JOINT INSTITUTE FOR NUCLEAR RESEARCH

Laboratory of Theoretical Physics

V.G. Soloviev, Ten Geun

ON AN APPLICATION OF THE STATISTICAL VARIATIONAL PRINCIPLE TO THE THEORY OF ATOMIC NUCLEUS

V.G. Soloviev, Ten Geun

ON AN APPLICATION OF THE STATISTICAL VARIATIONAL PRINCIPLE

TO THE THEORY OF ATOMIC NUCLEUS

The statistical variational principle[1], which is a generalization of the variational principle suggested by Bogoliubov[2], enables us to calculate thermodynamic quantities at zero temperature as well as at non-zero one.

In the present paper with the aid of the statistical variation principle the superfluid state of the nucleus at nonzero temperature was investigated: the temperature of the phase transition from the superfluid state into the normal one was obtained, the behaviour of thermodynamic quantities at the temperature close to $\theta \approx 0$ as well as at the temperature close to a critical one $\theta \approx \theta_0$ was investigated.

Just as in the previous papers of one of the authors[3,4] basing on the shell model of the nucleus we consider residual interactions of nucleons near the Fermi energy surface, i.e.

$$E_F - \delta \leq E(\Delta, m) \leq E_F + \Delta.$$

Write the Hamiltonian of the residual nucleon interactions in the form:

$$H = \sum_{\Delta, m, \rho} \left\{ E(\Delta, m) - \lambda \right\} a_{\rho m}(\Delta)^+ a_{\rho m}(\Delta) +$$
$$+ \frac{1}{4} \sum_{\substack{\Delta_1, \Delta_2, \Delta_1', \Delta_2' \\ m_1, m_2, m_1', m_2' \\ \rho_1, \rho_2, \rho_1', \rho_2'}} J(\Delta_1, \Delta_2; \Delta_1', \Delta_2' | m_1, m_2; m_1', m_2') a_{\rho_1 m_1}(\Delta_1)^+ a_{\rho_2 m_2}(\Delta_2)^+ a_{\rho_1' m_1'}(\Delta_1') a_{\rho_2' m_2'}(\Delta_2') \tag{1}$$

$$\rho_1 m_1 + \rho_2 m_2 = \rho_1' m_1' + \rho_2' m_2', \quad \rho_1 m_1 \neq \rho_1' m_1'$$

The chemical potential λ is determined from the condition:

$$n = \sum_{\Delta, m, \rho} \left\langle a_{\rho m}(\Delta)^+ a_{\rho m}(\Delta) \right\rangle \tag{2}$$

the notations are the same as in[4]. Let us perform the canonical transformation

$$a_{\rho m}(\Delta) = U_m(\Delta) \alpha_{m, -\rho}(\Delta) + \rho V_m(\Delta) \alpha_{m \rho}(\Delta)^+ \tag{3}$$

under the condition

$$\gamma_m(\Delta) = U_m(\Delta)^2 + V_m^+(\Delta)^2 - 1 = 0 \tag{4}$$

and determine a new vacuum state Ψ_0 :

In order to avoid the transition from an even nucleus to an odd one we examine the excited states of the type:

$$\Psi_2 = \alpha_{m+}(\Delta)^+ \alpha_{m-}(\Delta)^+ \Psi_0. \tag{5}$$

The Hamiltonian is represented by us in the form:

$$H = \langle H \rangle + H_0 + H_1 \tag{6}$$

where

$$H_0 = \sum_{\Delta, m, \rho} E_m(\Delta) \alpha_{m\rho}(\Delta)^+ \alpha_{m\rho}(\Delta)$$

and

$$E_m(\Delta) = 2 \left\{ E(\Delta, m) - \lambda \right\} \left\{ U_m(\Delta)^2 - V_m(\Delta)^2 \right\} - 4 \sum_{\Delta', m'} J(\Delta, \Delta' | m, m') \cdot U_m(\Delta) V_m(\Delta) U_{m'}(\Delta) V_{m'}(\Delta).$$

