JOINT INSTITUTE FOR NUCLEAR RESEARCH

STEKLOV MATHEMATICAL INSTITUTE OF THE USSR ACADEMY OF SCIENCES

N.N. BOGOLUBOV, V.V. TOLMACHOV, D.V. SHIRKOV

A NEW METHOD IN THE THEORY OF SUPERCONDUCTIVITY

Dubna, June, 1958.

JOINT INSTITUTE FOR NUCLEAR RESEARCH

STEKLOV MATHEMATICAL INSTITUTE OF THE USSR ACADEMY OF SCIENCES

==========

N.N. BOGOLUBOV, V.V. TOLMACHOV, D.V. SHIRKOV

A NEW METHOD IN THE THEORY OF SUPERCONDUCTIVITY

Dubna, June, 1958.

This manuscript represents a shortened translation, intended for publication in the "Fortschritte der Physik", of the russian original published as a preprint by the Joint Institute for Nuclear Research in January 1958 and in book form by the Publishing House of the Academy of Sciences of the USSR in June 1958.

The translation has been carried out by M.E. Mayer, V.D. Sutula, D.V. Shirkov and M.K. Polivanov and was edited by M.E. Mayer.

ABSTRACT

The method of canonical transformations proposed by one of the authors ten years ago in conneotion with a mioriscopic theory of superfluidity for Bose systems, is generalised here to Fermi systems, and forms the basis of a method for investigating the problem of superconductivity.

Starting from Fröhlich's Hamiltonian, the energy of the superconducting ground state and the one-Fermion and collective excitations corresponding to this state are obtained. It turns out that the final formulae for the ground state and one-Fermion excitations recently obtained by Bardeen, Cooper and Schrieffer are correct in the first approximation. The physical picture appears to be closer to the one proposed by Schafroth, Butler and Blatt.

The effect on superconductivity of the Coulomb interaction between the electrons is analyzed in detail. A criterion for the superfluidity of a Fermi system with a four-line vertex Hamiltonian is established.

C O N T E N T S

§I. Introduction 6 - I0

I.I An outline of the present status of the theory of superconductivity 6

2.2 Resume of the microscopic theory of superfluidity 7

§2. SUPERCONDUCTIVITY IN FRÖHLICH'S MODEL II - 20

2.I The principle of compensation of dangerous graphs • II

2.2 Simplification of the expressions . I5

2.3 The energy difference between the normal and superconducting states I8

2.4 Superconductivity I9

§3. THE SPECTRUM OF COLLECTIVE EXCITATIONS OF THE SUPERCONDUCTING STATE 2I - 34

3.I Application of the method of approximate second quantization to a system with Coulomb interaction 2I

3.2 Collective excitations in Fröhlich's model 23

3.3 The resolution of the secular equations. Longitudinal excitations 26

3.4 The resolution of the secular equations. Transversal excitations 3I

§4. THE COULOMB INTERACTION BETWEEN THE ELECTRONS 35 - 46

4.I Outline of the problem 35

4.2 The compensation and renormalization conditions 36

4.3 The final form of the compensation equation for electron graphs 38

4.4 The ground state energy and the energy of one electron excitations 4o

4.5 Transformation of the kernel $Q(K,K')$ 4I

4.6 The determination of λ, μ and $\tilde{\omega}$. 43

4.7 The connection with a model problem 44

§5. QUALITATIVE DESCRIPTION OF THE EFFECTS OF THE COULOMB INTERACTION 47 - 54

5.I Approximate determination of the renormalized $\tilde{\omega}$ and $\tilde{g}$. 47

5.2 Discussion of the properties of Q_c and Q_{ph} 5o

5.3 General properties of the fundamental compensation equation 5I

5.4 A criterion for superconductivity 53

§6. FERMI SYSTEMS WITH WEAK INTERACTION 55 - 66

6.I The formulation of the BCS theory 55

6.2 The compensation equation 57

6.3 Collective excitations. The influence of Coulomb interaction 63

§7. CONCLUSION 67 - 72

7.1 The thermodynamics of the superconducting state 67

7.2 The electrodynamics of the superconducting state 69

7.3 A qualitative picture of superconductivity 71

APPENDIX I. On a criterion for superfluidity in the theory of nuclear matter 73

APPENDIX II. On a variational principle in the many-body problem 79

REFERENCES 82

I. INTRODUCTION

I.I An outline of the present status of the theory of superconductivity

Except for some results of a particular character, all attempts at constructing a microscopic theory of superconductivity have failed for a long time.

An essential contribution to the development of the theory is due to Fröhlich ([I]), who in 1950 was the first to point out that the phenomenon of superconductivity is mainly due to the interaction of the electrons with the phonons of the crystal lattice, i.e. to the same interaction which under normal conditions accounts for the usual resistivity of the metal. On the basis of this assumption and using dimensional considerations, Fröhlich succeded to predict the very important isotope-effect, which was soon afterwards discovered experimentally.

After the discovery of the isotope-effect it became evident, that the electron-phonon interaction must form the basis of any attempt of constructing a microscopic theory of superconductivity. However, due to the extreme mathematical intricacy of the problem, the first attempts to obtain correct solutions were doomed to failure ([I,2]). Nevertheless, it is important to note the important role played by these attempts in the investigation of the applicability of perturbation theory to this problem.

Most instructive in this respect was the one-dimensional model, analyzed by Fröhlich ([3]), where the problem could be solved exactly. It turned out, that these results could not have been obtained by means of perturbation theory, since the energy difference between the normal and superconducting states depends nonanalytically on the coupling constant, the dependence being of the type $\exp(-I/g)$. As a result of a detailed investigation we now know that just this situation was the cause of the difficulties in the three-dimensional case.

A new important physical idea has been introduced by Schafroth, Butler and Blatt ([4]), who drew attention to and discussed in detail, the role of pair correlations, especially for electrons near the Fermi surface. These correlated pairs appeared to be essential in connection with the phenomenon of Bose-Einstein condensation of such structures. The appearance of the condensate was treated by the authors as the formation of the superconducting state. We stress the fact, that a pair of electrons within the Bose-Einstein condensate has zero total momentum. In the conception of Schafroth, Butler and Blatt the Fröhlich attraction between two electrons near the Fermi surface is the main factor assuring the formation of such correlated pairs. As we will show below, the ideas of Schafroth, Butler and Blatt are entirely correct.

A further move in the development of the theory, in which the framework of perturbation theory was surpassed, has been made recently in the work of Cooper, and Bardeen, Cooper and Schrieffer ([5]). These authors considered a simplified model, in which the interaction between electrons and phonons is replaced by an attraction between the electrons, which acts near the Fermi surface, and only the terms which correspond to the interaction of electron pairs with opposite momenta are retained in the Hamiltonian. Physically this corresponds to taking into account only those pairs

which belong to the Bose-Einstein condensate.

Starting from the idea that particles with opposite momenta form bound pairs, the above-mentioned authors take the wave function of the ground state in the form of a product of pair wave functions, containing parameters. The latter are determined by means of a variational principle for the energy minimum. It must be observed, that the method of Bardeen, Cooper and Schrieffer leaves unanswered a series of questions as e.g. the fundamentation of the whole procedure, the role of Coulomb interaction etc. which may render their results less convincing.

However, after solving correctly the problem of the interaction between phonons and electrons, we can see that the final formulae obtained by Bardeen, Cooper and Schrieffer for the ground state and one-fermion excitations are correct in the first approximation.

On the other hand their scheme does not lead to a branch in the spectrum, corresponding to collective excitations and the effect of Coulomb forces has not been taken into account correctly. A complete solution of the problem in the initial formulation of Fröhlich, as well as with the supplementary complications which arise, for example from taking into account the Coulomb interaction, could be obtained by means of a new method, developed by one of the authors (6).

This method is founded on the deep-lying physical, as well as mathematical analogy with the phenomenon of superfluidity and is a direct generalization of the method proposed in 1947 (7) for the development of a microscopic theory of superfluidity.

In the present work we give a systematic treatment of this method applying it to the study of the ground state and of the elementary excitations, both one-fermion and collective.

I.2 Resumé of the microscopic theory of superfluidity

We start with a short review of the fundamental principles of the microscopic theory of superfluidity for Bose-systems.

It is well known that all the particles of an ideal Bose gas at absolute zero, have their momenta strictly equal to zero and are in the so called condensate.

However in the absence of interaction such a condensate does not form a bound assembly, and therefore it cannot posses superfluid properties.

Indeed, let it be subjected to a motion, such that all the particles of the gas have velocities equal to $\underline{u}$. Then the total energy will be

$$E = \frac{1}{2} N m u^2$$

where N is the number of particles and m is the mass.

Suppose that one of the particle collides with an impurity or with the wall of the container, and hence its velocity $\underline{u}$ changes into the smaller one u_I. Evidently, the total energy

$$E = \frac{1}{2}(N-1)mu^2 + \frac{mu_I^2}{2}$$

diminishes. Therefore, from the energetic point of view it is convenient for the individual particles to leave the condensate and to slow down because of collision processes, and this will

lead to a gradual slowing down of the gas as a whole.

Thus an ideal Bose-gas is not suitable as a model for the study of superfluidity. The situation changes completely for a nonideal Bose gas, the particles of which are subject to interactions, however small. The second-quantized Hamiltonian of a weakly non-ideal Bose gas has the form

$$H = \sum_p \frac{|p|^2}{2m} a_p^+ a_p + \frac{1}{2V} \sum_{\substack{p_1,p_2,p_1',p_2' \\ p_1+p_2=p_1'+p_2'}} \nu(p_1-p_1')\, a_{p_1}^+ a_{p_2}^+ a_{p_2'} a_{p_1'} \qquad (I.1)$$

where p is the momentum of the particle, $\nu(p)$ the Fourier transform of the interaction energy between a pair of particles, which will be considered as proportional to a small parameter, a^+, a are the boson creation and annihilation operators, respectively, and V is the volume of the system.

An essential factor, ensuring the possibiblity of solving the problem, is the existence of the condensate, i.e. the fact that the overwhelming majority of the molecules is in the lowest energy state. Due to the fact that the condensate contains a microscopically large number of particles N_0, one may neglect the noncommutativity of the creation and annihilation operators for particles out of the vacuum, a_0^+; a_0 and replaoe them by c-numbers. Then, introducing the new Bose-operators

$$b_p = a_0^+ N_0^{-1/2} a_p \qquad b_p^+ = a_p^+ N_0^{-1/2} a_0$$

one may transform (I.I) to the form

$$H = H_0 + H_{int}$$

$$H_{int} = \frac{N^2}{2V}\nu(0) + \frac{N_0}{2V}\sum \nu(p)(b_p^+ b_{-p}^+ + b_p b_{-p} + 2 b_p^+ b_p) + H' \qquad H_0 = \sum \frac{|p|^2}{2m} b_p^+ b_p$$

where H' is an expression containing ternary and quaternary forms in the b^+, b .

Standard perturbation theory is not applicable to the Hamiltonian in this form, as the matrix elements corresponding to virtual creation of particles from the vacuum contain energy denominators of the form

$$\frac{p_1^2}{2m} + \dots + \frac{p_s^2}{2m}$$

Such denominators are, in general, non-dangerous and do not lead to divergences upon integrating with respect to $p_1, \dots, p_s$, except in the case when two virtual particles with opposite momenta $\pm p$ are created. In this case the higher-order approximations will contain denominators of the form:

$$\left(\frac{p^2}{2m}\right)^n$$

which lead to divergences. This means physically that even for infinitely small ν the interaction between particles with opposite momenta will be extremely intense, if these momenta are sufficiently small.

In the method of the paper (7) this difficulty is eliminated, by separating from the Hamiltonian the part quadratic in b^+, b , and diagonalizing it by means of the canonical transformation

$$b_k = u_k \xi_k + v_k \xi_{-k}^+ \qquad (I.3)$$

where ξ^+, ξ are the new Bose-operators and u_k, v_k are real c-number functions of k, which are subject to the condition

$$u_k^2 - v_k^2 = 1 \qquad u_k = u_{-k} \qquad v_k = v_{-k} \tag{I.4}$$

The diagonalization mixes up the particle creation and annihilation operators and in fact means going over to a new ground state which takes into account the interaction. Note, that in lowest order in ν this procedure is equivalent to the following. Choose u_k and v_k such, that the contribution of all diagrams of the form represented in fig. I, corresponding to the creation of a pair of particles with opposite momenta out of the vacuum, vanish. As this choice of u_k, v_k makes it unnecessary to take into account such processes there is no longer any obstacle to the application of perturbation theory.

In the paper referred to, the following expression has been obtained for the spectrum of elementary excitations of a non-ideal Bose-gas

$$E(p) = \sqrt{\frac{N_0}{V}\frac{|p|^2}{m}\nu(p) + \frac{|p|^4}{4m^2}} \tag{I.5}$$

or, approximately, for small and large p, respectively

$$E(p) = \sqrt{\frac{N_0}{Vm}\nu(0)}\,|p| \tag{I.6}$$

$$E(p) = \frac{|p|^2}{2m} + \frac{N_0\nu(p)}{V} \tag{I.7}$$

In first approximation the ground state C_0 is characterized by zero occupation numbers $\xi_p^+\xi_p$. From (I.6) it follows, that for the stability of C_0 it is necessary that

$$\nu(0) = \int \Phi(r)\,dr > 0 \tag{I.8}$$

since this ensures the reality of E(p). Condition (I.8) exhibits the predominancy of the repulsive forces. It is now easy to show, that the considered model possesses superfluid properties. In order to see this, note that due to the translational invariance of the dynamical system under consideration, there exists covariance with respect to a change of the origin of momenta and velocities.

Making the transformation

$$p \to p - mu$$

and constructing in the new reference frame the ground state C_0, which we will denote by C_u, it can be seen that in the old "rest" system the state C_u will be the state in which the particles have the average velocity u. It is easy to observe, that the energy of an elementary excitation for the "moving" state C_u will be in the usual "rest" system

$$E(p) - (p.u)$$

Let the velocity u be smaller than a certain critical velocity

$$|u| < u_{cr} = \min_p \frac{E(p)}{|p|} \tag{I.9}$$

For such velocities the energy of elementary excitations is positive. Therefore, the slowing down of individual particles due to their leaving the assembly, or, equivalently, the generation

of elementary excitations is not energetically favourable. Therefore the state C_u will be metastable for $\mu < \mu_c$. Thus we have a bound assembly, exhibiting the property of superfluidity.

Eq. (I.9) implies, that in the case of ideal gas, when $E(p) = p^2/2m$

$$\min_p \frac{E(p)}{|p|} = 0$$

and there will be no superfluidity.

We note in conclusion, that the method of ref. (7) has been recently extended in the papers of Brueckner and Sawada (8), in which a more realistic model of Helium II is considered.

Let us now pass on to an exposition of the theory of superconductivity.

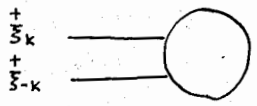

Fig.I

§2. SUPERCONDUCTIVITY IN FRÖHLICH'S MODEL+)

+) This Section follows the papers (6) and (10)

2.I The principle of compensation of dangerous graphs.

We take as our starting point a model proposed by Fröhlich, in which the Coulomb interaction is not introduced explicitly and the dynamical system is described by the Hamiltonian++)

++) We use here a system of units with $\hbar = 1$.

$$H_{Fr} = \sum_{k,s} E(k)\, a^{+}_{ks} a_{ks} + \sum_{q} \omega(q)\, b^{+}_{q} b_{q} + H_{int} \qquad (2.1)$$

$$H_{int} = \sum_{\substack{k,k',q,s \\ k'-k=q}} g(q) \left(\frac{\omega(q)}{2V}\right)^{1/2} a^{+}_{ks} a_{k's} b^{+}_{q} + \text{conj.} \qquad |q| < q_M \qquad (2.2)$$

where E(k) is the electron energy, $\omega(q)$ is the phonon energy, g is the coupling constant, V is the volume of the system, K, K', q are the wave vectors, a^{+}, a, b^{+}, b the creation and annihilation operators for electrons and phonons, respectively and s is a spin index with the values + or-. In a rigorous treatment one should have included in H_{int} the repulsion between the electrons, at least in the form of a strongly screened Coulomb interaction, since the electron- phonon interaction alone is unable to ensure the stability of the electron system. Nevertheless, when using perturbation theory, we are allowed not to take into account explicitly such stabilizing interactions, considering them as sufficiently small.

Let us show that this model leads in fact to the appearance of superconductivity.

It is a well established fact, that usual perturbation theory, i.e. an expansion in powers of the coupling constant, is not applicable since, despite its smallness, the electron-phonon interaction becomes very important near the Fermi surface. Moreover, as already shown by Fröhlich for the one-dimensional case, the energy is not an analytic function of the coupling constant, as it possesses singularities in the neighbourhood of the origin.

Therefore, we first perform a canonical transformation, starting from considerations similar to those made in the Introduction for the theory of superfluidity.

Note that the matrix elements corresponding to the virtual creation of "particles" from the vacuum always contain energy denominators

$$\varepsilon(k_1) + \dots + \varepsilon(k_{ss}) + \omega(q_1) + \dots + \omega(q_s)$$

in which $\varepsilon(k) = |E(k) - E_F|$ is the energy of an electron $(E(k) > E_F)$ or hole $(E(k) < E_F)$ which becomes small near the Fermi surface.

In general such denominators are not "dangerous" and the integration over the momenta does not lead to divergences, except in the case of the virtual creation of a single pair without phonons. In virtue of the conservation laws the momenta of the particles of this pair will be equal and opposite and the energy denominator $2\varepsilon(k)$ will then become "dangerous" for the integration. It may also be mentioned that the spins of the particles will likewise have opposite directions.

It must be stressed that in ordinary perturbation theory, applied directly to the normal state these denominators cannot appear because of the conservation of the number of real electrons. But if we mix electron and hole states by means of a canoninal transformation, the conservation law no longer applies and such den minators can appear.

Generalizing the transformation (I.3) from the theory of superfluidity, we introduce here the new Fermi amplitudes +)

$$\begin{aligned} \alpha_{k0} &= u_k a_{k,+} - v_k a^{+}_{-k,-} \\ \alpha_{k1} &= u_k a_{-k,-} + v_k a^{+}_{k,+} \end{aligned} \qquad (2.3)$$

or

$$\begin{aligned} a_{k,+} &= u_k \alpha_{k0} + v_k \alpha^{+}_{k1} \\ a_{-k,-} &= u_k \alpha_{k1} - v_k \alpha^{+}_{k0} \end{aligned} \qquad (2.4)$$

where U_k, V_k are real numbers, subject to the condition

$$u_k^2 + v_k^2 = 1 \qquad (2.5)$$

and symmetric with respect to the substitution

It is not difficult to verify that this transformation retains all commutation properties of the Fermi operators and is therefore canonical. Note also that it is a generalization of the usual transformation employed to introduce creation and annihilation operators for holes inside the Fermi surface or for electrons outside it. Indeed, if

$$\begin{aligned} u_k = 1 \,,\ v_k = 0 \qquad & E(k) > E_F \\ u_k = 0 \,,\ v_k = 1 \qquad & E(k) < E_F \end{aligned}$$

we obtain

$$\begin{aligned} \alpha_{k0} = a_{k,+} \quad \alpha_{k1} = a_{-k,-} \qquad & E(k) > E_F \\ \alpha_{k0} = -a^{+}_{-k,-} \quad \alpha_{k1} = a^{+}_{k,+} \qquad & E(k) < E_F \end{aligned}$$

so that α_{k0}, for example, will be the annihilation operator for an electron of momentum k and spin I/2 outside the Fermi sphere and the annihilation operator of a hole with momentum - k and spin -I/2 inside it. In the general case when $(u_k, v_k) \neq (0,1)$ a superposition of a hole and an electron is encountered.

+) We learned recently, that on the basis of the paper (6) Valatin (25) has shown that by means of this transformation the theory of Bardeen, Cooper and Schrieffer can be put into a clearer and more elegant form.

It will be more convenient not to be tied up with the relation expressing the constancy of the number of particles

$$\sum_{k,s} a^{+}_{ks} a_{ks} = N_0$$

where N_0 is the total number of electrons, but as usual, to introduce a parameter λ playing the role of a chemical potential.

Thus in place of H_{Fr} we shall consider the Hamiltonian

$$H = H_{Fr} - \lambda N \tag{2.6}$$

In due course the parameter λ will be determined form the condition that in the state under consideration

$$\bar{N} = N_0 \tag{2.7}$$

We could introduce from the outset an expansion in powers of the coupling constant of the electron-phonon interaction. In fact such an expansion would be in powers of the dimensionless parameter

$$\rho = g^2 \frac{dn}{dE}$$

Note that there exists another small parameter in this problem, namely, the ratio of the energy of the sound quantum to the Fermi energy: $\frac{\omega}{E_F}$. The parameter ρ is not strictly small and according to an estimate of Pines' (II) takes on values between 0,I and 0,49, whereas the ratio $\frac{\omega}{E_F}$ is indeed small. As the coupling constant g enters the Hamiltonian together with the factor $\sqrt{\omega}$ and as E_F is the natural unit for measuring energies, it would seem that the expansion parameter is of the form $\rho \frac{\omega}{E_F}$. However, the computation of the following approximation shows that the results of the first order approximation are modified by quantities of the order ρ , rather than $\rho \frac{\omega}{E_F}$.

This is due to the fact that in the virtual creation of a phonon pair without Fermions, the energy denominator is proportional to ω so that in the resulting formulas ρ appears without the accompanying factor $\frac{\omega}{E_F}$.

Thus the accuracy of the approximation can be considerably increased, if one carries out a renormalization of the phonon energy, or, equivalently, uses the method of graph compensation in order to compensate the contribution of the virtual phonon pair creation. Then the processes of virtual creation of a large number of phonons, appearing in the higher approximations, will not deteriorate the convergence as the corresponding matrix elements will contain $\rho \frac{\omega}{E_F}$ in a sufficiently high power, and the reduction of one factor $\frac{\omega}{E_F}$ with an energy denominator will no longer destroy the whole small factor in front of ρ .

In accord with the sketched programme we carry out the following canonical transformation of the quantized Boson amplitudes (cf. (I.3)):

$$b_q = \lambda_q \beta_q + \mu_q \beta^{+}_{-q} \tag{2.8}$$

where λ_q, μ_q are real numbers subject to the relation

$$\lambda_q^2 - \mu_q^2 = 1 \tag{2.9}$$

From the transformed Hamiltonian we separate the part

$$H_0 = \sum_k \tilde{\varepsilon}(k)\left(\overset{+}{\alpha}_{k0}\alpha_{k0} + \overset{+}{\alpha}_{k1}\alpha_{k1}\right) + \sum_q \tilde{\omega}(q)\overset{+}{\beta}_q\beta_q$$

in which $\tilde{\varepsilon}(k)$, $\tilde{\omega}(q)$ are the " renormalized" energies of the Fermion and Boson excitations respectively, and include the other terms in the interaction Hamiltonian. We thus obtain from (2.I)

$$H = U + H_0 + H' + H''$$

where

$$U = \text{const} = 2\sum_k (E(k)-\lambda)v_k^2 + \sum_q \omega(q)\mu_q^2$$

$$H' = \sum_{\substack{k,k',q \\ k'-k=q}} g(q)\left(\frac{\omega(q)}{2V}\right)^{1/2}(u_k v_{k'} + u_{k'}v_k)\left(\overset{+}{\alpha}_{k0}\overset{+}{\alpha}_{k'1} + \alpha_{k1}\alpha_{k'0}\right)(\lambda_q+\mu_q)\left(\overset{+}{\beta}_q + \beta_{-q}\right) +$$

$$+ \sum_{\substack{k,k',q \\ k'-k=q}} g(q)\left(\frac{\omega(q)}{2V}\right)^{1/2}(u_k u_{k'} - v_k v_{k'})\left(\overset{+}{\alpha}_{k0}\alpha_{k'0} + \overset{+}{\alpha}_{k'1}\alpha_{k1}\right)(\lambda_q+\mu_q)\left(\overset{+}{\beta}_q + \beta_{-q}\right)$$

$$H'' = \sum_k \left\{(E(k)-\lambda)(u_k^2-v_k^2) - \tilde{\varepsilon}(k)\right\}\left(\overset{+}{\alpha}_{k0}\alpha_{k0} + \overset{+}{\alpha}_{k1}\alpha_{k1}\right) + 2\sum_k (E(k)-\lambda)u_k v_k\left(\overset{+}{\alpha}_{k0}\overset{+}{\alpha}_{k1} + \alpha_{k1}\alpha_{k0}\right) +$$

$$+ \sum_q \left\{\omega(q)(\lambda_q^2+\mu_q^2) - \tilde{\omega}(q)\right\}\overset{+}{\beta}_q\beta_q + \sum_q \omega(q)\lambda_q\mu_q\left(\overset{+}{\beta}_q\overset{+}{\beta}_{-q} + \beta_{-q}\beta_q\right)$$

Let us apply the principle of compensation of dangerous graphs in order to compensate away the processes of virtual creation of Fermion $\overset{+}{\alpha}_{k0}\overset{+}{\alpha}_{k1}$ and Boson $\overset{+}{\beta}_q\overset{+}{\beta}_{-q}$ pairs out of the vacuum. To the second order we obtain the following equations

$$\begin{aligned} 2(E(k)-\lambda)u_k v_k - \langle \overset{*}{C}_v \alpha_{k1}\alpha_{k0} H' H_0^{-1} H' C_v\rangle = 0 \\ \omega(q)\lambda_q\mu_q - \langle \overset{*}{C}_v \beta_{-q}\beta_q H' H_0^{-1} H' C_v\rangle = 0 \end{aligned} \qquad (2.\text{I0})$$

where C_v is the vacuum state vector corresponding to zero occupation numbers.

