JOINT INSTITUTE FOR NUCLEAR RESEARCH

Laboratory of Theoretical Physics

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A B S T R A C T

In this paper the transformation properties of spinors and pseudospinors with respect to inversions of space and time are investigated in some detail. In particular the various possible twocomponent theories are investigated. A purely geometrical approach to the theorem of Lüders und Pauli is pointed out.

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I. Introduction

Inversions of space and time have gained particular importance at the time when non-conservation of parity become an experimentally established fact. A great number of papers is devoted to this subject, nevertheless it seems to me worth while to discuss it once more from a somewhat different point of view to that usually assumed, namely from a purely geometrical point of view.

To my knwledge most of the work done recently on space time inversions starts with either the field equations or Heisenbergs equations of motion (in Heisenbergs picture) or with Schroedingers equations (in Schroedinger picture) and derives the various properties of spinors and tensors from these equations. Thus a dynamical point of view is assumed to derive kinematical properties of the fields.

In this paper we shall assume as the basis for our considerations the connection between spinors and tensors and derive the consequences of this connection. The various possible field equations may be regarded as one of the consequences. It turns out that some known results of apparently profound physical meaning become trivial from the point of view of geometry. Some other results may be better understood being related to the group properties of spinors. One obtaines a complete review of all the equations corresponding to first rank spinors (spin 1/2); in particular of the twocomponent equations.

2. The connection of spinors and tensors

Spinors are usually defined by their transformation properties

$$\begin{aligned} c_1' &= \alpha c_1 + \beta c_2 \\ c_2' &= \gamma c_1 + \delta c_2 \end{aligned} \qquad \alpha\delta - \beta\gamma = 1 \tag{2.1}$$

One may introduce also a dual spinor space

$$\begin{aligned} c_{\dot 1}' &= \alpha^* c_{\dot 1} + \beta^* c_{\dot 2} \\ c_{\dot 2}' &= \gamma^* c_{\dot 1} + \delta^* c_{\dot 2} \end{aligned} \qquad \alpha^*\delta^* - \beta^*\gamma^* = 1 \tag{2.2}$$

The transformation coefficients in this dual space are complex conjugate (" * ") to those of the original space.

The two spaces may be regarded as independent. One may, however, introduce a connection between the dotted and the undotted quantities. The most general connection, consistent with the transformation properties (2.1-2), is of the form

$$c_{\dot\alpha} = \varkappa\, c_\alpha^* \qquad (\alpha = 1, 2) \tag{2.3}$$

where $\varkappa$ is an arbitrary complex number (scalar).

The connection with tensors is obtained by the intermediary of second rank spinors $C_{\alpha\dot\beta}$ and first rank tensors x_μ ($\mu = 1, 2, 3, 4$; $x_4 = i x_0$)

$$\begin{aligned} C_{1\dot 1} &= x_3 - x_0 & C_{1\dot 2} &= x_1 - i x_2 \\ C_{2\dot 1} &= x_1 + i x_2 & C_{2\dot 2} &= -x_3 - x_0 \end{aligned} \tag{2.4}$$

Another kind of spinors may be introduced, possessing the same transformation properties (2.1-2) and connected with pseudovectors $y_\mu = t_{[\nu\varrho\sigma]}$ (the bracket denoting antisymmetrization with respect to all three indices)

$$d_{11} = y_3 - y_0 \qquad d_{12} = y_1 - iy_2$$
$$d_{21} = y_1 + iy_2 \qquad d_{22} = -y_3 - y_0 \tag{2.5}$$

We shall call the quantities $d_{\alpha\beta}$ pseudospinors.

Relations (2.4-5) are the basis for the determination of the transformation properties of spinors and pseudospinors with respect to inversions.

