta \ldots$

and on the weight $m^{(o)}$. It is clear from D that $\ldots + \delta + \gamma + \beta + \alpha = 0$. Therefore at least one of the roots must be positive. Let us say that γ is the first positive root (from the right). Replacing $E_\gamma E_\beta$ by $E_\beta E_\gamma + [E_\gamma E_\beta]$ and so on until E_γ acts directly on u_o, and remembering that $E_\gamma u_o = 0$, we obtain a sum of terms with fewer matrices E than (89) but still of weight $m^{(o)}$. Continuing this process until there are no more operators of positive weight, we arrive at a sum of products of H_i acting on u_o, and these are finally converted into a polynomial of the components of $m^{(o)}$ multiplying u_o. The coefficients in this polynomial depend, evidently, only on the set of roots $\alpha, \beta, \gamma, \delta, \ldots$, and not on the particular representation.

Theorem 2. Two irreducible representations are equivalent if their highest weights are equal.

Proof: We distinguish the two representations D and D' by using unprimed quantities for D and primed ones for D'. Let u_o and u'_o be the vectors of the highest weight $m^{(o)}$, which is assumed to be the same for both D and D', and construct all possible vectors $u_j = \ldots E_\gamma E_\beta E_\alpha u_o$ and correspondingly $u'_j = \ldots E'_\gamma E'_\beta E'_\alpha u'_o$. It was shown in D of §1 that these vectors span the whole space and that each has a definite weight. The equivalence of the two representations will be proved if we show that to any linear relation which exists between the unprimed vectors there corresponds a linear relation with the same coefficients between the corresponding primed vectors. Assume there is a relation

$$\gamma_1 u_1 + \gamma_2 u_2 + \ldots = 0 ; \tag{90}$$

then, using the same coefficients γ we can construct a vector

$$\gamma_1 u_1' + \gamma_2 u_2' + \dots = w'. \qquad (90')$$

The vectors w' for all possible relations (90') form a subspace R_1' of R', and it is easily seen that R_1' is an invariant subspace under the operations of the group. Since D' is irreducible we must have $R_1' = 0$ unless R_1' consists of the whole space R'. The last alternative is excluded since u_0' certainly is not in R_1'. For if $w' = u_0'$, according to C of §1, the left-hand side of (90) contains only vectors of weight $m^{(o)}$. Theorem 1 would then lead to a relation

$$\gamma_1 k_1 + \gamma_2 k_2 + \dots \neq 0$$

which however, is incompatible with the corresponding relation

$$\gamma_1 k_1 + \gamma_2 k_2 + \dots = 0$$

derived from (90), the k being the same for the two representations.

The connection between highest weights and irreducible representations is completed when we show that there exists an irreducible representation which has any dominant weight as its highest weight. Indeed, Cartan has proved that

(A) For every simple group of rank ℓ there are ℓ fundamental dominant weights $L^{(1)}, \dots, L^{(\ell)}$ such that if a dominant weight L is given, it is a linear combination

$$L = \sum_{i=1} x_i L^{(i)} \qquad (91)$$

with non-negative integral coefficients;

(B) There exist ℓ fundamental irreducible representations $g_1, g_2, \dots, g_\ell$ which have the fundamental weights as their highest weights.

Since it is easy to see that the weights of the Kronecker product $A \times B$ of two representations are all the sums of one weight of A and one weight of B, the Kronecker product representation

$$G = \underbrace{g_1 \times g_1 \times \dots}_{x_1 \text{ times}} \times \underbrace{g_2 \times g_2 \dots}_{x_2 \text{ times}} \times \dots \tag{92}$$

has as highest weight exactly the weight L. G will, in general, be reducible, but one of its irreducible constituents will have L as its highest weight.

Cartan proved (A) and (B) for every simple group separately.* We shall here sketch as an example the proofs for the groups A_ℓ and B_ℓ.

A_ℓ. The components of a dominant weight satisfy the relation $L_1 \geq L_2 \geq \dots \geq L_{\ell+1}$. If we assume as fundamental weights

$$\begin{array}{ll} L^{(1)}: & \frac{\ell}{\ell+1}, -\frac{1}{\ell+1} \dots\dots -\frac{1}{\ell+1} \\ L^{(2)}: & \frac{\ell-1}{\ell+1}, \frac{\ell-1}{\ell+1}, -\frac{2}{\ell+1} \dots\dots -\frac{2}{\ell+1} \\ \dots & \dots\dots\dots\dots\dots\dots \\ L^{(\ell)}: & \frac{1}{\ell+1}, \frac{1}{\ell+1}, \dots\dots \frac{1}{\ell+1}, \frac{-\ell}{\ell+1} \end{array} \tag{93}$$

it can be verified that (91) is satisfied by setting

$$x_i = L_i - L_{i+1}\ . \tag{94}$$

The fundamental representations corresponding to the highest weights (93) are the linear unimodular group in $\ell+1$ dimensions itself and the transformations induced by this group on the antisymmetric tensors of rank $2, 3 \dots \ell$.

* Chevalley, Compt. Rend. 227, 1136 (1948) has given a proof of the whole theorem which does not make use of the particular structure of the different groups.

It may be shown that a tensor of rank f in the $\ell+1$ dimensional space which has the symmetry defined by the partition $(f_1, f_2, \ldots, f_{\ell+1})$ with $\sum_{i=1}^{\ell+1} f_i = f$ is a basis of the representation whose highest weight has the components $L_i = f_i - \frac{f}{\ell+1}$.

B_ℓ. The components of a dominant weight satisfy the relation $L_1 \geq L_2 \geq \ldots \geq L_\ell \geq 0$. If we take as fundamental weights

$$
\begin{array}{lccccccc}
L^{(1)}: & \frac{1}{2} & \frac{1}{2} & \cdot & \cdot & \cdot & \cdot & \frac{1}{2} \\
L^{(2)}: & 1 & 0 & 0 & \cdot & \cdot & \cdot & 0 \\
L^{(3)}: & 1 & 1 & 0 & 0 & \cdot & \cdot & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
L^{(\ell)}: & 1 & 1 & 1 & \cdot & \cdot & 1 & 0
\end{array}
\qquad (95)
$$

it is easy to see that (91) is satisfied by setting

$$
\begin{aligned}
x_1 &= 2L_\ell \\
x_i &= L_{i-1} - L_i \qquad (i > 1).
\end{aligned}
\qquad (96)
$$

The fundamental representations corresponding to the highest weights (95) are the double-valued representation of degree 2^ℓ, the orthogonal group in $2\ell+1$ dimensions, and the transformations induced by this group on the anti-symmetric tensors of rank $2, 3, \ldots, \ell-1$.

It may be shown that a tensor of rank f with vanishing trace in the $2\ell+1$ dimensional space which has a symmetry defined by the partition $(f_1 \ldots f_\ell, 0, \ldots 0)$ is the basis of the representation whose highest weight has components $L_i = f_i$.

§3. The Problem of Full Reducibility.

Having classified the irreducible representations of a group, we are in a position to classify all its representations if we know that every reducible representation is fully reducible, i.e. decomposable into its irreducible constituents.

It is well known that the representations of finite groups are fully reducible, and that the proof of this is based on the possibility of summing over all elements of a group representation. For continuous groups the analog of this summation is an integration for which, however, the question of convergence arises. Weyl has proved that if we impose some particular reality condition (this is called the unitary restriction) on the coefficients e^ρ of the general infinitesimal element $e^\rho X_\rho$ of a semi-simple group, the group is restricted to a subgroup for which the integrations converge and full reducibility may be proved. It follows from the full reducibility of any infinitesimal representation $D_1 \ldots D_r$ that the general element $e^\rho D_\rho$ is fully reducible even if the e^ρ no longer obey the unitary restriction. The representations of every semi-simple group are therefore fully reducible.

Under the unitary restriction the linear group becomes the unitary group, and the orthogonal group becomes that of real rotations. Weyl's proof involves integration over the entire group. A purely infinitesimal proof of the full reducibility was given by Casimir for the three-dimensional orthogonal group O_3. He considered the operator*

$$G = J_x^2 + J_y^2 + J_z^2 \qquad (97)$$

---

* Hereafter, to avoid confusion, we shall use J_x, J_y, J_z instead of L_1, L_2, L_3 (see p.20).

which is known to commute with J_x, J_y, and J_z. If the representation is irreducible, then Schur's lemma states that G is of the form

$$G = \lambda 1, \tag{98}$$

where

$$\lambda = j(j+1) \qquad (j = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \dots). \tag{98'}$$

If the representation is reducible, and has for example two irreducible constituents, the infinitesimal operators may be brought to the form (79), so that G can be written

$$G = \begin{pmatrix} \lambda 1 & K \\ 0 & \lambda' 1 \end{pmatrix}. \tag{99}$$

If $\lambda \neq \lambda'$, then by application of the transformation

$$T = \begin{pmatrix} 1 & \frac{K}{\lambda - \lambda'} \\ 0 & 1 \end{pmatrix} \tag{100}$$

we obtain

$$TGT^{-1} = \begin{pmatrix} \lambda 1 & 0 \\ 0 & \lambda' 1 \end{pmatrix}.$$

The same transformation also decomposes J_x, J_y, and J_z, since they commute with G. The decomposition fails if $\lambda = \lambda'$, but in this case the two irreducible constituents of the representation are equivalent and full reducibility may be proved by quite simple considerations. We shall see in the next section how this proof may be generalized so as to apply to any semi-simple group.

§4. Casimir's Operator and its Generalization.

We have seen in §2 that every irreducible representation is characterized by its highest weight $L \equiv (L_1 \ldots L_\ell)$. But in the group O_3, j is not only *the highest value of m, i.e. the highest eigenvalue of J_z for the given* representation, but is also connected with the eigenvalues of G, which are common to the whole basis of an irreducible representation. The connection is one-to-one, since it follows from (98') that

$$j = \pm\sqrt{\lambda + \frac{1}{4}} - \frac{1}{2} ,$$

but only the upper sign gives a j which is a dominant weight.

The generalization of G for any semi-simple group was given by Casimir, who introduced the operator

$$G = g^{\rho\sigma} X_\rho X_\sigma , \tag{101}$$

which commutes with every X_τ :

$$[G\, X_\tau] = g^{\rho\sigma} X_\rho [X_\sigma X_\tau] + g^{\rho\sigma} [X_\rho\, X_\tau]\, X_\sigma$$
$$= (c^\rho{}_\tau{}^\lambda + c^\lambda{}_\tau{}^\rho)\, X_\rho\, X_\lambda = 0$$

by the antisymmetry of the structure constants. The eigenvalues of G may be calculated if we use the standard basis and write

$$G = g^{ik} H_i H_k + \sum_\alpha E_\alpha E_{-\alpha} . \tag{102}$$

Let L be the highest weight of an irreducible representation and u be a vector of this weight in the space R. Then $E_\alpha u = 0$ for positive roots α, and

$$G\, u = g^{ik} L_i L_k u + \sum_{\alpha^+} [E_\alpha E_{-\alpha}] u = [(LL) + \sum_{\alpha^+} (\alpha L)] u \tag{103}$$

where $\sum_{\alpha^+}$ denotes summation over positive roots only. By introducing the vectors

$$R = \frac{1}{2} \sum_{\alpha+} \alpha \tag{104}$$

and

$$K = L + R, \tag{105}$$

we can write for the eigenvalues of G

$$\lambda = L^2 + 2(R\ L) = K^2 - R^2. \tag{106}$$

It is easy to see that while a highest weight determines an eigenvalue of the Casimir operator, the converse is not generally true, and the fact is not surprising as we cannot expect that the single number λ is sufficient to determine ℓ numbers L_i.

Casimir used the operator G in order to extend to any semi-simple group his proof of full reducibility, but was unable to apply it to the cases where inequivalent representations belong to the same eigenvalue of G. The latter case was treated by van der Waerden by the use of considerations entirely foreign to Casimir's original approach.

Another way of doing it is to generalize Casimir's operator by constructing a complete set of operators which commute with every operator of the group and whose eigenvalues characterize the irreducible representations.

A possible generalization of G is provided by the operators

$$\gamma_{\alpha_1 \alpha_2 \ldots \alpha_n} X^{\alpha_1} X^{\alpha_2} \ldots X^{\alpha_n}$$

with

$$\gamma_{\alpha_1 \alpha_2 \ldots \alpha_n} = c_{\alpha_1 \beta_1}^{\beta_2} c_{\alpha_2 \beta_2}^{\beta_3} \ldots c_{\alpha_n \beta_n}^{\beta_1},$$

and it is easy to verify that each of these operators commutes with every X_ρ.

But these still do not suffice, since it is found for example that for irreducible representations contragredient to each other and inequivalent, they have the same eigenvalues.

We therefore examine the conditions imposed on a general function of the infinitesimal operators, $F(X^{\rho})$, by the requirement that it commute with every operator of the group:

$$[X_{\sigma} F] = 0 \, . \tag{107}$$

It is well known that this expression can be written as

$$[X_{\sigma} X^{\tau}] \frac{\partial F}{\partial X^{\tau}} = c_{\sigma}^{\tau\lambda} X_{\lambda} \frac{\partial F}{\partial X^{\tau}} = c_{\lambda\sigma}^{\tau} X^{\lambda} \frac{\partial F}{\partial X^{\tau}} \, ,$$

where the products $X^{\lambda}\partial F/\partial X^{\tau}$ are suitably ordered. Comparison of this expression with (19) shows that the functions satisfying (107) may be constructed from the invariants of the adjoint group, which are characterized by

$$E_{\sigma} F(e^{\rho}) = 0 \, , \tag{108}$$

by substituting e^{ρ} for X^{ρ} and ordering the terms.