To the same order the renormalized energies $\tilde{\varepsilon}(k)$, $\tilde{\omega}(k)$ are determined by

$$\begin{aligned} (E(k)-\lambda)(u_k^2-v_k^2) - \tilde{\varepsilon}(k) - \langle \overset{*}{C}_v \alpha_{k0} H'(H_0-\tilde{\varepsilon}(k))^{-1} H' \overset{+}{\alpha}_{k0} C_v\rangle = 0 \\ \omega(q)(\lambda_q^2+\mu_q^2) - \tilde{\omega}(q) - \langle \overset{*}{C}_v \beta_q H'(H_0-\tilde{\omega}(q))^{-1} H' \overset{+}{\beta}_q C_v\rangle = 0 \end{aligned} \qquad (2.\text{II})$$

It must be stressed, that in the above equations the expressions for the matrix elements take into account only connected graphs. Due to the symmetry with respect to the pemutation (I) $\rightleftarrows$ (0) nothing would change in Eq. (2.II) if $\alpha_{k0}, \overset{+}{\alpha}_{k0}$ were replaced by $\alpha_{k1}, \overset{+}{\alpha}_{k1}$. From Eqs. (2.I0) and (2.II) we obtain

$$\left\{E(k) - \frac{1}{2V}\sum_{k'}\frac{g^2(k-k')\omega(k-k')}{\tilde{\omega}(k-k') + \tilde{\varepsilon}(k) + \tilde{\varepsilon}(k')}(\lambda_{k'-k}+\mu_{k'-k})^2(u_{k'}^2-v_{k'}^2) - \lambda\right\}u_k v_k = \frac{u_k^2-v_k^2}{2V}\sum_{k'}\frac{g^2(k-k')\omega(k-k')}{\tilde{\omega}(k-k') + \tilde{\varepsilon}(k) + \tilde{\varepsilon}(k')}(\lambda_{k'-k}+\mu_{k'-k})^2 u_{k'}v_{k'} \quad (2.\text{I2})$$

$$\lambda_q\mu_q - (\lambda_q+\mu_q)^2\frac{g^2(q)}{2V}\sum_{\substack{k,k' \\ k'-k=q}}\frac{(u_k v_{k'} + u_{k'}v_k)^2}{\tilde{\omega}(q) + \tilde{\varepsilon}(k) + \tilde{\varepsilon}(k')} = 0 \qquad (2.\text{I3})$$

$$\tilde{\varepsilon}(k) = \Big\{ E(k) - \frac{1}{2V}\sum_{k'} \frac{g^2(k-k')\,\omega(k-k')}{\tilde{\omega}(k-k') + \tilde{\varepsilon}(k) + \tilde{\varepsilon}(k')} \left(\lambda_{k-k'} + \mu_{k-k'}\right)^2 \left(u_{k'}^2 + v_{k'}^2\right) - \lambda \Big\} \left(u_k^2 - v_k^2\right) - \tag{2.I4}$$

$$- \frac{1}{V}\sum_{k'} \tilde{\varepsilon}(k') \frac{g^2(k-k')\,\omega(k-k')}{\left[\tilde{\omega}(k-k') + \tilde{\varepsilon}(k')\right]^2 - \left[\tilde{\varepsilon}(k)\right]^2} \left(\lambda_{k-k'} + \mu_{k-k'}\right)^2 \left(u_k^2 u_{k'}^2 + v_k^2 v_{k'}^2\right) +$$

$$+ 2u_k v_k \frac{1}{V}\sum_{k'} \frac{g^2(k-k')\,\omega(k-k')\left[\tilde{\omega}(k-k') + \tilde{\varepsilon}(k')\right]}{\left[\tilde{\omega}(k-k') + \tilde{\varepsilon}(k')\right]^2 - \left[\tilde{\varepsilon}(k)\right]^2} \left(\lambda_{k-k'} + \mu_{k-k'}\right)^2 u_{k'} v_{k'}$$

$$\tilde{\omega}(q) = \omega(q)\left(\lambda_q^2 + \mu_q^2\right) - \frac{\omega(q)\,g^2(q)\left(\lambda_q + \mu_q\right)^2}{2V} \sum_{\substack{k,k' \\ k'-k=q}} \left(u_k v_{k'} + u_{k'} v_k\right)^2 \left\{ \frac{1}{\tilde{\varepsilon}(k) + \tilde{\varepsilon}(k') - \tilde{\omega}(q)} + \frac{1}{\tilde{\varepsilon}(k) + \tilde{\varepsilon}(k') + \tilde{\omega}(q)} \right\} \tag{2.I5}$$

2.2.- Simplification of the expressions.

We undertake now a simplification of the above expressions, which on the one hand are extremely complicated, and on the other hand still contain terms which are of a higher order of smallness than the chosen approximation. We shall retain only terms of the first order with respect to the adopted small parameter $\rho \frac{\omega}{E_F}$.

We start with the solutions of Eqs. (2.I3) and(2.I5)

$$\left(\lambda_q + \mu_q\right)^2 = \left\{ 1 - \frac{2}{V} \sum_{\substack{k,k' \\ k'-k=q}} g^2(q) \frac{\left(u_k v_{k'} + u_{k'} v_k\right)^2}{\tilde{\omega}(q) + \tilde{\varepsilon}(k) + \tilde{\varepsilon}(k')} \right\}^{-1/2}$$

$$\tilde{\omega}(q) = \omega(q) \left\{ 1 - \frac{g^2(q)}{2V} \sum_{\substack{k,k' \\ k'-k=q}} \left(u_k v_{k'} + u_{k'} v_k\right)^2 \left[\frac{1}{\tilde{\varepsilon}(k) + \tilde{\varepsilon}(k') + \tilde{\omega}(q)} + \frac{3}{\tilde{\varepsilon}(k) + \tilde{\varepsilon}(k') - \tilde{\omega}(q)} \right] \right\} \left\{ 1 - \frac{2g^2(q)}{V} \sum_{\substack{k,k' \\ k'-k=q}} \frac{\left(u_k v_{k'} + u_{k'} v_k\right)^2}{\tilde{\omega}(q) + \tilde{\varepsilon}(k) + \tilde{\varepsilon}(k')} \right\}^{-1/2} \tag{2.I6}$$

Since u and v can differ from their normal values 0, I only in a layer of thickness $< \omega$ in the neighbourhood of the Fermi surface and since $\tilde{\varepsilon}(k) \sim |E(k) - E_F|$, we obtain from (2.I6), retaining only the principal terms

$$\left(\lambda_q + \mu_q\right)^2 = \left\{ 1 - \frac{4g^2(q)}{V} \sum_{\substack{k,k' \\ k'-k=q}} \frac{\theta_G(k)\,\theta_F(k')}{E(k) - E(k')} \right\}^{-1/2}$$

$$\tilde{\omega}(q) = \omega(q) \left\{ 1 - \frac{4g^2(q)}{V} \sum_{\substack{k,k' \\ k'-k=q}} \frac{\theta_G(k)\,\theta_F(k')}{E(k) - E(k')} \right\}^{1/2}$$

where θ_G and θ_F are defined by

$$\theta_G(k) = \begin{cases} 1 & E(k) > E_F \\ 0 & E(k) < E_F \end{cases} \qquad \theta_F(k) = \begin{cases} 0 & E(k) > E_F \\ 1 & E(k) < E_F \end{cases} \tag{2.I7}$$

For the simplification of Eq. (2.I2) we introduce the renormalized function $\tilde{g}(q)$

$$\tilde{g}^2(q) = g^2(q) \left\{ 1 - \frac{4g^2(q)}{V} \sum_{\substack{k,k' \\ k'-k=q}} \frac{\theta_G(k)\,\theta_F(k')}{E(k) - E(k')} \right\}^{-1} \tag{2.I8}$$

This renormalization is possible as long as the values of the coupling parameter are sufficiently small. In any case they are not allowed to exceed a value for which the renorma-

lized sound frequency $\tilde{\omega}(q)$ and the renormalized $\tilde{g}(q)$ become imaginary. The same limiting value for the coupling constant has been obtained by means of another method in the investigation of the stability of the lattice with respect to the electron-phonon interaction in paper by S.V. Tyablikov and one of authors (I2).

We put further

$$\xi(k) = E(k) - \frac{1}{2V}\sum_{k'} \frac{\tilde{g}^2(k-k')\,\tilde{\omega}(k-k')}{\tilde{\omega}(k-k') + \tilde{\varepsilon}(k) + \tilde{\varepsilon}(k')}\,(u_{k'}^2 - v_{k'}^2) - \lambda \tag{2.I9}$$

$$C(k) = \frac{1}{V}\sum_{k'} \frac{\tilde{g}^2(k-k')\,\tilde{\omega}(k-k')}{\tilde{\omega}(k-k') + \tilde{\varepsilon}(k) + \tilde{\varepsilon}(k')}\,u_{k'}v_{k'} \tag{2.20}$$

Then, by (2.I2) we obtain

$$u_k^2 = \frac{1}{2}\left\{1 + \frac{\xi(k)}{\sqrt{C^2(k)+\xi^2(k)}}\right\} \qquad v_k^2 = \frac{1}{2}\left\{1 - \frac{\xi(k)}{\sqrt{C^2(k)+\xi^2(k)}}\right\} \tag{2.2I}$$

and

$$C(k) = \frac{1}{2V}\sum_{k'} \frac{\tilde{g}^2(k-k')\,\tilde{\omega}(k-k')}{\tilde{\omega}(k-k') + \tilde{\varepsilon}(k') + \tilde{\varepsilon}(k)}\,\frac{C(k')}{\sqrt{C^2(k')+\xi^2(k')}} \tag{2.22}$$

Consider now the last term in the L.h.s of Eq. (2.I4), which is always small (for small ρ it will be of the order $\omega e^{-1/\rho}$). Furthermore, when $\tilde{\varepsilon}(k)$ increases to the order ω the factor $u_k v_k$, vanishes praotically. Therefore, in the chosen approximation, we may replace the denominator of this term by

$$[\tilde{\omega}(k-k') + \tilde{\varepsilon}(k')]^2$$

Thus Eqs. (2.I9) - (2.2I) imply

$$\tilde{\varepsilon}(k)\left\{1 + \frac{1}{V}\sum_{k'} \frac{\tilde{g}^2(k-k')\,\tilde{\omega}(k-k')}{[\tilde{\omega}(k-k') + \tilde{\varepsilon}(k')]^2 - [\tilde{\varepsilon}(k)]^2}\,(u_k^2 u_{k'}^2 + v_k^2 v_{k'}^2)\right\} = \sqrt{\xi^2(k)+C^2(k)} \tag{2.23}$$

For the normal state, when

$$u_k = \theta_G(k) \qquad v_k = \theta_F(k) \tag{2.24}$$

we obtain

$$\xi_n(k) = E(k) - \lambda - \frac{1}{2V}\sum_{k'} \frac{\tilde{g}^2(k-k')\,\tilde{\omega}(k-k')\,(\theta_G(k') - \theta_F(k'))}{\tilde{\omega}(k-k') + |E(k)-E_F| + |E(k')-E_F|}$$

$$\tilde{\varepsilon}_n(k) = (1-\eta_n(k))\,|\xi_n(k)| \qquad \eta_n(k) = \frac{1}{V}\sum_{k'} \frac{\tilde{g}^2(k-k')\,\tilde{\omega}(k-k')}{[\tilde{\omega}(k-k') + \tilde{\varepsilon}_n(k')]^2 - [\tilde{\varepsilon}_n(k)]^2}\,(\theta_G(k)\theta_G(k') + \theta_F(k)\theta_F(k'))$$

Here the subscript n denotes quantities in the state (2.24). The state corresponding to the nonvanishing solution of Eq. (2.22) will be denoted by the subscript s. In both cases the expressions $\xi(k)-(E(k)-E_F)$, $\eta(k)$ will be of the first order of smallness but the differences $\xi_n(k)-\xi_s(k)$ $\eta_n(k)-\eta_s(k)$ will be of higher order of smallness. Therefore, (2.23) yields

$$\tilde{E}_s(k) = \sqrt{(1-\eta_s)^2\xi_s^2(k) + (1-\eta_s)^2 C^2(k)} \approx \sqrt{(1-\eta_n)^2\xi_n^2(k) + C^2(k)}$$

so that in the chosen approximation

$$\tilde{E}_s(k) = \sqrt{\tilde{\xi}_n^2(k) + C^2(k)}$$

Note, further, that in Eqs. (2.21) and (2.22) $\tilde{E}(k)$, $|\xi(k)|$ may be replaced by $|E(k)-E_F|$.

We are thus led to

$$C(k) = \frac{1}{2(2\pi)^3}\int \frac{\tilde{g}^2(k-k')\,\tilde{\omega}(k-k')}{\tilde{\omega}(k-k') + |\xi(k)| + |\xi(k')|}\,\frac{C(k')}{\sqrt{\xi^2(k') + C^2(k')}}\,d\vec{k}' \qquad (2.25)$$

where

$$\xi(k) = E(k) - E_F$$

This result has been derived without the restriction to spherical symmetry.

By means of a simple renormalization of $g(q)$ and $\omega(q)$ we have considerably improved the degree of accuracy of the theory, since we have used asymptotic approximations not in powers of ρ but in powers of $\rho\,\frac{\omega}{E_F}$. The quantity

$$\rho = \tilde{g}^2\,\frac{dn}{dE}$$

must no longer be very small. It suffices that the solution $C(k)$ of Eq. (2.25) be small compared with ω (i.e.only, "exponential" smallness is required).

Eq. (2.25) has a peculiar singularity: for $g^2 \to 0$ the solution C vanishes as $\exp(-A/g^2)$ where $A = \text{const} > 0$. This is due to the fact that the integral in the r.h.s. of (2.25) becomes logarithmically divergent in the neighbourhood of the surface $\xi(k)=0$, if we put $C = 0$ under the square root in the integrand. In this situation one can easily obtain the asymptotic form of the solution, for small g:

$$C(k) = \omega\, e^{-\frac{1}{\rho}}\,\frac{1}{2g^2}\int_{-1}^{+1}\frac{\tilde{g}^2(k_F\sqrt{2(1-t)})\,\tilde{\omega}(k_F\sqrt{2(1-t)})}{\tilde{\omega}(k_F\sqrt{2(1-t)}) + |\xi(k)|}\,dt$$

$$\rho = g^2\,\frac{1}{2\pi^2}\left(\frac{k^2}{\frac{dE(k)}{dk}}\right)_{k=k_F} \qquad g^2 = \frac{1}{2}\int_{-1}^{+1}\tilde{g}^2(k_F\sqrt{2(1-t)})\,dt$$

$$\ln\omega = \int_0^\infty \ln\left(\frac{1}{2\xi}\right)\frac{d}{d\xi}\left\{\frac{1}{2g^2}\int_{-1}^{+1}\frac{\tilde{g}^2(k_F\sqrt{2(1-t)})\,\tilde{\omega}(k_F\sqrt{2(1-t)})}{\tilde{\omega}(k_F\sqrt{2(1-t)}) + \xi}\,dt\right\}^2 d\xi$$

The cut-off momentum q_M does not appear explicitly in these expressions, due to the fact that, formally, one may consider that $\omega(q)=0$ for $|q|>q_M$.

2.3.- The energy difference between the normal and superconducting states.

We now evaluate the difference

$$H_s(\lambda) - H_n(\lambda) \tag{2.26}$$

between the eigenvalues of the Hamiltonian in the eigenstates $\underline{s}$ and $\underline{n}$ for the same value of $\lambda \sim E_F$. In the chosen approximation we have

$$H(\lambda) = U - \langle C_\nu^* H' H_0^{-1} H' C_\nu \rangle = 2\sum_k (E(k)-\lambda) v_k^2 + \sum_q \omega(q)\mu_q^2 - \langle C_\nu^* H' H_0^{-1} H' C_\nu \rangle$$

or, after some manipulations

$$H(\lambda) = 2\sum_k (E(k)-\lambda) v_k^2 + \sum_q \omega(q)\mu_q^2 - \frac{1}{V}\sum_{\substack{k,k' \\ k\neq k}} \frac{\tilde{g}^2(k-k')\tilde{\omega}(k-k')}{\tilde{\omega}(k-k') + \tilde{\varepsilon}(k) + \tilde{\varepsilon}(k')} \left(u_k^* v_{k'}^2 + u_k v_k u_{k'} v_{k'}\right) \tag{2.27}$$

For the difference (2.26) we obtain from (2.27) after some simplifications and retaining only terms of the desired order of smallness

$$H_s(\lambda) - H_n(\lambda) = -\sum_k \theta_G(k) \frac{v_k^3}{u_k} C(k) - \sum_k \theta_F(k) \frac{u_k^3}{v_k} C(k)$$

or, by (2.2I)

$$\frac{H_s(\lambda) - H_n(\lambda)}{V} = -\frac{1}{2(2\pi)^3}\int dk \sqrt{C^2(k)+\xi^2(k)} \left\{ \theta_G(k)\left[1 - \frac{\xi(k)}{\sqrt{C^2(k)+\xi^2(k)}}\right]^2 + \theta_F(k)\left[1 + \frac{\xi(k)}{\sqrt{C^2(k)+\xi^2(k)}}\right]^2 \right\} \tag{2.28}$$

In order to determine the difference $E_s^N - E_n^N$ between the energies of the two states, for the same number of electrons $N = N_0$, note that

$$H(\lambda) = E(\lambda) - \lambda N(\lambda) \quad - \frac{\partial H(\lambda)}{\partial \lambda} = N(\lambda)$$

We obtain, consequently

$$E_s^N - E_n^N = E_s[\lambda_s(N)] - E_n[\lambda_n(N)] = H_s[\lambda_s(N)] - H_n[\lambda_n(N)] + [\lambda_s(N) - \lambda_n(N)]N =$$

$$= H_s[\lambda_s(N)] - H_n[\lambda_s(N)] + H_n[\lambda_s(N)] - H_n[\lambda_n(N)] + [\lambda_s(N) - \lambda_n(N)]N =$$

$$= H_s[\lambda_s(N)] - H_n[\lambda_s(N)] + H_n[\lambda_s(N)] - H_n[\lambda_n(N)] - [\lambda_s(N) - \lambda_n(N)]\left(\frac{\partial H_n(\lambda)}{\partial \lambda}\right)_{\lambda=\lambda_n(N)} =$$

$$= H_s(\lambda_s) - H_n(\lambda_s) + \frac{1}{2}[\lambda_s(N) - \lambda_n(N)]^2 \frac{\widetilde{\partial^2 H_n}}{\partial \lambda^2}$$

The second term of the last expression will be of second order of smallness as compared with the first, so that, in the approximation under consideration

$$E_s^N - E_n^N = H_s(\lambda_s) - H_n(\lambda_s) \tag{2.29}$$

Thus the energy difference is determined by the same Eq.(2.28) as $H_s(\lambda) - H_n(\lambda)$.

2.4.- Superconductivity.

We now establish the existence of superconductivity, since we do not take into account the action of the magnetic field it is more correct to say that we establish the existence of superfluidity for the electron fluid in Frohlich's model. This is achieved by showing that there exists a state with nonvanishing average total electron momentum, in which the energies of the elementary excitations are all positive. In this way we establish the possibility of exitence of a current-carrying state, which is stable with respect to weak perturbations.

In order to remain formally within the class of states with vanishing total momentum and to be able to use the foregoing results, we perform a translation of the origin in momentum space

$$k \to k + p \tag{2.30}$$

Evidently, the state with vanishing total momentum in the new frame of reference will have in the original frame a total momentum N_p.

On the other hand, the translation (2.30) replaces $E(k)$ in the original Hamiltonian H by

$$E(k-p) = E(k) - \left(p \cdot \frac{\partial E(k)}{\partial k}\right) + \cdots \tag{2.31}$$

The additional term $-\left(p \cdot \frac{\partial E(k)}{\partial k}\right)$ leads to the following additional term in H

$$-\sum_{k,s}\left(p \cdot \frac{\partial E(k)}{\partial k}\right) \overset{+}{\alpha}_{ks} a_{ks} = -\sum_{k,s}\left(p \cdot \frac{\partial E(k)}{\partial k}\right)\left(\overset{+}{\alpha}_{k0}\alpha_{k0} - \overset{+}{\alpha}_{k1}\alpha_{k1}\right)$$

which to be included in H". Supposing that the momentum p is so small that

$$\left(p \cdot \frac{\partial E(k)}{\partial k}\right) \quad \text{and} \quad E(k) \sim E_F$$

are of the order $C(k)$, the influence of the additional term on U_k, V_k, λ, μ in H may be neglected due to its smallness and the only effect will be a modification of the energy of the elementary excitations:

$$\tilde{\varepsilon}(k) \to \tilde{\varepsilon}(k) \pm \left(p \cdot \frac{\partial E(k)}{\partial k}\right)$$

We introduce the notation Δ for the energy gap separating the non current-carrying ground state from the elementary excitations, i.e. Δ denotes the value of $C(k)$ on the Fermi surface.

Since $\tilde{\varepsilon}(k) > \Delta$ the elementary excitations remain positive for the current-carrying state if the average velocity of the electrons is so small that

$$\left| p \frac{\partial E(k)}{\partial k} \right| < \Delta \ , \quad E(k) \sim E_F$$

or, in the spherically symmetric case

$$|u k_F| < \Delta$$

where u is the average velocity.

Consequently, for sufficiently small velocities the current-carrying state remains stable with respect to small perturbations.

Of course, for $\Delta = 0$ this is no longer true, since in this case the energies of the elementary excitations could take on negative values. It must be stressed, that insofar as we do not take into account the action of a magnetic field, the current-carrying state must be considered as metastable. Its energy differs from the energy of the "rest" state by an amount proportional to u^2. This increase of the energy can be calculated formally by intoducing into (2.3I) the supplementary term

$$\frac{1}{2} \sum_{\alpha,\beta=1,2,3} p^\alpha p^\beta \frac{\partial^2 E(k)}{\partial k^\alpha \partial k^\beta}$$

and taking into account its influence upon the energy of the current-carrying ground state.

We want to stress again the analogies with the case of a Bose gas. In that case the Fermi-sphere is replaced by a condensate in the lowest energy state. In the absence of interaction, i.e. for an ideal Bose gas, there will be no superfluidity. It appears only in the presence of an at least weak interaction. At the same time there appears an essential interaction of particles outside the condensate with opposite small momenta k, a fact which is responsible for the inapplicability of the usual perturbation theory.

In the case of an ideal electron gas, considered above the situation is quite analogous. Without interaction there is no superconductivity (superfluidity) and only the electron-phonon coupling, responsible for the interaction between pairs of electrons with momenta $\pm k$ and spins $\pm$ I/2 leads to the appearance of superconductivity. One can reduce approximately this interaction to an equivalent coupling between the Fermions, which pushes the analogy with the Bose gas still further.

Until recently it was thought that only Bose-Einstein particles can exhibit superfluidity. Due to the situation that arises now in the theory of superconductivity this point of view must be revised.

It is possible that nuclear matter is the real case on which the superfluidity of Fermi systems should be investigated.

§3. THE SPECTRUM OF COLLECTIVE EXCITATIONS OF THE SUPERCONDUCTING STATE[x)]

3.1.- Application of the method of approximate second quantization to a system with Coulomb interaction.

Until now we studied only elementary excitations of the simplest "individual" type-the appearance of a fermion of energy $\tilde{\varepsilon}(k)$ or a phonon of energy $\tilde{\omega}(k)$. Now let us proceed to the investigation of the more complicated branch of the spectrum corresponding to collective excitations of Fermions. A typical example of such excitations in Fermion systems are the plasma oscillations of a dense electron gas.

A correct and sufficiently simple analysis of these may be carried out by means of the method developed by Gell-Mann, Brueckner, Sawada, Brout and Fukuda [13]. In order to obtain the fundamental approximation, one may restrict oneself to the summation of only those graphs which do not lead to the destruction of the elementary particle-hole complex (Fig.2).

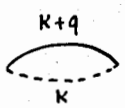

Fig. 2

From the physical point of view the importance of this kind of graphs is due to the existence of the Coulomb attraction between a particle and a hole. In order to obtain the secular equation for the energy spectrum $E(q)$ of the plasma oscillations one must sum over all graphs of the type represented in Fig. 3

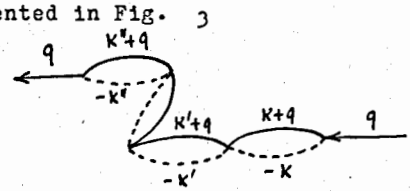

Fig. 3

For this summation one can use the method of approximate second quantization and thus construct a simplified Hamiltonian which allows of exact diagonalization so that only the desired diagrams are obtained, yielding however the same contribution as the exact Hamiltonian.

---

x) This Section is based on the investigations carried out by N.N. Bogolubov and in part (subsection 3.4) by V.V. Tolmachov.

Under these conditions the diagonalization of the simplified Hamiltonian naturally yields the same results as the direct summation of diagrams.

In order to obtain such a simplified "equivalent" Hamiltonian the following requirements must be fulfilled: the complex of Fig 2 is to be connected, the vertex parts of the simplified Hamiltonian must give the same contribution as the vertex parts of the exact one, taken into account in the graphs we considered and, finally, the energy denominators are to be the same in both cases.

The first condition may be satisfied by describing the creation and annihilation of the particle-hole complex not by means of the products of Fermi-amplitudes,

$$a^{+}_{k+q}\, b^{+}_{k} \quad , \quad b_{k}\, a_{k+q}$$

as in the exact case, but by means of Bose-amplitudes[x)]

$$\beta^{+}_{q}(k) \quad , \quad \beta_{q}(k)$$

with two indices k,q.

The considered graps contain also vertex parts like those of Fig. 4

Fig. 4

The corresponding terms in the exact Hamiltonian are

$$P_{q}(k,k')\, a^{+}_{k+q}\, b^{+}_{k}\, a^{+}_{k'-q}\, b^{+}_{k'}$$

$$P^{*}_{q}(k,k')\, b_{k'}\, a_{k'-q}\, b_{k}\, a_{k+q}$$

$$Q_{q}(k,k')\, a^{+}_{k+q}\, b^{+}_{k}\, b_{k'}\, a_{k'+q}$$

---

x) Notice that in the graphs of Fig. 3 all the complexes are different: By using Bose-amplitudes there may arise diagrams containing several identical complexes, for instance with k = k', which cannot be reduced to one. But such redundant elements will lead only to infinitesimal contributions to the elementary excitation, since the integration is carried out over all momenta k,k',k'' and we have a linear chain of complexes.

An interesting situation may arise sometimes when calculating the ground state, when it is necessary to forbid explicitly the repetition of identical complexes in graphs. In these cases [14] one should consider the amplitudes β^{+}, β not to be of the Bose, but of the Pauli type. In other words we have to prescribe for all of them Bose type commutation relations, except for the replacement of the relation

$$\beta\beta^{+} - \beta^{+}\beta = 1$$

by

$$\beta\beta^{+} + \beta^{+}\beta = 1$$

Therefore the interaction Hamiltonian in the simplified model is to be constructed of terms of the form

$$P_q(k,k')\,\beta^+_q(k)\,\beta^+_{-q}(k') \qquad P^*_q(k,k')\,\beta_{-q}(k')\,\beta_q(k) \qquad Q_q(k,k')\,\beta^+_q(k)\,\beta_q(k')$$

Finally, in order to obtain correct energy denominators it is necessary to choose the expression

$$\sum_{k,q}\left\{\varepsilon_+(k+q)+\varepsilon_-(k)\right\}\beta^+_q(k)\,\beta_q(k)$$

as H_0, the self energy of the particle-hole complexes. Here ε_+, ε_- stand for the energies of a particle and a hole respectively.

Thus we obtain as the simplified total Hamiltonian a quadratic form in Bose operators. Writing down the corresponding secular equation we obtain an equation for determining the energy $E(q)$ of collective excitations, for the plasma oscillations considered.

For fixed q $E(q)$ will be an isolated root of this equation corresponding so-to-say to, a "bound state" of a particle-hole pair, where as the continuous spectrum is a spectrum of ordinary one-fermion excitations.

We have just outlined the characteristic features of the approximate second quantization method as applied to the well known case of plasma oscillations of a dense electronic gas[15], since this method will be the basis of our investigation of collective excitations of the superconducting state.

3.2.- Collective excitations in Fröhlich's model.

At first we notice that in Fröhlich's model, which is being investigated, there exists an attraction between electrons with opposite spins and momenta $\pm k$ with k in neighbourhood of the Fermi-sphere. An attraction of this kind also exists between holes. It is clear that such an attraction must exist also when the corresponding momenta of the particles of a pair are not exactly opposite, but, say, equal to $k+q$, $-k$ with sufficiently small q.

Let us proceed, as usual, to fermions $(k,0)$, (k,I) characterized by the amplitudes

$$\alpha^+_{k0},\ \alpha_{k0};\ \alpha^+_{k1},\ \alpha_{k1}$$

Note, that the creation of the pair

$$\alpha^+_{k+q,0}\,\alpha^+_{k1}$$

does not modify the spin but modifies the total momentum by the vector q. Thus, the existence of an effective attraction between the fermions with $(k+q,o),(k,1)$ for sufficiently small q, is easily detected.

This gives us an important hint as to a programme of applying the method of approximate second quantization.