3. Inversions of space and time

For an inversion $I_{\vec{r}}$ of all three space components we have

$$I_{\vec{r}}: \quad x_i' = -x_i, \quad x_0' = x_0 \quad (i = 1, 2, 3) \tag{3.1}$$
$$y_i' = y_i, \quad y_0' = -y_0 \tag{3.2}$$

For time inversion we have

$$I_t: \quad x_i' = x_i, \quad x_0' = -x_0 \quad (i = 1, 2, 3) \tag{3.3}$$
$$y_i' = -y_i, \quad y_0' = y_0 \tag{3.4}$$

The corresponding transformation properties of the spinors $c_{\alpha\beta}$ are

$$I_{\vec{r}}: \quad c_{11}' = c_{22} \qquad c_{12}' = -c_{12}$$
$$c_{21}' = -c_{21} \qquad c_{22}' = c_{11} \tag{3.5}$$

$$I_t: \quad c_{11}' = -c_{22} \qquad c_{12}' = c_{12}$$
$$c_{21}' = c_{21} \qquad c_{22}' = -c_{11} \tag{3.6}$$

It is easily seen from (3.1-4) that the pseudospinors $d_{\alpha\beta}$ transform with respect to $I_{\vec{r}}$ according to the formulae (3.6) and with respect to I_t according to the formulae (3.5).

The corresponding most general transformations of first rank spinors are

$$I_{r}: \quad \begin{array}{ll} c_1' = a c_2 & c_2' = -a c_1 \\ c_{\dot 1}' = a^{-1} c_{\dot 2} & c_{\dot 2}' = -a^{-1} c_{\dot 1} \end{array} \qquad (3.7)$$

$$I_{t}: \quad \begin{array}{ll} c_1' = a c_2 & c_2' = -a c_1 \\ c_{\dot 1}' = -a^{-1} c_{\dot 2} & c_{\dot 2}' = a^{-1} c_{\dot 1} \end{array} \qquad (3.8)$$

where a is an arbitrary complex number.

For pseudospinors, the roles of I_t and I_r are again interchanged.

So far we have considered the two dual spaces C_α and $C_{\dot\alpha}$ as independent. Inversions of space and time interchange the dotted and the undotted quantities. These inversions are thus represented by linear transformation of the four-component quantities (bispinors) C_α ; $C_{\dot\alpha}$ or d_α , $d_{\dot\alpha}$ respectively.

The question arises whether it is possible to obtain a representation of inversions by means of two-component spinors. For this purpose it is necessary to introduce some relations between the components of the bispinor C_α , $C_{\dot\alpha}$, We have already stated that the only relation consistent with the transformation properties of spinors is relation (2.3)

$$C_{\dot\alpha} = \varkappa C_\alpha^* \qquad (3.9)$$

For pseudospinors we may put analogously

$$d_{\dot\alpha} = \varkappa d_\alpha^* \qquad (3.10)$$

These relations are of course independent of the particular frame of reference due to the scalar character of the coefficient $\varkappa$.

Introducing (3.9) into (3.7) we get a set of four transformation equations for only two components C_α. The consistency of these equations demands

$$|a|^2|\varkappa|^2 = 1 \tag{3.11}$$

This condition may always be satisfied by a proper choice of one of the constants.

Introducing (3.9) into (3.8) we get another set of four transformation equations for the two components of $C_{\dot{\alpha}}$. The consistency of these equations demands

$$|a|^2|\varkappa|^2 = -1 \tag{3.12}$$

an equation which can never be satisfied.

In other words we obtain the result that inversions of space[1] can be represented by transformations of two-component spinors whereas inversions of time cannot. It may be emphasised however that the representation is not linear since it transforms the original quantities into their complex conjugates.

Due to the fact that the pseudospinors transform with respect to I_t like spinors with respect to I_r and vice versa, we obtain immediately the result that inversions of time can be represented by (non-linear) transformations of two-component pseudospinors whereas inversions of space cannot.

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[1] This is shown here for simultaneous inversion of the three axes x_1, x_2, x_3, but it follows directly also for an inversion of each of the axes separately.