By applying an operator F which satisfies (107) to any vector of the space R of an irreducible representation we obtain, according to Schur's lemma, $Fu = \lambda u$, where λ is independent of the particular choice of u. If the vector of highest weight is chosen, we find from §2, Theorem 1, that

$$\lambda = \varphi(L_1 L_2 \ldots L_{\ell}) \equiv \varphi(L). \tag{109}$$

In order to characterize the representation we need ℓ operators of this kind such that the system of equations

$$\lambda_i = \varphi_i(L) \qquad (i = 1 \ldots \ell) \tag{109'}$$

has not more than one solution L which is a dominant weight. To prove the existence of such a set of operators it suffices to prove that

A) If we express the λ_i as functions of K instead of L, the functions

$$\lambda_i = f_i(K) \qquad (110)$$

are invariant under the transformations of the group (S) defined on page 36.

B) For any simple (or semi-simple) group there exists a set of ℓ (polynomial) invariants of the adjoint group such that the product of the degrees of these polynomials equals the order of (S).

According to A), the system (110) has, together with a solution K, any solution SK (which means the vector obtained from K by an operation S of the group (S)), and according to B), the number of solutions exactly equals the number of vectors SK(*); hence the system (110) has only one solution which is a dominant vector. Also (109') has only one solution L which is a dominant vector, because if a solution K of (110) is not dominant and is lower, say, than SK, then also K - R is not dominant, since

$$K - R < SK - R < SK - SR = S(K - R).$$

We shall prove A) by making use of the properties of the whole group, since it has not been possible so far to construct a proof which uses only the infinitesimal group. In any representation, (5) reads

$$U(\delta a)\, U(a) = U(a + da).$$

As the D_ρ of (81) are the infinitesimal elements of the representation we may

(*) From the definition of R it follows easily that no SK coincides with K, and therefore no two SK coincide.

write this as

$$\sum_t (q\,|\,1 + \delta a^\rho D_\rho\,|\,t)(t\,|U(a^\sigma)\,|\,s) = (q\,|\,U(a^\sigma + \mu_\rho{}^\sigma \delta a^\rho)\,|\,s) \quad \text{by (8)}$$

$$= (q|U(a^\sigma)\,|\,s) + \mu_\rho{}^\sigma \delta a^\rho \frac{\partial}{\partial a^\sigma}(q|\,U(a^\sigma)\,|\,s).$$

Comparison with (15) shows that

$$A_\rho(q|U(a^\sigma)\,|\,s) = \sum_t (q|D_\rho|t)(t|U(a^\sigma)\,|\,s),$$

where A_ρ is the infinitesimal operator of the first parameter group. Consequently for any function $f(X_\rho)$ we have

$$f(A_\rho)(q|U(a^\sigma)\,|\,s) = \sum_t (q|f(D_\rho)|t)(t|U(a^\sigma)\,|\,s). \tag{111}$$

In particular, if $f(X_\rho)$ is $F(X_\rho)$ satisfying (107), then $F(D_\rho)$ is diagonal according to Schur's lemma, and we get

$$F(A_\rho)(q|U(a^\sigma)|s) = \lambda(q|U(a^\sigma)|s), \tag{112}$$

so that each matrix element of the representation is an eigenfunction of $F(A_\rho)$. It follows that the trace of the matrix $(q|U(a^\sigma)|s)$, which is called the <u>character</u>, χ, of the representation, is also an eigenfunction corresponding to the same eigenvalue. Since the trace of a matrix is invariant under similarity transformations, it follows that the character is not a function of the individual elements of the group, but rather of the classes of conjugate elements. The classes of a semi-simple group of rank ℓ depend on ℓ parameters; by choosing them as a suitable set $\varphi^1 \ldots \varphi^\ell$, Weyl* has given a general formula,

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* Reference (6), p.389

$$\chi(L;\varphi) = \frac{\xi(K)}{\xi(R)} \tag{113}$$

for the characters of a unitary restricted semi-simple group, where K is defined by (105) and

$$\xi(K) = \sum_{S} \delta_S \, e^{i(SK)_j \varphi^j} \, ; \tag{114}$$

δ_S is plus or minus one depending on the parity of the element S.

If we now apply to K an operation S, the character is left invariant except for a possible change of sign ; hence the eigenvalue (110) to which the character belongs as eigenfunction is invariant under the operations of (S). Q.E.D.

As an example consider the group of rotations in three dimensions, R_3. Any element of this group can be obtained by a similarity transformation from the diagonal matrix with elements $e^{im\varphi}(-\ell \leq m \leq \ell)$ where φ is an angle of rotation around a properly chosen axis. Thus, φ is a function of the class and the character of R_3' is

$$\sum_{m=-\ell}^{\ell} e^{im\varphi} = \frac{e^{i(\ell+\frac{1}{2})\varphi} - e^{-i(\ell+\frac{1}{2})\varphi}}{e^{i\frac{\varphi}{2}} - e^{-i\frac{\varphi}{2}}} \, , \tag{115}$$

which is the value given by (114) where $k = \ell+\frac{1}{2}$.

In order to construct invariants of the adjoint group we construct the determinant

$$\Delta = \det(q|e^{\rho} D_{\rho} - \omega|s) = \det a_{qs} \, , \tag{116}$$

where ω is an arbitrary number and $(q|D_{\rho}|s)$ an arbitrary representation. The determinant is an invariant of the adjoint group, for

$$E_\sigma \Delta = e^\rho c_{\rho\sigma}^{\ \ \tau} \frac{\partial \Delta}{\partial e^\tau} = e^\rho c_{\rho\sigma}^{\ \ \tau} \sum_{qs} \frac{\partial \Delta}{\partial a_{qs}} \frac{\partial a_{qs}}{\partial e^\tau}$$

$$= e^\rho c_{\rho\sigma}^{\ \ \tau} \sum_{qs} \frac{\partial \Delta}{\partial a_{qs}} (q|D_\tau|s) =$$

$$= e^\rho \sum_{qst} \frac{\partial \Delta}{\partial a_{qs}} [(q|D_\rho|t)(t|D_\sigma|s) - (q|D_\sigma|t)(t|D_\rho|s)] \quad \text{by (81)}$$

$$= \sum_{qst} \frac{\partial \Delta}{\partial a_{qs}} [(a_{qt} + \omega \delta_{qt})(t|D_\sigma|s) - (q|D_\sigma|t)(a_{ts} + \omega \delta_{ts})]$$

$$= \sum_{qst} \frac{\partial \Delta}{\partial a_{qs}} [a_{qt}(t|D_\sigma|s) - (q|D_\sigma|t)\, a_{ts}]$$

$$= \sum_{st} \Delta \delta_{st}(t|D_\sigma|s) - \sum_{qt} \Delta \delta_{qt}(q|D_\sigma|t) = 0.$$

Δ is a polynomial in ω, and evidently the coefficient of each power of ω is separately an invariant. That this method yields a set of invariants which satisfy the conditions stated in B) is shown separately for each simple group in reference (14).

In conclusion, we can now state that for every semi-simple group there exists a set of ℓ functions $F_i(X_\rho)$ which commute with every operator of the group and whose eigenvalues characterize the irreducible representations. They constitute the extension to every semi-simple group of the operator (97) for the three-dimensional rotation group.

§5. Miscellaneous Problems.

Finally, we know a number of general properties of the irreducible representation of O_3, and we want to see to what extent they may be generalized to all semi-simple groups.

1. The dimension of an irreducible representation is 2j+1 for O_3. By calculating the value of (113) for the identity element, Weyl has found that the dimension of any irreducible representation is given by

$$\prod_{\alpha^+} \frac{(\alpha K)}{(\alpha R)} \qquad (117)$$

2. In O_3 the eigenvalues of J_z are non-degenerate and hence suffice to label the basis of the representations. The natural extension of the eigenvalues of J_z are the weights, but in general they are not simple. If γ_m is the multiplicity of the weight m, Weyl has shown that the character of the representation has the form

$$\chi = \sum_m \gamma_m e^{im_j \varphi^j}, \qquad (118)$$

so that the coefficients of the Fourier expansion of expression (113) give the multiplicities.

If the multiplicity is different from unity we need some additional operators $k(X_\rho)$, all commuting with each other and with H_i, whose eigenvalues will enable us to distinguish the different eigenvectors of a given weight; we must first find out how many such operators will be needed.

If the basis is chosen so that not only H_i but also $k(X_\rho)$ are diagonal, then by setting $f(X_\rho) = k(X_\rho)$ in (111) we obtain

$$k(A_\rho)(q|U(a^\sigma)|s) = k_q(q|U(a^\sigma)|s) \qquad (119)$$

where k_q is the eigenvalue of $k(X_\rho)$ corresponding to the row q; then $(q|U(a^\sigma)|s)$ is an eigenfunction of $k(A_\rho)$ corresponding to this eigenvalue. Similarly by considering the second parameter group it may be shown that $(q|U(a^\sigma)|s)$ is also an eigenfunction of $k(B_\rho)$ corresponding to the eigenvalue k_s.

In order to identify the functions $(q|U(a^{\sigma})|s)$ of the r parameters completely, we need a set of at least r commuting operators acting on these parameters.

We are already in possession of the ℓ commuting operators $F_i(A_\rho) = F_i(B_\rho)$. Hence we still need $\frac{r-\ell}{2}$ *) operators $k(X_\rho)$, in order to have $\frac{r-\ell}{2}$ operators $k(A_\rho)$ and the same number of $k(B_\rho)$. However, ℓ such operators $k(X_\rho)$ are already known to us; they are the H_i themselves. Hence for the set of commuting operators to be complete we need at least to construct $\frac{r-3\ell}{2}$ operators $k(X_\rho)$.

In the particular case of the group O_3, $\frac{r-3\ell}{2} = 0$, and it is well known that the operators J_z and J^2 form the complete set. The problem of finding the complete set of operators $k(X_\rho)$ has so far been solved only for some types of simple groups.

3. Explicit construction of the irreducible representations. The representations of the infinitesimal operators of O_3 are the diagonal matrix J_z and the matrices $J_x \pm iJ_y$ whose only non-vanishing matrix elements are given by

$$(j\ m \pm 1\ |\ J_x \pm i\ J_y\ |\ j\ m) = \sqrt{(j \pm m + 1)(j \mp m)}. \qquad (120)$$

In the general case, the corresponding formula should be

$$(L\ M + \alpha\ k_q^{(h)} | E_\alpha | L\ M\ k_s^{(h)}) = f(L\ M \alpha\ k_q^{(h)}\ k_s^{(h)}), \qquad (121)$$

but as long as the $k^{(h)}$ are not known it is impossible to give an explicit form to the function f. Later on we shall present some special methods for

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*) The structure of semi-simple groups assures that this number is always an integer.

solving this problem in particular cases in which we are interested.

4. Decomposition of the Kronecker product.

In O_3 this is done by the Clebsch-Gordan series:

$$\mathcal{D}(J_1) \times \mathcal{D}(J_2) = \sum_{J=|J_1-J_2|}^{J_1+J_2} \mathcal{D}(J). \qquad (122)$$

In general we have seen that

$$\mathcal{D}(L^{(1)}) \times \mathcal{D}(L^{(2)}) = \mathcal{D}(L^{(1)}+L^{(2)}) + \ldots \qquad (123)$$

but we were not in a position to say anything about the other terms of the series. The coefficients in this series have been given by Brauer and Weyl* by using the characters of the representations.

But this is only the first part of the problem, since we not only need to know which irreducible representations are contained in a Kronecker product, but we also want to calculate the matrix which actually decomposes the Kronecker product.

For O_3 this problem has been solved in several different ways.

The classical Clebsch-Gordan method exploits the homomorphism between O_3 and the unimodular group in two dimensions (which is the basis of spinor calculus), but this method is applicable only to this particular case and is not capable of generalization. Wigner solved the same problem by performing integrations over the whole group, but actually it is sufficient to consider the infinitesimal representation, as will be indicated here.

The transformation coefficients $(j_1m_1j_2m_2|j_1j_2JM) \equiv (m_1m_2|JM)$ are defined by the relation

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* Reference (13), p.229.

$$\sum_{m_1'' m_2''} (J''M''|m_1''m_2'')\left\{(m_1''|j_{1x} \pm ij_{1y}|m_1') + (m_2''|j_{2x} \pm ij_{2y}|m_2')\right\}(m_1'm_2'|JM)$$

$$= (J''M''|J_x \pm iJ_y|JM)\ \delta_{JJ''}\ .$$

Matrix multiplication of this equation from the left by $(m_1m_2|J''M'')$ gives

$$\sum_{m_1'm_2'}\left\{(m_1|j_{1x} \pm i\ j_{1y}|m_1') + (m_2|j_{2x} \pm i\ j_{2y}|m_2')\right\}(m_1'm_2'|JM)$$

$$= \sum_{M''}(m_1m_2|J\ M'')(J\ M''|J_x \pm i\ J_y|JM)$$

which, using (120), becomes the recursion formula:

$$(m_1|j_{1x} \pm ij_{1y}|m_1 \mp 1)(m_1 \mp 1\ m_2|JM) + (m_2|j_{2x} \pm ij_{2y}|m_2 \mp 1)(m_1\ \ m_2 \mp 1|JM)$$

$$= (JM \pm 1|J_x \pm iJ_y|JM)(m_1m_2|JM \pm 1). \qquad (124)$$

If we take the upper sign and set $M = J$, we see that the right hand side vanishes and we find a set of equations which determines the different $(m_1m_2|JJ)$ apart from a common factor whose absolute value is fixed by normalization and whose phase is fixed by the convention that $(j_1\ J-j_1|JJ)$ be real and positive. Taking now the lower sign we obtain $(m_1m_2|J\ M-1)$ from $(m_1m_2|JM)$; hence by a 'ladder' procedure starting from $M=J$ we get all the transformation coefficients.

This method would probably be the one best suited for extension to the other groups provided the right hand side of (121) were known explicitly.

§6. The Full Linear Group and the Unitary Group.