Correspondingly we introduce as an "indecomposable element" of the graphs the complex which is charactrized in the exact treatment by the products of Fermi amplitudes:

$$\alpha^{+}_{k+q\,0}\,\alpha^{+}_{k1} \qquad\qquad \alpha_{k1}\,\alpha_{k+q\,0}$$

In our approximate model we associate to this complex the Bose-amplitudes

$$\beta^{+}_{q}(k) \qquad\qquad \beta_{q}(k)$$

with two indices. Now we should obtain the Hamiltonian Γ' describing the interaction between two different complexes in the form[x)]

$$\Gamma' = -\sum_{\substack{k,k',p\\ k\neq k'}} \beta^{+}_{p}(k)\,\beta_{p}(k')\,A_{p}(k,k') - \frac{1}{2}\sum_{\substack{k,k',p\\ k\neq k'}} B_{p}(k,k')\left\{\beta^{+}_{-p}(k')\,\beta^{+}_{p}(k) + \beta_{p}(k)\,\beta_{-p}(k')\right\}$$

Then adding the pair self energy we obtain the simplified model Hamiltonian

$$\Gamma = \sum_{k,p}\left\{\tilde{\varepsilon}(k+p) + \tilde{\varepsilon}(k)\right\}\beta^{+}_{p}(k)\,\beta_{p}(k) + \Gamma' \tag{3.2}$$

The diagonalization of a such quadratic form may be reduced to solving a system of linear homogeneous equations with respect to the c-number quantities φ_k, χ_k /15/ .

$$\begin{aligned}\left\{\tilde{\varepsilon}(k+p) + \tilde{\varepsilon}(k) - E\right\}\varphi_{p}(k) &= \sum_{k'} A_{p}(k,k')\,\varphi_{p}(k') + \sum_{k'} B_{p}(k,k')\,\chi_{p}(k'),\\ \left\{\tilde{\varepsilon}(k-p) + \tilde{\varepsilon}(k) + E\right\}\chi_{p}(k) &= \sum_{k'} B_{-p}(k,k')\,\varphi_{p}(k') + \sum_{k'} A_{-p}(k,k')\,\chi_{p}(k')\end{aligned} \tag{3.3}$$

with the normalization condition

$$\sum_{k,p}\left\{\varphi^{2}_{p}(k) - \chi^{2}_{p}(k)\right\} = 1 \tag{3.4}$$

For a given fixed momentum p the energy of collective excitations $E = E_c(p)$ is determined by an isolated root of the secular equation corresponding to (3.3). In order to obtain explicit expressions for A and B we remark that in the simplified model

$$A_{p}(k,k') = \left\langle \beta_{p}(k)\,\Gamma'\,\beta^{+}_{p}(k')\right\rangle_{0}$$

$$B_{p}(k,k') = \left\langle \beta_{-p}(k')\,\beta_{p}(k)\,\Gamma'\right\rangle_{0}$$

where the expectation vatues are with respect to the "B-vacuum". In the exact Fröhlich model such vertex parts are brought into existence only through the exchange of phonons.

---

x) Without any calculations one can see that the quantities $A_p(k,k')$, $B_p(k,k')$ must be real, because the Hamiltonian of Fröhlich's model is invariant with respect to time inversion and contains only real coefficients.

Taking into account, in first approximation, only the exchange of one phonon we put

$$-A_p(k,k') = \langle \alpha_{k_1}\, \alpha_{k+p\,0}\, H' H_0^{-1} H' \overset{+}{\alpha}_{k'+p\,0}\, \overset{+}{\alpha}_{k'_1} \rangle_0 \qquad (3.5)$$

$$-B_p(k,k') = \langle \alpha_{k'_1}\, \alpha_{k'-p\,0}\, \alpha_{k_1}\, \alpha_{k+p\,0}\, H' H_0^{-1} H' \rangle_0$$

where H' is the electron-phonon interaction Hamiltonian (cf. the formulae of § 2), which may be written in the form:

$$H' = \sum_{\substack{k,k',q \\ k'-k=q}} \tilde{g}(q) \left(\frac{\tilde{\omega}(q)}{2V}\right)^{\frac{1}{2}} (u_k v_{k'} + u_{k'} v_k)(\overset{+}{\alpha}_{k_0}\, \overset{+}{\alpha}_{k'_1} + \alpha_{k_1}\, \alpha_{k'_0})(\overset{+}{\beta}_q + \beta_{-q}) +$$

$$+ \sum_{\substack{k,k',q \\ k'-k=q}} \tilde{g}(q) \left(\frac{\tilde{\omega}(q)}{2V}\right)^{\frac{1}{2}} (u_k u_{k'} - v_k v_{k'})(\overset{+}{\alpha}_{k_0}\, \alpha_{k'_0} + \overset{+}{\alpha}_{k'_1}\, \alpha_{k_1})(\overset{+}{\beta}_q + \beta_{-q})$$

Substituting these expressions into (3.5) we obtain

$$A_p(k,k') = \tilde{g}^2(k-k')\, \frac{\tilde{\omega}(k-k')}{2V} (u_{k'+p} u_{k+p} - v_{k'+p} v_{k+p})(u_{k'} u_k - v_k v_{k'}) \left\{ \frac{1}{\tilde{\varepsilon}(k+p) + \tilde{\varepsilon}(k') + \tilde{\omega}(k'-k)} + \frac{1}{\tilde{\varepsilon}(k'+p) + \tilde{\varepsilon}(k) + \tilde{\omega}(k'-k)} \right\} +$$

$$+ \tilde{g}^2(p)\, \frac{1}{2V} (u_{k'+p} v_{k'} + u_{k'} v_{k+p})(u_{k+p} v_k + u_k v_{k+p}) + \tilde{g}^2(p)\, \frac{\tilde{\omega}(p)}{2V}\, \frac{(u_{k'+p} v_{k'} + u_{k'} v_{k'+p})(u_{k+p} v_k + u_k v_{k+p})}{\tilde{\varepsilon}(k'+p) + \tilde{\varepsilon}(k') + \tilde{\varepsilon}(k+p) + \tilde{\varepsilon}(k) + \tilde{\omega}(p)} \qquad (3.6)$$

$$B_p(k,k') = -\tilde{g}^2(k'-k-p)\, \frac{\tilde{\omega}(k'-k-p)}{2V}\, \frac{(u_{k+p} v_{k'} + u_{k'} v_{k+p})(u_{k'-p} v_k + u_k v_{k'-p})}{\tilde{\varepsilon}(k') + \tilde{\varepsilon}(k+p) + \tilde{\omega}(k'-k-p)} -$$

$$- \tilde{g}^2(k-k'+p)\, \frac{\tilde{\omega}(k-k'+p)}{2V}\, \frac{(u_{k+p} v_{k'} + u_{k'} v_{k+p})(u_{k'-p} v_k + u_k v_{k'-p})}{\tilde{\varepsilon}(k) + \tilde{\varepsilon}(k'-p) + \tilde{\omega}(k-k'+p)} + \tilde{g}^2(p)\, \frac{\tilde{\omega}(p)}{2V}\, \frac{(u_{k+p} v_k + u_k v_{k+p})(u_{k'-p} v_{k'} + u_{k'} v_{k'-p})}{\tilde{\varepsilon}(k+p) + \tilde{\varepsilon}(k) + \tilde{\omega}(p)} +$$

$$+ \tilde{g}^2(p)\, \frac{\tilde{\omega}(p)}{2V}\, \frac{(u_{k+p} v_k + u_k v_{k+p})(u_{k'-p} v_{k'} + u_{k'} v_{k'-p})}{\tilde{\varepsilon}(k'-p) + \tilde{\varepsilon}(k') + \tilde{\omega}(p)} \qquad (3.7)$$

To simplify these complicated expressions, we suppose that for small values of p we can put

$$g(p) = 0$$

This assumption seems to be reasonable from the physical point of view.(II).

Let us now simplify these formulae, in order to get rid of terms of smaller order than that taken into account in the chosen approximation. First we notice that in the process of collective oscillations only large momentum transfers $k - k'$ are essential, as may be seen from the equations (3.3). Therefore in quantities of type $\tilde{\omega}(k-k'-p)$, $\tilde{g}(k-k'-p)$ which depend smoothly on the momentum, we may neglect the additional momentum p. Further, since the energy layer, which in fact determines the effect, contains only energies $\tilde{\varepsilon}(k)$ small compared with $\tilde{\omega}$, we replace

(as was done in the derivation of the equation for C(k)) the quantities

$$\tilde{\varepsilon}(k),\ \tilde{\varepsilon}(k'-p),\ldots$$

by

$$|\xi(k)|,\ |\xi(k')|$$

where as before:

$$\xi(k) = |E(k) - E_F|.$$

We thus obtain

$$A_p(k,k') = \frac{J(k,k')}{V}\left(u_{k'+p}u_{k+p} - v_{k'+p}v_{k+p}\right)\left(u_{k'}u_k - v_{k'}v_k\right)$$

$$B_p(k,k') = \frac{J(k,k')}{V}\left(u_{k+p}v_{k'} + u_{k'}v_{k+p}\right)\left(u_{k'-p}v_k + v_{k'-p}u_k\right) \tag{3.8}$$

The expression $J(k,k')$ here is the same as in the equation for C(k):

$$C(k) = \frac{1}{2V}\sum_{k'} J(k,k')\frac{C(k')}{\sqrt{C^2(k') + \xi^2(k')}} \qquad J(k,k') = \tilde{g}^2(k-k')\frac{\tilde{\omega}(k-k')}{|\xi(k)| + |\xi(k')| + \tilde{\omega}(k-k')} \tag{3.9}$$

We put for abbreviation

$$u_{k'}u_k - v_{k'}v_k = L(k,k')$$

$$u_{k'}v_k + v_{k'}u_k = M(k,k')$$

Then Eq. (3.8) may be represented in the form

$$A_p(k,k') = \frac{J(k,k')}{V}L(k+p,k'+p)L(k,k')$$

$$B_p(k,k') = -\frac{J(k,k')}{V}M(k+p,k')M(k,k'-p)$$

3.3. The resolution of the secular equations. Longitudinal excitations. Considering the secular equations (3.3) and introducing the new variables

$$\varphi_p(k) + \chi_p(k) = \vartheta_p(k)$$

$$\varphi_p(k) - \chi_p(k) = \theta_p(k)$$

One obtains

$$\left\{\frac{\tilde{\varepsilon}(k+p)+\tilde{\varepsilon}(k-p)}{2} + \tilde{\varepsilon}(k)\right\}\theta_p(k) - \frac{1}{V}\sum_{k'}J(k,k')\left\{\frac{L(k+p,k'+p)+L(k-p,k'-p)}{2}L(k,k') + \frac{M(k+p,k')M(k,k'-p)+M(k-p,k')M(k,k'+p)}{2}\right\}\theta_p(k') =$$

$$= \left\{E - \frac{\tilde{\varepsilon}(k+p)-\tilde{\varepsilon}(k-p)}{2}\right\}\vartheta_p(k) + \frac{1}{V}\sum_{k'}J(k,k')\left\{\frac{L(k+p,k'+p)-L(k-p,k'-p)}{2}L(k,k') - \frac{M(k+p,k')M(k,k'-p)-M(k-p,k')M(k,k'+p)}{2}\right\}\vartheta_p(k') \tag{3.10}$$

$$\left\{\frac{\tilde{\varepsilon}(k+p)+\tilde{\varepsilon}(k-p)}{2} + \tilde{\varepsilon}(k)\right\}\vartheta_p(k) - \frac{1}{V}\sum_{k'}J(k,k')\left\{\frac{L(k+p,k'+p)+L(k-p,k'-p)}{2}L(k,k') - \frac{M(k+p,k')M(k,k'-p)+M(k-p,k')M(k,k'+p)}{2}\right\}\vartheta_p(k') =$$

$$= \left\{E - \frac{\tilde{\varepsilon}(k+p)-\tilde{\varepsilon}(k-p)}{2}\right\}\theta_p(k) + \frac{1}{V}\sum_{k'}J(k,k')\left\{\frac{L(k+p,k'+p)-L(k-p,k'-p)}{2}L(k,k') + \frac{M(k+p,k')M(k,k'-p)-M(k-p,k')M(k,k'+p)}{2}\right\}\theta_p(k') \tag{3.11}$$

Consider first the case $p = 0$. Then (3.10), (3.11) will simplify and we obtain:

$$2\sqrt{\xi^2(k)+c^2(k)}\;\theta_0(k) - \frac{1}{V}\sum_{k'} J(k,k')\theta_0(k') = E\vartheta_0(k) \tag{3.12}$$

$$2\sqrt{\xi^2(k)+c^2(k)}\;\vartheta_0(k) + \frac{1}{V}\sum_{k'} \frac{c(k)c(k')-\xi(k)\xi(k')}{\sqrt{\xi^2(k)+c^2(k)}\sqrt{\xi^2(k')+c^2(k')}} J(k,k')\vartheta_0(k') = E\,\theta_0(k) \tag{3.13}$$

Note that according to (3.9), this system admits the solution

$$\vartheta_0(k)=0 \qquad \theta_0(k)= S\frac{c(k)}{\sqrt{\xi^2(k)+c^2(k)}} \qquad S=\text{const} \qquad E=0$$

We thus reach the important conclusion that the spectrum of collective excitations starts from zero.

Let us turn now to the investigation of the collective excitations for small, but nonvanishing p. To avoid complicated calculations we restrict ourselves to the case of radial symmetry.

First of all we must expand the coefficients which enter our equations in powers of p. Since in the interesting energy domain $E(k)$ is extremely close to E_f, we admit that

$$\xi(k+p) = s(p\cdot e) + \xi(k)$$

where S is the absolute value of the electron velocity on the Fermi-sphere and e is the unit-vector in the direction of K. It is more convenient to choose the direction of p as a reference axis for e.

Notice further, that in the expressions containing $\xi^2 + c^2(\xi)$ we may neglect the variations of $c(\xi)$ as compared with ξ, since $c(\xi)$ is a slowly varying function. We obtain

$$u(k+p) = u(\xi) + spe_0\frac{\partial u}{\partial \xi} + \dots$$

$$\tilde{\varepsilon}(k+p) = \tilde{\varepsilon}(\xi) + spe_0\frac{\partial \tilde{\varepsilon}(\xi)}{\partial \xi} + \dots$$

It is convenient to introduce the number ξ and the unit vector e as new arguments in place of the vector k. Thus we are led to asymptotic formulae of the type.

$$\frac{L(k+p,k'+p)+L(k-p,k'-p)}{2}L(k,k') = \frac{1}{2} + \frac{\xi\xi'-cc'}{2\sqrt{\xi^2+c^2}\sqrt{\xi'^2+c'^2}} + spe_0P_1(\xi,\xi') + spe'P_2(\xi,\xi') + \dots$$

$$\frac{L(k+p,k'-p)-L(k-p,k'-p)}{2}L(k,k') = spe_0Q_1(\xi,\xi') + spe'Q_2(\xi,\xi')$$

$$\frac{M(k+p,k')M(k,k'-p)+M(k-p,k')M(k,k'+p)}{2} = \frac{1}{2} - \frac{\xi\xi'-cc'}{2\sqrt{\xi^2+c^2}\sqrt{\xi'^2+c'^2}} + (sp)^2\{e_0^2R_1(\xi,\xi')+e_0e_0'R_2(\xi,\xi')+e_0'^2R_3(\xi,\xi')+R_4(\xi,\xi')$$

$$\frac{M(k+p,k')M(k,k'-p)-M(k-p,k')M(k,k'+p)}{2} = spe_0S_1(\xi,\xi') + spe'S_2(\xi,\xi') + \dots$$

We now apply ordinary perturbation theory to the secular equations (3.10), (3.11), considering the quantities p and E as small of the first order. We put

$$\theta(k) = \theta_0(\xi) + spe_0\theta_1(\xi) + \dots$$

$$\vartheta(k) = E\vartheta_1(\xi) + spe_0\bar{\vartheta}_1(\xi) + \dots \qquad \theta_0(\xi) = S\frac{c}{\sqrt{\xi^2+c^2}} \qquad S=\text{const} \tag{3.14}$$

Substituting these expressions into (3.10),(3.11) and separating terms of different orders, we may express $\theta_I(\xi), \vartheta_1(\xi), \bar{\vartheta}_1(\xi)$ in terms of the $\theta_0(\xi)$. Using the resulting expressions, we expand the equation (3.10) to the second order of smallness. Thus we are led to the relation

$$2\tilde{\varepsilon}(k)\theta(k) - \frac{1}{V}\sum_{k'} J(k,k')\theta(k') = F(E, sp, k) \qquad (3.15)$$

where $F(E,sp,k)$ is a quadratic form with respect to E, sp. We multiply (3.15) by $C(k)[2V\tilde{\varepsilon}(k)]^{-1}$ and take the sum over k. Taking into account Eq. (3.9) we obtain

$$\frac{1}{V}\sum_{k} F(E, sp, k)\ \frac{C(k)}{\tilde{\varepsilon}(k)} = 0$$

hence

$$E = \alpha sp$$

where α is a numerical coefficient. So, if we omit terms vanishing together with the parameter $\rho = \tilde{g}^2 \frac{dn}{dE}$, we have

$$E\vartheta_1(\xi) + sp\alpha\bar{\vartheta}_1(\xi) = \frac{E - e_0 ps}{2\tilde{\varepsilon}(\xi)}\theta_0(\xi) \qquad \alpha^2 = \frac{1}{3}$$

Consequently, we obtain two roots for E. In order to determine the sign we turn to the normalization condition (3.4), which we write in the form

$$\sum_{k} \vartheta_\gamma(k)\,\theta_p(k) = 1$$

Hence

$$E\sum \frac{\vartheta_0^2(\xi)}{2\tilde{\varepsilon}(\xi)} = 1$$

and therefore

$$E > 0$$

Also, in the chosen approximation

$$E_c(p) = \frac{sp}{\sqrt{3}}$$

Let us note now, that colleotive excitations will exist, as long as

$$E_c(p) < 2\Delta$$

where Δ is the magnitude of the energy gap (i.e. the value of C at $\xi = 0$). Indeed, in the opposite case $E = E_c(p)$ will overlap somewhat with the Fermion spectrum

$$\tilde{\varepsilon}(k) + \tilde{\varepsilon}(k+p)$$

and ceases to be an isolated root of the secular equation. This intuitive consideration may be confirmed by a direct calculation. For simplification we retain only the principal terms with respect to the parameter ρ and replace the function $J(k,k')$ by a quantity J which is constant within the energy layer $E_F \pm \omega_D$ and zero outside it.

In this approximation we obtain from (3.II):

$$\vartheta_p(k) = \frac{E - \frac{\tilde{\varepsilon}(k+p) - \tilde{\varepsilon}(k-p)}{2}}{\tilde{\varepsilon}(k) + \frac{\tilde{\varepsilon}(k+p) + \tilde{\varepsilon}(k-p)}{2}}\ \theta_p(k)$$

Substituting this expression into (3.IO) and restricting ourselves as before, to the principal terms only, we obtain:

$$\left\{ \tilde{\varepsilon}(k) + \frac{\tilde{\varepsilon}(k+p) + \tilde{\varepsilon}(k-p)}{2} - \frac{\left(E - \frac{\tilde{\varepsilon}(k+p) - \tilde{\varepsilon}(k-p)}{2}\right)^2}{\tilde{\varepsilon}(k) + \frac{\tilde{\varepsilon}(k+p) + \tilde{\varepsilon}(k-p)}{2}} \right\} \theta_p(k) = \frac{J}{V} \sum_{k'} \theta_p(k')$$

The corresponding secular equation is

$$1 = \frac{J}{2V} \sum_k \left\{ \frac{1}{\tilde{\varepsilon}(k) + \tilde{\varepsilon}(k+p) - E} + \frac{1}{\tilde{\varepsilon}(k) + \tilde{\varepsilon}(k-p) + E} \right\}$$

On the other hand the equation for C in the considered approximation has the form

$$C = \frac{J}{2V} C \sum \frac{1}{\tilde{\varepsilon}(k)}$$

Therefore our secular equation may be written in the form

$$\sum_k \left\{ \frac{1}{\tilde{\varepsilon}(k) + \tilde{\varepsilon}(k+p) - E} + \frac{1}{\tilde{\varepsilon}(k) + \tilde{\varepsilon}(k-p) + E} - \frac{1}{\tilde{\varepsilon}(k)} \right\} = 0 \qquad (3.I7)$$

For fixed p this equation has a continuous spectrum

$$E = \tilde{\varepsilon}(k) + \tilde{\varepsilon}(k+p) + O\left(\frac{1}{V}\right)$$

where $O\left(\frac{1}{V}\right) \to 0$, $V \to \infty$

The continuous spectrum starts from $E = 2C = 2\Delta$ and is charactrized for a given $\underline{p}$ by the vector index $\underline{k}$. It corresponds to the excitation of two "individual" fermions. The discrete spectrum, corresponding to collective excitations, exists only for those values of $\underline{p}$ for which the equation (3.I7) has an isolated root

$$E < 2\Delta$$

It is convenient to write (3.I7) in an integral form. We have

$$\int_{-1}^{+1} dt \int_0^\infty d\xi \left\{ \frac{1}{\sqrt{\xi^2 + C^2} + \sqrt{(\xi + spt)^2 + C^2} - E} + \frac{1}{\sqrt{\xi^2 + C^2} + \sqrt{(\xi - spt)^2 + C^2} + E} - \frac{1}{\sqrt{\xi^2 + C^2}} \right\} = 0$$

or

$$\phi\left(\frac{E}{C}, \frac{sp}{C}\right) + \phi\left(-\frac{E}{C}, \frac{sp}{C}\right) = 0$$

where

$$\phi(\xi, \nu) = \int_{-1}^{+1} dt \int_0^\infty d\xi \left\{ \frac{1}{\sqrt{1+\xi^2} + \sqrt{1 + (\pm\nu + \xi)^2} - \xi} - \frac{1}{2\sqrt{1+\xi^2}} \right\}$$

For $sp \ll c$ we obtain for E the value (3.16). But the ratio $\frac{E}{sp}$ will be of the order of unity up to $E = 2c$.

p_{max} is determined from the equation

$$\phi\left(2, \frac{sp}{c}\right) + \phi\left(-2, \frac{sp}{c}\right) = 0$$

Hence

$$s p_{max} = \gamma c$$

where γ is a numerical coefficient.

By the way, we remark that these equations might have been obtained without simplifying the structure of the function $J(k,k')$. Their applicability is determined only by the smallness of the parameter ρ . Due to the fact that oollective oscillations exist only for momenta $p < p_{max}$, the nonvanishing of $E_c(p)$ in the absence of electron-phonon interaction is not paradoxal. Indeed in this case the formula (3.16) will have no domain of applicability, since $\Delta = 0$.

Till now we considered the collective excitations only for the superconducting state. Let us now analyse the role they play for a normal state. For simplicity we consider the case $\underset{\sim}{p} = 0$. Then from Eqs.(3.12), (3.13) we obtain:

$$2|\xi(k)|\theta(k) - \frac{1}{V}\sum_{k'} J(k,k')\theta(k') = E\vartheta(k), \tag{3.18}$$

$$2|\xi(\bar{\xi})|\vartheta(k) - \frac{1}{V}\sum_{k'} \frac{\xi(k)\xi(k')}{|\xi(k)||\xi(k')|} J(k,k')\theta(k') = E\theta(k) \tag{3.19}$$

Consider the radially-symmetric solutions and rewrite Eqs. (3.18), (3.19) in integral form. We obtain

$$2|\bar{\xi}|\theta(\bar{\xi}) - \int_{-\infty}^{+\infty} \rho(\bar{\xi}\bar{\xi}')\theta(\bar{\xi}')d\bar{\xi}' = E\vartheta(\bar{\xi}) \tag{3.20}$$

$$2|\bar{\xi}|\vartheta(\bar{\xi}) - \int_{-\infty}^{+\infty} \frac{\bar{\xi}\bar{\xi}'}{|\bar{\xi}||\bar{\xi}'|}\rho(\bar{\xi}\bar{\xi}')\vartheta(\bar{\xi}')d\bar{\xi}' = E\theta(\bar{\xi}) \tag{3.21}$$

where

$$\rho(\bar{\xi}\bar{\xi}') = \frac{1}{4\pi^2}\int_{-1}^{+1} \frac{k_F^2}{\left(\frac{\partial E}{\partial k}\right)_{k=k_F}} \tilde{q}^2\left(k_F\sqrt{2(1-t)}\right) \frac{\tilde{\omega}\left(k_F\sqrt{2(1-t)}\right)}{|\bar{\xi}| + |\bar{\xi}'| + \tilde{\omega}\left(k_F\sqrt{2(1-t)}\right)} dt$$

We introduced the infinite limits of integration in the equations (3.20),(3.21), since their exact values are unimportant, the integrals being practically determined by the contribution of the interval $|\bar{\xi}| \lesssim \tilde{\omega}$.

We put

$$C_1(\bar{\xi}) = \int_{-\infty}^{+\infty} \rho(\bar{\xi}\bar{\xi}')\theta(\bar{\xi}')d\bar{\xi}' \tag{3.22}$$

$$C_2(\xi)=\int_{-\infty}^{+\infty}\frac{\xi'}{|\xi'|}\rho(\xi\xi')\vartheta(\xi')d\xi' \qquad (3.23)$$

Then from Eqs. (3.20), (3.21) we obtain

$$\theta(\xi)=\frac{2|\xi|C_1(\xi)+\frac{\xi}{|\xi|}C_2(\xi)E}{4\xi^2-E^2}$$

Substitute this expression into (3.22) and note that $C_1(\xi), C_2(\xi), \rho(\xi\xi')$ are even functions of ξ, ξ'. Consequently,

$$C_1(\xi)=\int_0^\infty \rho(\xi\xi')\frac{4\xi'}{4\xi'^2-E^2}C_1(\xi')d\xi'$$

whence, for small $\rho=\rho(0,0)$ we obtain an asymptotic formula of the type

$$-E^2\sim 4\omega_0^2 e^{-\frac{2}{\rho}}$$

where ω_0 is an average ω. So, the energy of collective excitations turns out to be a purely imaginary quantity, a fact that points to the instability of the normal state.

This kind of collective excitations is responsible for the instability, for positive energies, of the fermion and phonon excitations of the normal state.

3.4. The Resolution of the Secular Equations. Transversal Excitations.

We have investigated only those solutions of the secular equations (3.10), (3.11), which may be represented in the form of the series (3.14). More exactly, we have restricted our attention only to those expressions for $\theta_p(k)$ and $\vartheta_p(k)$, for which

$$\theta_p(k)=\theta(|p|,|k|,(kp))$$
$$\vartheta_p(k)=\vartheta(|p|,|k|,(kp)) \qquad (3.24)$$

These solutions correspond to longitudinal waves. But besides the longitudinal waves there exists a class of solutions of another kind corresponding to transversal waves. In other words, for these equations:

$$\theta_p(k)=\theta(|p|,|k|,(k.p))[k.p]_n$$
$$\vartheta_p(k)=\vartheta(|p|,|k|,(k.p))[k.p]_n \qquad (3.25)$$

where the index n denotes the component of the vector product in the direction of n.