The two-component theory of spinors (3.9) is exactly the theory of Majorana (1937) (cf. also Section 4). In this theory there does not exist the notion of time inversions.

The two-component theory of pseudospinors (3.10) was to my knowledge not yet discussed. In this theory there does not exist the notion of space inversions. Both theories may be used to describe processes which do not conserve parity (cf. Sections 4 and 5).

Another way to obtain a two-component theory of spinors was proposed independently by many authors (Lee and Yang 1957), Salam 1957, Landau 1957). It consists in neglecting either the dotted or the undotted quantities altogether (this may be formally achieved by putting the $\varkappa$ of equations (3.9) and (3.10) equal to zero or to ∞). In frames of such a theory there is no place for inversions of space as well as of time (cf, conditions (3.11-12), cf. also Section 5). We may consider here only rotations of the four-dimensional manifold.

4. Invariant equations.

To write the various types of invariant equations it is convenient to introduce spinors and pseudospinors with upper indices according to the rule

$$\begin{aligned} c_2 &= c^1, \quad c_1 = -c^2; & d_{\dot 2} &= d^1, \quad d_1 = -d^2 \\ c_{\dot 2} &= c^{\dot 1}, \quad c_{\dot 1} = -c^{\dot 2}; & d_{\dot 2} &= d^1, \quad d_{\dot 1} = -d^2 \end{aligned} \tag{4.1}$$

Relations of the type

$$a_\alpha \; b^\alpha \qquad \text{or} \qquad a_{\dot\alpha} \; b^{\dot\alpha} \tag{4.2}$$

are then invariant with respect to transformations (2.1-2).

We may now write invariant relations of the form

$$C_{\alpha\dot{\beta}} C^{\dot{\beta}} = m c_{\alpha} \, , \quad C^{\dot{\alpha}\beta} c_{\beta} = - m C^{\dot{\alpha}} \tag{4.3}$$

for spinors, and

$$C_{\alpha\dot{\beta}} d^{\dot{\beta}} = i m d_{\alpha} \, , \quad C^{\dot{\alpha}\beta} d_{\beta} = i m d^{\dot{\alpha}} \tag{4.4}$$

for pseudospinors. The coefficients in each pair are chosen in such a way that for real m the equations become complex conjugate when relation (2.3) is assumed with $|\varkappa|^2 = 1$. It may be easily shown that equations (4.3-4) are invariant with respect to inversions of space and time if we put

$$a^2 = 1 \qquad (a = \pm 1) \tag{4.5}$$

($C_{\alpha\dot{\beta}}$, c_{α} and $c_{\dot{\alpha}}$ are transformed like spinors, d_{α} and $d_{\dot{\alpha}}$ like pseudosponors with $a = \pm 1$).

Introducing in $C_{\alpha\dot{\beta}}$ the derivative vector ∂_{μ} instead of x_{μ} we get two sets of differential equations. They may be conveniently written with help of the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{4.6}$$

in the form

$$\left.\begin{aligned} \sigma_{\mu\alpha\dot{\beta}} \partial_{\mu} c^{\dot{\beta}} &= m c_{\alpha} \, , & \sigma'^{\,\dot{\alpha}\beta}_{\mu} \partial_{\mu} c_{\beta} &= - m c^{\dot{\alpha}} \\ \sigma_{\mu\alpha\dot{\beta}} \partial_{\mu} d^{\dot{\beta}} &= i m d_{\alpha} \, , & \sigma'^{\,\dot{\alpha}\beta}_{\mu} \partial_{\mu} d_{\beta} &= i m d^{\dot{\alpha}} \end{aligned}\right\} \tag{4.7}$$

where $\sigma'_i = - \sigma_i$, $\sigma'_0 = \sigma_0$.