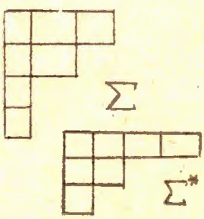
We saw on P.29 that the full linear group in k dimensions (as well as its unitary subgroup) is not semi-simple. But, since it is the direct product of a semi-simple group with an Abelian group, the full linear group shares many properties with the semi-simple groups, including the possibility of bringing the commutation relations to the standard form (50) and all the results of §§ 1 and 2. It is also clear from p.29 that, as the unimodular condition is omitted, H_i has now to be identified with X_{ii} and not with X'_{ii} defined by (69'); the components of the weights are now always integers, and the relations (72) and (88') which were obtained for the unimodular group do not hold for the full linear group.

It can be shown that a tensor of rank f in the k-dimensional space which has the symmetry defined by the partition $\Sigma = (f_1, f_2, \ldots, f_k)$, with

$$f = f_1 + f_2 + \ldots + f_k$$

is a basis of the representation whose highest weight has the components f_i.

We may conveniently illustrate a partition Σ by a Young diagram such as the one at the left, consisting of f boxes in k rows, the i^{th} row containing f_i boxes.



A partition Σ^* is said to be dual to Σ if its diagram is obtained by interchanging the rows and columns of Σ.

In particular, the partition

$$f = \underbrace{1 + 1 + \ldots + 1}_{f} + \underbrace{0 + 0 + \ldots + 0}_{k-f}$$

which is possible if $k \geq f$, characterizes a representation $\mathcal{R}_f$ of degree $\binom{k}{f}$

whose basis is formed by the totally antisymmetrical tensors of rank f.

The irreducible representations of the full linear group do not decompose if we restrict the group to its unimodular subgroup, but the representations which belong to the partitions $(f_1, \ldots, f_k)$ and $(f_1 + e, f_2 + e, \ldots, f_k + e)$ become equivalent.

All the properties which we have stated for the full linear group hold also for its unitary subgroup.

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Lectures 7 and 8.

THE CALCULATION OF THE ENERGY MATRIX.

§ 1. The Interaction of Two Particles.

Since the interaction matrix for n particles is calculated according to (132) in terms of that for n - 1 particles, we must start by calculating the interaction energy for two particles. Let us first assume for simplicity that there is an ordinary spin-independent interaction (Wigner interaction), given by $J(r_{12}) = J\left(\sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\omega_{12}}\right)$ between the two particles. We can expand this in Legendre polynomials of $\cos\omega_{12}$:

$$J(r_{12}) = \sum_k J_k(r_1,r_2)\, P_k(\cos\omega_{12}) \tag{193}$$

so that, by the addition theorem (156), they can be expressed in terms of scalar products of tensors:

$$J(r_{12}) = \sum_k J_k(r_1,r_2)\, (\mathbf{C}_1^{(k)} \cdot \mathbf{C}_2^{(k)}) \tag{194}$$

where

$$C_{iq}^{(k)} = \sqrt{\frac{4\pi}{2k+1}}\; Y_{kq}(\theta_i\varphi_i) \tag{195}$$

is the q^{th} component of $\mathbf{C}_i^{(k)}$.

The matrix giving the interaction of two particles is in general

$$\begin{aligned}(n_1\ell_1n_2\ell_2LM|J(r_{12})|n_1\ell_1n_2\ell_2LM) &= \\ &= \sum_k (\ell_1\ell_2LM|(\mathbf{C}_1^{(k)}\cdot\mathbf{C}_2^{(k)})|\ell_1\ell_2LM)F^k \\ &= \sum_k (-)^{\ell_1+\ell_2-L}(\ell_1\|C^{(k)}\|\ell_1)(\ell_2\|C^{(k)}\|\ell_2)\cdot \\ &\qquad \cdot W(\ell_1\ell_2\ell_1\ell_2;Lk)\,F^k.\end{aligned} \tag{196}$$

Lectures 5 and 6

THE EIGENFUNCTIONS OF THE NUCLEAR SHELLS

§1. Introduction.

If we wish to calculate the energy levels of a system of many particles, the fact that we cannot solve directly the Schrödinger equation for the many-body problem forces us to proceed by successive approximations.

In atomic spectroscopy we assume that in the "zeroth approximation" every electron moves independently of the others in a central field which is the superposition of the fields of the nucleus and of the mean field produced by the other electrons. In this approximation we may assign to every electron four quantum numbers $n\,\ell\,m_s\,m_\ell$; as the zeroth order energy depends only on n and ℓ, the electrons appear to be distributed in different shells, each characterized by a pair of values $n\ell$. Such a distribution is called a configuration.

The next step is to take as a perturbation the interaction between electrons in shells which are not closed, neglecting in first approximation the matrix elements which connect different configurations.

It is well known that applied to atomic spectroscopy this method gives good results. It is also well known that the theoretical arguments for using this method in nuclear spectroscopy are very weak, but that there is on the other hand some empirical evidence that the nucleons also are ordered in shells. We shall not, however, discuss here the validity of the nuclear shell model.

It is the purpose of the remaining lectures to show some applications of group theory to the classification of the levels of a nuclear shell and to the calculation in first approximation of the perturbation energy.

As is customary in dealing with problems of spectroscopy we shall use the standard notation of reference 15.

§2. The Coefficients of Fractional Parentage.

If a shell contains one particle, the quantum numbers $m_\tau\ m_s\ m_\ell$ describe the state completely. If a shell contains two particles, one can use the quantum numbers $m_\tau^{(1)} m_s^{(1)} m_\ell^{(1)} m_\tau^{(2)} m_s^{(2)} m_\ell^{(2)}$ or, alternatively, $T\ S\ L\ M_T\ M_S\ M_L$ of which the second scheme is the more useful, since it diagonalizes the energy; the transformation leading from the one of these schemes to the other is given by the Clebsch-Gordan coefficients. A further advantage of the second scheme is that in it the states are either symmetrical or antisymmetrical, depending on the parity of T + S + L; the exclusion principle simply removes the states for which T + S + L is even, without changing the scheme.

If we add to the allowed states of ℓ^2 a third ℓ-particle, we obtain a set of wave functions

$$\psi(\ell^2(T^{(12)}S^{(12)}L^{(12)})\,\ell,\ T\ S\ L\ M_T\ M_S\ M_L)\,, \qquad (125)$$

which are in general antisymmetrical only with respect to the first two particles, but not with respect to the third. If to (125) we apply the transformation

$$\psi(\ell^2(T^{(12)}S^{(12)}L^{(12)})\ell,\ T\ S\ L\ M_T\ M_S\ M_L) =$$

$$= \sum_{T^{(23)}S^{(23)}L^{(23)}} \psi(\ell,\ \ell\ell(T^{(23)}S^{(23)}L^{(23)}),\ T\ S\ L\ M_T\ M_S\ M_L)\cdot$$

$$\cdot(\tfrac{1}{2},\ \tfrac{1}{2}\tfrac{1}{2}(T^{(23)}),\ T|\tfrac{1}{2}\tfrac{1}{2}(T^{(12)})\tfrac{1}{2},T)(\tfrac{1}{2},\ \tfrac{1}{2}\tfrac{1}{2}(S^{(23)}),\ S|\tfrac{1}{2}\tfrac{1}{2}(S^{(12)})\tfrac{1}{2},\ S)\cdot$$

$$(\ell,\ \ell\ell(L^{(23)}),\ L|\ell\ell(L^{(12)})\ell,\ L), \qquad (126)$$

we see that in the expansion appear some terms which will be symmetrical rather than antisymmetrical in the last two particles.

The eigenfunctions of the configuration ℓ^3, which have to be antisymmetrical in all three particles, span a subspace of the space spanned by the functions (125) and will thus be linear combinations of them:

$$\psi(\ell^3\alpha\ T\ S\ L) = \sum_{T^{(12)}S^{(12)}L^{(12)}} \psi(\ell^2(T^{(12)}S^{(12)}L^{(12)})\ell,\ T\ S\ L)\cdot$$

$$\cdot(\ell^2(T^{(12)}S^{(12)}L^{(12)})\ell,\ T\ S\ L\,|\}\ \ell^3\alpha\ T\ S\ L). \qquad (127)$$

We have omitted $M_T\ M_S\ M_L$ from the notation because they play no role in the transformation; α distinguishes independent states of ℓ^3 which have the same values of T S L. The notation $(|\})$ is a reminder that this transformation matrix is not square, since on the left side we have all states which are antisymmetrical in (1,2), while on the right we have only those states which are antisymmetrical in (1,2,3). The coefficients of this linear combination are called coefficients of fractional parentage or, for short, c.f.p.

If (127) is to be antisymmetrical in all three particles, this requires that when (126) is substituted into (127), all those coefficients which belong to forbidden wave functions shall vanish, and it is easy to see that the necessary and sufficient condition for this is

$$\sum_{T^{(12)}S^{(12)}L^{(12)}} (\tfrac{1}{2}, \tfrac{1}{2}\tfrac{1}{2}(T^{(23)}), T \,|\, \tfrac{1}{2}\tfrac{1}{2}(T^{(12)})\tfrac{1}{2}, T) \cdot$$

$$\cdot (\tfrac{1}{2}, \tfrac{1}{2}\tfrac{1}{2}(S^{(23)}), S \,|\, \tfrac{1}{2}\tfrac{1}{2}(S^{(12)})\tfrac{1}{2}, S)(\ell, \ell\ell(L^{(23)}), L \,|\, \ell\ell(L^{(12)})\ell, L) \cdot$$

$$(\ell^2(T^{(12)}S^{(12)}L^{(12)})\ell, T\,S\,L \,|\}\, \ell^3\alpha\, T\,S\,L) = 0 \qquad (128)$$

when $T^{(23)} + S^{(23)} + L^{(23)}$ is even.

This system of equations contains all the information we need for the configuration ℓ^3, since the number of independent solutions for given T S L, which are distinguished by the parameter α, is the number of allowed states of this kind, and since we can also use the c.f.p. to calculate the interaction energy for these three-particle states:

$$(\ell^3\alpha\, T\,S\,L \,|\, E \,|\, \ell^3\alpha'\, T\,S\,L) = 3\sum_{T^{(12)}S^{(12)}L^{(12)}} (\ell^3\alpha\, T\,S\,L \,\{|\, \ell^2(T^{(12)}S^{(12)}L^{(12)})\ell, T\,S\,L)$$

$$\cdot E(T^{(12)}S^{(12)}L^{(12)}) \cdot (\ell^2(T^{(12)}S^{(12)}L^{(12)})\ell, T\,S\,L \,|\}\, \ell^3\alpha'\, T\,S\,L\,); \qquad (129)$$

$E(T^{(12)}S^{(12)}L^{(12)})$ is the interaction energy for the two-particle system, and the factor 3 enters because there are three pairs of particles in the configuration.

The extension of these methods to a shell which contains n particles is in principle immediate. We start from a shell with n-1 particles, for which we suppose the c.f.p. to be calculated. Then

$$\psi(\ell^n\alpha\, T\,S\,L) = \sum_{\alpha_1 T_1 S_1 L_1} \psi(\ell^{n-1}(\alpha_1 T_1 S_1 L_1)\ell, T\,S\,L) \cdot$$

$$\cdot (\ell^{n-1}(\alpha_1 T_1 S_1 L_1)\ell, T\,S\,L \,|\}\, \ell^n\alpha\, T\,S\,L) \qquad (130)$$

where, analogously to (128), the c.f.p. satisfy the system of equations

$$\sum_{\alpha_1 T_1 S_1 L_1} (T_2, \tfrac{1}{2}\tfrac{1}{2}(T'), T \mid T_2 \tfrac{1}{2}(T_1)\tfrac{1}{2}, T)(S_2, \tfrac{1}{2}\tfrac{1}{2}(S'), S \mid S_2 \tfrac{1}{2}(S_1)\tfrac{1}{2}, S)$$

$$\cdot (L_2, \ell\ell(L'), L \mid L_2\ell(L_1)\ell, L)(\ell^{n-2}(\alpha_2 T_2 S_2 L_2)\ell, T_1 S_1 L_1 \|\} \ell^{n-1}\alpha_1 T_1 S_1 L_1)$$

$$\cdot (\ell^{n-1}(\alpha_1 T_1 S_1 L_1)\ell, T\ S\ L \{\| \ell^n \alpha T\ S\ L) = 0 \qquad (131)$$

for every value of T_2, S_2, L_2 and $T' + S' + L'$ even.

The interaction energy is given by

$$(\ell^n \alpha T\ S\ L \mid E \mid \ell^n \alpha' T\ S\ L) = \frac{n}{n-2} \sum_{\alpha_1 \alpha_1' T_1 S_1 L_1} (\ell^n \alpha T\ S\ L \{\| \ell^{n-1}(\alpha_1 T_1 S_1 L_1)\ell, T\ S\ L)$$

$$\cdot (\ell^{n-1}\alpha_1 T_1 S_1 L_1 \mid E \mid \ell^{n-1}\alpha_1' T_1 S_1 L_1)(\ell^{n-1}(\alpha_1' T_1 S_1 L_1)\ell, T\ S\ L \|\} \ell^n \alpha' T\ S\ L). \qquad (132)$$

Although this procedure has been used successfully to calculate all atomic configurations d^n and the configuration f^3, it becomes extremely laborious for the higher configurations, and it is at this point that group theory comes to our aid in the following three ways:

a) The hitherto unspecified variable α will be replaced by a set of quantum numbers which is almost complete. The choice of these quantum numbers, suggested by group theory, will greatly simplify the calculations.

b) The c.f.p. will be calculated without the use of the cumbersome equations (131).

c) The summations in (132) will be simplified.

§3. The Classification of the States of ℓ^n.