For the sake of simplicity we consider the case p = 0. In this case it is sufficient to examine the equations (3.12) and (3.13). But now we shall look for solutions whioh are not spherically symmetrical, but have the form

$$\theta_0(k)=\theta_0(|k|)e_x$$
$$\vartheta_0(k)=\vartheta_0(|k|)e_x$$

Without loss of generality we may consider for convenience p as directed along the z-axis and n -along the x-axis; ℓ_x is the x-component of the unit vector of k. For $\theta_0(|k|)$ and $\vartheta_0(|k|)$ we obtain the equations

$$2\tilde{\varepsilon}(k)\theta_0(k) - \frac{1}{V}\sum_{k'}\tilde{J}(k,k')\theta_0(k') = E\vartheta_0(k)$$

$$2\tilde{\varepsilon}(k)\vartheta_0(k) + \frac{1}{V}\sum_{k'}\tilde{J}(k,k')\frac{C(k)C(k') - \xi(k)\xi(k')}{\tilde{\varepsilon}(k)\tilde{\varepsilon}(k')}\vartheta_0(k') = E\theta_0(k) \qquad (3.26)$$

where

$$\tilde{J}(k,k') = \frac{1}{2}\int_{-1}^{+1} J\left(|k|,|k'|,\sqrt{|k|^2+|k'|^2-2|k||k'|t}\right)t\,dt \qquad (3.27)$$

$$\tilde{\varepsilon}(k) = \sqrt{\xi^2(k)+C^2(k)}$$

Further, it will be convenient to restrict ourselves to the case when $\tilde{J}(k,k')$ is localized in the neighbourhood of the Fermisphere and may be replaced inside the layer $E_F \pm \tilde{\omega}$ by a constant $\tilde{J}$. We admit a similar approximation also for $J(k,k')$ and replace it by a constant "J" inside the layer $E_F \pm \omega$ and by zero outside. Under these assumptions the secular equations (3.26), (3.27) may be solved without difficulty, if we introduce new variables in place of $\theta_0(k)$ and $\vartheta_0(k)$

$$x = \frac{\tilde{J}}{V}\sum\theta_0(k) \qquad y = \frac{\tilde{J}}{V}\sum\frac{C(k)}{\tilde{\varepsilon}(k)}\vartheta_0(k) \qquad z = \frac{\tilde{J}}{V}\sum\frac{\xi(k)}{\tilde{\varepsilon}(k)}\vartheta_0(k)$$

where the summation is extended over the domain $E_F \pm \tilde{\omega}$ in the neighbouhood of the Fermi sphere. The equation for E takes the form:

$$\begin{vmatrix} \frac{\tilde{J}}{V}\sum\frac{2\xi^2(k)}{D(k)} - 1 & -\frac{\tilde{J}}{V}\sum\frac{C(k)E}{D(k)} & \frac{\tilde{J}}{V}\sum\frac{\xi(k)E}{D(k)} \\ \frac{\tilde{J}}{V}\sum\frac{C(k)E}{D(k)} & -\frac{\tilde{J}}{V}\sum\frac{2C^2(k)}{D(k)} - 1 & \frac{\tilde{J}}{V}\sum\frac{2\xi(k)C(k)}{D(k)} \\ \frac{\tilde{J}}{V}\sum\frac{\xi(k)E}{D(k)} & -\frac{\tilde{J}}{V}\sum\frac{2\xi(k)C(k)}{D(k)} & \frac{\tilde{J}}{V}\sum\frac{2\xi^2(k)}{D(k)} - 1 \end{vmatrix} = 0 \qquad (3.28)$$

where

$$D(k) = \tilde{\varepsilon}(k)\left[4\xi^2(k) - E^2\right]$$

We perform an asymptotic expansion of the determinant (3.28) for small C and restrict ourselves to terms that do not vanish for $C \to 0$. In fact this expansion will be with respect to the parameter $\frac{C}{\omega}$ or $\frac{C}{\tilde{\omega}}$. The final expression is

$$\left\{\left(\tilde{\rho}\ln\frac{2\tilde{\omega}}{C} - 1 + \varepsilon^2\right)\left(\frac{\tilde{\rho}}{\varepsilon\sqrt{1-\varepsilon^2}}\operatorname{arctg}\frac{\varepsilon}{\sqrt{1-\varepsilon^2}} + 1\right) - \varepsilon^2\right\}\left\{\tilde{\rho}\ln\frac{2\tilde{\omega}}{C} - \tilde{\rho}\frac{\sqrt{1-\varepsilon^2}}{\varepsilon}\operatorname{arctg}\frac{\varepsilon}{\sqrt{1-\varepsilon^2}} - 1\right\} = 0 \qquad (3.29)$$

where

$$\tilde{\rho} = \frac{\tilde{J}}{2\pi^2}\frac{k_F^2}{E'(k_F)} \qquad \varepsilon = \frac{E}{2C}$$

In writing down Eq. (3.29) we have for simplicity omitted a term which leads to superfluous complications. This omission corresponds to the neglect in the initial equation (3.27) of the term $\gamma(k)\gamma(k')$ in comparison with $C(k)C(k')$, since, for momenta k and k' near the Fermi surface the former vanishes, whereas the latter remains finite.

The examination of the roots of Eq. (3.29) is not difficult and yields the following result. For small $\frac{c}{\omega}$ Eq. (3.28) has a single root for $\tilde{\rho}$ in the interval

$$-1 < \tilde{\rho} < \frac{1}{\ln \frac{2\omega}{c}}$$

For $\tilde{\rho}$ near to -1 the root starts of from zero. Its value is

$$E = 2C\sqrt{\tfrac{1}{2}(1+\tilde{\rho})}$$

For $\tilde{\rho}$ increasing from -1 to 0 the root increases and reaches the value $2C$, where it goes over into the continuous spectrum. For $\tilde{\rho}$ close to zero

$$E = 2C\left(1 - \tfrac{\pi^2}{8}\tilde{\rho}^2 \ln^2 \tfrac{2\omega}{c}\right)$$

For $\tilde{\rho}$ increasing further from 0 to $\left(\ln \frac{2\omega}{c}\right)^{-1}$ the root runs through the interval $(0, 2c)$ in the reverse direction and for $\tilde{\rho}$ close to $\left(\ln \frac{2\omega}{c}\right)^{-1}$ it becomes

$$E = 2C\sqrt{\ln \tfrac{2\omega}{c}\left(1 - \tilde{\rho} \ln \tfrac{2\omega}{c}\right)}$$

The root we considered appears from the first factor of (3.29). The second factor may have a root only in a quite narrow domain of values

$$\frac{1}{\ln \frac{2\omega}{c}} < \tilde{\rho} < \frac{1}{\ln \frac{2\omega}{c} - 1}$$

which for $C \to 0$ asymptotically tends a point. Therefore the consideration of this root is of no interest. It should be noted that if $\tilde{\rho}$ goes outside the limits (3.30) there appears a purely imaginary root which indicates the instability of the ground state in this case. Some remarks on behaviour of the root for nonvanishing p are in place. Without any calculation one may conclude, only by inspection of the secular equations (3.10), (3.11), that for small p

$$E(p) = \sqrt{E^2 + \alpha p^2 s^2}$$

where S is the velocity on the Fermi sphere and α is a numerical factor.

Finally, some words about the physical meaning of the transversal solutions. Let us consider the physical quantity

$$\operatorname{rot} V(r) = -i \sum_j \left[\frac{\partial}{\partial r} \delta(r_j - r) \cdot \frac{\partial}{\partial r_j}\right] \qquad (3.31)$$

which represents the curl of the velocity field $V(r)$. In the second quantization representation

Eq. (3.3I) will be

$$m\,rot\,v(r) = \frac{i}{V}\sum_{k,k'}\left(a^{+}_{k,+}a_{k',+} + a^{+}_{-k,-}a_{-k',-}\right)e^{i(k-k').r}\,[k.k']$$

Going over from the operators a^{+}, a to α^{+}, α by means of the u,v-transformation we obtain

$$m\,rot\,v(r) = \frac{i}{V}\sum_{k,k'} M(k,k')\left(\alpha^{+}_{k'1}\alpha_{k1} - \alpha^{+}_{k0}\alpha_{k'0}\right)e^{i(k-k').r}\,[k.k'] + \frac{i}{V}\sum_{k,k'} L(k,k')\left(\alpha_{k1}\alpha_{k'0} - \alpha^{+}_{k0}\alpha^{+}_{k'1}\right)e^{i(k-k').r}\,[k.k'] \qquad (3.32)$$

We attempt to represent this operator in terms of the collective Bose amplitudes $\beta^{+}_{q}(k), \beta_{q}(k)$. It is not difficult to observe that this procedure may by carried out by substituting for the operator (3.32) the following "model" operator:

$$m\,rot\,v(r) = \frac{i}{V}\sum_{q,k} L(k+q,k)\,\beta^{+}_{q}(k)\,e^{iq.r}\,[k.q] + conj. \qquad (3.33)$$

Transforming the operator (3.33) to the new Bose amplitudes which diagonalize the quadratic form (3.2), we obtain:

$$m\,rot\,v(r) = \frac{i}{2V}\sum_{\mu,q}\xi^{+}_{\mu}\left\{\sum_{k} L(k+q,k)\left(\vartheta_{q}(k';\mu) + \vartheta_{q}(k';\mu)\right)[k.q]\right\}e^{iq.r} + \frac{i}{2V}\sum_{\mu,q}\xi_{\mu}\left\{\sum_{k} L(k+q,k)\left(\vartheta_{q}(k;\mu) - \vartheta_{q}(k;\mu)\right)[k.q]\right\}e^{iq.r} \qquad (3.34)$$

We recall, that the new Bose amplitudes ξ^{+}, ξ are connected with the old amplitudes β^{+}, β by the transformation:

$$\beta_{q}(k) = \sum_{\mu}\left\{\xi_{\mu}\varphi_{q}(k;\mu) + \xi^{+}_{\mu}\chi_{-q}(k;\mu)\right\}$$

According to (3.24) the longitudal waves considered before turn into zero the sums in the curly brackets in (3.34). A quite different situation arises with the transversal waves. On account of (3.25) the expressions in curly brackets no longer vanish. Thus, the transversal collective oscillations represent curl-structures. It is quite probable that their properties will have many features equivalent with those which were investigated for rotons in the microscopic theory of superfluidity. But it has not been established that a special branch of the spectrum characterizing these excitations exists.

Thus we complete the investigation of the ground state of one-Fermion and collective excitations in Frohlich's model. It should be emphasized that the method used for the computation of the ground state and one-Fermion, excitations is a regular one. On the contrary, the method of computation of the collective excitations is rather to be interpreted as an approximate summation of the most important diagrams. The question of the excitation of various collective oscillations in a system represents an interesting but very complicated problem. Particularly it would be desirable to consider collective excitations, based not on pairs but say on quadruples of Fermi amplitudes.

§4. THE COULOMB INTERACTION BETWEEN THE ELECTRONS[x)]

4.1. Outline of the problem. Up to this moment we considered explicitly only the electron-phonon interaction. The repulsive Coulomb interaction between the electrons has not been included into the Hamiltonian.

It is easy to see that all our preceding results may be trivially generalized to the case when H_{int} contains a screened Coulomb interaction term which we can consider as small, so that the perturbation theory approximation be valid for the whole H_{int}.

Such an approach gives us essentially the same results as above with suitably modified numerical factors, such as, for example, ρ .

However this procedure is not satisfactory from the physical point of view and does not give an essential improvement of the Frohlich model.

Firstly, the electrostatic Coulomb repulsion is more intensive that the feeble attraction caused by the exchange of virtual phonons. Secondly, as we shall see later the screening of the Coulomb interaction essentially modifies the stucture of the energy spectrum of the longitudinal collective oscillations.

So we shall extend our previous considerations to the more realistic model described by the Hamiltonian

$$H = \sum_{k,s} (E(k) - \lambda)\, a^+_{ks} a_{ks} + \sum_{q} \omega(q)\, b^+_q b_q + H_{ph} + H_c \tag{4.1}$$

where

$$H_{ph} = \sum_{\substack{(k,k',q,s \\ k'-k=q)}} g(q) \sqrt{\frac{\omega(q)}{2V}}\; a^+_{ks} a_{k's}\, b^+_q + \text{Herm. conj.} \tag{4.2}$$

and H_c is the Coulomb interaction between the electrons

$$H_c = \sum_{\substack{(k_1,k_2,k_1',k_2',s_1,s_2) \\ k_1+k_2=k_1'+k_2'}} \frac{1}{V} I(k_1,k_2,k_2',k_1')\, a^+_{k_1 s_1} a^+_{k_2 s_2} a_{k_2' s_2} a_{k_1' s_1} \tag{4.3}$$

with

$$I(k_1,k_2,k_2',k_1') = \frac{e^2}{|k_1 - k_1'|^2}$$

We shall use the method developed in §2 and shall carry out the compensation of both the two-boson and the two-fermion graphs. Therefore we can use the ratio ω/E_F as a small parameter and retain only the main terms in the resulting asymptotic formulas. In our method it is not

---

x) This part of the paper is based on results of D.V. Shirkov.

necessary to consider the Coulomb interaction as small, and therefore we have to take into account graphs of all orders with respect to H_c .

In order to simplify the calculations, we suppose that the canonical transformation parameters u, v differ practically from their normal values only in a very narrow spherical layer near the Fermi surface. On the basis of this assumption we can retain in our formulae only the main terms with respect to the effeotive thickness of this energetic layer. Note here, that, according to the results of §2 this quantity is "expontially" small as $\omega \exp(-\frac{1}{\rho})$ in the case when the Coulomb interaotion is omitted.

This assumption seems to be quite reasonable, as the Coulomb interaction can only reduce the effective value of the parameter ρ. Besides, after obtaining explicitly the approximate equation for u and v we shall be able to evaluate post factum the effective dimension of the energetic layer in which u, v differ from their normal values and at the same time confirm the correctness of the assumption.

4.2. The Compensation and Renormalization Conditions. Let us carry out in (4.I) our canonical transformation of the Fermi and Bose operators (2.4), (2.8). In the transformed Hamiltonian we choose for the "free Hamiltonian" H_0 the following expression

$$H_0 = U + \sum_k \tilde{\varepsilon}(k)(\alpha^+_{k0}\alpha_{k0} + \alpha^+_{k1}\alpha_{k1}) + \sum_q \tilde{\omega}(q)\beta^+_q\beta_q$$

where $\tilde{\varepsilon}(k)$, $\tilde{\omega}(k)$ are the renormalized energies of electron and phonon excitations and U is the energy of the ground state. Then we have

$$H = H_0 + H_{int}$$

where the "interaction Hamiltonian" H_{int} is

$$H_{int} = U' + H' + H'' + H_{ph} + H_c \qquad (4.4)$$

where

$$U' = const = 2\sum_k (E(k) - \lambda)u_k^2 + \sum_q \omega(q)\mu_q^2 - U$$

$$H' = \sum_k \{(E(k) - \lambda)(u_k^2 - v_k^2) - \tilde{\varepsilon}(k)\}(\alpha^+_{k0}\alpha_{k0} + \alpha^+_{k1}\alpha_{k1}) + 2\sum_k (E(k) - \lambda)u_k v_k (\alpha^+_{k0}\alpha^+_{k1} + \alpha_{k1}\alpha_{k0})$$

$$H'' = \sum_q \{\omega(q)(\lambda_q^2 + \mu_q^2) - \tilde{\omega}(q)\}\beta^+_q\beta_q + \sum_q \omega(q)\lambda_q\mu_q(\beta^+_q\beta^+_{-q} + \beta_{-q}\beta_q)$$

In (4.4) H_c and H_{ph} represent the expressions (4.3) and (4.2) respectively, transformed to the new amplitudes.

In order to write down the compensation and renormalization conditions it is convenient to introduce the more compact "time-dependent" formulation of the basic expressions, which is quite analogous to the one used in quantum field theory. For this purpose note that in terms of

the "S-matrix"

$$S_- = T\left(e^{-i\int_{-\infty}^{0} H_{int}(t)dt}\right)$$

the sum

$$R(E) = \sum_{m\geq 1} R_m(E) = H_{int} + H_{int}\frac{1}{E-H_0}H_{int} + \dots$$

can be represented as an eigenvalue

$$R(E)C_E = RC_E$$

of the energy operator R

$$R = H_{int}\cdot S_- = T\left(H_{int}(0)\,e^{-i\int_{-\infty}^{0} H_{int}(t)dt}\right) \qquad (4.5)$$

Here $H_{int}(t)$ is the interaction Hamiltonian in the interaction picture

$$H_{int}(t) = e^{-iH_0 t} H_{int}\, e^{iH_0 t}$$

Let us consider now the compensation and renormalization conditions. The compensation conditions imply the vanishing of the sums of terms corresponding to graphs with two external outgoing electron lines (fig. 5) and two external outgoing phonon lines (fig. 6).

Fig. 5

Fig. 6

The equations for determining U, $\tilde{\varepsilon}(\kappa)$ and $\tilde{\omega}(q)$ represent the conditions for the vanishing of the contributions from electron self-energy graphs (fig. 7), phonon self-energy graphs fig. 8) and vacuum graphs (fig. 9).

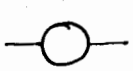

Fig. 7

Fig. 8

Fig. 9

In figs. 5–9 the circle designates strongly connected graphs which cannot be represented as two parts linked by one or two fermion lines or by one or two boson lines.

In terms of R the compensation and renormalization conditions can be written as

$$\langle \alpha_{\kappa_1}\alpha_{\kappa_0} R\rangle_c = 0 \qquad (4.6)$$

$$\langle \beta_{-q}\beta_q R\rangle_c = 0 \qquad (4.7)$$

$$\langle \alpha_{\kappa_0} R\, \alpha^+_{\kappa_0}\rangle_c = 0 \qquad (4.8)$$

$$\langle \beta_q R\, \beta^+_q\rangle_c = 0 \qquad (4.9)$$

$$\langle R\rangle_c = 0 \qquad (4.10)$$

The subscript "c" indicates that here we deal only with terms corresponding to the above mentioned strongly connected graphs. Eqs. (4.6)-(4.IO) may be written in a clearer form in terms of functional derivatives of R with respect to Fermi and Bose operators in the interaction picture

$$\alpha_{k\nu}(t) = \alpha_{k\nu}\, e^{-i\tilde{\varepsilon}(k)t}, \qquad \overset{+}{\alpha}_{k\nu}(t) = \overset{+}{\alpha}_{k\nu}\, e^{i\tilde{\varepsilon}(k)t} \qquad (\nu = 0, 1)$$

$$\beta_q(t) = \beta_q\, e^{-i\tilde{\omega}(q)t}, \qquad \overset{+}{\beta}_q(t') = \overset{+}{\beta}_q\, e^{i\tilde{\omega}(q)t'}$$

Performing in Eqs. (4.6)-(4.9) the commutations of $\alpha, \overset{+}{\alpha}, \beta, \overset{+}{\beta}$ with R, we are led to expressions containing second order functional derivatives of R. Taking into account the condition of strong connection, one can perform explicitly the functional differentiation of the terms H'(o) and H"(o) in the Hamiltonian $H_{int}(o) = H_{int}$. Eqs. (4.6)-(4.IO) are thus replaced by:

$$2(E(k)-\lambda)u_k v_k + \int \Big\langle \frac{\delta^2 R'}{\delta \overset{+}{\alpha}_{k_1}(t)\, \delta \overset{+}{\alpha}_{k_0}(t')} \Big\rangle_c e^{i\tilde{\varepsilon}(k)(t+t')}\, dt\, dt' = 0 \qquad (4.II)$$

$$\omega(q)\lambda_q/\mu_q + \int \Big\langle \frac{\delta^2 R'}{\delta \overset{+}{\beta}_{-q}(t)\, \delta \beta_q(t')} \Big\rangle_c e^{i\tilde{\omega}(q)(t+t')}\, dt\, dt' = 0 \qquad (4.I2)$$

$$(E(k)-\lambda)(u_k^2 - v_k^2) - \tilde{\varepsilon}(k) - \int \Big\langle \frac{\delta^2 R'}{\delta \overset{+}{\beta}_{-q}(t)\, \delta \beta_q(t')} \Big\rangle_c e^{i\tilde{\varepsilon}(k)(t-t')}\, dt\, dt' = 0 \qquad (4.I3)$$

$$\omega(q)(\lambda_q^2 + \mu_q^2) - \tilde{\omega}(q) - \int \Big\langle \frac{\delta^2 R'}{\delta \overset{+}{\beta}_q(t)\, \delta \beta_q(t')} \Big\rangle_c e^{i\tilde{\omega}(q)(t-t')}\, dt\, dt' = 0 \qquad (4.I4)$$

$$2\sum_k (E(k)-\lambda) v_k^2 + \sum_q \omega(q)\mu_q^2 - U + \langle R' \rangle_c = 0 \qquad (4.I5)$$

Here

$$R' = T\Big[(H_e(o) + H_{ph}(o))\, e^{-i\int_{-\infty}^{\infty} H_{int}(t)\,dt} \Big]$$

$\delta/\delta\alpha$ is the right functional derivative, $\delta/\delta\overset{+}{\alpha}$ - the left functional derivative, which anticommute with each other and with $\alpha, \overset{+}{\alpha}$.

4.3. The final form of the compensation equation for electron graphs. Here we shall simplify the basic compensation equation (4.II). First of all let us make the transition from the α-representation to the a-representation in the functional derivatives

$$\frac{\delta}{\delta \overset{+}{\alpha}_{k_0}(t)} = u_k \frac{\delta}{\delta \overset{+}{a}_{k,+}(t)} - v_k \frac{\delta}{\delta a_{-k,-}(t)}, \qquad \frac{\delta}{\delta \overset{+}{\alpha}_{k_1}(t)} = u_k \frac{\delta}{\delta \overset{+}{a}_{-k,-}(t)} + v_k \frac{\delta}{\delta a_{k,+}(t)}$$

In the limiting case, when u, v take their limiting values (2.24) the operators $\hat{a}^{+}(t), a(t)$ introduced here may be written as in the form

$$\left.\begin{aligned} a_{k,\sigma}(t) &= a_{k\sigma}\, e^{-i\tilde{\varepsilon}(k)t} \\ \hat{a}^{+}_{k\sigma}(t) &= \hat{a}^{+}_{k\sigma}\, e^{i\tilde{\varepsilon}(k)t} \end{aligned}\right\} \quad \text{for} \quad k > k_F$$

$$\left.\begin{aligned} a_{k,\pm}(t) &= \pm \hat{a}^{+\prime}_{-k,\mp}\, e^{i\tilde{\varepsilon}(k)t} \\ \hat{a}^{+}_{k,\pm}(t) &= \pm a'_{-k,\mp}\, e^{-i\tilde{\varepsilon}(k)t} \end{aligned}\right\} \quad \text{for} \quad k < k_F \qquad (4.16)$$

where $\hat{a}^{+\prime}$, a' are the creation and annihilation operators for holes inside the Fermi sphere. In fact we have introduced here another representation for the electron operators which we call the a'- representation.

We obtain

$$2\left[E(k)-\lambda\right] u_k v_k + u_k v_k \int dt\,dt'\, e^{i\tilde{\varepsilon}(k)(t+t')} \left\{ \left\langle \frac{\delta^2 R'}{\delta a_{k,+}(t)\,\delta \hat{a}^{+}_{k,+}(t')} \right\rangle_c - \left\langle \frac{\delta^2 R'}{\delta \hat{a}^{+}_{-k,-}(t)\,\delta a_{-k,-}(t')} \right\rangle_c \right\} =$$

$$= \int dt\,dt'\, e^{i\tilde{\varepsilon}(k)(t+t')} \left\{ u_k^2 \left\langle \frac{\delta^2 R'}{\delta \hat{a}^{+}_{k,+}(t')\,\delta \hat{a}^{+}_{-k,-}(t')} \right\rangle_c + v_k^2 \left\langle \frac{\delta^2 R'}{\delta a_{k,+}(t)\,\delta a_{-k,-}(t')} \right\rangle_c \right\} \qquad (4.17)$$

Now we can use the assumption that the width of the energy layer in which u, v strongly differ from their normal values (2.24) is very small and retain in (4.17) only the main terms.

In the left hand side this procedure replaces the expectation values of the second functional derivatives over C_ν by expectation values over C_0, the Fermi sphere state.

The application of such a procedure to the right hand side would turn it into zero. Therefore it is necessary to make a preliminary transformation of the coefficients of u^2, v^2 in order to separate a small parameter of the order of magnitude of the product uv. This transformation can be performed by means of a procedure well known in quantum field theory under the name of the "generalized Wick theorem" (cf. /I7/, §34.2), using the following formula for the chronological contraction

$$\overbrace{a_{k',+}(\tau)\, a_{-k',-}(\tau')} \equiv \left\langle T\, a_{k',+}(\tau)\, a_{-k',-}(\tau') \right\rangle_c = -\, u_{k'} v_{k'}\, e^{-i\tilde{\varepsilon}(k')|\tau-\tau'|} \qquad (4.18)$$

Taking into account that u_k^2 and v_k^2 differ from their limiting values (2.24) only in the neighbourhood of the Fermi surface we can write down Eq.(4.17) in the form

$$2\,\mathfrak{J}(k)\, u_k v_k = (u_k^2 - v_k^2) \sum_{k'} u_{k'} v_{k'}\, Q(k,k') \qquad (4.19)$$

where

$$\mathfrak{Z}(\kappa) = E(\kappa) - \lambda - \frac{1}{2}\int dt dt' e^{i\tilde{\varepsilon}(\kappa)(t'+t)} \left\{ \left\langle \frac{\delta^2 R'}{\delta \hat{a}^+_{\kappa,+}(t')\,\delta a_{\kappa,+}(t)} \right\rangle_0 + \left\langle \delta \hat{a}^+_{-\kappa,-}(t) \right. \right. \tag{4.20}$$

$$Q(\kappa,\kappa') = \begin{cases} \int \left\langle \dfrac{\delta^4 R}{\delta a_{\kappa',+}(\tau)\,\delta a_{-\kappa',-}(\tau')\,\delta \hat{a}^+_{\kappa,+}(t')\,\delta \hat{a}^+_{-\kappa,-}(t)} \right\rangle_0 e^{i\tilde{\varepsilon}(\kappa)(t+t') - i\tilde{\varepsilon}(\kappa')|\tau-\tau'|} dt dt' d\tau d\tau' & \text{for } \kappa > \kappa_F \\ \int \left\langle \dfrac{\delta^4 R}{\delta \hat{a}^+_{\kappa',+}(\tau)\,\delta \hat{a}^+_{-\kappa',-}(\tau')\,\delta a_{\kappa,+}(t)\,\delta a_{-\kappa,-}(t')} \right\rangle_0 e^{i\tilde{\varepsilon}(\kappa')(t+t') - i\tilde{\varepsilon}(\kappa')|\tau-\tau'|} dt dt' d\tau d\tau' & \text{for } \kappa < \kappa_F \end{cases} \tag{4.21}$$

The subscript "o" means that the expectation values are taken in the state C_0. We also recall that, according to the sence of the limiting process, the functionals R and R' depend here on the Hamiltonians in the a'-representation. For example

$$H_{ph} = \sum_{\substack{(\kappa,\kappa',q,s) \\ \kappa'-\kappa=q}} g(q)\sqrt{\frac{\omega(q)}{2V}}\, \hat{a}^+_{\kappa s} a_{\kappa' s} (\beta^+_q + \beta_{-q})(\lambda_q + \mu_{-q}) \tag{4.22}$$

Introducing a new undnown function

$$C(\kappa) = \sum_{\kappa'} u_{\kappa'} v_{\kappa'} Q(\kappa,\kappa') \tag{4.23}$$

with due account of Eq.(2.2I); one can write (4.I9) in the form (2.22):

$$C(\kappa) = \frac{1}{2}\sum_{\kappa'} Q(\kappa,\kappa') \frac{C(\kappa')}{\sqrt{C^2(\kappa') + \mathfrak{Z}^2(\kappa')}} \tag{4.24}$$

Before studying carefully the effect of Coulomb interaction on Q and $\mathfrak{Z}$, let us establish the connection between the ground-state and one electron excitation energies with the solution of (4.24) for the case under consideration.

4.4. The Ground State Energy and the Energy of One-Fermion Excitations. The ground state energy must be determined from the vacuum renormalization equation (4.I5). Using twice the "generalized Wick theorem", taking into account (4.I8) and

$$\overbrace{\hat{a}^+_{\kappa,+}(t)\, a_{\kappa,+}(t')} = \begin{cases} v_\kappa^2\, e^{-i\tilde{\varepsilon}(\kappa)(t-t')} & t > t' \\ -u_\kappa^2\, e^{-i\tilde{\varepsilon}(\kappa)(t'-t)} & t < t' \end{cases} \tag{4.25}$$

$$\hat{a}^+_{\kappa,+}(t)\,\hat{a}^+_{-\kappa,-}(t') = u_\kappa v_\kappa\, e^{-i\tilde{\varepsilon}(\kappa)|t-t'|} \tag{4.26}$$

and going over to the a'-representation, we obtain

$$U = 2\sum_\kappa (E(\kappa)-\lambda) v_\kappa^2 + \sum_q \omega(q)\mu_q^2 - \sum_{\kappa,s} v_\kappa^2 \int_{t'<t} \left\langle \frac{\delta^2 R'}{\delta \hat{a}^+_{\kappa s}(t)\,\delta a_{\kappa s}(t')} \right\rangle_0 e^{-i\tilde{\varepsilon}(\kappa)(t-t')} dt dt' + \tag{4.27}$$

$$+ \sum_{\kappa,s} u_\kappa^2 \int_{t'>t} \left\langle \frac{\delta^2 R'}{\delta \hat{a}^+_{\kappa s}(t)\,\delta a_{\kappa s}(t')} \right\rangle_0 e^{-i\tilde{\varepsilon}(\kappa)(t'-t)} dt dt' + \sum_{\kappa,\kappa'} u_\kappa v_\kappa u_{\kappa'} v_{\kappa'} \int dt dt' d\tau d\tau' e^{i\tilde{\varepsilon}(\kappa)|t-t'| - i\tilde{\varepsilon}(\kappa')|\tau-\tau'|} \left\langle \frac{\delta^4 R}{\delta \hat{a}^+_{\kappa,+}(t)\,\delta \hat{a}^+_{-\kappa,-}(t')\,\delta a_{\kappa',+}(\tau)\,\delta a_{-\kappa',-}(\tau')} \right\rangle_0$$

For u and v differing from their normal values this formula represents the energy of the superconducting ground state U_s. For $u_k = \theta_G(k)$, $v_k = \theta_F(k)$ Eq. (4.27) describes the energy of the normal ground state U_n.