Making use of the Bogoliubov's variational theorem we get an upper limit for the thermodynamic potential of nucleus in the following form

$$\Psi = -\Theta \sum_{s,m} \ln \left\{ 2 \operatorname{ch} \frac{E_m(s)}{\Theta} \right\} + \sum_{s,m} \left\{ E(s,m) - \lambda \right\} -$$

$$- 2 \sum_{s,s',m,m'} J(s,s'|m,m') \operatorname{th} \frac{E_m(s)}{2\Theta} \, U_m(s) \, V_m(s) \, U_{m'}(s') \, V_{m'}(s') +$$

$$+ \sum_{s,s',m,m'} J(s,s'|m,m') \operatorname{th} \frac{E_m(s)}{2\Theta} \operatorname{th} \frac{E_{m'}(s')}{2\Theta} \, U_m(s) \, V_m(s) \, U_{m'}(s') \, V_{m'}(s'). \tag{8}$$

From the condition of the thermodynamic potential minimum we obtain equation for the determination of U, V:

$$\left\{ E(s,m) - \lambda \right\} U_m(s) V_m(s) + \frac{8}{\Theta} \frac{U_m(s)^2 V_m(s)^2}{\operatorname{sh} \frac{E_m(s)}{\Theta}} \sum_{s',m'} J(s,s'|m,m') V_{m'}(s') U_{m'}(s') V_{m'}(s') +$$

$$+ \frac{4}{\Theta} \frac{U_m(s) V_m(s)}{\operatorname{sh} \frac{E_m(s)}{\Theta}} \left\{ U_m(s)^2 - V_m(s)^2 \right\} \sum_{s',m'} J(s,s'|m,m') U_{m'}(s') V_{m'}(s') \sum_{s'',m''} J(s,s'|m,m'') V_{m''}(s'') U_m(s) V_{m''}(s'') +$$

$$+ \frac{2}{\Theta} \left\{ U_m(s)^2 - V_m(s)^2 \right\} \operatorname{cth} \frac{E_m(s)}{2\Theta} \sum_{s',m'} J(s,s'|m,m') \frac{U_{m'}(s')^2 V_{m'}(s')^2}{\operatorname{ch}^2 \frac{E_m(s)}{2\Theta}} \sum_{s'',m''} J(s,s'|m,m'') V_{m''}(s') U_{m''}(s') V_{m''}(s') +$$

$$+ \frac{U_m(s)^2 - V_m(s)^2}{2} \sum_{s',m'} J(s,s'|m,m') \operatorname{th} \frac{E_{m'}(s')}{2\Theta} U_{m'}(s') V_{m'}(s') = 0 \tag{9}$$

where

$$V_m(s) = \frac{1}{1 + e^{E_m(s)/\Theta}}.$$

It is easily seen that (9) assumes a trivial solution corresponding to the normal state, viz.

$$U_m(s) = 1 - \Theta_F(s,m), \qquad V_m(s) = \Theta_F(s,m)$$

where $\Theta_F(s,m) = 1$ for $E(s,m) < E_F$ and $\Theta_F(s,m) = 0$ for $E(s,m) > E_F$.

Considering the second variation of the thermodynamic potential, as in[3], we obtain the following equation for finding the phase transition temperature

$$2 |E(s,m) - \lambda| \operatorname{cth} \frac{|E(s,m) - E_F|}{\Theta_a} \mathcal{G}_m(s) + \sum_{s',m'} J(s,s'|m,m') \mathcal{G}_{m'}(s') = 0. \tag{10}$$

The phase transition temperature Θ_0 is determined from the condition: the equation (10) should have a non-zero solution. Taking into account the smallness of $|E(s,m) - E_F|/\Theta_0$ and assuming the approximation[4]

$$J = \text{const.}, \qquad \rho = \text{const.} \tag{11}$$

we get the phase transition temperature Θ_0 equal to:

$$\Theta_0 = - \frac{J\rho}{2} (\Delta + \delta). \tag{12}$$

This expression differs by a factor 2 from the corresponding expression in[3]. This is re-

lated to that we have considered the excited states of the type (5). It should be noted that the concept temperature of finite nucleus begins to assume a physical sense under rather strong excitation of the nucleus, i.e. near and above the phase transition temperature.