The states of a single particle in a given shell are characterized by the set of quantum numbers $m_\tau\, m_s\, m_\ell$. There are $4(2\ell+1)$ independent states to which correspond the $4(2\ell+1)$ eigenfunctions $\phi(m_\tau\, m_s\, m_\ell)$. If we have n particles in the same shell, the configuration has $\binom{4(2\ell+1)}{n}$ independent antisymmetrical states to which correspond the eigenfunctions $\psi(\ell^n\Gamma)$, where Γ is a set of quantum numbers which may assume $\binom{4(2\ell+1)}{n}$ different values.

If we consider the $\phi(m_\tau\, m_s\, m_\ell)$ as the basis vectors of the $4(2\ell+1)$-dimensional space of the states of a single particle in a given shell, the $\psi(\ell^n\Gamma)$ will form a complete set of antisymmetrical tensors of rank n in this space. This means that a unitary transformation

$$\phi'(m_\tau'\, m_s'\, m_\ell') = \sum_{m_\tau m_s m_\ell} \phi(m_\tau\, m_s\, m_\ell)\, c\,(m_\tau m_s m_\ell\, ;\, m_\tau'\, m_s'\, m_\ell') \tag{133}$$

on the ϕ's will induce in the ψ's the transformation

$$\psi'(\ell^n\Gamma') = \sum_{\Gamma} \psi(\ell^n\Gamma)\, c\,(\Gamma, \Gamma'). \tag{134}$$

The $\psi(\ell^n\Gamma)$ are therefore the basis of a representation $\mathcal{A}_n$ of degree $\binom{4(2\ell+1)}{n}$ of the unitary group $U_{4(2\ell+1)}$, characterized by the partition

$$n = 1+1+1+\ldots+1+0+0+\ldots+0.$$

In order to obtain a set of functions $\psi(\ell^n\Gamma)$ which will make the matrix of the perturbation energy as nearly diagonal as possible, we have to restrict the group $U_{4(2\ell+1)}$ to its largest subgroup under which the perturbation energy is invariant. If we assume that the interaction between particles is central and charge-independent, this group is the group of three independent rotations in the coordinate space, spin space, and isotopic spin space. If we proceeded in this way, which is traditional in the application of group theory to quantum mechanics, we should obtain the group theoretical definitions of only the six quantum numbers T S L M_T M_S M_L.

But since we want to obtain a more nearly complete set of quantum numbers, even though they may not be <u>good</u> quantum numbers, we shall rather carry out the transition from $U_{4(2\ell+1)}$ to $R_3 \times R_3 \times R_3$ by successive steps. We shall therefore impose successive restrictions on the group $U_{4(2\ell+1)}$ to obtain subspaces of the $\binom{4(2\ell+1)}{n}$-dimensional space of the representation $\mathcal{A}_n$ which will be invariant with respect to different subgroups. These subspaces are characterized by the highest weights of the representations to which they belong, and these highest weights will be our new quantum numbers.

We shall start by considering the subgroup of $U_{4(2\ell+1)}$ which consists of those transformations (133) which are of the form

$$c(m_\tau m_s m_\ell \,;\, m'_\tau m'_s m'_\ell) = \gamma(m_\tau m_s \,;\, m'_\tau m'_s)\, \bar{c}\,(m_\ell m'_\ell), \tag{135}$$

with γ and $\bar{c}$ unitary; this subgroup is the direct product $U_4 \times U_{2\ell+1}$. If we denote by $\mathcal{H}_\Sigma$ the irreducible representations of $U_{2\ell+1}$ and by $\mathcal{G}_{\Sigma'}$ those of U_4, the irreducible representations of $U_4 \times U_{2\ell+1}$ will be the Kronecker products

$$\mathcal{G}_{\Sigma'} \times \mathcal{H}_\Sigma . \tag{136}$$

Every irreducible representation of $U_{4(2\ell+1)}$ is a reducible representation of $U_4 \times U_{2\ell+1}$ and breaks up into representations (136); the general law of decomposition is somewhat complicated, but for the particular case of the representation $\mathcal{A}_n$ it is very simple: only those representations (136) appear in the decomposition of $\mathcal{A}_n$ for which Σ' is Σ^*, the partition dual to Σ, and every representation of this kind occurs only once. Since the Young diagram which illustrates the partition Σ has not more than $2\ell+1$ rows, the length of a row in the diagram of Σ^* cannot exceed $2\ell+1$. Similarly, the length of a row of Σ cannot exceed four.

If the elements of the basis of $\mathcal{G}_{\Sigma^*}$ are characterized by a set of quantum numbers θ and those of $\mathcal{H}_\Sigma$ by a set Δ, the elements of the basis of $\mathcal{G}_{\Sigma^*} \times \mathcal{H}_\Sigma$ will be characterized by the set $\theta\,\Delta$, and the states of ℓ^n by the set $\Sigma\,\theta\,\Delta$.

As a second step we restrict $U_{2\ell+1}$ to the orthogonal subgroup $R_{2\ell+1}$ which leaves invariant the bilinear symmetric form

$$\sum_{m_\ell} (-)^{m_\ell} \phi_1(m_\ell)\, \phi_2(-m_\ell); \tag{137}$$

$R_{2\ell+1}$ has R_3 as a subgroup because (137) is proportional to the eigenfunction of the S-state of ℓ^2, which is left invariant by R_3. Let the irreducible representation of $R_{2\ell+1}$ whose highest weight is W be $\mathcal{B}_W$. Then in $R_{2\ell+1}$,

$$\mathcal{H}_\Sigma = \sum_W b_W\, \mathcal{B}_W\,. \tag{138}$$

The possibility that we may have $b_W > 1$ gives rise to a running index β to number the equivalent representations, but in practice β takes on only small values. (For the states d^n, $b_W \leq 2$.)

The next step of the reduction is to restrict the orthogonal matrices $\bar{c}(m_\ell\, m'_\ell)$ to the particular matrices belonging to the representation $\mathcal{D}_\ell$ of R_3. When this is done, every $\mathcal{B}_W$ becomes a representation of R_3 and will in general decompose as

$$\mathcal{B}_W = \sum_L c_L\, \mathcal{D}_L\,. \tag{139}$$

where L is the highest weight of $\mathfrak{D}_L$. If $c_L > 1$, another running index γ distinguishes the various $\mathfrak{D}_L$ belonging to a given L. We have thus arrived at the following scheme:

$$\psi(\ell^n \Sigma \theta \beta W \gamma L M_L) \,. \tag{140}$$

For the nuclear configuration d^n, c_L is never larger than three, but for higher values of ℓ it is expected to be much larger. In the particular case of the configuration f^n we may avoid such large values for γ if we avail ourselves of the fortunate coincidence that when $\ell = 3$ there exists another group, contained in R_7 and containing the representation $\mathfrak{D}_3$ of R_3 which is a realization of G_2 and may be used to introduce a new subclassification.*

In order to complete the scheme (140), we must now perform an analogous reduction for U_4. If we restrict it to its unimodular subgroup, we obtain the semi-simple group belonging to the vector diagram A_3, which we have seen to be the same as D_3. Therefore the unitary unimodular group in four dimensions is isomorphic with R_6. Applying to $\Sigma^* = [\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4]$ the transformation (68), we obtain as highest weight of the representation of R_6

$$P = \tfrac{1}{2}(\Lambda_1 + \Lambda_2 - \Lambda_3 - \Lambda_4), \quad P' = \tfrac{1}{2}(\Lambda_1 - \Lambda_2 + \Lambda_3 - \Lambda_4), \quad P'' = \tfrac{1}{2}(\Lambda_1 - \Lambda_2 - \Lambda_3 + \Lambda_4). \tag{141}$$

This transformation, introduced by Wigner, corresponds to considering instead of the unimodular unitary group in four dimensions the homomorphic group R_6 which is the group of rotations in the six-dimensional space of the spin and isotopic spin.

From R_6 we go on to the subgroup $R_3 \times R_3$ by restricting the transformations γ of (135) to those which are of the form

* See reference <u>23</u>, section 4, subsection 3.

$$\gamma(m_\tau\, m_s;\, m'_\tau\, m'_s) = \gamma_1(m_\tau\, ;\, m'_\tau\,)\ \gamma_2(m_s;\, m'_s)\ . \tag{142}$$

In this subgroup the representations $\mathcal{G}_{\Sigma^*}$ may be decomposed:

$$\mathcal{G}_{\Sigma^*} \equiv \mathcal{G}_{PP'P''} = \sum_{T\,S} a_{TS}\ \mathcal{D}_T \times \mathcal{D}_S\ .$$

In case $a_{TS} > 1$ we have to introduce a new running index α ; this gives us finally a scheme for the wave functions of the entire shell

$$\psi(\ell^n \Sigma\, \alpha\, T\, S\, M_T\, M_S\, \beta\, W\, \gamma\, L\, M_L). \tag{143}$$

Except for the presence of the running indices $\alpha\, \beta\, \gamma$, we have found a complete set of quantum numbers, and we have achieved the first of the three purposes set forth in §1.

§4. The Factorization of the Coefficients of Fractional Parentage.

The wave functions on the right of (130) transform according to $\mathcal{A}_{n-1} \times \mathcal{A}_1$; that on the left transforms according to $\mathcal{A}_n$. Thus it is evident that the c.f.p. are a rectangular part of the matrices which perform the decomposition

$$\mathcal{A}_{n-1} \times \mathcal{A}_1 = \mathcal{A}_n + \ldots\ldots \tag{144}$$

The calculation of these matrices is simplified by the following considerations:

We have seen that if a group g has a subgroup h, an irreducible representation $U_A(s)$ of g will in general be reducible in the subgroup h, consisting of elements t. Let us assume that the matrices $U_A(t)$ have been reduced:

$$(\beta\, B\, b\,/\, U_A(t)\, /\, \beta' B' b') = (b\,|\, V_B(t) | b')\ \delta_{BB'} \delta_{\beta\beta'}, \tag{145}$$

where B specifies the different representations of h, b denotes their rows and columns, and β is a running index distinguishing equivalent irreducible representations. The Kronecker product $U_{A_1} \times U_{A_2}$ of two irreducible representations A_1 and A_2 of g can be completely reduced:

$$U_{A_1} \times U_{A_2} = \sum_{A} c_A U_A \tag{146}$$

by a similarity transformation with a matrix

$$(A_1 \beta_1 B_1 b_1; A_2 \beta_2 B_2 b_2 | A_1 A_2 \alpha A \beta B b) \tag{147}$$

where the parameter α is a running index which enumerate the A's whenever a c_A is greater than 1 in (146). We shall now state without proof* a corollary to Schur's lemma which will enable us to express this matrix in a simpler way.

The matrix elements of the transformation (147) are the products of the matrix elements of the transformation which reduces the Kronecker product $U_{B_1}(t) \times U_{B_2}(t)$ in h and coefficients which are independent of the b's:

$$(A_1 \beta_1 B_1 b_1; A_2 \beta_2 B_2 b_2 | A_1 A_2 \alpha A \beta B b) =$$

$$= (B_1 b_1 B_2 b_2 | B_1 B_2 B b)(A_1 \beta_1 B_1; A_2 \beta_2 B_2 | A_1 A_2 \alpha A \beta B). \tag{148}$$

If we take the representation of $\mathcal{R}_n$ in the scheme (143) and apply this lemma to each subgroup of the chain which was constructed in the preceding section, we can bring the matrix which reduces the direct product (144) into the form

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* For the proof see reference 23, section 3.

$$(\ell^{n-1}(\Sigma_1\alpha_1 T_1 S_1 M_{T_1} M_{S_1} \beta_1 W_1 \gamma_1 L_1 M_{L_1}); \tfrac{1}{2}\tfrac{1}{2}\ell\, m_\tau m_s m_\ell | \ell^n \Sigma \alpha\, T\, S\, M_T M_S \beta W \gamma L\, M_L) =$$

$$= (T_1 M_{T_1} \tfrac{1}{2} m_\tau | T_1 \tfrac{1}{2} T\, M_T)\cdot(S_1 M_{S_1} \tfrac{1}{2} m_s | S_1 \tfrac{1}{2} S\, M_S)\cdot(\Sigma^*_1 \alpha_1 T_1 S_1; [1]\, \tfrac{1}{2}\, \tfrac{1}{2} | \Sigma^* \alpha\, T\, S)\cdot$$

$$\cdot(L_1 M_{L_1} \ell m_\ell | L_1 \ell\, L\, M_L)\cdot(W_1 \gamma_1 L_1;\ (1)\ell | W \gamma L)\cdot$$

$$\cdot(\Sigma_1 \beta_1 W_1;\ [1](1) | \Sigma \beta W)\cdot(\ell^{n-1} \Sigma_1; \ell [1] | \ell^n\, \Sigma), \qquad (149)$$

where the symbol (1) means $W = (1\ 0\ \ldots\ 0)$ and [1] means $\Sigma = [1\ 0\ \ldots\ 0]$. This expression as it stands is not exactly the c.f.p., since the wave functions on the right hand side of (130) contain already the M-dependent factors of (149); hence we have

$$(\ell^{n-1}(\Sigma_1 \alpha_1 T_1 S_1 \beta_1 W_1 \gamma_1 L_1)\ell,\ T\ S\ L |\} \ell^n \Sigma \alpha T\ S \beta W \gamma L) =$$

$$= (\Sigma^*_1 \alpha_1 T_1 S_1;\ [1]\ \tfrac{1}{2}\ \tfrac{1}{2} | \Sigma^* \alpha\ T\ S)(W_1 \gamma_1 L_1;\ (1)\ell | W \gamma L)\cdot$$

$$\cdot(\Sigma_1 \beta_1 W_1;\ [1](1) | \Sigma \beta W)(\ell^{n-1} \Sigma_1; \ell [1] | \ell^n \Sigma). \qquad (150)$$

Thus the problem of calculating the c.f.p. is reduced to the separate calculation of the different factors which appear in this equation; but before we solve this last problem we have to develop a new mathematical tool.