Taking into account the identity

$$v_k^2 - \theta_F(k) = -\left[u_k^2 - \theta_G(k)\right] = \frac{\theta_G(k)}{2}\left\{1 - \frac{\mathcal{J}(k)}{\sqrt{C^2(k)+\mathcal{J}^2(k)}}\right\} - \frac{\theta_F(k)}{2}\left\{1 + \frac{\mathcal{J}(k)}{\sqrt{C^2(k)+\mathcal{J}^2(k)}}\right\}$$

which follows from (2.2I), and Eqs. (4.20),(4.23), in the limit of small $\tilde{\varepsilon}(k)$, $\tilde{\varepsilon}(k')$(corresponding to the neighbourhood of the Fermi surface), the difference $U_s - U_n$ turns out to be

$$U_s - U_n = -\frac{1}{2}\sum_k \sqrt{C^2(k)+\mathcal{J}^2(k)}\left\{\theta_G(k)\left(1 - \frac{\mathcal{J}(k)}{\sqrt{C^2(k)+\mathcal{J}^2(k)}}\right)^2 + \theta_F(k)\left(1 + \frac{\mathcal{J}(k)}{\sqrt{C^2(k)+\mathcal{J}^2(k)}}\right)^2\right\} \qquad (4.28)$$

Now we shall analyze the energy spectrum of one-Fermion excitations $\tilde{\varepsilon}(k)$ for the superconducting state. The renormalized energy $\tilde{\varepsilon}$ is determined by Eq. (4.I3). Carrying out the transformation from the α -representation to the a' -representation in the functional derivatives in the last term of Eq. (4.I3) and using the " generalized Wick theorem", one can express $\tilde{\varepsilon}$ as the sum of three terms

$$\tilde{\varepsilon}(k) = u_k^2\,\varepsilon_1(k) + v_k^2\,\varepsilon_2(k) + u_k v_k\,\varepsilon_3(k) \qquad (4.29)$$

In the neighbourhood of the Fermi surface, where $\tilde{\varepsilon}$ is small, we have

$$\varepsilon_1(k) = -\varepsilon_2(k) = \mathcal{J}(k)$$

and by Eqs.(4.2I),(4.23) and (2.2I)

$$u_k v_k\,\varepsilon_3(k) = 2u_k v_k \sum_{k'} Q(k,k')\,u_{k'}v_{k'} = \frac{C^2(k)}{\sqrt{C^2(k)+\mathcal{J}^2(k)}}$$

Note that on the Fermi surface itself, where $u_F^2 = v_F^2 = 1/2$ and $\mathcal{J}(k_F)$ is equal to zero, we obtain from(4.29)

$$\tilde{\varepsilon}(k_F) = C(k_F)$$

Thus the one Fermion excitation energy is separated from the ground state energy by the gap $C(k_F) = \Delta$.

4.5. Transformation of the Kernel Q(k,k'). We shall now take into account the fact that, as a consequemce of the compensation of graphs in fig. 9the functional R depends on g^2 only through the small parameter $g^2\tilde{\omega}/E_F$. Hence the r.h.s. of Eqs. (4.20), (4.2I) can be expanded in powers of g^2 and only the first two terms retained. It is more convenient to expand R. The first term of its expansion, independent of g^2, will be denoted by R_c. It has

the form

$$R_c = T(H_c(0) S_c) \tag{4.30}$$

where

$$S_c = T\left(e^{-i\int_{-\infty}^{\circ} H_c(t)dt}\right) \tag{4.31}$$

The second term R_{ph}, proportional to g^2, can be represented as

$$R_{ph} = R_1 + R_2 \tag{4.32}$$

where

$$R_1 = -iT\left(H_{ph}(0)\int_{-\infty}^{\circ} dt\, H_{ph}(t) S_c\right), \qquad R_2 = (-i)^2 T\left(H_c(0)\int_{-\infty}^{\circ} dt\, H_{ph}(t)\int_{-\infty}^{\circ} dt'\, H_{ph}(t') S_c\right)$$

Taking the expectation value in the phonon vacuum we obtain

$$R_1 = -i\sum_{\substack{(l',l,p',p,s,\sigma)\\ l'-l=p'-p=q}} \frac{g^2(q)\omega(q)(\lambda_q+\mu_q)^2}{2V}\int_{-\infty}^{\circ} dt\, e^{i\tilde{\omega}(q)t} T\left(a^+_{p's}(0) a_{ps}(0) a^+_{l\sigma}(t) a_{l'\sigma}(t) S_c\right)$$

$$R_2 = -\sum \frac{g^2(q)\omega(q)(\lambda_q+\mu_q)^2}{2V}\int_{-\infty}^{\circ} dt dt' e^{i\tilde{\omega}(q)(t'-t)} T\left(H_c(0)\, a^+_{p's}(t) a_{ps}(t) a^+_{l\sigma}(t') a_{l'\sigma}(t') S_c\right)$$

We can carry out another step in the simplification of the kernel Q_{ph}. Note that the maximal energy of phonons is small with respect to the Fermi energy. On the other hand, according to (4.19) only momenta k,k' in the neighbourhood of the Fermi surface, where $\tilde{\varepsilon}(k)$, $\tilde{\varepsilon}(k')$ are also small, is essential. So the sum $\tilde{\omega}(k-k')+\tilde{\varepsilon}(k)+\tilde{\varepsilon}(k')$ is small, and the contributions to Q containing this sum in a deniminator are large. It is evident that such contributions correspond to graphs (fig.10) which can be split into two parts by a vertical line, intersecting only two fermion lines (k), (k') and one phonon line (k-k').

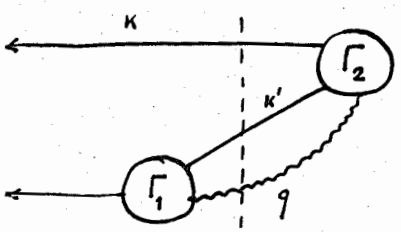

fig. 10

The circles Γ_1 and Γ_2 in fig.10 stand for generalized vertices including Coulomb corrections to all orders. From the point of view of the "time-dependent representation" all ver-

tices in Γ_2 must be later in time than each of the vertices in Γ_1.

Thus in the approximation under consideration, the kernel Q(k,k') of Eq.(4.19) can be reprexented as a sum of two terms

$$Q(k,k') = Q_c(k,k') + Q_{ph}(k,k') \tag{4.33}$$

The first term Q_c corresponds to pure Coulomb interaction (4.30), while the main part of the second term, described by the graphs of fig. 10 may be represented as follows

$$Q_{ph}(k,k') = \frac{g^2(q)\omega(q)(\lambda_q+\mu_q)^2}{\tilde{\omega}(q)+\tilde{\varepsilon}(k)+\tilde{\varepsilon}(k')} \cdot \frac{\Lambda(q,k,k')}{V} \quad ; \quad q = \pm(k-k') \tag{4.34}$$

Λ is a product of factors arising from the generalized vertices Γ_1 and Γ_2. In the limit of switching off the Coulomb interaction we have $\Lambda = 1$. In general

$$\frac{\Lambda(q,k,k')}{V\{\tilde{\omega}(q)+\tilde{\varepsilon}(k)+\tilde{\varepsilon}(k')\}} = \sum_\sigma (M_\sigma + N_\sigma) P_\sigma \tag{4.35}$$

The quantity P_σ corresponds to Γ_1. It equals

$$P_\pm = -i\int_{-\infty}^{0} dt\,dt'\,d\theta\; e^{i\theta\tilde{\omega}(q)+it\tilde{\varepsilon}(k)+it'\tilde{\varepsilon}(k')} \left\langle \frac{\delta^2 R_{-q}(\theta)}{\delta a^+_{\pm k,\pm}(t)\,\delta a_{\pm k',\pm}(t')} \right\rangle_0 \tag{4.36}$$

where

$$R_{-q}(\theta) = T(H_q(\theta)\,S_c), \quad H_q(t) = \frac{1}{2V}\sum_{p,s} a^+_{p+q,s}\,a_{ps}(t)$$

The functions M and N correspond to the vertex Γ_2 for R_I and R_2. It can be shown that they have the form

$$M_\pm = \int_{-\infty}^{0} d\tau\,d\tau' \left\langle \frac{\delta^2 R_{-q}(0)}{\delta a^+_{\pm k,\pm}(\tau)\,\delta a_{\pm k',\pm}(\tau')} \right\rangle_0 \tag{4.37}$$

$$N_\pm = -i\int_{-\infty}^{0} d\theta\,d\tau\,d\tau' \left\langle \frac{\delta^2 [H_c(0)R_q(\theta)]}{\delta a^+_{\pm k,\pm}(\tau)\,\delta a_{\pm k',\pm}(\tau')} \right\rangle_0 \tag{4.38}$$

4.6. The determination of $\lambda+\mu$ and $\tilde{\omega}$. We shall now determine the sum $\lambda+\mu$ from the condition of compensation (4.12) for the phonon graphs represented in fig. 6. Performing the functional differentiation and neglecting all terms except those of the order g^2 we get

$$2\omega(q)\lambda_q\mu_q - g^2(q)\omega(q)(\lambda_q+\mu_q)^2 Z(q) = 0 \tag{4.39}$$

where

$$Z(q) = \frac{1}{2V}\int_{-\infty}^{0} dt\,dt'\; e^{i\tilde{\omega}(q)(t+t')} \sum \left\langle T(H_c(0)\,a^+_{ks}(t)\,a_{k's}(t)\,a^+_{\ell\sigma}(t')\,a_{\ell'\sigma}(t')\,S_c) \right\rangle_0 +$$

$$+ \frac{i}{V}\int_{-\infty}^{0} dt\; e^{i\tilde{\omega}(q)t} \sum \left\langle T(a^+_{k\sigma}(0)\,a_{k's}(0)\,a^+_{\ell\sigma}(t)\,a_{\ell'\sigma}(t)\,S_c) \right\rangle_0 \tag{4.40}$$

Solving Eq. (4.39) with respect to $(\lambda+\mu)^2$, we find

$$(\lambda_q+\mu_q)^2 = (1-2g^2(q)Z(q))^{-1/2} \tag{4.41}$$

Consider now Eq.(4.14) for the renormalized phonon energy $\tilde{\omega}$. Carrying out a similiar transformation, we get

$$\omega(q)(\lambda_q^2+\mu_q^2) - \tilde{\omega}(q) - \tilde{\omega}(q)g^2(q)(\lambda_q+\mu_q)^2 Y(q) = 0 \tag{4.42}$$

where

$$Y(q) = \frac{1}{2V}\int dt\,dt'\, e^{i\tilde{\omega}(q)(t-t')}\sum\langle T(H_c(0)\hat{a}^+_{ks}(t)a_{k's}(t)\hat{a}^+_{l\sigma}(t')a_{l'\sigma}(t')S_c)\rangle_0 +$$

$$+\frac{i}{2V}\int dt\,(e^{i\tilde{\omega}(q)t}+e^{-i\tilde{\omega}(q)t})\sum\langle T(\hat{a}^+_{ks}(0)a_{k's}(0)\hat{a}^+_{l\sigma}(t)a_{l'\sigma}(t)S_c)\rangle_0 \tag{4.43}$$

Solving Eq. (4.42) with respect to $\tilde{\omega}$ with due account of (4.41), we find

$$\tilde{\omega}(q) = \frac{1-g^2(q)\{Y(q)+Z(q)\}}{\sqrt{1-2g^2(q)Z(q)}}\,\omega(q) \tag{4.44}$$

4.7. The connection with a model problem. Now we shall show that $Z(q), Y(q), (\lambda+\mu)^2, \tilde{\omega}$ as well as Λ in Eq. (4.34) can be approximately expressed in terms of the solution of a model problem.

Consider the model system, described by the Hamiltonian

$$H = H_0 + H_{int}\ ,\quad H_0 = \sum_{k>k_F,s}\tilde{\varepsilon}(k)\hat{a}^+_{ks}a_{ks} + \sum_{k<k_F,s}\tilde{\varepsilon}(k)a_{ks}\hat{a}^+_{ks}$$

$$H_{int} = H_c + \frac{\delta}{\sqrt{2V}}\sum_{k,s}(\hat{a}^+_{k+q,s}a_{ks} + \hat{a}^+_{ks}a_{k+q,s}) \tag{4.45}$$

Here H_c is the Hamiltonian of the Coulomb interaction in the form (4.3), and δ a small parameter. The ground state energy U of this system can be expanded in powers of δ

$$U = U_0 + \delta^2 U_2^q + \ldots \tag{4.46}$$

and the coefficient U_2^q has the form

$$U_2^q = -\frac{1}{2V}\int dt\,dt'\sum_{k,l,s,\sigma}\langle T(H_c(0)\hat{a}^+_{k+q,s}(t)a_{ks}(t)\hat{a}^+_{l-q,s}(t')a_{l\sigma}(t')S_c)\rangle_0 -$$

$$-\frac{i}{V}\int dt\sum_{k,l,s,\sigma}\langle T(\hat{a}^+_{k+q,s}(0)a_{ks}(0)\hat{a}^+_{l-q,\sigma}(t)a_{l\sigma}(t)S_c)\rangle_0 \tag{4.47}$$

Comparing Eqs. (4.40), (4.43) and (4.47) it follows that in the limit of small $\tilde{\omega}$, one has the identity

$$Z(q) = Y(q) = -U_2^q \tag{4.48}$$

as a consequence of which Eqs. (4.4I),(4.44) become

$$(\lambda_q + \mu_q)^2 = \frac{1}{\sqrt{1 + 2g^2(q) U_2^q}} \tag{4.49}$$

$$\tilde{\omega}(q) = \sqrt{1 + 2g^2(q) U_2^q}\ \omega(q) \tag{4.50}$$

Now we want to construct the quantities which from the point of view of the considered model, correspond to M,N and P in Eq. (4.35).

Consider the matrix elements

$$V_\pm = \langle a_{\pm\kappa,\pm} R \dot{a}_{\pm\kappa',\pm} \rangle_0 \tag{4.5I}$$

$$W_\pm = \langle a_{\pm\kappa,\pm} S^0_{-\infty} \dot{a}_{\pm\kappa',\pm} \rangle_0 \tag{4.52}$$

which correspond to the graph of fig. II

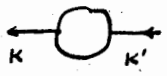

Fig. II

These matrix elements can be expanded in powers of δ

$$V_\sigma = \delta v_\sigma + \ldots \qquad , W_\sigma = \delta w_\sigma + \ldots \tag{4.53}$$

The coefficients v_σ, w_σ have the form

$$v_\sigma = \langle a_{\sigma\kappa,\sigma} T([H_q(0) + H_{-q}(0)] S_c) \dot{a}_{\sigma\kappa',\sigma} \rangle_0 -$$

$$-i \langle a_{\sigma\kappa,\sigma} \int_{-\infty}^{0} H_c(0) T([H_q(\theta) + H_{-q}(\theta)] S_c) \dot{a}_{\sigma\kappa';\sigma} \rangle_0 d\theta' = 2(M_\sigma + N_\sigma)$$

$$w_\sigma = -i \int \langle a_{\sigma\kappa,\sigma} T(\{H_q(\theta) + H_{-q}(\theta)\} S_c) \dot{a}_{\sigma\kappa,\sigma} \rangle_0 =$$

$$= 2 \frac{\tilde{\varepsilon}(\kappa) + \tilde{\varepsilon}(\kappa') + \tilde{\omega}(\kappa - \kappa')}{\hat{\varepsilon}(\kappa) + \tilde{\varepsilon}(\kappa')} P_\sigma$$

Substituting these coefficients into (4.34), (4.35) one obtains

$$Q_{ph}(k,k') = \frac{g^2(q)\omega(q)(\lambda_q + \mu_q)^2}{\tilde{\omega}(q) + \hat{\varepsilon}(\kappa) + \tilde{\varepsilon}(\kappa')} \frac{\sum_\sigma v_\sigma \tilde{w}_\sigma}{4}$$

The expression $\tilde{\omega}_\sigma$ introduced here

$$\tilde{\omega}_\sigma = \left(\tilde{\varepsilon}(k) + \tilde{\varepsilon}(k') \right) \omega_\sigma$$

is in fact independent of $\tilde{\varepsilon}(k) + \tilde{\varepsilon}(k')$.

By Eqs. (4.49) and (4.50) Q_{ph} can be rewritten in the form

$$Q_{ph}(k,k') = \frac{\tilde{g}^2(q)\tilde{\omega}(q)}{\tilde{\omega}(q) + \tilde{\varepsilon}(k) + \tilde{\varepsilon}(k')} \frac{\sum_\sigma v_\sigma \tilde{\omega}_\sigma}{4} \tag{4.55}$$

where $\tilde{g}$ - is the renormalized value of g

$$\tilde{g}(q) = \frac{g(q)}{\sqrt{1 + 2g^2(q)V_2^q}} \tag{4.56}$$

§ 5. QUALITATIVE DESCRIPTION OF THE EFFECTS COULOMB INTERACTION[x)]

5.I. Approximate determination of the renormalized $\tilde{\omega}$ and $\tilde{q}$. In subsection 4.7 we have reduced the problem of the determination of $\lambda+\mu, \tilde{\omega}$, Q_{ph} to the solution of a model problem with the Hamiltonian containing only the "kinetic energy", the energy of the Coulomb interaction and a term describing the interaction with a weak external field.

Here we shall estimate the quantities $\lambda+\mu, \tilde{\omega}, \ldots$ in the approximation of a "strongly compressed" electron gas, where the Coulomb energy can be considered as small as compared with the kinetic energy.

It is well known, that in problems with Coulomb interaction one cannot use direct by expansions in powers of e^2, as these contain divergences in the region of small momenta and leads to a situation similar to the "infrared catastrophe" in quantum field theory.

The procedure for correcting such expansions became completely clear due to the results of Gell-Mann, Brueckner and Sawada/I3/. It follows from their paper, that in order to improve the lowest order approximation (with respect to e^2) one has to sum over the graphs composed of complexes particle – hole mentioned in § 3. Now we follow this procedure, giving it, for convenience, the form of an approximate second quantization.

First of all we introduce by means of the canonical transformation (2.4) the Fermi amplitudes for the particles and holes with trivial values of the parameters u, v :

$$u_k = \theta_c(k) \ , \ v_k = \theta_f(k) \qquad (5.I)$$

The representation (2.4), (5.I) is in faot the a' - representation (4.I6), in which all previous results were obtained.

Consider now the Fourier components of the space density of electrons

$$\rho(q) = \sum_{k,s} a^{+}_{k+q,s}\, a_{k,s} \qquad (q \neq 0)$$

Transform them into the a'- representation and retain in the resulting expression only these terms whioh do not vanish after their operating either from the left or from the right on the "vacuum state" C_0^{*} or C_0 respectively

$$C_0^{*}\, \alpha^{+}_{\nu} = 0 \ , \ \alpha_{\nu} C_0 = 0 \qquad (\nu = 0,1)$$

Then we obtain approximately

$$\rho(q) = \sum_{k} \left(M(k, k+q)\, \alpha^{+}_{k+q,0}\, \alpha^{+}_{k,1} + M(k, k-q)\, \alpha_{k,1}\, \alpha_{k-q,0} \right)$$

where

$$M(k,q) = u_k v_q + u_q v_k$$

---

x) This Section is based on the investigations carried out by N.N. Bogolubov (§§ 5.I, 5.2) and V.V. Tolmachov (§§ 5.3, 5.4).

According to the rules of the approximate second quantization method, we replace the products of Fermi amplitudes by Bose amplitudes

$$\overset{+}{\alpha}_{k+q,0}\,\overset{+}{\alpha}_{k,1} \to \overset{+}{\beta}_q(k) \quad , \quad \alpha_{k,1}\,\alpha_{k-q,0} \to \beta_{-q}(k)$$

and obtain

$$\rho(q) = \sum_k M(k,k+q)\,\overset{+}{\beta}_q(k)$$

The substitution of this expression in the Coulomb interaction energy

$$H_c = \frac{4\pi e^2}{2V} \sum_{(q,q\neq 0)} \frac{\rho(q)\rho(q)}{|q|^2}$$

yields

$$H_c = \frac{2\pi e^2}{V} \sum_{\substack{(k,k';q)\\ q\neq 0}} \frac{1}{|q|^2} \Big\{ M(k,k+q)M(k';k'-q)\,\overset{+}{\beta}_{-q}(k')\,\overset{+}{\beta}_q(k) + M(k,k-q)M(k';k'+q)\,\beta_{-q}(k)\beta_q(k') +$$
$$+ M(k,k+q)M(k';k'+q)\,\overset{+}{\beta}_q(k)\beta_q(k') + M(k,k-q)M(k';k'-q)\,\overset{+}{\beta}_{-q}(k')\beta_{-q}(k) \Big\}$$

In the same way, in this approximation, the interaction term with a "classical field" in (4.45) becomes

$$\frac{\delta}{2V} \sum_k \Big\{ M(k,k+p)\,\overset{+}{\beta}_p(k) + M(k,k-p)\,\overset{+}{\beta}_{-p}(k) + M(k,k-p)\,\beta_{-p}(k) + M(k,k+p)\,\beta_p(k) \Big\}$$

As shown in Sec.3, in order to obtain the correct energy denominators we must take for the self energy the following expression

$$\sum_{k,q} \Big\{ \tilde{\varepsilon}(k) + \tilde{\varepsilon}(k+q) \Big\}\,\overset{+}{\beta}_q(k)\beta_q(k)$$

Thus the complete Hamiltonian of the problem under consideration, is in the method of approximate second quantization,

$$H = \sum_{k,q} \Big\{ \tilde{\varepsilon}(k) + \tilde{\varepsilon}(k+q) \Big\}\,\overset{+}{\beta}_q(k)\beta_q(k) +$$
$$+ \frac{2\pi e^2}{V} \sum_{(k,k';q\neq 0)} \frac{1}{|q|^2} \Big\{ M(k,k+q)M(k';k-q)\,\overset{+}{\beta}_q(k)\,\overset{+}{\beta}_{-q}(k') + M(k,k+q)M(k';k'-q)\,\beta_q(k)\beta_q(k') + 2M(k,k+q)M(k';k'+q)\,\overset{+}{\beta}_q(k)\beta_q(k') \Big\} +$$
$$+ \frac{\delta}{2V} \sum_k \Big\{ M(k,k+p)\big(\overset{+}{\beta}_p(k) + \beta_p(k)\big) + M(k,k-p)\big(\overset{+}{\beta}_{-p}(k) + \beta_{-p}(k)\big) \Big\} \qquad (5.2)$$

We see that this Hamiltonian consists of a quadratic form of Bose amplitudes and a linear form proportional to δ .

In order to evaluate the effect of this linear form on the shift of the ground state energy level and to calculate U_2' , one can use the well known method of translation of the Bose amplitudes

$$\beta_q(k) \to \beta_q(k) + C_q(k) \quad , \quad \beta_q^+(k) \to \beta_q^+(k) + C_q^*(k) \quad , \quad q = \pm p$$

where C and C* - are c -numbers to be determined from the condition of the vanishing of the linear forms

$$\frac{\partial H}{\partial \beta_q} = 0 \quad , \quad \frac{\partial H}{\partial \beta_q^+} = 0 \quad , \quad q = \pm p$$

Explicitly these conditions are

$$\{\tilde{\varepsilon}(k) + \tilde{\varepsilon}(k+q)\} C_q(k) + \frac{2\pi e^2}{V|q|^2} 2M(k,k+q) X(q) + \frac{\delta}{\sqrt{2V}} M(k,k+q) = 0$$

$$\{\tilde{\varepsilon}(k) + \tilde{\varepsilon}(k+q)\} C_{-q}^*(k) + \frac{2\pi e^2}{V|q|^2} 2M(k,k-q) X(q) + \frac{\delta}{\sqrt{2V}} M(k,k-q) = 0$$

where

$$X(q) = \sum_k \{M(k,k+q) C_q(k) + M(k,k-q) C_{-q}^*(k)\}$$

These condition yield for X the expression

$$X(q) = -\delta \sqrt{\frac{V}{2}} \frac{F(q)}{1 + \frac{4\pi e^2}{|q|^2} F(q)} \tag{5.3}$$

where

$$F(q) = \frac{1}{V} \sum_k \left\{ \frac{M^2(k,k+q)}{\tilde{\varepsilon}(k) + \tilde{\varepsilon}(k+q)} + \frac{M^2(k,k-q)}{\tilde{\varepsilon}(k) + \tilde{\varepsilon}(k-q)} \right\}$$

Note here, that in the first approximation with respect to δ the energy shift equals

$$\Delta U = \frac{\delta}{\sqrt{2V}} \sum_k \{M(k,k+p)(C_p^*(k) + C_p(k)) + M(k,k-p)(C_{-p}^*(k) + C_{-p}(k))\} =$$

$$= \frac{\delta}{\sqrt{2V}} (X(p) + X(-p))$$

By (5.3) this expressions can be written in the form

$$\Delta U = -\frac{\delta^2}{2} \frac{F(p)}{1 + \frac{4\pi e^2}{|p|^2} F(p)}$$

The quantity F(q) may be reduced to the form

$$F(q) = \frac{2}{V} \sum_k \frac{M^2(k,k+q)}{\tilde{\varepsilon}(k) + \tilde{\varepsilon}(k+q)} = \frac{4}{V} \sum_{k'-k=q} \frac{\theta_G(k)\,\theta_F(k')}{\tilde{E}(k) - \tilde{E}(k')} \tag{5.4}$$

here E(k) is the energy of the electron elementary exitation with respect to the Fermi surface, so that

$$\hat{\varepsilon}(k) = |\tilde{E}(k) - E_F|$$

Comparing Eqs.(4.46) and (5.4) we obtain

$$v_2^p = -\frac{1}{2}\frac{F(p)}{1 + \frac{4\pi e^2}{|p|^2}F(p)} \tag{5.5}$$

Substituting (5.5) into Eqs. (4.50) and (4.56) we obtain

$$\tilde{\omega}(q) = \omega(q)\left\{1 - \frac{g^2(q)F(q)}{1 + \frac{4\pi e^2}{|q|^2}F(q)}\right\}^{1/2} \tag{5.6}$$

$$\tilde{g}(q) = g(q)\left\{1 - \frac{g^2(q)F(q)}{1 + \frac{4\pi e^2}{|q|^2}F(q)}\right\}^{-1/2} \tag{5.7}$$

Comparing (5.6) and (5.7) with the corresponding expressions from 2.3 it follows that for small q the Coulomb forces practically destroy the renormalization.

5.2. Discussion of the properties of Q_c and Q_{ph}. Now let us turn to the quantities Q_c (Eqs. (4.2I), (4.33)) and $v, \tilde{\omega}$ in Q_{ph} (Eq. (4.55)). One could carry out an approximate analysis of these quantities in order to obtain their properties in the region of the "infrared Coulomb catastrophe", where the Coulomb interaction is not small. However, inasmach as such an investigation may give only a qualitative picture and as the correct analysis is much more complicated (such a correct analysis is now being corried out), we shall not undertake here such an investigation, and limit ourselves to a qualitative discussion of the properties of Q_c and Q_{ph}.