We find a non-trivial solution for the Eq.(9). This can be done in the two limitting cases: when the temperature is close to zero and when the temperature is close to that of the phase transition. In both cases the Eq. (9) takes the following form:

$$
\xi_m(s)\, U_m(s)\, V_m(s) + \frac{U_m(s)^2 - V_m(s)^2}{2} \sum_{s',m'} J(s,s'|m,m')\, th\,\frac{E_{m'}(s')}{2\theta}\, U_{m'}(s')\, V_{m'}(s') = 0 \tag{13}
$$

where $\qquad \xi_m(s) = E(s,m) - \lambda$

Let us introduce a new function:

$$
C_m(s) = \sum_{s',m'} J(s,s'|m,m')\, th\,\frac{E_m(s)}{2\theta}\, U_{m'}(s')\, V_{m'}(s') \tag{14}
$$

and assume

$$
U_m(s)^2 = \frac{1}{2}\left\{ 1 + \frac{\xi_m(s)}{C_m(s)^2 + \xi_m(s)^2} \right\} \quad,
$$

$$
V_m(s)^2 = \frac{1}{2}\left\{ 1 - \frac{\xi_m(s)}{C_m(s)^2 + \xi_m(s)^2} \right\}
$$

then, from the Eq.(13) we obtain $\qquad U_m(s)\, V_m(s) = - \dfrac{C_m(s)}{2\left\{C_m(s)^2 + \xi_m(s)^2\right\}^{1/2}}$

and the equation for $C_m(s)$ takes the form similar to (12) in[5].

$$
C_m(s) = -\frac{1}{2}\sum_{s',m'} J(s,s'|m,m')\, th\,\frac{E_{m'}(s')}{2\theta}\, \frac{C_{m'}(s')}{\left\{C_{m'}(s')^2 + \xi_{m'}(s')^2\right\}^{1/2}} \tag{15}
$$

In the approximation of (11) the Eq. (15) is got in the form:

$$
1 = -\frac{J\rho}{2}\int_{E_F-\lambda-\delta}^{E_F-\lambda+\Delta}\frac{th\,\frac{E(\xi)}{2\theta}}{\sqrt{C^2 + \xi^2}}\, d\xi \tag{16}
$$

where $\quad E(\xi) = E_m(s)$

$$
= \frac{2\xi^2}{\sqrt{C^2+\xi^2}} - \frac{J\rho C^2}{\sqrt{C^2+\xi^2}}\int_{E_F-\lambda-\delta}^{E_F-\lambda+\Delta}\frac{d\xi'}{\sqrt{C^2+\xi'^2}} \tag{17}
$$

The Eq.(2) describing the chemical potential in this approximation is written as

$$
n = \rho\int_{E_F-\lambda-\delta}^{E_F-\lambda+\Delta}\left\{ 1 - \frac{\xi}{\sqrt{C^2+\xi^2}} \right\}\, d\xi \tag{18}
$$

and the thermodynamic potential of the system in the following form:

$$\Psi = \rho \int_{E_F-\lambda-\delta}^{E_F-\lambda+\Delta} \left\{ -\theta \ln \left(1+e^{-\frac{E(\xi)}{\theta}}\right) - \frac{E(\xi)}{2} + \xi + \frac{c^2}{\sqrt{c^2+\xi^2}} \right\} d\xi + \frac{c^2}{J}$$

(19)

Note that for $\theta = 0$ the eqs. (16), (18) are solved in a simple manner and the magnitudes of C_0 and λ_a are obtained in [4].

We find an explicit form of C and λ for each limitting case.