§5. The Algebra of Tensor Operators.

The algebra of vector operators and of their representation by matrices was developed by Güttinger and Pauli* and is presented in standard form in Chapter III of Condon and Shortley (reference 15). The possibility of extending it to tensors was indicated by Eckart and Wigner**. We shall outline it here following reference 21, § 3, where the problem is treated by the standard methods of Condon and Shortley.

We define an irreducible tensor $\underset{\sim}{T}^{(k)}$ of degree k to be a set of 2k + 1 quantities $T_q^{(k)}$, $-k \leq q \leq k$, which under rotations in three-dimensional space transform like the 2k + 1 spherical harmonics of degree k. If the operators J_x, J_y, J_z operate on these quantities, we have

$$J_z\, T_q^{(k)} = T_q^{(k)} (kq|J_z|kq) = q\, T_q^{(k)} \tag{151a}$$

$$(J_x \pm iJ_y)\, T_q^{(k)} = T_{q\pm1}^{(k)} (kq \pm 1|J_x \pm iJ_y|kq) = \sqrt{(k \pm q + 1)(k \mp q)}\, T_{q\pm1}^{(k)} \tag{151b}$$

If the $T_q^{(k)}$ are themselves operators, the left side of (151) must be replaced by commutators

$$[J_z, T_q^{(k)}] = T_q^{(k)} (kq|J_z|kq) = q\, T_q^{(k)} \tag{152a}$$

$$[J_x \pm iJ_y, T_q^{(k)}] = T_{q\pm1}^{(k)} (kq \pm 1|J_x \pm iJ_y|kq) = \sqrt{(k \pm q + 1)(k \mp q)}\, T_{q\pm1}^{(k)}. \tag{152b}$$

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* Zeits. fur Physik, 67, 743, (1931).

** See references 17 and 18.

As in vector algebra, it is possible to define many kinds of tensorial products. Guided by the example of the vector addition law in quantum mechanics we shall define the tensor product of order K by the equation

$$X^{(K)}_{Q} = \sum_{q_1 q_2} T^{(k_1)}_{q_1} \; U^{(k_2)}_{q_2} \; (k_1\, q_1\, k_2\, q_2 | k_1\, k_2\, K\, Q), \tag{153}$$

and it is easy to verify that this satisfies (152). The unitarity of the Clebsch-Gordan coefficients permits us to solve this equation

$$T^{(k_1)}_{q_1} \; U^{(k_2)}_{q_2} = \sum_{K Q} X^{(K)}_{Q} (k_1\, k_2\;\; K\, Q | k_1\, q_1\, k_2\, q_2). \tag{154}$$

According to the definition (153) it would be logical to define a scalar product as $X^{(o)}_{o}$. However, it is traditional to define as scalar product the quantity

$$(\underset{\sim}{T}^{(k)} \cdot \; \underset{\sim}{U}^{(k)}) = \sum_{q} (-)^{q} \, T^{(k)}_{q} \;\; U^{(k)}_{-q} \; = (-)^{k} \; \sqrt{2k + 1} \; X^{(o)}_{o} \; . \tag{155}$$

An example of this formula is the addition theorem for spherical harmonics

$$P_k \,(\cos \omega_{12}) = \frac{4\pi}{2k + 1} \; \sum_{q} (-)^{q} \, Y_{kq} \, (\theta_1 \varphi_1) \; Y_{k-q} \, (\theta_2 \varphi_2), \tag{156}$$

where

$$\cos \omega_{12} = \cos \theta_1 \, \cos \theta_2 + \sin \theta_1 \, \sin \theta_2 \, \cos \, (\varphi_1 - \varphi_2). \tag{156'}$$

If we represent the components of a tensor $T^{(k)}_{q}$ in the scheme αjm and write down (152) in the form of relations between matrices, (152a) tells us that the non-vanishing elements of $(\alpha\, jm | T^{(k)}_{q} | \alpha' j' m')$

satisfy the selection rule $m - m' = q$ and (152b) reduces to (124) if we replace $(\alpha jm|T_q^{(k)}|\alpha' j' m')$ by $(j'm'kq|j'kjm)$. Since (124) was sufficient to determine $(j'm'kq|j'kjm)$ apart from a normalization factor, we obtain

$$(\alpha jm|T_q^{(k)}|\alpha j'm') = A(j'm'kq|j'kjm), \tag{157}$$

with A independent of m and q.

In order to bring out the symmetries of the Clebsch-Gordan coefficients, it will be convenient to introduce the notation

$$(j_1m_1j_2m_2|j_1j_2jm) = (-)^{j+m}\sqrt{2j+1}\;\; V(j_1j_2j;\;\; m_1m_2 - m) \tag{158}$$

where

$$V(abc;\alpha\beta\gamma) = \delta_{\alpha+\beta+\gamma}\Delta(abc)\cdot$$

$$\cdot \sum_z (-)^{c-\gamma+z} \frac{[(a+\alpha)!(a-\alpha)!(b+\beta)!(b-\beta)!(c+\gamma)!(c-\gamma)!]^{\frac{1}{2}}}{z!(a+b-c-z)!(a-\alpha-z)!(b+\beta-z)!(c-b+\alpha+z)!}\cdot$$

$$\cdot (c-a-\beta+z)!\,, \tag{159}$$

and

$$\Delta(abc) = \left[\frac{(a+b-c)!(a+c-b)!(b+c-a)!}{(a+b+c+1)!}\right]^{\frac{1}{2}}. \tag{160}$$

The $V(abc;\alpha\beta\gamma)$ thus defined have the symmetries

$$V(abc;\alpha\beta\gamma) = (-)^{a+b-c}\, V(bac;\beta\alpha\gamma) = (-)^{a+b+c}\, V(acb;\alpha\gamma\beta) =$$

$$\tag{161a}$$

$$= (-)^{a-b+c}\, V(cba;\gamma\beta\alpha) = (-)^{2b}\, V(cab;\gamma\alpha\beta) = (-)^{2c}\, V(bca;\beta\gamma\alpha)$$

and

$$V(abc;\alpha\beta\gamma) = (-)^{a+b+c}\, V(abc;\,-\alpha\,-\beta\,-\gamma), \tag{161b}$$

and they vanish if a, b, c do not satisfy the triangle inequality, or if one of the numbers $a - |\alpha|$, $b - |\beta|$, $c - |\gamma|$ is negative. Further, they satisfy the orthogonality relations

$$\sum_{\alpha,\beta} V(abc;\alpha\beta\gamma)\, V(abc';\alpha\beta\gamma') = \frac{1}{2c+1}\, \delta_{cc'}\, \delta_{\gamma\gamma'} \quad \text{or } 0, \tag{162}$$

$$\sum_{c\gamma} (2c+1)\, V(abc;\alpha\beta\gamma)\, V(abc;\alpha'\beta'\gamma) = \delta_{\alpha\alpha'}\, \delta_{\beta\beta'} \quad \text{or } 0, \tag{163}$$

the zeros occuring if any of the above conditions are violated by the parameters on which there is no summation.

In terms of the V's thus defined, we write (157) as

$$(\alpha jm|T_q^{(k)}|\alpha' j'm') = (-)^{j+m} (\alpha j\|T^{(k)}\|\alpha' j') V(jj'k; -mm'q) \tag{164}$$

This equation divides the physical properties of the tensor, which are described by $(\alpha j\|T^{(k)}\|\alpha' j')$ from its geometrical properties as described by the V's.

As an example of the utility of this separation we calculate the matrices of the scalar product (155) and find out by (162) that

$$(\alpha jm|(\underset{\sim}{T}^{(k)} \cdot \underset{\sim}{U}^{(k)})|\alpha' j'm') =$$

$$= \frac{1}{2j+1}\, \delta_{jj'}\delta_{mm'} \sum_{\alpha'' j''} (-)^{j-j''} (\alpha j\| T^{(k)}\| \alpha'' j'')\, (\alpha'' j''\| U^{(k)}\|\alpha' j'), \tag{165}$$

which is, as required for a scalar, diagonal in j and m and independent of m.

In the practical applic tions, the most important scalar products are those in which the two tensors operate on different parts of a system.

(Examples are $\underset{\sim}{L}_1 \cdot \underset{\sim}{L}_2$, describing a coupling of the orbital angular momentum of two particles, $P_k(\cos\omega_{12})$ expressed by (156), or $\underset{\sim}{L} \cdot \underset{\sim}{S}$, in which space and spin functions belonging to the same system are coupled). If $\underset{\sim}{T}^{(k)}$ operates on part 1 of the system and $\underset{\sim}{U}^{(k)}$ operates on part 2, then expressed in the scheme $\alpha_1 \alpha_2 j_1 j_2 jm$, such a product is

$$(\alpha_1 \alpha_2 j_1 j_2 jm | (\underset{\sim}{T}^{(k)} \cdot \underset{\sim}{U}^{(k)}) | \alpha_1' \alpha_2' j_1' j_2' j' m') =$$

$$= \sum_{m_1 m_1' m_2 m_2' q} (-)^q (j_1 j_2 jm | j_1 m_1 j_2 m_2)(\alpha_1 j_1 m_1 | T_q^{(k)} | \alpha_1' j_1' m_1') \cdot$$

$$\cdot (\alpha_2 j_2 m_2 | U_{-q}^{(k)} | \alpha_2' j_2' m_2')(j_1' m_1' j_2' m_2' | j_1' j_2' j' m'). \qquad (166)$$

With (158) and (164), this involves sums over the products of four V's; it is found in general that

$$\sum_{\alpha\beta\gamma\delta\varphi} (-)^{f+\varphi} V(abe; \alpha\beta - \epsilon)\, V(acf; -\alpha\gamma\varphi)\, V(bdf; -\beta\delta - \varphi)\, V(cdg; \gamma\delta - \eta) =$$

$$= \frac{(-)^{e + c + f + d - b}}{2e + 1} W(abcd; ef)\, \delta_{eg}\, \delta_{\epsilon\eta}, \qquad (167)$$

where

$$W(abcd; ef) = \Delta(abe)\,\Delta(cde)\,\Delta(acf)\,\Delta(bdf) \cdot$$

$$\cdot \sum_z (-)^z \frac{(a + b + c + d + 1 - z)!}{(a + b - e - z)!(c + d - e - z)!(a + c - f - z)!(b + d - f - z)! \cdot}$$

$$\cdot\, z!\,(e + f - a - d + z)!(e + f - b - c + z)! \qquad (168)$$

Using (167) we obtain for (166) the expression

$$(\alpha_1\alpha_2 j_1 j_2 jm|(\underset{\sim}{T}^{(k)}\cdot\underset{\sim}{U}^{(k)})|\alpha_1'\alpha_2 j_1' j_2' j'm') =$$

$$= (-)^{j_1+j_2'-j}(\alpha_1 j_1\|T^{(k)}\|\alpha_1' j_1')(\alpha_2 j_2\|U^{(k)}\|\alpha_2' j_2')\cdot$$

$$\cdot W(j_1 j_2 j_1' j_2';\ jk)\delta_{jj'}\delta_{mm'}\ . \tag{169}$$

The geometrical interpretation of this formula is the following. If $\underset{\sim}{T}^{(k)}$ is a 2^k-pole moment whose average (expectation value) in the direction of j_1 is $(\alpha_1 j_1\|T^{(k)}\|\alpha_1 j_1)/\sqrt{2j_1+1}$ and similarly for $\underset{\sim}{U}^{(k)}$ with j_2, then the diagonal elements of their scalar product are given in the limit of large j_1 and j_2 and small k by the product of these average values with $P_k(\widehat{j_1 j_2})$ where $\widehat{j_1 j_2}$ is the angle between j_1 and j_2; indeed, the asymptotic value of $(-)^{j_1+j_2-j}\,W(j_1 j_2 j_1 j_2;\ jk)$ in (169) is just equal to $P_k(\widehat{j_1 j_2})/\sqrt{(2j_1+1)(2j_2+1)}$.

Also, the W's have many symmetries,

$$\begin{aligned} W(abcd;\ ef) &= W(badc;\ ef) = W(cdab;\ ef) = W(acbd;fe) \\ &= (-)^{e+f-a-d}\,W(ebcf;ad) = (-)^{e+f-b-c}\,W(aefd;\ bc)\,. \end{aligned} \tag{170}$$

The W's are useful also for expressing in the scheme $\alpha j_1 j_2 jm$ the components of a tensor which operates on part 1 or part 2. The matrix elements of $T_q^{(k)}$ are

$$(\alpha j_1 j_2 jm|T_q^{(k)}|\alpha' j_1' j_2' j'm') = \sum_{m_1 m_1' m_2}(j_1 j_2 jm|j_1 j_2 m_1 m_2)\cdot$$

$$\cdot(\alpha j_1 m_1|T_q^{(k)}|\alpha' j_1' m_1')\,(j_1' j_2 m_1' m_2|j_1' j_2 j'm');$$

using (164), (167) and the orthogonality relations of the V's, we get

$$(\alpha j_1 j_2 j \| T^{(k)} \| \alpha' j_1' j_2 j') =$$
$$= (-)^{j_2+k-j_1'-j} \sqrt{(2j+1)(2j'+1)} \quad (\alpha j_1 \| T^{(k)} \| \alpha' j_1') W(j_1 j j_1' j'; j_2 k). \tag{171}$$

Analogously for $\underset{\sim}{U}^{(k)}$

$$(\alpha j_1 j_2 \| U^{(k)} \| \alpha' j_1 j_2' j') =$$
$$= (-)^{j_1+k-j_2-j'} \sqrt{(2j+1)(2j'+1)} \quad (\alpha j_2 \| U^{(k)} \| \alpha' j_2') W(j_2 j j_2' j'; j_1 k). \tag{172}$$

The geometrical interpretation of (171) and (172) is the same as that of (169).