Note, for this, that the main result of the preceding subsection 5.I consists of the fact, that the Coulomb radiative corrections are summed as a geometric progression yielding a formula of the type (5.5). This result is not an accidental consequence of the approximative second quantization method used, but follows from the general structure of the Coulomb interaction. Another well known consequence of this general structure is the property of Coulomb screening. So we can state that, taking into account all Coulomb corrections to a given vertex of the graph describing the electron scattering with momentum transfer q, leads to a factor of the form

$$\frac{1}{1 + \frac{4\pi e^2}{|q|^2}\Phi(q)} \tag{5.8}$$

with $\Phi(q) > 0$.

Just the same factor occurs in the Eq. (5.5). It is also clear that inserting (5.8) into the usual Coulomb vertex $4\pi e^2/V|q|^2$, one obtains the screening

$$\frac{4\pi e^2}{V|q|^2}\cdot\frac{1}{1+\frac{4\pi e^2}{|q|^2}\Phi(q)}=\frac{4\pi e^2/V}{|q|^2+4\pi e^2\Phi(q)} \tag{5.8}$$

At the same time the appearance of the factor (5.8) in a non-Coulomb vertex (for instance in a phonon vertex) yields a cut-off effect for small $|q|^2$ (Cf. (5.6), (5.7)).

Now turn to the Q_c, Q_{ph}. In the lowest order with respect to e^2 the function Q_c reduces to the usual Coulomb vertex. Thus the complete expression for Q_c is finite for small $/q/^2$ as a consequence of the screening effect. The main part of Q_{ph} to the first order, corresponds to the usual phonon vertices. So the complete expression for Q_{ph} must vanish for $q = 0$.

5.3. General Properties of the Fundamental Compensation Equation. Consider, in conclusion, the compensation equation in the form (4.24) in the simple case of radial symmetry. In this case one can reduce this equation to the one-dimentional form, taking as the new independent variable

$$C(\xi)=\frac{1}{2}\int Q(\xi,\xi')n(\xi')\frac{C(\xi')}{\sqrt{\xi'^2+C^2(\xi')}}d\xi' \tag{5.10}$$

with

$$n(\xi)=\frac{1}{2\pi^2}\left(\kappa^2\frac{d|\kappa|}{d|\xi|}\right)$$

and

$$Q(\xi,\xi')=\frac{1}{4\pi}\int Q(|\kappa|,|\kappa'|\vec{e})d\vec{e}$$

For the analysis of this equation near the Fermi surface we replaqe $C^2(\xi')$ under the square root in the dencminator by the constant $\Delta^2=C^2(0)$. Such a simplification is correct from an asymphtotical point of view, as for very small ξ the approximation

$$\sqrt{\xi'^2+C^2(\xi')}\sim\sqrt{\xi'^2+\Delta^2}$$

is quite good.

Thus we obtain the "quasilinear" equation

$$C(\xi)=\frac{1}{2}\int Q(\xi,\xi')n(\xi')\frac{C(\xi')}{\sqrt{\xi'^2+\Delta^2}}d\xi' \tag{5.11}$$

$$C(0)=\Delta \tag{5.12}$$

As was shown in subsection 4.5, the kernel Q consists of the pure Coulomb part Q_c and the phonon part Q_{ph}

$$Q=Q_c+Q_{ph}$$

It follows from subsection 5.2 that Q_c may be approximately represented in the form (5.9), corresponding to a screened Coulomb interaction. According to (4.55) the term Q_{ph} is essential only in the neighbourhood of the Fermi surface for $|\xi| \lesssim \tilde{\omega}$.

Consider now the following auxiliary integral equation with the kernel proportional to Q_c .

$$u(\xi) - \frac{1}{2}\int Q_c(\xi,\xi')n(\xi')\frac{u(\xi')}{\sqrt{\xi'^2+\Delta^2}}\,d\xi' = f(\xi) \qquad (5.13)$$

and introduce the corresponding resolvent G_Δ

$$u(\xi) = f(\xi) - \frac{1}{2}\int G_\Delta(\xi,\xi'')f(\xi'')\,d\xi'' \qquad (5.14)$$

Due to the singularity of the kernel of Eq. (5.13) for $\Delta = 0$, the resolvent G_Δ has also the same singularity and can be represented as follows

$$G_\Delta(\xi,\xi'') = G_\Delta^{reg}(\xi,\xi'') + \frac{z(\xi)}{\sqrt{\xi''^2+\Delta^2}} \qquad \text{for } \xi''\to 0 \qquad (5.15)$$

$z(\xi)$ is the solution of the equation

$$z(\xi) - \frac{1}{2}\int Q_c(\xi,\xi')n(\xi')\frac{z(\xi')}{\sqrt{\xi'^2+\Delta^2}}\,d\xi' = -Q_c(\xi,0)n(0) \qquad (5.16)$$

Now note that Eq. (5.11) may be reduced to the form (5.13) with the aid of the substitution

$$u(\xi) = C(\xi)$$

$$f(\xi) = \frac{1}{2}\int Q_{ph}(\xi,\xi')n(\xi')\frac{C(\xi')}{\sqrt{\xi'^2+\Delta^2}}\,d\xi'$$

So, taking into account Eqs. (5.14), (5.15) one has

$$C(\xi) = \frac{1}{2}\int\Big\{Q_{ph}(\xi,\xi') - \frac{1}{2}\int G_\Delta^{reg}(\xi,\xi'')Q_{ph}(\xi'',\xi')\,d\xi'' - \frac{1}{2}\int\frac{z(\xi)}{\sqrt{\xi''^2+\Delta^2}}Q_{ph}(\xi'',\xi')\,d\xi''\Big\}n(\xi')\frac{C(\xi')}{\sqrt{\xi'^2+\Delta^2}}\,d\xi'$$

As was noticed, the function Q_{ph} is important only in a small neighbourhood of the Fermi surface where both its arguments are very small. Hence the integral term

$$\int G_\Delta^{reg}(\xi,\xi'')Q_{ph}(\xi'',\xi')\,d\xi''$$

is small and may be omitted. For the same reasons one must replace $n(\xi')$ and $z(\xi)$ by their values on the Fermi surface $n(0)$ and $z(0)$. So we obtain

$$C(\xi) = \frac{1}{2}\int\Big\{Q_{ph}(\xi,\xi') - \frac{z(0)}{2}\int\frac{Q_{ph}(\xi'',\xi')}{\sqrt{\xi''^2+\Delta^2}}\,d\xi''\Big\}n(0)\frac{C(\xi')}{\sqrt{\xi'^2+\Delta^2}}\,d\xi' \qquad (5.17)$$

The analysis of (5.17) can be carried through by means of the asymptotic method from sec.2. However, for the sake of simplicity, we use here a more crude version, replacing $Q_{ph}(\xi'',\xi')$

by the constant value $Q_{ph}(0,0)$ inside some region $|\xi| < \tilde{\omega}$ and by zero outside this region.

With due account of $\tilde{\omega}/\Delta \gg 1$, it yields

$$C(\xi) = \frac{\rho}{2}\int_{-\tilde{\omega}}^{\tilde{\omega}} \left\{1 - z(0)\ln\frac{2\tilde{\omega}}{\Delta}\right\} \frac{C(\xi')}{\sqrt{\xi'^2+\Delta^2}}\, d\xi' \qquad \text{for } |\xi| < \tilde{\omega} \qquad (5.18)$$

where

$$\rho = Q_{ph}(0,0)n(0)$$

It follows from (5.18)

$$C(\xi) = \Delta \qquad \text{for } -\tilde{\omega} < \xi < \tilde{\omega}$$

Δ is to be determined from the equation

$$1 = \rho\left(1 - z(0)\ln\frac{2\tilde{\omega}}{\Delta}\right)\ln\frac{2\tilde{\omega}}{\Delta} \qquad (5.19)$$

So one can see that the Coulomb interaction changes the effective parameter ρ

$$\rho \to \rho\left(1 - z(0)\ln\frac{2\tilde{\omega}}{\Delta}\right)$$

where $z(0)$ is determined from (5.16)

For the crude estimation of the value $z(0)$ replace the function $-Q_c(\xi,\xi')n(\xi')$ in (5.16) by its average value

$$\rho_c = \frac{4\pi e^2}{\kappa_c^2}\bar{n} \quad , \quad \kappa_c \sim \kappa_F$$

inside the region $|\xi| < \tilde{E}_F$, $\tilde{E}_F \sim E_F$ and by zero outside.

Under this condition the equation (5.19) has the following simple solution (for $\tilde{E}_F \gg \Delta$)

$$z(0) = \frac{\rho_c}{1 + \rho_c \ln\left(2\tilde{E}_F/\Delta\right)}$$

Substitution of this value into Eq. (5.19) yields

$$1 = \rho\left(1 - \frac{\rho_c \ln\frac{2\tilde{\omega}}{\Delta}}{1 + \rho_c \ln\frac{2\tilde{E}_F}{\Delta}}\right)\ln\frac{2\tilde{\omega}}{\Delta} \qquad (5.20)$$

5.4. A Criterion for Superconductivity. As was shown, the Coulomb interaction effectively reduces the parameter ρ and thus counteracts the appearance of the superconducting state. The superconducting state can not exist for arbitrarily small ρ. The criterion for the existence of superconductivity has the form

$$\rho > \frac{\rho_c}{1 + \rho_c \ln\left(\tilde{E}_F/\tilde{\omega}\right)} \qquad (6.21)$$

Note here that in the theory of BCS /5/, in place of Eqs (5.20), (5.21) one obtains the formula

$$\left.\begin{aligned} 1 &= (\rho - \rho_c)\ln\frac{2\tilde{\omega}}{\Delta} \\ \rho &> \rho_c \end{aligned}\right\} \qquad \text{BCS}$$

In our theory the effect of the Coulomb interaction, counteracting the appearance of superconductivity, is essentially reduced by the "large logarithm" $\ln(\tilde{E}_F/\tilde{\omega})$.
Note also that in the case when

$$\rho_c > \rho > \frac{\rho_c}{1 + \rho_c \ln(\tilde{E}_F/\tilde{\omega})}$$

the kernel $Q = Q_c + Q_{ph}$ may be negative everywhere but the superconducting state exist.

Thus the negative sign of Q is not sufficient for the absence of superconductivity in contradiction to the basic statement of the theory of BCS.

The above qualitative considerations may certainly be surely refined on the basis of Eqs (5.I6) and (5.I7). However completely convincing quantitive results may be expected only on the basis of explicitly taking into account the crystalline structure of the metal (this is now under investigation).

6. FERMI-SYSTEMS WITH WEAK INTERACTION[x)]

6.I The Formulation of the Bardeen-Cooper-Schrieffer Theory

In the preceding sections we considered the electron-phonon interaction only by means of Frohlich's Hamiltonian. However, the principal results of the treatment with Frohlich's Hamiltonian might have been obtained by operating with a model Hamiltonian from which the phonons are eliminated.

So, Bardeen, Cooper, Schrieffer[5], in agreement with the earlier results of Bardeen and Pines[18], start directly from a Hamiltonian in which the interaction between electrons and phonons is replaced by a direct electron-electron interaction. In terms of our notations their original Hamiltonian is

$$H = \sum_{k,s} E(k) \overset{+}{a}_{ks} a_{ks} + H'$$

where

$$H' = -\frac{1}{2V} \sum_{s_1,s_2} \sum_{\substack{k_1,k_2,k_1',k_2' \\ k_1+k_2=k_1'+k_2'}} \frac{g^2(k_1'-k_2)\,\omega^2(k_1'-k_1)}{\omega^2(k_1'-k_1)-(E(k_1)-E(k_1'))^2} \overset{+}{a}_{k_2's_2} a_{k_2s_2} \overset{+}{a}_{k_1's_1} a_{k_1s_1} \qquad (6.I)$$

Bardeen, Cooper and Schrieffer carried out a further simplification of this Hamiltonian. As a fundamental approximation they omitted in Eq. (6.I) all the terms, which lead to the destruction of a pair of particles with opposite momenta $\pm k$ and spins $\pm\frac{1}{2}$. The Hamiltonian thus obtained is:

$$H'_{red} = -\frac{1}{V} \sum_{k,k'} \frac{g^2(k-k')\,\omega^2(k-k')}{\omega^2(k-k')-(E(k)-E(k'))^2} \overset{+}{a}_{-k',-} a_{-k,-} \overset{+}{a}_{k',+} a_{k,+} \qquad (6.2)$$

The treatment of the Hamiltonian (6.2) by means of a variational principle forms the essential part of the quoted paper.

The consistency of the choice of the original Hamiltonian (6.2) as well as the correctness of the approximation (6.2) is insufficiently investigated in the mentioned paper. In the present section we shall show that, as long as one deals with the energy of the ground state or the Fermi branch of the elementary excitation spectrum, such a reduction of Frohlich's Hamiltonian to a model of the type (6.2) is in fact valid. According to our calculations, however, the model Hamiltonian is to be chosen in a slightly different way, namely

$$H'_{red} = -\frac{1}{V} \sum_{k,k'} \frac{g^2(k'-k)\,\omega(k'-k)}{\omega(k-k')+|E(k')-E_F|+|E(k)-E_F|} \overset{+}{a}_{-k',-} a_{-k,-} \overset{+}{a}_{k',+} a_{k,+} \qquad (6.3)$$

---

x) This section is based on investigations carried out by V.V. Tolmachov.

Matters are much more involved, when the special Boson branch of the elementary excitations spectrum of Frohlich's Hamiltonian, connected with the collective effects of the electron-phonon interaction is investigated. For it the mentioned reduction no longer occurs. More exactly, in this case we can still use a Hamiltonian of the type (6.I), but the Bardeen-Cooper -Schrieffer approximation, which leads us to Hamiltonians of types (6.2) or (6.3) is no longer applicable.

The fact that the phonon operators may be excluded from the Fröhlich Hamiltonian is not astonishing. Indeed, by means of Feynman's procedure, well known in Quantum Field Theory, we can always carry out a functional integration over the virtual phonons and arrive at a fourth-order form in the Fermi amplitudes of electron states. Such a fourth-order form, however, would be non-local, since it would contain another time integration. Physically this means that the fourth-order form would automatically inolude retardation effects of the electron-phonon interaction.

A Hamiltonian with electron-electron interaction of the type (6.I) is only an approximation to the mentioned fourth-order form including the retardation effects, when the latter are neglected. From the point of view of energetical relations this means that one may use the local Hamiltonian only for the calculation of excitation energies, small compared with the average energy of the interaction-transmitting agent.

Exactly such a situation arises in the calculation of the energy of the superconducting ground state. As established in the preceding sections, in this case the difference between the energies of the normal and the superconducting states is small compared with the average energy of a phonon x). When calculating the Fermi branch of elementary excitations one also may neglect the effects of retardation. Indeed, in the preceding sections we found that the influence of the interaction on this branch is limited to energies much smaller than the average energy of a phonon. At higher energies the Fermi branch turns into the usual Fermi excitations of an ideal gas. Thus, for the evaluation of the influence of interaction on the Fermi branch of the spectrum it may be admitted that phonons possess infinite energies and therefore the effects of retardation may be neglected.

The retardation effects also have no influence on the special Boson branch of Frohlich's Hamiltonian, since all the Boson excitations possess sufficiently small energies, much less than the average phonon energies.

So for the calculation of the energy of the ground state of the Fermi branch and the special Bose branch of the spectrum of elementary excitations we may replace Fröhlich's Hamiltonian by one of the type (6.I) with a direct electron-electron interaction. We emphasize

---

x) We are entitled to speak about the average energy of a phonon, since (as may be seen from a detailed consideration of Frohlich's Hamiltonian) phonons of all possible frequencies have the same importance in the effects of the electron-phonon interaction.

once more that here we speak about the general form of a local four-field interaction and not about a simpified one as (6.2) or (6.3). As for the latter simplified form, it does not include collective interaction effects and does not exhibit the special Bose branch of elementary excitations.

6.2. The compensation equation. In order to take into acoount this collective interact ion we shall from the start consider the Hamiltonian

$$H = \sum_{k,s} (E(k)-\lambda)\, a^{+}_{ks}\, a_{ks} + H' \tag{6.4}$$

$$H' = -\frac{1}{2V} \sum_{s_1,s_2} \sum_{\substack{k_1,k_2,k_1',k_2' \\ k_1+k_2=k_1'+k_2' \quad k_1 \neq k_1'}} J(k_1',k_2';k_1,k_2)\, a^{+}_{k_2',s_2}\, a_{k_2,s_2}\, a^{+}_{k_1',s_1}\, a_{k_1,s_1}$$

where, as in the preceding sections, the parameter λ, introduced into the Hamiltonian plays the role of a chemical potential. Considering the potential of interaction of two particles as invariant with respect to a transposition of both particles and also invariant under space reflections we obtain that $J(k_1',k_2';k_1,k_2)$ is a real function and has the properties:

$$J(k_1',k_2';k_1,k_2) = J(k_1,k_2;k_1',k_2')$$

$$J(k_1',k_2';k_1,k_2) = J(-k_1',-k_2';-k_1,-k_2)$$

$$J(k_1',k_2';k_1,k_2) = J(k_2',k_1';k_2,k_1)$$

Now as in the case of Frohlich's Hamiltonian we go over to new Fermi amplitudes by means of the same canonical transformation. In the present section it will be, however, convenient to write it in a somewhat different form:

$$a_{2sk,s} = u_k \alpha_{k,-s} + 2s v_k \alpha^{+}_{k,s} \tag{6.5}$$

$$u_k^2 + v_k^2 = 1 \qquad u_k, v_k \text{ real}$$

(For comparison with the preceding sections we remark that $\alpha_{k,-\frac{1}{2}} \equiv \alpha_{k0}$, $\alpha_{k,\frac{1}{2}} \equiv \alpha_{k1}$). The transformed Hamiltonian will be

$$H = U + H_0 + H' + H'',$$

where

$$U = 2 \sum_k (E(k)-\lambda) v_k^2$$

$$H_0 = \sum_{k,s} (E(k)-\lambda)(u_k^2 - v_k^2)\, \alpha^{+}_{k,s}\, \alpha_{k,s}$$

$$H' = -\frac{1}{2V} \sum_{s_1,s_2} \sum_{\substack{k_1,k_2,k_1',k_2' \\ k_1+k_2=k_1'+k_2' \quad k_1 \neq k_1'}} J(2s_1k_1, 2s_2k_2; 2s_1k_1', 2s_2k_2')\,(u_{k_1'}\alpha^{+}_{k_1',-s_1} + 2s_2 v_{k_1'}\alpha_{k_1',s_1})(u_{k_2}\alpha_{k_2,-s_2} + 2s_2 v_{k_2}\alpha^{+}_{k_2,s_2}) \times$$
$$\times (u_{k_1'}\alpha^{+}_{k_1',-s_1} + 2s_1 v_{k_1'}\alpha_{k_1',s_1})(u_{k_1}\alpha_{k_1,-s_1} + 2s_1 v_{k_1}\alpha^{+}_{k_1,s_1})$$

$$H'' = \sum_{k,s} (E(k)-\lambda) 2s\, u_k v_k\, (\alpha^{+}_{k,s}\alpha^{+}_{k,-s} + \alpha_{k,-s}\alpha_{k,s})$$

Applying to this Hamiltonian the principle of compensation of "dangerous" diagrams in order to forbid the creation of a pair of fermions $\alpha^{+}_{k,-\frac{1}{2}}\ \alpha^{+}_{k,\frac{1}{2}}$ out of vacuum, we obtain the following equation for u_k, v_k :

$$2(E(k)-\lambda)u_k v_k - \langle C_\nu^{*}\, \alpha_{k,\frac{1}{2}}\, \alpha_{k,-\frac{1}{2}}\, H'\, C_\nu \rangle = 0 \qquad (6.5)$$

where C_ν is the vacuum amplitude with zero occupation numbers $\alpha^{+}_{ks}\alpha_{ks}$. There will be no change in Eq. (6.5) if we transpose the spin indices + and = , since the original Hamiltonian is invariant under this transformation. From (6.5) we obtain

$$2\xi(k)u_k v_k = \frac{1}{V}\sum_{k'} J(k,-k\,;\,k',-k')\,u_{k'}v_{k'}\,(u_k^2 - v_k^2) \qquad (6.6)$$

where

$$\xi(k) = E(k)-\lambda - \frac{1}{2V}\sum_{k'} J(k',k\,;\,k,k')\,(u_{k'}^2 - v_{k'}^2) \qquad (6.7)$$

Introducing the new function

$$C(k) = \frac{1}{V}\sum_{k'} J(k,-k\,;\,k',-k')\,u_{k'}v_{k'}$$

Eq. (6.6) may be transformed into that for C(k)

$$C(k) = \frac{1}{V}\sum_{k'} J(k,-k\,;\,k',-k')\,\frac{C(k')}{2\tilde{\varepsilon}(k')} \qquad (6.8)$$

where

$$\tilde{\varepsilon}(k) = \sqrt{C^2(k)+\xi^2(k)}$$

Besides we obtain

$$u_k^2 = \frac{1}{2}\left\{1+\frac{\xi(k)}{\tilde{\varepsilon}(k)}\right\} \qquad v_k^2 = \frac{1}{2}\left\{1-\frac{\xi(k)}{\tilde{\varepsilon}(k)}\right\} \qquad (6.9)$$

$$u_k v_k = \frac{C(k)}{\tilde{\varepsilon}(k)}$$

The equation always admits the trivial solution C(k) = 0 and correspondingly

$$u_k = \begin{cases} 0 & E(k)<\lambda \\ 1 & E(k)>\lambda \end{cases} \qquad v_k = \begin{cases} 1 & E(k)<\lambda \\ 0 & E(k)>\lambda \end{cases}$$

This solution describes not the superfluid state but a normal one. Apart from this trivial solution Eq. (6.8) can have another nontrivial solution, which leads to the superfluid state.

For simplicity we restrict ourselves to the consideration of the case of spherically symmetrical solutions of the equation (6.8). Then, replacing in the latter the sum by an integral, we obtain

$$C(k) = \int_0^\infty K(k,k')\,\frac{C(k')\,k'^2\,dk'}{\sqrt{C^2(k')+\xi^2(k')}} \qquad (6.9)$$

where

$$K(k,k') = \frac{1}{2(2\pi)^2}\int_{-1}^{+1} J(k,-k;k',-k')dt \equiv \frac{1}{2(2\pi)^2}\int_{-1}^{+1} J(|k|,|k'|,\sqrt{|k|^2+|k'|^2-2|k||k'|t})\,dt \quad (6.10)$$

As has been mentioned above, the equation (6.9) has nontrivial solutions for certain $K(k,k')$. In order to write down the condition on $K(k,k')$ for the existence of such a solution, we proceed as follows. Let the kernel $K(k,k')$ vary continuously from the form for which the nontrivial solution of the equation (6.9) is absent to that for which it is present. Because of the continuous dependence of the nontrivial solution on the form of the kernel, during the change of the form of $K(k,k')$, the nontrivial solution must smoothly depart from the trivial solution. Therefore it is sufficient to restriot our attention to the following equation, instead of the old one (6.9):

$$C(k) = -2\ln\frac{C(k_F)}{E'(k_F)}K(k,k_F)k_F^2\frac{C(k_F)}{E'(k_F)} - \int_0^\infty \frac{d}{dk'}\left[K(k,k')k'^2C(k')\frac{k'-k_F}{|\xi(k')|}\right]\ln 2|k'-k_F|\,dk' \quad (6.11)$$

The right hand side of this equation coincides asymptotically, for small C with that of Eq.(6.9).

Introduce now a new function

$$f(k) = -\frac{C(k)}{C(k_F)\ln\frac{C(k_F)}{E'(k_F)}}$$

The inverse relation is

$$C(k) = f(k)\frac{E'(k_F)}{f(k_F)}e^{-\frac{1}{f(k_F)}}$$

The solution $C(k)$ will be close to zero if $f(k_F)$ goes to zero through positive values. The equation for $f(k)$ is

$$\frac{E'(k_F)}{k_F^2}f(k) = 2K(k,k_F) - \int_0^\infty \frac{d}{dk'}\left[K(k,k')f(k')\frac{k'^2E'(k_F)}{k_F^2}\frac{k'-k_F}{|\xi(k')|}\right]\ln 2|k'-k_F|\,dk' \quad (6.12)$$

It should be noted that the equation (6.12) is a linear integral equation.

For a certain form of the kernel $K(k,k')$ the solution of Eq. (6.12) will be such that $f(k_F)>0$. At the same time the nonlinear equation (6.9) will have a nontrivial solution.

For another form of the kernel $K(k,k')$ the solution $f(k)$ will be such that $f(k_F)<0$ and Eq. (6.9) will have no nontrivial solution. Thus, the condition for the existence of a nontrivial solution will be

$$f(k_F; K(k,k')) > 0 \quad (6.13)$$

where the second argument denotes a functional dependence of on the kernel $K(k,k')$.

Notice that the oriterion for superconductivity in presence of the Coulomb interaction,

obtained in subsection 4.3, may be derived without difficulties from the condition (6.I3). Let us discuss qualitatively, what is the form of interaction $K(k,k')$ favouring the appearance of a nontrivial solution of the equation (6.9). First, for positive and sufficiently small $K(k_F, k_F)$ (corresponding to an attraction between electrons in the neighbourhood of the Fermi sphere) $f(k_F)$ will be also small and of the order of $K(k_F,k_F)$, and consequently, the second term in (6.I2) will be of higher order of smallness as compared with the first. The system will be in the superfluid state.

Another case when a nontrivial solution exists, is the case of an interaction which is localized on the Fermi sphere. In this case the second term of the right hand side of (6.I2) will contain a small parameter which represents the ratio between the length of localization of interaction and the radius of the Fermi sphere. The system turns out again to be superfluid.

Notice that the superfluidity may be proper to Fermi-systems with negative kernels $K(k',k)$ corresponding to repulsive forces. It is only necessary that there exists a domain in k-space where the kernel varies rapidly. Then in this domain the derivative which enters the integral term of the right hand side of (6.I2) will be large, and the positive second integral term may exceed the negative first term.

Without considering in detail such nonregular interactions it should be noted that the superfluidity is chiefly proper to Fermi systems with predominance of attractive forces. In the microscopic theory of the superfluidity of Bose systems[7]) it has been shown that for the appearance of superfluidity in such systems the opposite situation, namely the predominance of repulsive forces, is necessary.

Thus, the criteria for superfluidity of Bose- and Fermi systems exclude each other. This circumstance is in a good agreement with the fact that a system like He_3 is not superfluid. Indeed, it is hard to believe that the intermolecular forces in He_3 are essentially different from those in He_4. The latter being a Bose system is superfluid.

Let us return now to the equation (6.II). For small C it has the following approximate solution

$$C(k) = \omega \frac{K(k,k_F)}{K(k_F,k_F)}\, e^{-\frac{1}{\rho}} \qquad (6.I4)$$

where

$$\rho = \frac{2k_F^2\, K(k_F,k_F)}{E'(k_F)}$$

$$\ln \omega = -\frac{1}{2}\int_0^\infty \frac{d}{dk'}\left[\left(\frac{K(k_F,k')}{K(k_F,k_F)}\right)^2 \frac{k'^2}{k_F^2}\,\frac{(k'-k_F)\,E'(k_F)}{|E(k')-E(k_F)|}\right] \ln 2E'(k_F)|k'-k_F|\,dk' \qquad (6.I5)$$

These formulae describe the solution of the nonlinear integral equation (6.II) the better, the closer we are to the point of appearance of the nontrivial solution. From the direct consideration of Eqs.(6.I4), (6.I5) we see, that for the asymptotic form of the superconductive solution only a comparatively small part $J(k,-k;k',-k')$ of the total form $J(k_1,k_2;k_1',k_2')$ is essential. Thus, we should obtain exactly the same formulae, if in the equation (6.8) we put from the outset

$$\tilde{\varepsilon}(k) = \sqrt{C^2(k) + (E(k)-E_F)^2} \tag{6.I6}$$

or, what is absolutely equivalent, if we started not from the Hamiltonian (6.4), but from the simplified one:

$$H'_{red} = -\frac{1}{V}\sum_{\substack{k,k' \\ k\neq k'}} J(k,-k;k',-k')\, a^{+}_{-k',-}\, a_{-k,-}\, a^{+}_{k',+}\, a_{k,+} \tag{6.I7}$$

In addition, Eqs.(6.8), (6.I6) completely coincide with the corresponding equations for Fröhlich's Hamiltonian, if we put

$$J(k,-k;k',-k') = \frac{\tilde{g}^2(k-k')\,\tilde{\omega}(k-k')}{\tilde{\omega}(k-k') + |E(k)-E(k_F)| + |E(k')-E(k_F)|} \tag{6.I8}$$

Thus the reduction of Fröhlich's Hamiltonian to the simplified model Hamiltonian (6.I7),(6.I8) is justified with respect to the function C and the quantities conneoted with it.