Case I: $\theta \approx 0$ i.e. the temperature θ is close to zero. Expanding

$$th \frac{E(\xi)}{2\theta} \approx 1 - e^{-\frac{2\sqrt{C_0^2+\xi^2}}{\theta}}$$

We obtain

$$1 = -\frac{J\rho}{2} \int_{E_F-\lambda-\delta}^{E_F-\lambda+\Delta} \frac{d\xi}{\sqrt{c^2+\xi^2}} - K(\theta)$$

where

$$K(\theta) = -\frac{J\rho}{2} \int_{E_F-\lambda_0-\delta}^{E_F-\lambda_0+\Delta} \frac{e^{-\frac{2\sqrt{C_0^2+\xi^2}}{\theta}}}{\sqrt{C_0^2+\xi^2}} d\xi$$

Solving (16), (18) in this approximation we get

$$C = \frac{(2\Omega-n)n}{\rho(e^{2/G}-1)} e^{1/G} \left\{ 1 - \frac{K(\theta)}{G} cth\frac{2}{G} \right\}$$ (20)

$$E_F-\lambda = \frac{\Omega-n}{\rho(e^{2/G}-1)} \left\{ 1 - 2\frac{K(\theta)}{G} \frac{1}{1-e^{-2/G}} \right\}$$ (21)

where $G = -J\rho$, n - is the number of nucleons and Ω- is the number of levels near the Fermi surface.

Case II: $\theta \approx \theta_0$ i.e. temperature θ is somewhat less than the phase transition temperature θ_0.

In this case (13) takes the form:

$$\xi_m(s)U_m(s)V_m(s) + \frac{1}{4\theta}\left\{U_m(s)^2-V_m(s)^2\right\}\sum_{s'm'} J(ss'|mm')U_{m'}(s)V_{m'}(s) = 0$$

$$C_m(s) = \frac{1}{2\theta}\sum_{s'm'} J(s,s'|m,m') E_{m'}(s)U_{m'}(s)V_{m'}(s)$$

(22)

In the approximation of (11) for determining C we obtain the equation

$$1 = \frac{G}{2\theta} \int_{E_F-\lambda-\delta}^{E_F-\lambda+\Delta} \frac{\xi^2}{C^2+\xi^2} d\xi$$ (23)

from where

$$\left(\frac{\partial c^2}{\partial \theta}\right)_{\theta=\theta_o} = - \frac{1}{4\theta_o}\, \frac{n(2\Omega - n)}{\rho^2}$$

Remaining only the linear term in the expression $C(\theta)$ near $\theta = \theta_0$ we obtain

$$c^2 = \frac{1}{4\,\rho^2}\; n(2\Omega - n)\cdot(1 - \frac{\theta}{\theta_0})$$

The entropy S and the free energy E of the system are connected with the thermodynamical potential ψ by the following relations:

$$S = - \frac{\partial \psi}{\partial \theta} - \frac{d\psi}{dc^2}\, \frac{\partial c^2}{\partial \theta}$$

$$E = \theta S + \psi$$

It is easy to show that the derivative $\frac{d\psi}{dc^2}$ vanishes in the point θ_0 and owing to this fact the entropy S suffers no jump in the point θ_0, therefore the phase transition of the system from the superfluid state into the normal one is a phase transition of the second kind.

R e f e r e n c e s

1.   I.A. Kvasnikov, V.V. Tolmachev. DAN SSSR 120, 273 (1958).

2.   N.N. Bogoliubov. DAN SSSR 119, 244 (1958).

3.   V.G. Soloviev. JETP, 35, 823 (1958), 36, 293 (1959). DAN SSSR, 123,652 (1958).
        Nucl. Phys. 9, 655 (1958/1959).

4.   V.G. Soloviev. JETP, 36, 1869 (1959).

5.   N.N. Bogoliubov, D.N. Zubarev, J.A. Tserkovnikov. DAN SSSR 117, 788 (1957).