A further use of the W's is to express the transformation connecting different schemes of parentage:*

$$(j_1 j_2 (j_{12}) j_3, J \mid j_1, j_2 j_3 (j_{23}) J) =$$
$$= \sqrt{(2j_{12}+1)(2j_{23}+1)} W(j_1 j_2 J j_3; j_{12} j_{23}). \tag{173}$$

In general, every quantity which is invariant under rotations in three dimensions and therefore does not depend on the choice of axes or on m can be expressed in terms of the double-barred matrices and the W's.

§ 6. Tensor Operators and Lie Groups.

In lecture 4 we were not able to construct the matrices which decompose the Kronecker product of two representations because we did

* Ref. 22, Equ.(4)

not even possess a complete scheme, except, of course, in the case of the group O_3. Now that we have a nearly complete scheme, the way is open to a further attempt. But the scheme we have achieved is not that of the weights, in which the H_i are diagonal, but it is a new scheme characterizing the physical problem, and one in which $T\ S\ L\ M_T\ M_S\ M_L$ are diagonal. Together with the diagonality of the H_i we have also lost the selection rules for the operators E_α; it is, therefore, convenient also to change to a basis of the infinitesimal operators of the group which fits the new scheme better. We shall see that such a basis can be given in terms of an appropriate set of tensor operators.

Let us consider the unit tensor operators defined by

$$(n\ell \| u^{(k)} \| n'\ell') = \delta_{nn'}\, \delta_{\ell\ell'} \tag{174}$$

which connect only states within the same shell. The matrices of these operators are, by (164)

$$(\ell m | u^{(k)}_q | \ell m') = (-)^{\ell + m}\ V(\ell\ell k;\ -mm'q); \tag{175}$$

for every value of k there are 2k + 1 matrices of this kind with $2\ell + 1$ rows and columns; since V vanishes for $k > 2\ell$, this gives a total of $(2\ell + 1)^2$ matrices for each ℓ.

It is easy to verify that the tensor product of two u's, as defined by (153), is given by a tensor X which satisfies

$$(\ell \| X^{(K)} \| \ell') = (-)^{k_1+k_2-K} \sqrt{2K+1}\ W(k_1\ell k_2\ell;\ \ell K) \quad ,$$

and hence

$$X^{(K)}_Q = (-)^{k_1+k_2-K} \sqrt{2K+1}\ \ W(k_1\ell k_2\ell;\ \ell K)\ u^{(K)}_Q \ , \tag{176}$$

while the commutator of two u's is given by

$$[u^{(k_1)}_{q_1}\ u^{(k_2)}_{q_2}] = 2\sum_{KQ}{}'(-)^{k_1+k_2-K}\sqrt{2K+1}\ W(k_1\ell k_2\ell;\ell K)(k_1k_2KQ|k_1q_1k_2q_2)u^{(K)}_Q \qquad (177)$$

where the prime on the summation indicates that, owing to the symmetries of the Clebsch-Gordan coefficients, the sum is to be taken only over values of K for which $k_1 + k_2 - K$ is odd. (177) is of the form (14) and hence defines the structure of a Lie group.

In virtue of the orthogonality relations (162), the $(2\ell + 1)^2$ matrices (175) are linearly independent. Since they are of degree $2\ell + 1$, they form a linearly complete set of matrices of this degree; it follows that the structure defined by (177) is that of the full linear group in $2\ell + 1$ dimensions and of its unitary subgroup $U_{2\ell+1}$.

For a system of n particles we can define a set of $\underset{\sim}{u}^{(k)}_i$ (i = 1,2 .... n), each operating on one particle, and we can construct the symmetrical tensors

$$\underset{\sim}{U}^{(k)} = \sum_{i=1}^{n} \underset{\sim}{u}^{(k)}_i \qquad (178)$$

operating on the whole system. It is evident that the $\underset{\sim}{U}^{(k)}_q$ also satisfy the commutation relations (177). The matrices of the $\underset{\sim}{U}^{(k)}_q$ in the scheme (143) will therefore be the representations $\mathcal{H}_\Sigma$ of the infinitesimal operators of $U_{2\ell+1}$.

From (177) it is also seen that commutators of tensors of odd degree are linear combinations of again only such tensors; hence, the tensors $\underset{\sim}{U}^{(k)}$ of odd degree are the infinitesimal operators of a subgroup of the group $U_{2\ell+1}$. It is easy to see that this subgroup is the orthogonal subgroup $R_{2\ell+1}$ which leaves invariant the bilinear form (137), the eigenfunction of the S-state of ℓ^2; indeed the matrix elements $(\ell^2 LM | U_q^{(k)} | \ell^2 S0)$ vanish according to the triangular condition unless $k = L$, and vanish for odd L because the two states have different parity. The matrices of $U_q^{(k)}$ with odd k in the scheme (143) will therefore be the representations $\mathcal{B}_W$ of the infinitesimal operators of $R_{2\ell+1}$.

According to (164) the problem of the construction of the representations of $R_{2\ell+1}$ and $U_{2\ell+1}$ is reduced to the construction of the double-barred matrices of the $\underset{\sim}{U}^{(k)}$, and the problem of constructing the factors $(W_1\gamma_1L_1;(1)\ell|W\gamma L)$ and $(\Sigma_1\beta_1W_1;\ [1](1)|\ \Sigma\beta W)$ of (150) is reduced to the construction of the similarity transformation which decomposes these matrices for odd and even k respectively.

§ 7. Calculation of the Coefficients of Fractional Parentage.

In order to calculate the factors $(W_1\gamma_1L_1;(1)\ell|W\gamma L)$ we have to construct for every odd $k < 2\ell$ the matrices

$$(W_1\gamma_1L_1\ell L \| \underset{\sim}{U}^{(k)} \| W_1\gamma_1' L_1' \ell L'), \tag{179}$$

where $\underset{\sim}{U}^{(k)} = \underset{\sim}{U}_1^{(k)} + \underset{\sim}{u}^{(k)}$. This can be done by using equations (171) and (172) if we already know $(W_1\gamma_1 L_1 \| U_1^{(k)} \| W_1\gamma_1' L_1')$. The transformation matrix which decomposes (179) is $(W_1\gamma_1L_1;\ (1)\ell|\ W\gamma L)$.

If we are not interested in the matrices of $\underset{\sim}{U}^{(k)}$ per se but only in these transformation coefficients, it is sufficient to choose one particular odd value of $k > 1$, e.g., $k = 3$. $k = 1$ does not serve our purpose because $U^{(1)}$ is proportional to L and is therefore already diagonal in our scheme.

As an example we shall calculate the coefficients

$$((20)L_1;\ (1)d \mid WL) \tag{180}$$

for the configuration d^n. We first construct

$$(d^2 L \parallel u_1^{(3)} + u_2^{(3)} \parallel d^2 L') \tag{181}$$

for which, using (174), (171), (172) and Table I, we obtain the matrix

	S	P	D	F	G
[illegible]	[illegible]	0	0	0	0
[illegible]	0	[illegible]	0	$\sqrt{\frac{6}{5}}$	0
[illegible]	[illegible]	[illegible]	$-\frac{8}{7}$	0	$\frac{3\sqrt{[illegible]}}{7}$
[illegible]	[illegible]	[illegible]	0	$-\sqrt{\frac{3}{5}}$	0
[illegible]	[illegible]	[illegible]	$\frac{3\sqrt{[illegible]}}{7}$	0	$\frac{3\sqrt{11}}{7}$

(182)

which decomposes by [illegible] of rows and columns into matrices of [illegible]:

[illegible]

$$[illegible] = \begin{vmatrix} [illegible] & \sqrt{[illegible]} \\ \sqrt{6/5} & [illegible] \end{vmatrix}$$

$$[illegible] = \begin{vmatrix} -8/7 & 3\sqrt{10}/7 \\ 3\sqrt{10}/7 & 3\sqrt{11}/7 \end{vmatrix}$$

where the identification of the values of W to which these constituents belong has been made by the use of the <u>branching laws</u> as explained in detail in reference 26.

Now it is possible to obtain by the same method

$$((20)L_1\ dL \| U_1^{(3)} + u^{(3)} \| (20)L_1'\ dL') \tag{184}$$

but we shall see that in order to calculate (180) it is sufficient to know the last row of (184) which is

$$((20)GdI \| U_1^{(3)} + u^{(3)} \| (20)L_1'\ dL') . \tag{185}$$

By the use of (183) and Table II we obtain (185) as the sum of

$$A = ((20)GdI \| U_1^{(3)} \| (20)L_1' dL')$$

and

$$B = ((20)\ GdI \| u^{(3)} \| (20)G\ dL'):$$

L'	D					G					
L'	S	P	D	F	G	D	F	G	H	I	
A	0	0	0	$\sqrt{\frac{39}{7}}$	$\sqrt{\frac{13}{7}}$	0	$\frac{\sqrt{13}}{7\sqrt{30}}$	$\frac{3\sqrt{13}}{\sqrt{770}}$	$\sqrt{\frac{13}{15}}$	$2\sqrt{\frac{26}{55}}$	
B	0	0	0	0	0	0	$-\sqrt{\frac{13}{30}}$	$\sqrt{\frac{91}{110}}$	$-\sqrt{\frac{13}{15}}$	$\sqrt{\frac{26}{55}}$	
A + B	0	0	0	$\sqrt{\frac{39}{7}}$	$\sqrt{\frac{13}{7}}$	0	$-\frac{\sqrt{78}}{7\sqrt{5}}$	$\sqrt{\frac{130}{77}}$	0	$3\sqrt{\frac{26}{55}}$	(185')

It is possible to deduce from the branching laws that $\mathfrak{P}_{(20)} \times \mathfrak{B}_{(10)}$ decomposes into $\mathfrak{B}_{(10)} + \mathfrak{B}_{(21)} + \mathfrak{B}_{(30)}$, and that to these three representations belong the states D, P D F G H, and S F G I respectively. Hence, the transformation matrix which decomposes (184) by bringing it into the form

$$(WL \| U^{(3)} \| WL') \tag{186}$$

will have the structure

$$
\begin{array}{ll|cccccccccc|}
 & & S & P & D & F & G & D & F & G & H & I \\
(10) & D & & & * & & & * & & & & \\
 & P & & 1 & & & & & & & & \\
 & D & & & * & & & * & & & & \\
(21) & F & & & & * & & & * & & & \\
 & G & & & & & * & & & * & & \\
 & H & & & & & & & & & 1 & \\
 & S & 1 & & & & & & & & & \\
(30) & F & & & & * & & & * & & & \\
 & G & & & & & * & & & * & & \\
 & I & & & & & & & & & & 1
\end{array}
\qquad (187)
$$

where the stars denote the non-vanishing matrix elements which have to be calculated. It follows from the form of (187) that the row (30)I of (186) is obtained simply by multiplying the row (185), which has the elements (185'), with the columns of the transpose of (187). The selection rule which follows from the requirement that (186) is to be decomposed, in conjunction with all the available orthogonality and reciprocity relations, permits us to determine, apart from arbitrary phases, all elements of the matrix (187). They are contained in Table III.

By reciprocity we mean the relation

$$
(W\gamma L\, W_1\gamma_1 L_1;\ell) \sqrt{\frac{2L_1 + 1}{2L + 1}\,\frac{g_W}{g_{W_1}}}\ (-)^{L - L_1 + x}\, (W_1\gamma_1 L_1 | W\gamma L;\ell) \qquad (188)
$$

where g_W and g_{W_1} are the degrees of the representations and x is a phase which may be chosen arbitrarily for every pair W, W_1 but which is independent of the L's. This relation is proved in reference 23, equation (46); since its proof is based on the fact that the identity representation appears in the decomposition of $\mathfrak{B}_W \times \mathfrak{B}_W$, such a relation does not hold for the unitary group.

The calculation of

$$(\Sigma\beta W|\Sigma_1\beta_1 W_1; \ [1](1)) \tag{189}$$

can be carried out in an analogous fashion, using $\underset{\sim}{U}^{(2)}$ rather than $\underset{\sim}{U}^{(3)}$. The result for the configuration d^3 is given in Table IV. For the coefficients of the spin functions, corresponding to the passage from R_6 to $R_3 \times R_3$, the infinitesimal operators of the two groups R_3 are T_ξ, T_η, T_ζ and σ_x, σ_y, σ_z; the infinitesimal operators of R_6 are, in addition to these, $X_{\rho r} = T_\rho \sigma_r$, which form a tensor, or better a double vector with one foot in the isotopic-spin space and one in the spin space. The construction of the matrices

$$(\Sigma^*\alpha \ \ TS\|X\|\Sigma^*\alpha_1 T_1 S_1) \tag{190}$$

and the decomposition of the corresponding Kronecker products can be done in the same way as before. (For n = 3, see Table V).

For the construction of the coefficients

$$(\ell^n \Sigma|\ell^{n-1}\Sigma_1; \ \ell[1]) \tag{191}$$

the calculation is based not on the properties of Lie groups, but rather on those of the permutation groups, π_n. The result, which we give here without proof, is very simple:

$$(\ell^n \Sigma \ \| \ell^{n-1}\Sigma_1; \ell[1]) = \pm\sqrt{\frac{g_{\Sigma_1}}{g_\Sigma}} \tag{192}$$

where g_Σ is the degree of the representation of π_n which is characterized by the partition Σ. The sign depends on the choice of sign made for the other coefficients. For $n = 3$ they are given in Table VI.

TABLE I. W(2L2L'; 23).