The above mentioned reduction may be carried out also for Fröhlich's Hamiltonian with Coulomb interaction. Indeed, according to §4.3 for that purpose it is only necessary to put

$$\frac{1}{V} J(k,-k;k',-k') = Q(k,k') \tag{6.I9}$$

A calculation of the ground state energy gives

$$2\sum_k (E(k)-\lambda)v_k^2 + \langle \vec{C}_\nu H' C_\nu \rangle =$$

$$= 2\sum_k (E(k)-\lambda)v_k^2 - \frac{1}{V}\sum_{\substack{k,k' \\ k\neq k'}} J(k',k;k,k')\, v_k^2 u_{k'}^2 - \frac{1}{V}\sum_{\substack{k,k' \\ k\neq k'}} J(k,-k;k',-k')\, u_k v_k u_{k'} v_{k'}$$

For the case of the Hamiltonian (6.I7) this formula acquires a very simple form. It may be transformed into

$$\sum_k \{E(k) - E(k_F) - \tilde{\varepsilon}(k)\}$$

where $\tilde{\varepsilon}(k)$ is given by Eq. (6.I6). For the difference between the energies of the normal and the superconducting states we obtain the following expression

$$E_n^N - E_s^N = \sum_k \{\tilde{\varepsilon}(k) - E(k) + E(k_F)\}$$

which coincide with the analogous expression for Fröhlich's Hamiltonian, given in § 2.3. There only remains to calculate the energy of the elementary excitation of the Fermi branch of the spectrum. This may be carried out by means of the following formula:

$$E_e(k) = (E(k)-\lambda)(u_k^2 - v_k^2) + \langle C_v^* \alpha_{k,\frac{1}{2}} H' \alpha^+_{k,\frac{1}{2}} C_v \rangle \qquad (6.20)$$

For the calculation of the vacuum expectation value in this formula, one should keep in mind that $\alpha_{k,\frac{1}{2}}$ is not to be paired with $\alpha^+_{k,\frac{1}{2}}$. Notice, further, that if we replaced in Eq.(6.20) the spin index $\frac{1}{2}$ by $-\frac{1}{2}$ we would, certainly, obtain the same result, because of the invariance of the Hamiltonian (6.4) under such transformations. After some manipulations on (6.20) we find:

$$E_e(k) = \xi(k)(u_k^2 - v_k^2) + \frac{2}{V}\sum_{k'} J(k,-k;\, k',-k')\, u_k v_k\, u_{k'} v_{k'}$$

and by the use of (6.9) this formula may be transformed into

$$E_e(k) = \tilde{\varepsilon}(k) \qquad (6.21)$$

Thus, the quantity $\tilde{\varepsilon}(k)$ we have introduced is, in fact, the energy of an elementary excitation. For the case of the Hamiltonian (6.17) Eq. (6.21) may be written in the form:

$$E_e^s(k) = \sqrt{(E_e^n(k))^2 + C^2(k)} \qquad (6.22)$$

which is entirely analogous to the relation for Frohlich's Hamiltonian in § 2.2. We have investigated (6.4) by means of the principle of compensation of "dangerous" graphs, restricting ourselves to the first order of perturbation theory and using this principle in the form (6.5). It may be shown, however, that the taking into aocount of higher orders of perturbations theory adds nothing new in principle to the equation for C(k). This situation was demonstrated in detail in a paper by S.V. Tyablikov and one of athors/19/.

Indeed, to the second order the principle of compensation of "dangerous" graphs acquires the form.

$$2(E(k)-\lambda)u_k v_k - \langle C_v^* \alpha_{k,\frac{1}{2}} \alpha_{k,-\frac{1}{2}} (H' - H' H_0^{-1} H') C_v \rangle_{comp} \qquad (6.23)$$

The suffix "comp" denotes that besides the graphs of first order drawn in Fig. 12, it is necessary to take care only of the second order graphs, Fig. 13 .

Fig. 12

Fig. 13

These describe the creation of four particles from the vacuum and the subsequent turning of three particles into one. The second or graphs, of fig. I4, do not need any special compensation, since, because of the stated compensation rule, they will be automatically compensated by the third order graphs of the same figure.

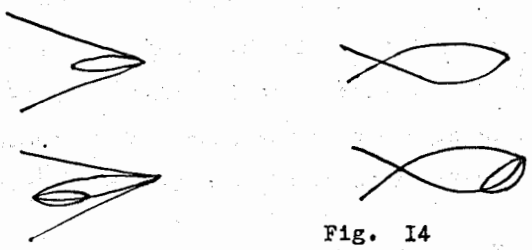

Fig. I4

The equations (6.23) yield, restricting ourselves for simplicity, to the case of a kernel $J(k_1,k_2;k_1',k_2')$ localized and constant in the neibourghood of Fermi-shell:

$$2\left\{\xi(k)-\left(\frac{J}{V}\right)^2\sum_{\substack{k_1,k_2,k_3\\ k_2-k_1+k_3=k}}\frac{\{u_{k_1}^2v_{k_2}^2-(u_{k_1}v_{k_1})(u_{k_2}v_{k_2})\}(u_{k_3}^2-v_{k_3}^2)}{|\xi(k)|+|\xi(k_1)|+|\xi(k_2)|+|\xi(k_3)|}\right\}u_kv_k=$$

$$=(u_k^2-v_k^2)\left\{\frac{J}{V}\sum_{k_1}u_{k_1}v_{k_1}+\left(\frac{J}{V}\right)^2\sum_{\substack{k_1,k_2,k_3\\ k_2-k_1+k_3=k}}\frac{u_{k_1}v_{k_1}\{(u_{k_1}v_{k_1})(u_{k_2}v_{k_3})-u_{k_2}^2v_{k_3}^2\}}{|\xi(k)|+|\xi(k_1)|+|\xi(k_2)|+|\xi(k_3)|}\right\}$$

where $\xi(k)=E(k)-E(k_F)$

Hence the corrective terms with J^2 are in fact somewhat smaller since they contain the products uv with the same index. The latter lead to an exponentially small contribution and, as may be seen, do not change the asymptotic quantities obtained above.

6.3. Collective excitations. The influence of Coulomb interaction.

Let us proceed now to the investigation of the collective branch of the Hamiltonian (6.4). Considerations analogous to those of subsection 3.2 for the collective branch of Fröhlich's Hamiltonian point to the necessity of diagonalizing the quadratic form

$$\Gamma=\sum_{k,p}\left\{\tilde{\varepsilon}(k+p)+\tilde{\varepsilon}(k)\right\}\beta_p^+(k)\beta_p(k)+\Gamma' \tag{6.24}$$

where

$$\Gamma'=\sum_{\substack{k,k',p\\ k\neq k'}}\beta_p^+(k)\beta_p(k')A_p(k,k')+\frac{1}{2}\sum_{\substack{k,k',p\\ k\neq k'}}\beta_p(k)\beta_{-p}(k')B_p(k,k')+\frac{1}{2}\sum_{\substack{k,k',p\\ k\neq k'}}\beta_{-p}^+(k')\beta_p(k)B_p(k,k') \tag{6.25}$$

In turn the coefficients $A_p(k,k')$ and $B_p(k,k')$ are connected with the matrix elements of the original Hamiltonian by means of the relations:

$$A_p(k,k') = \langle C_\nu^* \alpha_{k_1} \alpha_{k+p\sigma} H' \alpha^+_{k'+p\sigma} \alpha^+_{k'_1} C_\nu \rangle$$

$$B_p(k,k') = \langle C_\nu^* \alpha_{k'_1} \alpha_{k'-p\sigma} \alpha_{k_1} \alpha_{k+p\sigma} H' C_\nu \rangle \qquad (6.26)$$

for which, explicit expressions may be obtained in the same way as in subsection 3.2.

In the case of a Fermi gas with weak attraction we have in fact the same situation as in sec. 3. Repeating almost literally the considerations of that section we may prove the existence of collective excitations of different kinds, longitudinal as well as transversal. In the present section we shall discuss the influence of the Coulomb interaction between electrons on the collective excitations, investigated in the subsections 3.3, 3.4 for Frohlich's Hamiltonian. In order to get at least a rough idea about the situation, we restrict ourselves to the consideration of the Hamiltonian:

$$H = \sum_{k,\sigma} (E(k)-\lambda)\, a^+_{k\sigma} a_{k\sigma} + H'$$

$$H' = \frac{1}{2V} \sum_{\sigma_1,\sigma_2} \sum_{\substack{k_1,k_2,k'_1,k'_2 \\ k_1+k_2=k'_1+k'_2 \\ k_1 \neq k'_1}} \{-J(k_1,k'_1) + \nu(|k'_1-k_1|)\}\, a^+_{k'_2\sigma_2} a_{k_2\sigma_2} a^+_{k'_1\sigma_1} a_{k_1\sigma_1} \qquad (6.27)$$

the interaction $J(k,k')$ represents a direct attraction between electrons, which appears from the electron-phonon interaction and is localized in the neighbouhood of the Fermi surface. The interaction $\nu(|k'_1-k_1|)$ describes the Coulomb repulsion between the eleotrons.

Computing the matrix elements (6.26) for the Hamiltonian (6.27) we obtain

$$A_p(k,k') = \frac{1}{V}\{-J(k'+p,k+p)+\nu(|k'-k|)\}\, L(k,k')\, L(k+p,k'+p) +$$

$$+\frac{1}{V}\{-J(k,k+p)+\nu(|p|)\}\, M(k',k'+p)\, M(k,k+p)$$

$$B_p(k,k') = \frac{1}{V}\{-J(k',k'-p)+\nu(|p|)\}\, M(k,k+p)\, M(k',k'-p) -$$

$$-\frac{1}{V}\{-J(k,k'-p)+\nu(|k-k'|)\}\, M(k',k+p)\, M(k,k'-p)$$

These expressions are quite complicated. We can, however, consider the case of small p. In the terms which contain $\nu(|p|)$ one may retain this quantity only, since for small p the Coulomb $\nu(|p|)$ has a strong singularity. On the contrary the other terms $\nu(|k-k'|)$ may be included into the phonon interaction, since in the collective excitations, which are being considered, only large momentum transfers $k-k'$ are essential and in addition to this, as shown in subsection 5.3, in expressions, where the Coulomb repulsion is joined to the phonon attraction it is always screened.

Thus

$$A_p(k,k') = -\frac{J(k,k')}{V} L(k,k') L(k+p,k'+p) + \frac{\nu(p)}{V} M(k,k+p) M(k',k'+p) \qquad (6.28)$$

$$B_p(k,k') = \frac{J(k,k')}{V} M(k',k+p) M(k,k'-p) + \frac{\nu(p)}{V} M(k,k+p) M(k',k'-p)$$

The secular equations which correspond to (6.28) are of the form:

$$\left(\tilde{\varepsilon}(k) + \frac{\tilde{\varepsilon}(k+p)+\tilde{\varepsilon}(k-p)}{2}\right)\vartheta_p(k) - \frac{1}{V}\sum_{k'} J(k,k')\left\{L(k,k')\,\frac{L(k+p,k'+p)+L(k-p,k'-p)}{2} - \right.$$

$$\left. - \frac{M(k',k+p)M(k,k'-p) - M(k',k-p)M(k,k'+p)}{2}\right\}\vartheta_p(k') +$$

$$+ \frac{\nu(p)}{2V}\sum_{k'}\left\{M(k,k+p)M(k',k'+p) + M(k',k'-p)M(k,k-p) + M(k,k+p)M(k',k'-p) + M(k,k-p)M(k',k'+p)\right\}\vartheta_p(k') =$$

$$= \left(E - \frac{\tilde{\varepsilon}(k+p)-\tilde{\varepsilon}(k-p)}{2}\right)\theta_p(k) + \frac{1}{V}\sum_{k'} J(k,k')\left\{L(k,k')\,\frac{L(k+p,k'-p)-L(k-p,k'-p)}{2} + \right.$$

$$\left. + \frac{M(k',k+p)M(k,k'-p) - M(k',k-p)M(k,k'+p)}{2}\right\}\theta_p(k') -$$

$$- \frac{\nu(p)}{2V}\sum_{k'}\left\{M(k,k+p)M(k',k'+p) - M(k',k'-p)M(k,k-p) - M(k,k-p)M(k',k'-p) + M(k,k-p)M(k',k'+p)\right\}\theta_p(k') \qquad (6.29)$$

$$\left(\tilde{\varepsilon}(k) + \frac{\tilde{\varepsilon}(k+p)+\tilde{\varepsilon}(k-p)}{2}\right)\theta_p(k) - \frac{1}{V}\sum_{k'} J(k,k')\left\{L(k,k')\,\frac{L(k+p,k'+p)+L(k-p,k'-p)}{2} + \right.$$

$$\left. + \frac{M(k',k+p)M(k,k'-p) + M(k',k-p)M(k,k'+p)}{2}\right\}\theta_p(k') +$$

$$+ \frac{\nu(p)}{2V}\sum_{k'}\left\{M(k,k+p)M(k',k'+p) + M(k',k'-p)M(k,k-p) - M(k,k+p)M(k',k'-p) - M(k',k'+p)M(k,k-p)\right\}\theta_p(k') =$$

$$= \left(E - \frac{\tilde{\varepsilon}(k+p)-\tilde{\varepsilon}(k-p)}{2}\right)\vartheta_p(k) + \frac{1}{V}\sum_{k'} J(k,k')\left\{L(k,k')\,\frac{L(k+p,k'+p)-L(k-p,k'-p)}{2} - \right.$$

$$\left. - \frac{\nu(p)}{2V}\sum_{k'}\left\{M(k,k+p)M(k',k'+p) - M(k',k'-p)M(k,k-p) + M(k,k+p)M(k',k'-p) - M(k,k-p)M(k',k'+p)\right\}\vartheta_p(k')\right. \qquad (6.30)$$

These secular equations differ from Eqs. (3.10), (3.11) by additional terms with the Coulomb interaction.

From the structure of these terms it may be observed directly that the so called, transversal waves of subsection 4.4 turn these terms into zero. Thus, we reach the important conclusion, that the influence of the Coulomb interaction is reduced only to the modification of the effective interaction.

The situation with the longitudal collective excitations is somewhat more complicated.

Taking into account the Coulomb interaction explicitly, these excitations are modified in such a manner that there appears an ordinary plasma branch of the collective excitations. One may easily prove that this is true, by retaining in the secular equations (6.29), (6.30) only those terms, which are essential at sufficiently large p, when u_k, v_k may be replaced by their normal values:

$$u_k = \theta_F(k) \qquad v_k = \theta_F(k)$$

Besides it is suffioient to consider in (6.29), (6.30) only the terms with the Coulomb interaction. For this sake it is convenient to proceed from the functions ϑ, θ to the original φ, χ

They will satisfy the equations:

$$\left\{\tilde{\varepsilon}(k)+\tilde{\varepsilon}(k+p)-E\right\}\varphi_p(k) = \frac{v(p)}{V} M(k,k+p)\sum_{k'} M(k',k'+p)\varphi_p(k') + \frac{v(p)}{V} M(k,k+p)\sum_{k'} M(k',k'-p)\,\chi_p(k') \tag{6.31}$$

$$\left\{\tilde{\varepsilon}(k)+\tilde{\varepsilon}(k-p)+E\right\}\chi_p(k) = \frac{v(p)}{V} M(k,k+p)\sum_{k'} M(k',k'-p)\chi_p(k') + \frac{v(p)}{V} M(k,k-p)\sum_{k'} M(k',k'+p)\varphi_p(k') \tag{6.32}$$

which may be easily solved, and lead us to the following equation for the definition of E:

$$1 = \frac{2v(p)}{V}\left\{\sum_{\substack{k \\ |k|<k_F \\ |k-p|>k_F}} \frac{1}{E-E(k-p)+E(k)} + \sum_{\substack{k \\ |k|<k_F \\ |k+p|>k_F}} \frac{1}{-E-E(k+p)+E(k)}\right\} \tag{6.33}$$

whioh corresponds to the Sawada – Brout's plasma secular equation I3 .

For momenta p , smaller than those, corresponding to the energy gap Δ , we can no longer replace u_k, v_k by their normal values and the nontrivial behaviour of u_k, v_k beoomes essential. Here, physically, we encounter a very interesting dispersion of the plasma frequency at small p . This dispersion turns out to be a peculiar property of a superconductor and may be used as a basis for the experimental definition of the quantity C which enters the theory.

§ 7 CONCLUSION

7.I. The Thermodynamics of the Superconducting State.

Up to this point we considered only the ground state and elementary excitations. Now we want to analyse the thermodynamical aspects of our model. Let us first note that collective excitations are not important in this case. Due to the smallness of the maximal momentum of collective excitations its contribution to the thermodynamical functions (in the absence of current and magnetic field) may be neglected.

Indeed, the effective volume of fermion excitations in momentum space is proportional to

$$k_F c \Delta$$

(c -is the sound velocity on the Fermi surface). The corresponding volume for collective excitations is of the smaller order

$$\Delta^3/c^3$$

So, for thermodynamical purposes, one may use a Hamiltonian in which only pair interactions have been retained. Such a Hamiltonian, as shown above, gives a correct description of the ground state and the elementary excitations of one-fermion type. An interesting feature of this Hamiltonian is that the free energy may be evaluated exactly. This calculation was performed by Zubarev, Tserkovnikov and one of us (20). From their result one can get Bardeen, Cooper, and Schrieffer's formulae, originally obtained by means of a variational principle in the approximation when J is constant near the Fermi surface.

We reproduce briefly the mentioned calculus. Let us consider the Hamiltonian

$$H = H_0 + H_{int} \quad , \quad H_0 = \sum_{k,s} (E(k) - \lambda) a^+_{ks} a_{ks}$$

$$H_{int} = -\frac{1}{V} \sum a^+_{-k,-} a^+_{k,+} a_{k',+} a_{-k',-} J(k,k') \tag{7.I}$$

Here J is a real bounded function which prectically vanishes outside a certain finite range of the momenta k,k'.

As will be shown, one can construct the thermodynamic potential

$$\psi = F - \lambda \mathcal{N} = -\theta \ln Sp\, e^{-\frac{H}{\theta}} \tag{7.2}$$

which is asymptotically exact for $V \to \infty$.

Performing our canonical transformation (2.4) we obtain

$$H = H^{(0)} + H' \quad , \quad H^{(0)} = \tilde{V} + \sum_k H_k$$

$$H' = -\frac{1}{V} \sum J(k,k') B_k B_{k'} \tag{7.2}$$

where

$$\tilde{V} = const = 2\sum_{k}(E(k)-\lambda)v_k^2 - \frac{1}{V}\sum_{k,k'}J(k,k')u_k v_k u_{k'} v_{k'}$$

$$H_k = \left\{(E(k)-\lambda)(u_k^2-v_k^2) + 2u_k v_k \sum_{k'}\frac{J(k,k')}{V}u_{k'}v_{k'}\right\}(\overset{+}{\alpha}_{k0}\alpha_{k0} + \overset{+}{\alpha}_{k1}\alpha_{k1}) +$$

$$+\left\{2(E(k)-\lambda)u_k v_k - (u_k^2-v_k^2)\sum_{k'}\frac{J(k,k')}{V}u_{k'}v_{k'}\right\}(\overset{+}{\alpha}_{k0}\overset{+}{\alpha}_{k1} + \alpha_{k1}\alpha_{k0})$$

$$B_k = u_k v_k\left(\overset{+}{\alpha}_{k0}\alpha_{k0} + \overset{+}{\alpha}_{k1}\alpha_{k1}\right) - u_k^2\alpha_{k1}\alpha_{k0} + v_k^2\overset{+}{\alpha}_{k0}\overset{+}{\alpha}_{k1} \qquad (7.3)$$

The new operators $H_k, B_k, \overset{+}{B}_k$ all commute for different k.

Applying statistical perturbation theory to (7.2) we obtain after some transformations

$$\ln Sp\, e^{-\frac{H}{\theta}} - \ln Sp\, e^{-\frac{H^{(0)}}{\theta}} = \ln\left\{1 + \sum_{n\geq 1}\int_0^{1/\theta} dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{n-1}} dt_n\, \mathcal{U}_n\right\} \qquad (7.4)$$

where

$$\mathcal{U}_n = \frac{1}{V^n}\sum_{\substack{(k_1,\dots k_n)\\(k_1',\dots k_n')}} J(k_1,k_1')\dots J(k_n,k_n')\,\frac{Sp\left\{e^{-\frac{H^{(0)}}{\theta}}\tilde{B}_{k_1}(t_1)B_{k_1'}(t_1)\dots\tilde{B}_{k_n}(t_n)B_{k_n'}(t_n)\right\}}{Sp\left\{e^{-\frac{H^{(0)}}{\theta}}\right\}} \qquad (7.5)$$

$$B_k(t) = e^{-H^{(0)}t} B_k e^{H^{(0)}t} = e^{-H_k t} B_k e^{H_k t}$$

$$\tilde{B}_k(t) = e^{-H_k t}\overset{+}{B}_k e^{H_k t}$$

Now we use the supplementary statement that if

$$Sp\left(e^{-H_k/\theta}B_k\right) = 0 \qquad (7.6)$$

for all k, then each of the $\mathcal{U}_n$ tends to a finite limit as $V\to\infty$.

To prove this statement, note that if (7.6) holds, all the terms in the sum (7.5) for which there is at least one momentum k_q or k_q' which does not equal any of the rest of the k_j, k_j', drop out and really the sum (7.5) contains only the terms for which there are no more then n different among the $k_1, k_1', \dots k_n, k_n'$ momenta, and hence are proportional to V^n, yielding a finite value for $\mathcal{U}_n$. As both terms in the l.h.s of (8.6) are proportional to V (for $V\to\infty$) we obtain

$$\ln Sp\, e^{-H/\theta} \simeq \ln Sp\, e^{-\frac{H^{(0)}}{\theta}}$$

So we are led to the following expression for the thermodynamical potential of the mentioned type

$$\psi = U - \theta\sum_{k}\ln Sp\, e^{-\frac{H_k}{\theta}} \qquad (7.7)$$

So we have only to determine u_k, v_k from the condition (7.6) and then use (7.7).

Carrying this programme, we obtain

$$u_k v_k = \frac{C(k)}{2\Omega(k)} \frac{1 - e^{-\frac{\Omega(k)}{\theta}}}{1 + e^{-\frac{\Omega(k)}{\theta}}} \quad ; \quad C(k) = \frac{1}{V} \sum J(k,k') u_{k'} v_{k'}$$

$$\Omega(k) = \sqrt{(E(k)-\lambda)^2 + C^2(k)}$$

and hence $C(k)$must be determined from the equation

$$C(k) = \frac{1}{2V} \sum J(k,k') \, th \left(\frac{\Omega(k')}{2\theta} \right) \frac{C(k')}{\Omega(k')} \tag{7.8}$$

which always admits the trivial solution $C(k)=0$.

In explicit form (7.7), becomes

$$\psi = \sum_k \left\{ E(k) - \lambda + \frac{C^2(k)}{2\Omega(k)} th \frac{\Omega(k)}{2\theta} - \Omega(k) - 2\theta \ln \left(1 + e^{-\frac{\Omega(k)}{\theta}} \right) \right\}$$

Considering this expression as a function of $C^2(k)$ we have

$$\frac{\partial \psi}{\partial C^2(k)} = C^2(k) \frac{\partial \Omega(k)}{\partial C^2(k)} \left\{ \frac{\partial}{\partial \Omega(k)} \frac{1}{2\Omega(k)} th \frac{\Omega(k)}{2\theta} \right\} = \frac{C^2(k)}{4\theta} f\left(\frac{\Omega(k)}{\theta} \right)$$

where

$$f(x) = \frac{sh\, x - x}{2x^3 ch^2 \frac{x}{2}} > 0$$

Thus for $C^2 \neq 0$ ψ always has a lower value then for the trivial solution.

Hence the phase trensition will take place at the same tempereture, at which eq.(7.8) will have a nontrivial solution.

7.2. The Electrodynamics of the Superconducting State.

The question of obtaining the electrodynamical equations is more complicated. One must take into consideration that a systematic motion of electrons is always connected with a magnetic field and superconductors exhibit specific magnetic properties, e.g. the Meissner effect.

Restricting ourselves to the case of weak magnetic fields, we naturally lodk for a linear dependence of the current on the vector potential.

Two types of such dependences are known from phenomenological considerations, London's equations and Pippard's equations. London's equations are local, i.e. j(x) is determined by A(x) in the same point. In the more general Pippard equations the relation between j(x) and

A(x) is given by an integral formula.

It is easy to see without any calculation that in the linear approximation of our theory as a consequence of the spatial correlation between electrons, we shall obtain equations of the Pippard type.

The corresponding integral kernels must be smeared over a spatial region with linear dimensions characterized by the specifio correlation length of particles with opposite spins.

Here an essential difficulty appears because, in order to obtain the equations of electrodynamics one must take into account collective oscillations, especially transversal ones. On the other hand, and this is of the utmost importance, we must take into consideration the existence of boundaries since the spatial correlation between electrons amounts to 10^{-4} 10^{-5} cm, and the penetration depth of the magnetic field is of the order of 10^{-5}cm.

In order to clarify the statement about the range of spatial correlation, let us calculate the pair correlation function $F_2(x,x';\,1/2,-1/2)$ for electrons with opposite spins at absolute zero.

With the usual definition of the correlation function we have

$$F_2(x,x';\,1/2,-1/2) = \langle \psi^+_{1/2}(x)\,\psi^+_{-1/2}(x')\,\psi_{-1/2}(x')\,\psi_{1/2}(x)\rangle$$

Here

$$\psi_{1/2}(x) = \frac{1}{\sqrt{V}}\sum_k a_{k,1/2}\,e^{ikx} \qquad ; \qquad \psi_{-1/2}(x) = \frac{1}{\sqrt{V}}\sum_k a_{k,-1/2}\,e^{ikx}$$

are the second quantized wave functions. The average is taken with respect to the vacuum of the occupation numbers $\alpha^+_{k0}\alpha_{k0}$, $\alpha^+_{k1}\alpha_{k1}$. Expressing a_{ks} in terms of α_{k0} and α_{k1} we obtain

$$F_2(x-x';\,1/2,-1/2) = \frac{1}{V^2}\Big(\sum_k v_k^2\Big)^2 + \frac{1}{V^2}\Big|\sum_k u_k v_k\, e^{-ik(x-x')}\Big|^2 =$$

$$= \frac{1}{4}\left\{ n_0^2 + \frac{1}{V^2}\Big|\sum_k \frac{C(k)\,e^{-ik(x-x')}}{\sqrt{C^2(k)+J^2(k)}}\Big|^2\right\}.$$

where $n_0 = 2V^{-1}\sum_k v_k^2$ - is the eleotron density.

This expression shows that in the normal state where $C(k) = 0$ $(u_k v_k = 0)$ there is no correlation between electrons with opposite spins.