L \ L'	S	P	D	F	G
S	0	0	0.	$\frac{1}{\sqrt{35}}$	0
P	0	0	$\frac{\sqrt{2}}{5\sqrt{7}}$	$\frac{1}{\sqrt{70}}$	$\frac{-1}{\sqrt{210}}$
D	0	$\frac{\sqrt{2}}{5\sqrt{7}}$	$\frac{4}{35}$	$\frac{\sqrt{3}}{35\sqrt{2}}$	$\frac{-1}{7\sqrt{2}}$
F	$\frac{1}{\sqrt{35}}$	$\frac{1}{\sqrt{70}}$	$\frac{\sqrt{3}}{35\sqrt{2}}$	$\frac{-\sqrt{3}}{14\sqrt{5}}$	$\frac{-\sqrt{11}}{14\sqrt{5}}$
G	0	$\frac{-1}{\sqrt{210}}$	$\frac{-1}{7\sqrt{2}}$	$\frac{-\sqrt{11}}{14\sqrt{5}}$	$\frac{-\sqrt{11}}{42}$

TABLE II.

L' →	F	G	H	I
W(462L';23)	$\frac{-1}{\sqrt{210}}$	$\frac{-\sqrt{7}}{9\sqrt{10}}$	0	0
W(464L';23)	$\frac{-1}{3\sqrt{2310}}$	$\frac{-\sqrt{7}}{33\sqrt{10}}$	$\frac{-7}{33\sqrt{15}}$	$\frac{-14\sqrt{2}}{33\sqrt{65}}$
W(262L';43)	$\frac{-1}{\sqrt{210}}$	$\frac{-\sqrt{7}}{3\sqrt{110}}$	$\frac{-1}{\sqrt{165}}$	$\frac{-\sqrt{2}}{\sqrt{715}}$

TABLE III. $(WL|W_1L_1; (1)d)$.

W \ W_1	L \ L_1	(00) S	(10) D	(11) P	(11) F	(20) D	(20) G
(00)	S	0	1	0		0	
(10)	D	1	0	$\sqrt{\frac{3}{10}}$	$\sqrt{\frac{7}{10}}$	$\sqrt{\frac{5}{14}}$	$\sqrt{\frac{9}{14}}$
(11)	P	0	1	$-\sqrt{\frac{8}{15}}$	$-\sqrt{\frac{7}{15}}$	0	
	F		1	$-\sqrt{\frac{1}{5}}$	$\sqrt{\frac{4}{5}}$		
(20)	D	0	1	0		0	
	G		1				
(21)	P	0		$-\sqrt{\frac{7}{15}}$	$\sqrt{\frac{8}{15}}$	1	0
	D			$\sqrt{\frac{7}{10}}$	$-\sqrt{\frac{3}{10}}$	$\sqrt{\frac{9}{14}}$	$-\sqrt{\frac{5}{14}}$
	F			$-\sqrt{\frac{4}{5}}$	$-\sqrt{\frac{1}{5}}$	$-\sqrt{\frac{2}{7}}$	$-\sqrt{\frac{5}{7}}$
	G			0	-1	$-\sqrt{\frac{10}{21}}$	$\sqrt{\frac{11}{21}}$
	H			0	1	0	1
(30)	S	0				1	0
	F					$\sqrt{\frac{5}{7}}$	$-\sqrt{\frac{2}{7}}$
	G					$\sqrt{\frac{11}{21}}$	$\sqrt{\frac{10}{21}}$
	I					0	1

TABLE IV. $(\Sigma W|\Sigma_1 W_1;[1](1))$ for d^3.

Σ \ Σ_1	W \ W_1	[11] (11)	[20] (00)	[20] (20)
[111]	(11)	1	0	
[210]	(10)	1	$\sqrt{\frac{8}{15}}$	$-\sqrt{\frac{7}{15}}$
	(21)	1	0	1
[300]	(10)	0	$\sqrt{\frac{7}{15}}$	$\sqrt{\frac{8}{15}}$
	(30)		0	1

TABLE V. $(\Sigma^* TS|\Sigma_1^* T_1 S_1;\ [1]\frac{1}{2}\frac{1}{2})$ for n = 3.

$\Sigma^*(PP'P'')$ \ Σ_1^* $(P_1P_1'P_1'')$	$(2T+1,2S+1)$ \ $(2T_1+1, 2S_1+1)$	[20] (111) (11)	[20] (111) (33)	[11] (100) (13)	[11] (100) (31)
[300] $(\frac{3}{2}\frac{3}{2}\frac{3}{2})$	(22)	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	
	(44)	0	1		
[210] $(\frac{3}{2}\frac{1}{2}\frac{1}{2})$	(22)	$\frac{-1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$	$\frac{-1}{\sqrt{2}}$
	(24)	0	1	1	0
	(42)	0	1	0	1
[111] $(\frac{1}{2}\frac{1}{2}\frac{-1}{2})$	(22)	0		$\frac{-1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$

TABLE VI. $(\ell^3\ \Sigma \mid \ell^2 \Sigma_1; \ell[1])$.

Σ \ Σ_1	[11]	[20]
[111]	1	0
[210]	$\frac{-1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
[300]	0	1

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where the coefficients F^k are given by

$$F^k = \iint J_k(r_1, r_2)\, R^2_{n_1 \ell_1}(r_1)\, R^2_{n_2 \ell_2}(r_2)\, dr_1\, dr_2, \qquad (197)$$

which is called the generalized Slater integral.

It is interesting to note that the classical Slater integrals, which were defined for $J(r_{12}) = \frac{e^2}{r_{12}}$ are decreasing functions of k. But, if instead of a Coulomb interaction we have a short range interaction, F^k may no longer decrease with k. On the contrary, it is easy to see that for $J(r_{12}) = \delta(\vec{r_1} - \vec{r_2})$ which is the limiting case of short range interaction, one has

$$F^k = (2k + 1)F^0. \qquad (198)$$

For the particular case in which we are interested, of two particles in the same shell, (196) reduces to

$$(\ell^2 LM|J(r_{12})|\ell^2 LM) = \sum_k (-)^L (\ell \| C^{(k)} \| \ell)^2 W(\ell\ell\ell\ell;Lk)F^k. \qquad (199)$$

If, instead of a Wigner interaction we have some kind of exchange interaction, the sign of this expression has to be changed for some values of T, S, and L.

§ 2. The Group-Theoretical Classification of the Interactions.

The general formula for calculating the energy matrix for a system of n equivalent particles was given by (132), but since the α's stand for a set of many quantum numbers which may assume many different values, the summation of (132) is very long and has to be split up into a set of independent smaller summations. This is made possible by the factorization of the c.f.p. and by a similar factorization of the energy matrix which we shall discuss now.

We have seen in §5 of Lectures 5 and 6 that there is a relation between the Clebsch-Gordan coefficients and the matrix elements of the components of the irreducible tensor operators. But the relation (157), which is a property of the group R_3, may be generalized to any other group if we adopt the general standpoint of Eckart and Wigner.

If G is a group whose irreducible representations X have rows and columns characterized by χ, and $T(\Omega\omega)$ is an operator which has the same transformation properties with respect to the group as the element ω of the basis of the representation Ω of G, then, in analogy to (157), the matrix element $(X'\chi' \mid T(\Omega\omega) \mid X\chi)$ will be proportional to the matrix element $(X\Omega X'\chi' \mid X\chi\Omega\omega)$ of the transformation which decomposes the Kronecker product $X \times \Omega$. In the particular case that G is the group $U_{4(2\ell+1)}$ and X is $\mathcal{A}_n$ (cf. p. 62), we have

$$(\ell^n \Gamma' \mid T(\Omega\omega) \mid \ell^n \Gamma) = c(\mathcal{A}_n \Omega \mathcal{A}_n \Gamma' \mid \mathcal{A}_n \Gamma \Omega\omega), \qquad (200)$$

and, if we assume for $\mathcal{A}_n$ and Ω the scheme (143), this matrix element may be factorized according to (148).

Since a central and charge-independent interaction is a scalar with respect to the three-dimensional rotations in coordinate space, spin space, and isotropic-spin space, it follows that the energy matrix is diagonal with respect to T S L and is independent of $M_T\, M_S\, M_L$, as is well known.

Unfortunately, the interaction is not an irreducible tensor operator with respect to the group $U_{4(2\ell+1)}$ and to its subgroups which were used to classify the states of ℓ^n. We shall, therefore, as a first step decompose the interaction operator into a sum of interactions which are components of irreducible tensor operators, and then calculate the energy matrices of these particular interactions, using (150) and the factorization which follows from (200) and (148) to simplify the summations (132).

In general, the interaction operator will be a tensor of some kind which is reducible with respect to $U_{4(2\ell+1)}$, and, if for the time being we limit ourselves to a spin-independent interaction (Wigner or Majorana), it will be a scalar with respect to U_4 and a tensor with respect to $U_{2\ell+1}$.

In order to identify the irreducible parts of this tensor, we start by considering an operator which operates on the space coordinates of a single particle in a given shell. Since it has to be a linear transformation in the $2\ell+1$-dimensional space, it will be a tensor of the second rank with one covariant and one contravariant index. It was stated on p. 55 that the components of a (contravariant) vector are the basis of the representation $\mathcal{H}_{[10\ \ldots\ldots 0]}$; analogously, the components of a covariant vector are the basis of the representation $\mathcal{H}_{[00\ \ldots\ldots 0\ -1]}$; hence the components of a mixed tensor of rank two are the basis of the reducible representation

$$\mathcal{H}_{[10\ \ldots\ldots 0]} \times \mathcal{H}_{[00\ldots\ldots 0\ -1]}, \tag{201}$$

which decomposes into $\mathcal{H}_{[0\ \ldots\ldots 0]} + \mathcal{H}_{[10\ \ldots\ldots 0\ -1]}$. (This decomposition corresponds to separating the trace from the traceless part of the mixed tensor).

The interaction between two particles is expressed according to (196) as a sum of products of operators operating on the two particles, and will, therefore, belong to the basis of the reducible representation

$$(\mathcal{H}_{[0\ \ldots 0]} + \mathcal{H}_{[10\ \ldots 0\ -1]}) \times (\mathcal{H}_{[0\ \ldots\ 0]} + \mathcal{H}_{[10\ \ldots 0\ -1]}) \tag{202}$$

of $U_{2\ell+1}$.

If we decompose the representation (202) into its irreducible components and adopt a scheme in which W and L are diagonal, as we did in § 3 of Lectures 5 and 6 for the classification of the states, then, since the interaction is a scalar in the three-dimensional space, it will appear as a linear combination of the different basis elements which are classified as S-states in this scheme.

Since $\mathcal{H}_{[10 \ldots 0]}$ and $\mathcal{H}_{[0 \ldots 0-1]}$ are in R_3 the representation $\mathcal{D}_\ell$, (201) is the representation $\mathcal{D}_\ell \times \mathcal{D}_\ell$ which decomposes into $\sum_{L=0}^{2\ell} \mathcal{D}_L$, and it follows that in the basis of (202) there are $2\ell + 1$ independent invariants with respect to R_3 which have various tensorial characters in $U_{2\ell+1}$ and $R_{2\ell+1}$. It may be shown by the branching laws that two of them are invariants, also, with respect to $U_{2\ell+1}$ and $R_{2\ell+1}$. One is still an invariant with respect to $R_{2\ell+1}$, but with respect to $U_{2\ell+1}$ it belongs to the representation with highest weight [20 ... 0 - 2]. The other scalars are, in the scheme (143), of the following kinds: [20 ... 0 - 2] (22) S, [20 ... 0 - 2] (40) S, [110 ... 0 - 1 - 1] (22) S, [110 ... 0 - 1 -1] (1111) S.

The decomposition of the interaction (196) into its irreducible parts may be made in a general way based on the fractional parentages of the different representations, as was done for f^n in reference 23, § 6, 1, but we shall consider here only the configurations d^n and follow a more empirical method.

Any kind of spin-independent interaction $E^{(\lambda)}$ will be represented in the d^2 configuration by a diagonal matrix

$$(d^2LM|\,E^{(\lambda)}\,|\,d^2LM) = f^{(\lambda)}(L) \tag{203}$$

and we have to calculate $f^{(\lambda)}(L)$ for the different irreducible parts into which the interaction (199) decomposes. The result is tabulated here:

Interaction				States of d^2					
Name	Tensorial Character			Σ:	[11]	[11]	[20]	[20]	[20]
				W:	(11)	(11)	(00)	(20)	(20)
	Σ	W	L	L:	P	F	S	D	G
$E^{(\alpha)}$	[00000]	(00)	S		1	1	1	1	1
$E^{(\beta)}$	[00000]	(00)	S		-1	-1	1	1	1
$E^{(\gamma)}$	[2000-2]	(00)	S		0	0	-14	1	1
$E^{(\epsilon)}$	[2000-2]	(22)	S		0	0	0	-9	5
$E^{(\eta)}$	[110-1-1]	(22)	S		-7	3	0	0	0

This table was obtained as follows: according to Schur's lemma $f^{(\alpha)}$ and $f^{(\beta)}$ have to be constant for states belonging to the same value of Σ; any linear combination of them has this property and the choice is determined only by considerations of simplicity. $f^{(\gamma)}$ must be constant for states belonging to the same value of W; moreover, according to (200) and in virtue of the orthogonality of the transformation matrices, $f^{(\gamma)}$ has to be orthogonal to both $f^{(\alpha)}$ and $f^{(\beta)}$ (if we consider every value of L with its (2L + 1)-fold degeneracy). $f^{(\epsilon)}$ and $f^{(\eta)}$ must be orthogonal to $f^{(\alpha)}$, $f^{(\beta)}$ and $f^{(\gamma)}$; and, in addition $f^{(\epsilon)}$ has to vanish for Σ = [11] since $\mathcal{H}_{[11]}$ does not appear in the decomposition of the Kronecker product $\mathcal{H}_{[11]} \times \mathcal{H}_{[2000-2]}$ and $f^{(\eta)}$ has to vanish for Σ = [20] because $\mathcal{H}_{[20]}$ does not appear in the decomposition of the Kronecker product $\mathcal{H}_{[20]} \times \mathcal{H}_{[110-1-1]}$. (These selection rules are the analogs for $U_{2\ell+1}$ of the triangular conditions for R_3).