In the superconducting state u_k v_k essentially differs from zero only in a small neighbourhood Δk of the Fermi surface, with Δk defined by

$$|E(k+\Delta k) - E_F| \gtrsim C(k_F) \qquad \text{then} \qquad \Delta k \sim \frac{C(k_F)k_F}{E_F} \sim 10^{-4}\,cm^{-1}$$

and the uncertainty relation gives for the correlation length an amount

$$\Delta x \sim \frac{1}{\Delta k} \sim 10^{-4}\,cm$$

Note that the correlation function for electrons with parallel spins

$$F_2(x-x';\,1/2,\,1/2) = \sum_{k_1 k_2} v_{k_1}^2 v_{k_2}^2 \left(1 - e^{-i(k_1-k_2)(x-x')}\right)$$

is determined mainly by exchange effects and, practically, is the same for the normal and the superconducting states.

Neglecting boundary and collective oscillations effects, the Hamiltonian should have the form

$$H = \frac{1}{2m}\sum_{k,\delta}\left(k - \frac{e}{c}A\right)^2 + \mu H + H_{int}$$

with constant A and H . In this way one should be able to get equations of the London type with a non-linear dependence of the penetration depth on the magnetic field.

In order to improve the theory and to obtain not only qualitative information about the Pippard functions, a detailed investigation of the full Hamiltonian including interactions between any two particles, not only with antiparallel momenta and spins, is required, as well as the taking into aocount of the existence of the boundaries.

7.3. A Qualitative Picture of Superconductivity.

In conclusion let us say some words about the physical nature of the superfluid or superconducting state C_s . As α_k are the amplitudes for superpositions of particles and holes $(\pm k, \pm 1/2)$ the Fermi sphere expands in general. There appears a characteristic correlation between particles $(\pm k, \pm \frac{1}{2})$ and holes $(\pm k, \pm \frac{1}{2})$. One can present the picture in the following intuitive manner.

There is an attractive interaction both between particles $(\pm k, \pm \frac{1}{2})$ and holes $(\pm k, \pm \frac{1}{2})$. Then it is profitable from the point of view of interaction energy 'to dilute' the Fermi sea with holes $(\pm k, \pm \frac{1}{2})$. On the other hand the expansion is not profitable for the kinetic energy. The balancing of these two faotors leads to the lowest energy state.

In the normal state, using a sufficiently high approximation, one can always obtain the correlation between particles with momenta $k + q$, $-k+q$ but the value $q = 0$ presents no partioularities. In the state C_s on the contrary we have a gap. In connection with this it is clear, that for example an interaction term of the form

$$\frac{1}{V}\sum J(k,k')\, a^+_{k,+}\, a^+_{-k,-}\, a_{-k',-}\, a_{k',+}$$

which for $V \to \infty$ gives an infinitely small contribution in the normal state, plays an important role in the case of C_s .

Of oourse we are not allowed to simplify the picture too much and introduce the concept of bound pairs of particles. Indeed, taking this concept seriously and calculating the binding energy of this pair one obtains a quantity of the same order as the energy of interaction between different pairs.

In fact the system forms a bound ensemble of the same type as for Bose systems. If it is possible to make use of the terminology of quantum field theory which is not quite clear, but is nowadays in some sense "quasi-intuitive", then it is possible speak about virtual pairs and to consider C_s as a bound Bose condensate formed of such virtual pairs.

The analogy with Bose-systems may be continued. Indeed, besides excitations of one fermion type, which corresponds to the dropping out of the ensemble of single particles, there are also excitations of the ensemble as a whole.

The existence of correlations between the particles in momentum space naturally gives rise to a 'correlation cloud' in ordinary coordinate space.

This 'cloud' has an interesting structure in the case of the superconducting state. An electron with definite spin is surrounded by holes which effectively screen its charge in a range of the order of $10^{-\pi}$ cm. At much larger distances, of the order 10^{-4} cm, 10^{-5} cm, there is a weak predominance of electrons with opposite spins as a result of an attractive interaction.

The authors are much indebted to S.V. Tiablikov, Y.A. Tserkovnikov and D.N. Zubarev for clarifying discussions and helpful advice.

APPENDIX I. ON A CRITERION FOR SUPERFLUIDITY IN THE THEORY OF NUCLEAR MATTER x)

It was noted in Sec.6 that a system of Fermi-particles can possess superfluidity under certain conditions, which, roughly speaking, reduce to the predominance of attractive forces. So it is natural to consider the question fo superfluidity for nuclear matter.

This problem is complicated, due to the fact, that nuclear interactions are strong, so that perturbation theory expansions in powers of the interaction are not applicable. The most consistent method would be obtained by a generalization of the graph summation method of Sec.4 A rigorous realization of this program is sufficiently complicated and so we think it useful to consider a simplified model which allows exact integration.

In this theory of nuclear matter nucleons in the nuclear matter are considered approximately as free particles. The effect of the interaction reduces essentially to an effective alteration of dependence of the energy on the momentum of the nucleon. In nuclear matter the energy of the nucleon is not equal to $p^2/2M$, but is a function $E(p)$ which may be approximated in the usual way by an expression of the form

$$E(p) \sim \frac{p^2}{2M_{eff}} + V \quad , \quad V < 0$$

In this framework the ground state of nuclear matter is described by the wave function C_n which corresponds to the usual Fermi sphere, where all states with $E(p) < E_F$ are occupied, and all others are free.

In view of this fact, we consider a model dynamical system with the Hamiltonian

$$H = \sum_{k,\sigma} \{E(k) - E_F\} a^+_{k\sigma} a_{k\sigma} + \frac{1}{2V} \sum_{(k,k',\sigma)} J(k,k'/\sigma_1,\sigma_2,\sigma_2',\sigma_1') a^+_{k\sigma_1} a^+_{-k\sigma_2} a_{-k'\sigma_2'} a_{k'\sigma_1'} \qquad (A.I.1)$$

Here σ is a discrete index of spin and isotopic spin of the nucleon, E_F a parameter, which plays the role of a chemical potential; the subscript F is to remind us of the fact that in the normal state it is equal to the Fermi-energy, V is the volume of the system.

The model character of this Hamiltonian is due to the fact, that only interactions of particles with opposite momenta are taken into account here.

---

x) This Appendix is based on a paper (21)

It is easy to see that the interaction Hamiltonian H' is not effective in the state C_n. Indeed, calculating

$$\overline{\langle H'^2 \rangle_{C_n}}$$

we found it to be finite when $V \to \infty$, and the energy should be proportional to V in this limiting process[x)]

The model considers explicitly only the specific interactions which are effective in the special case of the "superfluid" state C_S. The "regular" part of the interaction is implicitly taken into account in the effective energy of the nucleon E(k).

Let us show that the state C_S can be found asymptotically exactly in the case $V \to \infty$. The conditions for its existense can also be obtained. It is convenient to introduce the abbreviation q for the pair $(k, -k)$; q and -q describe the same pair, the sum over q runs over different pairs. Now we need the new index $\rho = \pm 1$ in order to express k as (q, ρ). As a discrete index ρ and σ will be designated as $s = (\sigma, \rho)$. In the new notations the Hamiltonian under consideration, (A.I.I), takes the form

$$H = \sum_{q,s} \{ E(q) - E_F \} \hat{a}^+_{qs} a_{qs} +$$

$$+ \frac{1}{2V} \sum_{(q,q',\dots s,\dots)} I(q,q'|s_1,s_2,s_2',s_1') \hat{a}^+_{qs_1} a_{qs_2} a_{q's_2'} a_{q's_1'} \qquad \text{(A.I.2)}$$

Using a modification of the technique proposed by D.N. Zubarev and Iu.A. Tserkovnikov[20] (see also § 7.I), we introduce C - number functions $A_q(s_1, s_2)$ and rewrite the Hamiltonian (A.I.2) in the form

$$H = U_0 + H_0 + H_1$$

where

$$U = const = -\frac{1}{2V} \sum I(q,q'|s_1,s_2,s_2',s_1') A^*_q(s_1,s_2) A_q(s_1',s_2') \; ;$$

$$H_0 = \sum_q H_q \quad ; \quad H_1 = \frac{1}{2V} \sum I(q,q'|s_1,s_2,s_2',s_1') \hat{B}^+_q(s_1,s_2) B_{q'}(s_1',s_2')$$

and

$$H_q = (E(q) - E_F) \sum_s \hat{a}^+_{qs} a_{qs} + \frac{1}{2V} \sum \{ I(q,q'|s_1,s_2,s_2',s_1') A_{q'}(s_1',s_2') \hat{a}^+_{qs_1} \hat{a}^+_{qs_2} +$$
$$+ I(q',q|s_1,s_2,s_2',s_1') A^*_{q'}(s_1,s_2) a_{qs_2'} a_{qs_1'} \}$$

$$B_q(s_1,s_2) = a_{qs_2} a_{qs_1} - A_q(s_1,s_2) \qquad \text{(A.I.3)}$$

---

x) It can be shown, that applying perturbation theory to (A.I.I) with C_n as a solution of the unperturbed equation one finds that the corrections of any order are infinitely small in the limit of $V \to \infty$

As H_q is quadratic in the Fermi-operators its diagonalization is achieved elementarily by the linear canonical transformation

$$a_{qs} = \sum_{s'} \{ u(q,s,s')\alpha_{qs'} + v(q,s,s')\alpha^{+}_{qs'} \} \qquad \text{(A.I.4)}$$

The functions U,V must satisfy orthonormality conditions of the form

$$\mathcal{I} \equiv \sum_{s''} \{ u^{*}(q,s,s'')u(q,s',s'') + v^{*}(q,s,s'')v(q,s',s'') \} = \delta_{ss'}$$
$$\eta \equiv \sum_{s''} \{ u(q,s,s'')v(q,s',s'') + v(q,s,s'')u(q,s',s'') \} = 0 \qquad \text{(A.I.5)}$$

After determining U,V from the secular equations corresponding to (A.I.3) the expression for H_q takes the form

$$H_q = \Gamma_q + \sum_{s} \varepsilon_s(q)\alpha^{+}_{qs}\alpha_{qs}$$

From the latter it is obvious that the ground state C_0 of the Hamiltonian H_0 is a vacuum state for the new Fermi-amplitudes

$$\alpha_{ks} C_0 = 0$$

We choose the c-number functions A in such that

$$\langle C^{*}_0 B_q(s_1,s_2) C_0 \rangle = 0$$

and notice also that H_0, B_q, B^{*}_q with different q commute one with another. Using an argument of[20] it can be shown that the contribution to the ground state energy of H_I is negligibly small as compared to the contribution of $V_0 + H_0$ in the limiting case $V \to \infty$. Roughly speaking this fact is due to the finite character of H_I^2 in the limit $V \to \infty$ whereas the energy is proportional to V.

So with a corresponding choice of the functions U,V the expectation value $\bar{H} = \langle C^{*}_0 H C_0 \rangle$ gives an asymptotically exact expression for the ground state energy of the Hamiltonian under consideration.

This consideration provides a recipe for the practical definition of the functions U, V. Substituting Eqs. (A.I.4) into the expression for H we have

$$\bar{H} = \sum_{q,s} (E(q) - E_F) \sum_{s'} v^{*}(q,s,s')v(q,s,s') +$$
$$+ \frac{1}{2V} \sum_{(q,q'\neq 0)} I(q,q'|s_1,s_2,s_2',s_1') \{ \sum_{s} v^{*}(q,s_1,s_2)u^{*}(q,s_2,s_1) \} \{ \sum_{s} u(q,s_2',s)v(q,s_1',s) \} = \mathcal{E}(u,v)$$

Then u, v would be obtained by minimizing the form $\mathcal{E}(u,v)$ with the supplementary conditions (A.I.5). For these values of u and v, $\mathcal{E}$ gives the energy of the ground state.

The corresponding stationary condition has the form

$$\delta\tilde{\mathcal{E}} = \delta\Big\{\mathcal{E} + \sum_{q,s,s'}\big(\lambda(q,s,s')\xi(q,s,s') + \mu(q,s,s')\eta(q,s,s') + \mu^*(q,s,s')\eta^*(q,s,s')\big\} = 0 \tag{A.I.7}$$

where λ, μ are Lagrange multipliers.

Obviously this equation always admits a trivial solution

$$u_q = \theta_c(q)\delta_{ss'} \quad , \quad v_q = \theta_F(q)\delta_{ss'} \quad ,$$
$$\mu = 0 \quad , \quad \lambda = \theta_F(q)\,(E_F - E(q))\,\delta_{ss'} \tag{A.I.8}$$

In the corresponding state $C_0^{(n)}$ the interaction is not effective and only the first term of the Hamiltonian (A.I.I) contributes to the energy.

In order to determine when the energy of $C_0^{(n)}$ is not minimal and consequently the ground state $C_0^{(s)}$ is characterized by a nontrivial solution of (A.I.7) one must use a standard procedure of the calculus of variations. Constructing the second variation $\delta^2\tilde{\mathcal{E}}$ for the trivial solution one obtains

$$\delta^2\tilde{\mathcal{E}} = \sum_{q,s,s'} |E(q) - E_F|\,\psi^*(q,s,s')\psi(q,s,s') +$$
$$+ \frac{1}{2V}\sum_{(q,q',\dots)} I(q,q'|s_1,s_2,s_2',s_1')\,\psi(q,s_1,s_2)\,\psi^*(q',s_1',s_2')$$

with

$$\psi(q,s,s') = \theta_F(q)\,\delta u(q,s,s') + \theta_c(q)\,\delta v(q,s,s')$$

the functions ψ are subject only to the antisymmetry conditions resulting from the variation of the orthonormality conditions: $\psi(q,s,s') = -\psi(q,s',s)$.

After returning to the original system of indices (cf. (A.I.I)) the second variation takes the form

$$\delta^2\tilde{\mathcal{E}} = \sum_{k,\sigma,\sigma'} |E(k) - E_F|\,\psi^*(k,\sigma,\sigma')\psi(k,\sigma,\sigma') +$$
$$+ \frac{1}{2V}\sum_{(k,k',\dots)} J(k,k'|\sigma_1,\sigma_2,\sigma_2',\sigma_1')\,\psi^*(k,\sigma_1,\sigma_2)\,\psi(k',\sigma_1',\sigma_2')$$

and the antisymmetry condition takes the form

$$\psi(-k, \sigma_2, \sigma_1) = -\psi(k, \sigma_1, \sigma_2)$$

Obviously the second variation $\delta^2\tilde{\varepsilon}$ is negative if and only if the equation

$$2\,|E(k)-E_F|\,\psi(k,\sigma_1,\sigma_2) + \\ +\frac{1}{V}\sum_{(k',\sigma_1',\sigma_2')} J(k,k'|\sigma_1,\sigma_2,\sigma_2',\sigma_1')\,\Psi(k',\sigma_1',\sigma_2') = E\psi(k,\sigma_1,\sigma_2) \qquad \text{(A.I.9)}$$

admits an eigenfunction with the negative eigenvalue E.

In this case the energy of the state $c_0^{(n)}$ is not minimal and a new ground state $c_0^{(s)}$ arises. This state is characterised by the nontrivial solution of Eq. (A.I.7). It is interesting to notice that Eq. (A.I.9) written in configuration space (with a velocity independent interaction)

$$2\,|E(k)-E_F|\,\psi(\vec{r},\sigma_1,\sigma_2) + \\ +\sum_{\sigma_1',\sigma_2'} \Phi(\vec{r}|\sigma_1,\sigma_2,\sigma_2',\sigma_1')\,\psi(\vec{r},\sigma_1',\sigma_2') = E\,\psi(\vec{r}|\sigma_1,\sigma_2) \qquad \text{(A.I.IO)}$$

is very similar to the Schrödinger equation for the two body problem in the center of mass system. The difference lies only in the special form of the kinetic energy operator. This difference naturally disappears in the case of vanishing density when $E_F = 0$.

Then equation (I.IO) can be used for the investigation of the problem of superfluidity of nuclear matter as a criterion of the instability of the normal state.

For these purposes it is convenient to make use of a variational principle and to minimize the expression

$$2\sum_{\sigma_1,\sigma_2}\int |E(k)-E_F|\cdot|\psi(k,\sigma_1,\sigma_2)|^2\,d\vec{k} + \qquad \text{(A.I.II)}$$

$$+\sum_{(\ldots\sigma\ldots\sigma'\ldots)}\int \Phi(\vec{r}|\sigma_1,\sigma_2,\sigma_2',\sigma_1')\,\psi^*(\vec{r},\sigma_1,\sigma_2)\,\psi(\vec{r},\sigma_1',\sigma_2')\,d\vec{r}$$

where

$$\psi(k) = \frac{1}{(2\pi)^{3/2}}\int \psi(\vec{r})\,e^{-i\vec{k}\vec{r}}\,d\vec{r}$$

with the supplementary condition

$$\sum_{\sigma_1,\sigma_2}\int |\psi(r,\sigma_1,\sigma_2)|^2 d\vec{r} = 1$$

If the special choice of the trial function ψ leads to a negative sign of this expression, then in Eq.(A.I.IO) $E<0$ and our criterion is satisfied.

Let us note in conclusion that in the model under consideration it is possible to construct an asymptotically exact form for the free energy.

The equations here are nonlinear and sufficiently complicated but the equations which determines the oritical temperature of the phase transition into the normal state is linear. As shown by I. Kvasnikov and one of the authors[22] these linear equations differ from Eq.(A.I.IO) only by the fact that E = 0 and by the replacement of

$$2|E(k)-E_F|$$

by

$$2|E(k)-E_F| \operatorname{cth} \frac{|E(k)-E_F|}{2\theta}$$

where θ is the critical temperature.

APPENDIX II. ON A VARIATIONAL PRINCIPLE IN THE MANY BODY PROBLEM[x)]

We studied here only the spatially homogeneous problem. But in a number of cases it is very interesting to consider spatially inhomogenous problems. So in order to obtain exact electrodynamical equations in the theory of superconductivity we must take into account the boundary of the superconductor. It is also very important for the further development of the theory to take into account explicitly the crystal lattice of the metals.

Especially in nuclear theory the consideration of the matter as unbounded is a very rough simplification. For all these physically very different purposes we propose here a new approximate method, which is a natural generalization of the well known Fock method[24].

Consider a dynamical system of Fermi-particles with a Hamiltonian of the form

$$H = \sum \{T(f,f') - \lambda \delta_{ff'}\} \overset{+}{a}_f a_{f'} + \frac{1}{2} \sum J(f_1, f_2, f_2', f_1') \overset{+}{a}_{f_1} \overset{+}{a}_{f_2} a_{f_2'} a_{f_1'} \qquad (A.II.1)$$

here λ is the chemical potential, $a, \overset{+}{a}$ are the Fermi amplitudes and f is a set of indices characterising one particle states.

Let us perform a linear transformation of the Fermi amplitudes

$$a_f = \sum_\nu (u_{f\nu} \alpha_\nu + v_{f\nu} \overset{+}{\alpha}_\nu) \qquad (A.II.2)$$

In order to preserve the commutation properties of the Fermi amplitudes the transformation must be canonical and the c-number functions u, v must obey the following orthonormality conditions

$$\delta_{f,f'} \equiv \sum_\nu \{u_{f\nu} \overset{*}{u}_{f'\nu} + v_{f\nu} \overset{*}{v}_{f'\nu}\} = \delta_{ff'}$$
$$\eta_{f,f'} \equiv \sum_\nu \{u_{f\nu} v_{f'\nu} + v_{f\nu} u_{f'\nu}\} = 0 \qquad (A.II.3)$$

Substituting (A.II.2) into the expression (A.II.I) and taking the expectation value in the vacuum state C_0 :

$$\alpha_\nu C_0 = 0$$

corresponding to the new Fermi-amplitudes we obtain

$$\bar{H} = \sum \{T(f,f') - \lambda \delta_{ff'}\} F_1(f,f') + \frac{1}{2} \sum J(f_1, f_2, f_2', f_1') \overset{*}{\Phi}(f_1, f_2) \Phi(f_1', f_2') + F_1(f_1, f_1') F_1(f_2, f_2') - F_1(f_2, f_1') F_1(f_1, f_2') \equiv \mathcal{E}(u,v) \qquad (A.II.4)$$

---

x) This Appendix is based on a paper[23]

where

$$F_1(f,f') = \sum_\nu v^*_{f\nu} v_{f'\nu}$$

$$\Phi(f,f') = \sum_\nu v_{f\nu} u_{f'\nu}$$

The functions u, v are determined by minimizing the form $\mathcal{E}(u,v)$ with the subsidiary conditions (A.II.3).

The corresponding stationary equation has the form:

$$\delta \tilde{\mathcal{E}}(u,v) = 0$$

$$\tilde{\mathcal{E}}(u,v) = \mathcal{E}(u,v) + \sum_{f,f'} \{\lambda(f,f')\xi(f,f') + \mu(f,f')\eta(f,f') + \mu^*(f,f')\eta^*(f,f')\}$$

where λ, μ are Lagrange multipliers. The variation $\delta u, \delta v$ and $\delta u^*, \delta v^*$ are considered here as independent.

Now we come to the formulation of the new approximate method in the many body problem. In this method we take the functions u and v satisfying the stationary equations, which minimize the form $\mathcal{E}(u,v)$. For these functions the corresponding C_0 is treated as the wave function of the ground state and $\mathcal{E}(u,v)$ as the ground state energy.

The question of the fundamentation and limitations of the method is complicated. We shall restrict ourselves only to several remarks. We assert on the basis of the results of Appendix I, that the method provides the exact solution of the problem in the case when in the Hamiltonian only interactions between pairs with opposite momenta is taken into account.

On the other hand, we maintain, that among the solutions of the stationary equation there is always present the one exactly corresponding to the well-known Fock method[24].

Indeed let us take the set of functions $\varphi_{f\nu}$ orthonormalized in the usual sense

$$\zeta(f',f) \equiv \sum_\nu \varphi_{f'\nu} \varphi_{f\nu} = \delta_{ff'} \qquad \text{(A.II.6)}$$

and divide all ν into two parts F and G. A finite number of the indices N (N - is the number of particles) belongs to F - the Fermi sphere. The others belong to the complementary set G. Let us choose

$$\begin{aligned} u_{f\nu} &= 0 \quad , \quad v_{f\nu} = \varphi_{f\nu} \quad , \quad \nu \in F \\ u_{f\nu} &= \varphi_{f\nu} \quad , \quad v_{f\nu} = 0 \quad , \quad \nu \in G \end{aligned} \qquad \text{(A.II.7)}$$

Obviously all orthogonality conditions (A.II.3) are satisfied. If we substitute these u and v into the form $\mathcal{E}$ then Φ vanishes and it depends on F_I only and thus only on $\varphi_{f\nu}$ with

$\nu \in G$. Let us denote by ω the indices ν belonging to F and define $\varphi_{f\omega}$ from the minimum of the form $\mathcal{E}(\ldots\varphi_{f\omega}\ldots)$ with the subsidiary conditions (A.II.6).

The corresponding stationary equation has the form

$$\delta \tilde{\mathcal{E}}_F(0) = 0 \quad ; \quad \tilde{\mathcal{E}}_F = \mathcal{E}(\ldots\varphi_{f\omega}\ldots) + \sum_{f,f'} \lambda_{(ff')}\, \mathcal{Z}(f,f') \qquad \text{(A.II.8)}$$

It is easy to see that we formulated here just the usual Fock method. The wave function of the system C_0 corresponds to the state where individual particles occupy all the states $\varphi_{f\omega}$ the other states $\varphi_{f\nu}$ are free.

On the other hand it is clear that Eq. (A.II.5) always has a solution of the type (A.II.7) with $\varphi_{f\omega}$ chosen with the help of the Fock method as the solution of Eq. (A.II.8). Consequently our method may be considered as a generalization of the Fock method and in any case its domain of applicability is not narrower.

Caloulating as in Appendix I, the second variation $\delta^2\tilde{\mathcal{E}}(u,v)$ of the "normal solution" (A.II.7) we can obtain the instability condition. This condition is formulated as the eigenvalue problem for the corresponding system of nonlinear equations.

Practically this condition may be used e.g. for obtaining a superfluidity criterion in the model where the oristal lattice is explicitly taken into acoount.

In conclusion let us note that the method summarized here may be developed further by means of an investigation of the chain of equations for the "distribution functions":

$$\overline{a^+_{f_1}\ldots a^+_{f_s}\, a_{f'_2}\ldots a_{f'_1}} \equiv F_{s+2}(t, f_1\ldots f_s; f'_2\ldots f'_1)$$

For example in the stationary case, retaining only the functions $F_{0+2}(f_1,f_2)$ and $F_{2+0}(f_1,f_2)$ in the equations we shall obtain again the original equations of our method.

In the case of explicit time dependence of F_{0+2} , F_{2+0} in the approximation linear in the deviations

$$F_{0+2} - F^{st}_{0+2} \quad , \quad F_{2+0} - F^{st}_{2+0}$$

we shall obtain the equations determining the spectrum of oollective oscillations.

REFERENCES

I. H.Fröhlich, Phys.Rev., 79, 845 (I950); Proc.Roy.Soc., A 2I5, 29I (I952).

2. J.Bardeen, Phys.Rev., 79, I67 (I950); 80, 567 (I950); 8I, 829 (I95I).

3. H.Fröhlich, Proc. Roy. Soc. A 223, 296 (I954).

4. M.R.Schafroth, Phys.Rev., 96, I442 (I954); M.R. Schafroth, S.T. Butler, J.M. Blatt, Helv. Phys. Acta, 30, 93 (I957).

5. L.N. Cooper, Phys. Rev., I04, II89 (I956); J.Bardeen, L.N. Cooper, J.R. Schrieffer, Phys. Rev., I06, I62 (I957); I08, II75 (I957).

6. N.N. Bogolubov, JETP 34, 58 (I958); Nuovo Cimento 7, 794 (I958).

7. N.N. Bogolubov, Journ. of Phys. USSR, 9, 23 (I947), Vestnik MGU No.7, 43 (I947).

8. K.A. Brueckner , K. Sawada, Phys. Rev. I06, III7, II28 (I957).

9. N.M. Hugenholtz, Physica, 23, 48I (I957).

I0. N.N. Bogolubov, Doklady Vysshey Shkoly, No. I, (I958).

II.+ D.Pines, Phys. Rev. I09, 280 (I958).

I2. S.V. Tyablikov, V.V. Tolmachov, JETP, (I958) (in the press)

I3.+ M. Gell-Mann, K.A. Brueckner, Phys. Rev. I06, 364 (I957); K.Sawada, Phys.Rev. I06,372(I957); K.Sawada, K.A. Brueckner, N.Fukuda, R.Brout, Phys.Rev.I08,507 (I957); R.Brout, Phys.Rev. I08, 5I5 (I957).

I4. N.N. Bogolubov, JETP, 34, 73 (I958).

I5. N.N. Bogolubov, Lectures on quantum statistics (in Ucrainian) Kiev, I947.

I6.+ J. Goldstone, Proc. Roy. Soc. A 239, 267 (I957).

I7. N.N. Bogolubov, D.V. Shirkov, Introduction to quantized field theory (in russian) Gostehizdat, Moskow, I957.

I8. J. Bardeen, D. Pines, Phys. Rev. 99, II4I (I955).

I9. V.V. Tolmachov, S.V. Tyablikov, JETP, 34, 66 (I958).

20. N.N. Bogolubov, D.N. Zubarev, Yu.A. Tserkovnikov, Doklady Akad. Nauk SSSR, II7, 788 (I957).

2I. N.N. Bogolubov, Doklady Akad. Nauk SSSR II9, 52 (I958).

22. I.A. Kvasnikov, V.V. Tolmachov, Dokl. Akad. Nauk SSSR (in the press).

23. N.N. Bogolubov, Doklady Akad. Nauk SSSR II9, 244 (I958).

24. V.A. Fock, Z.Physik, 6I, I26 (I930).

25.+ J.G. Valatin, Nuovo Cimento, 7, 843 (I958).

+) The authors express their gratitude to M. Gell-Mann, K. Brueckner, K. Sawada, J. Goldstone, D.Pines and J.G. Valatin for sending them preprints.