The perturbation energy $E(L)$ of the configuration d^2 for an ordinary (Wigner) interaction may be obtained from (199). In order to avoid the appearance of fractional coefficients we introduce the standard normalization*

$$F_o = F^o, \; F_2 = F^2/49, \qquad F_4 = F^4/441, \tag{204}$$

and write for the energy**

$$\begin{aligned} E(S) &= F_o + 14\, F_2 + 126\, F_4 \\ E(P) &= F_o + 7F_2 - 84F_4 \\ E(D) &= F_o - 3\, F_2 + 36\, F_4 \\ E(F) &= F_o - 8F_2 - 9F_4 \\ E(G) &= F_o + 4\, F_2 + F_4 \end{aligned} \tag{205}$$

These results may be expressed in terms of the irreducible interactions by

$$V_W = E^{(\alpha)} F_o + \left(-\tfrac{7}{12} E^{(\alpha)} + \tfrac{35}{12} E^{(\beta)} - \tfrac{5}{6} E^{(\gamma)}\right)(F_2 + 9F_4) + \tfrac{1}{2}\left(E^{(\epsilon)} - 3E^{(\zeta)}\right) \cdot (F_2 - 5F_4). \tag{206}$$

The corresponding expression for the Majorana interaction is obtained by interchanging $E^{(\alpha)}$ and $E^{(\beta)}$ and changing the sign of $E^{(\zeta)}$:

$$V_M = E^{(\beta)} F_o + \left(\tfrac{35}{12} E^{(\alpha)} - \tfrac{7}{12} E^{(\beta)} - \tfrac{5}{6} E^{(\gamma)}\right)(F_2 + 9F_4) + \tfrac{1}{2}\left(E^{(\epsilon)} + 3E^{(\zeta)}\right) \cdot (F_2 - 5F_4). \tag{207}$$

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* Reference 15, p. 177.

** Reference 15, p. 202.

§ 3. The Calculation of the Energy Matrices.

When we go from d^2 to d^n, the summation over the different pairs of particles can be carried out very simply for the interactions $E^{(\alpha)}$ and $E^{(\beta)}$:

$$\sum_{i<k} E_{ik}^{(\alpha)} = \frac{1}{2} n(n-1) \qquad (208a)$$

$$\sum_{i<k} E_{ik}^{(\alpha)} = M \qquad (208b)$$

where M is the eigenvalue in the unperturbed state of the Majorana operator and can be expressed as a function of the partition $\Sigma^* = (\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_4)$:*

$$M = -\frac{1}{2}[\Lambda_1(\Lambda_1 - 1) + \Lambda_2(\Lambda_2 - 3) + \Lambda_3(\Lambda_3 - 5) + \Lambda_4(\Lambda_4 - 7)]. \qquad (209)$$

We can also obtain $\sum_{i<k} E_{ik}^{(\gamma)}$ in closed form by using the Casimir operator for the group R_5: it follows from (106) that for R_5 the eigenvalues of this operator are

$$g(W) = w_1(w_1 + 3) + w_2(w_2 + 1), \qquad (210)$$

so that in particular,

$$g(00) = 0, \quad g(10) = 4, \quad g(11) = 6, \quad g(20) = 10. \qquad (210')$$

We can therefore write

$$E^{(\gamma)} = \frac{3}{2}[g(W) - 2g(10)] - \frac{5}{2} E^{(\beta)} + \frac{1}{2} E^{(\alpha)} \qquad (211)$$

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* L. Rosenfeld, Nuclear Forces, Amsterdam, 1948, p. 211, Eq. (14).

and then find for d^n *

$$\sum_{i<k} E_{ik}^{(\gamma)} = \frac{3}{2}[g(W) - ng(10)] - \frac{5}{2}\sum_{i<k} E_{ik}^{(\beta)} + \frac{1}{2}\sum_{i<k} E_{ik}^{(\alpha)}$$

$$= \frac{3}{2}g(W) - \frac{5}{2}U + \frac{1}{4}n\,(n - 25). \qquad (212)$$

In a similar way it is easy to see that

$$E^{(\varepsilon)} + E^{(\zeta)} = L(L + 1) - \frac{3}{2}g(W), \qquad (213)$$

so that also for n d-particles,

$$\sum_{i<k} (E_{ik}^{(\varepsilon)} + E_{ik}^{(\zeta)}) = L(L + 1) - \frac{3}{2}g(W). \qquad (214)$$

The calculation of the energy matrices for the interactions $E^{(\varepsilon)}$ and $E^{(\zeta)}$ separately has to be made by the use of (132); actually, owing to (214) it is sufficient to calculate

$$X = \frac{1}{2}\sum_{i<k} (E_{ik}^{(\varepsilon)} - E_{ik}^{(\zeta)}), \qquad (215)$$

which will be of the form

$$(d^n \Sigma\beta W \gamma L|X|d^n \Sigma\beta' W' \gamma' L) = \sum_{\rho=1}^{r} (\Sigma\beta W|\Lambda_\rho|\Sigma\beta' W')(W|\Psi_\rho(\gamma\gamma' L)|W'). \qquad (216)$$

Although from (148) one might expect the factorization on the right hand side to be a complete one, this is not so because (148) did not represent the most general case. It contains the implicit assumption that in the decomposition of $U_{B_1} \times U_{B_2}$ the representation U_B appears only once, and this was actually the case when U_{B_2} was a representation which had as its basis the states of one particle in a given shell. However, now that U_{B_2} is the representation to which the interaction operator belongs, viz. $\mathfrak{B}_{(22)}$, we need (148) in its most general form,**

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*The proof is the same as that of the well known formula

$$\sum_{i<k} (\boldsymbol{l}_i \cdot \boldsymbol{l}_k) = \frac{1}{2}[L(L + 1) - n\,l(l + 1)].$$

** Reference 23, § 3.

and this still involves a summation. The number of terms in this summation, r, equals the number of times that $\mathfrak{B}_W$ appears in the reduction of $\mathfrak{B}_{W'} \times \mathfrak{B}_{(22)}$; and it follows from the branching laws that it can never exceed three.

Introducing (216) and (150) in (132), we obtain

$$(d^n \Sigma \beta W \gamma L | X | d^n \Sigma \beta' W' \gamma' L) =$$

$$\frac{n}{n-2} \sum_{\Sigma_1 \beta_1 \beta_1' W_1 W_1' \gamma_1 \gamma_1' L_1 \varsigma_1} (d^n \Sigma | d^{n-1} \Sigma_1; d[1])(\Sigma \beta W | \Sigma_1 \beta_1 W_1; [1](1))(W \gamma L | W_1 \gamma_1 L_1; (1)d)$$

$$\cdot (\Sigma_1 \beta_1 W_1 | A_{\varsigma_1} | \Sigma_1 \beta_1' W_1')(W_1 | \Psi_\varsigma(\gamma_1 \gamma_1' L_1) | W_1') \tag{217}$$

$$\cdot (W_1' \gamma_1' L_1; (1)d | W' \gamma' L)(\Sigma_1 \beta_1' W_1'; [1](1) | \Sigma \beta' W')(d^{n-1} \Sigma_1; d[1] | d^n \Sigma).$$

We perform at first the summation

$$\sum_{\gamma_1 \gamma_1' L_1} (W \gamma L | W_1 \gamma_1 L_1; (1)d)(W_1 | \Psi_{\varsigma_1}(\gamma_1 \gamma_1' L_1) | W_1')(W_1' \gamma_1' L_1; (1)d | W' \gamma' L) \tag{218}$$

which, owing to the tensorial properties of the Ψ_ς will be a linear combination of the $(W | \Psi_\varsigma(\gamma\gamma' L) | W')$ with coefficients that are independent of $\gamma\gamma'$ and L:

$$\sum_{\gamma_1 \gamma_1' L_1} (W \gamma L | W_1 \gamma_1 L_1; (1)d)(W_1 | \Psi_{\varsigma_1}(\gamma_1 \gamma_1' L_1) | W_1')(W_1' \gamma_1' L_1; (1)d | W' \gamma' L)$$

$$= \sum_{\varsigma} (W | x_\varsigma(W_1 W_1' \varsigma_1) | W')(W | \Psi_\varsigma(\gamma\gamma' L) | W'). \tag{218'}$$

When the $(W | \Psi_\varsigma(\gamma\gamma' L) | W')$ are known, in order to obtain the coefficients of the linear combination, it suffices to perform the summation (218) for only a few values of $\gamma\gamma'$ and L.

Then we calculate

$$(\Sigma\beta W|y_\zeta(\xi_1)|\Sigma\beta' W') = \sum_{\beta_1\beta_1' W_1 W_1' \xi_1} (\Sigma\beta W|\Sigma_1\beta_1 W_1;[1](1))\,.$$

$$\cdot(\Sigma_1\beta_1 W_1|A_{\xi_1}|\Sigma_1\beta_1' W_1')(W|x_\zeta(W_1 W_1'\xi_1)|W')(\Sigma_1\beta_1' W_1';[1](1)|\Sigma\beta W) \quad (219)$$

and obtain finally

$$(\Sigma\beta W|A_\zeta|\Sigma\beta' W') = \frac{n}{n-2}\sum_{\Sigma_1}(d^n\Sigma|d^{n-1}\Sigma_1;[1](1))^2(\Sigma\beta W|y_\zeta(\Sigma_1)|\Sigma\beta' W'). \quad (220)$$

In the particular case of a δ-interaction, which is the limit of forces with very short range, it follows from (198) and (204) that $F_2 - 5F_4$ vanishes and, therefore, the energies of the Wigner interaction may in this particular case be expressed in closed form:

$$V = \frac{5}{14} F_0 \,[n(n+3) + 4M - g(W)] \quad (221)$$

by introducing (208a), (208b) and (212) into (206). Further, Wigner and Majorana interactions become equal for a δ-interaction.

Even if the interaction is not a δ-function, but is still of short range (compared with the dimensions of the nuclei), as is the case for nuclear interactions, the most important contributions to the energy come from $E^{(\alpha)}$, $E^{(\beta)}$ and $E^{(\gamma)}$, and the lowest levels are those with the smallest values of $g(W)$. These levels belong to $W = (00)$ for even nuclei and to $W = (10)$ for odd nuclei. Since $\mathfrak{B}_{(00)} \times \mathfrak{B}_{(22)} = \mathfrak{B}_{(22)}$ and $\mathfrak{B}_{(10)} \times \mathfrak{B}_{(22)} = \mathfrak{B}_{(21)} + \mathfrak{B}_{(22)} + \mathfrak{B}_{(32)}$ it follows that for $W = W' = (00)$ or $W = W' = (10)$, r vanishes in (216), i.e., for the levels belonging to these values of W the diagonal element of X vanishes. It must be remembered here that the W are not good quantum numbers; however, they are fairly good quantum numbers for short range forces, so that it is possible to calculate the lowest level of a configuration d^n without calculating the matrix of X.

§ 4. Spin-Dependent Interactions.

As in our previous discussion, we shall limit ourselves to the d^n shell in discussing spin-dependent interactions of the Bartlett and Heisenberg types, although the method is applicable to any nuclear shell. In addition to the five spin-independent irreducible interactions tabulated on page 95, we have now five which depend on the spin and which may be obtained in the same manner:

Interaction					States of d^2									
Name	Tensorial Character				^{33}P	^{11}P	^{33}F	^{11}F	^{31}S	^{13}S	^{31}D	^{13}D	^{31}G	^{13}G
	Σ'	Σ	W	L										
$E^{(\eta)}$	[11-1-1]	[00000]	(00)	"S	0	0	0	0	1	-1	1	-1	1	-1
$E^{(\theta)}$	[200-2]	[00000]	(00)	"S	1	-9	1	-9	0	0	0	0	0	0
$E^{(\gamma)}$	[11-1-1]	[2000-2]	(00)	"S	0	0	0	0	-14	14	1	-1	1	-1
$E^{(\epsilon')}$	[11-1-1]	[2000-2]	(22)	"S	0	0	0	0	0	0	-9	9	5	-5
$E^{(\zeta')}$	[200 -2]	[110-1-1]	(22)	"S	-7	63	3	-27	0	0	0	0	0	0

In this table Σ' characterizes the representation $\mathcal{G}_{\Sigma'}$ of U_4 and Σ characterizes the representation $\mathcal{H}_{\Sigma}$ of U_5 to which the interactions belong.

The Bartlett and Heisenberg interactions must now be expressed in terms of these interactions, and it is easy to see that

$$V_B + V_H = 2\ E^{(\eta)}\ [F_o + \tfrac{7}{3}(F_2 + 9F_4)] - \tfrac{5}{3}E^{(\gamma')}(F_2 + 9F_4) + E^{(\varepsilon')}\ (F_2 - 5F_4) \quad (222)$$

$$V_B - V_H = \tfrac{2}{5}(E^{(\theta)} + 2E^{(\alpha)} - 2E^{(\beta)})[F_o - \tfrac{7}{2}(F_2 + 9F_4)] - \tfrac{3}{5}\ (E^{(\zeta')} + 4E^{(\zeta)})\ \cdot$$

$$\cdot\ (F_2 - 5F_4). \quad (223)$$

It is also easy to show that

$$2 \sum_{i<k} E_{ik}^{(\eta)} = S(S + 1) - T(T + 1) \quad (224)$$

and

$$\frac{2}{5} \sum_{i<k} (E_{ik}^{(\theta)} + 2E_{ik}^{(\alpha)} - 2E_{ik}^{(\beta)}) = S(S + 1) + T(T + 1) + \tfrac{1}{2}\,n\,(n - 4). \quad (225)$$

The calculation of the energy matrices for the interactions $E^{(\gamma')}$, $E^{(\varepsilon')}$, and $E^{(\zeta')}$ has to be made by the methods used in the proceeding section for the interaction X.

Bibliography.

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