Introducing the two-component spinors and pseudospinors

$$\varphi = \begin{vmatrix} c_1 \\ c_2 \end{vmatrix}, \quad \chi = \begin{vmatrix} c^{\dot{1}} \\ c^{\dot{2}} \end{vmatrix}, \quad \xi = \begin{vmatrix} d_1 \\ d_2 \end{vmatrix}, \quad \eta = \begin{vmatrix} d^{\dot{1}} \\ d^{\dot{2}} \end{vmatrix} \tag{4.8}$$

we may write further in operator form

$$(\sigma_i\partial_i + \sigma_0\partial_0)\chi = m\varphi$$
$$(\sigma_i\partial_i - \sigma_0\partial_0)\varphi = m\chi \tag{4.9}$$

$$(\sigma_i\partial_i + \sigma_0\partial_0)\eta = im\xi$$
$$(\sigma_i\partial_i - \sigma_0\partial_0)\xi = -im\eta \tag{4.10}$$

Finally we may write these equations in fourdimensional form introducing the fourdimensional spinors and pseudospinors

$$\psi = \begin{vmatrix}\varphi \\ \chi\end{vmatrix} \qquad \zeta = \begin{vmatrix}\xi \\ \eta\end{vmatrix}. \tag{4.11}$$

and the fourdimentsional Dirac matrices

$$\gamma_i = -\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & -i\sigma_0 \\ i\sigma_0 & 0 \end{pmatrix} \tag{4.12}$$

One obtains

$$(\gamma_\mu\partial_\mu + m)\psi = 0 \tag{4.13}$$

and

$$(\gamma_\mu\partial_\mu + im\gamma_5)\zeta = 0 \tag{4.14}$$

where

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} \tag{4.15}$$

Equations of the form (4.14) were proposed by Ivanienko (1957). Taking in accordance with (4.5) for a the value $a = \epsilon = \pm 1$, one easily obtains from (3.7-8), from the corresponding transformations formulae for pseudoscalars and from (4.1) the following transformation formulae for ψ and ζ

$$I_r: \ \psi'(x') = i\epsilon\gamma_4\psi(x), \qquad \zeta'(x') = i\epsilon\gamma_5\gamma_4\zeta(x) \tag{4.16}$$

$$I_t: \ \psi'(x') = i\epsilon\gamma_5\gamma_4\psi(x), \qquad \zeta'(x') = i\epsilon\gamma_4\zeta(x) \tag{4.17}$$

For the three inversions I_1, I_2, I_3 one easily verifies by the same meaint that for I_i: $\psi'(x') = i\epsilon\gamma_5\gamma_i\psi(x)$, $\zeta'(x') = i\epsilon\gamma_i\zeta(x)$.

In terms of twocomponent spinors fourmulae (4.16-17) may be written in the following way.

$$I_{\mathfrak{r}}: \quad \begin{matrix} \varphi' = \epsilon\chi & \xi' = \epsilon\eta \\ \chi' = -\epsilon\varphi & \eta' = \epsilon\xi \end{matrix} \tag{4.18}$$

$$I_t: \quad \begin{matrix} \varphi' = \epsilon\chi & \xi' = \epsilon\eta \\ \chi' = \epsilon\varphi & \eta' = -\epsilon\xi \end{matrix} \tag{4.19}$$

Let us go over now in equations (4.9–10) to the two-component theory according to equations (3.9–10). Written in terms of ψ and ζ these relations are

$$\psi = C\bar{\psi}, \qquad \zeta = C\bar{\zeta} \tag{4.20}$$

where

$$C = -\varkappa\gamma_1\gamma_3 = -\varkappa\begin{pmatrix}\sigma_2 & 0 \\ 0 & \sigma_2\end{pmatrix}, \ (|\varkappa|^2 = 1), \ \bar{\psi} = \psi^*\gamma_4, \quad \bar{\zeta} = \zeta^*\gamma_4 \tag{4.21}$$

C is in our representation just the matrix of charge conjugation. It satisfies, as may be easily verified, the wellknown relations

$$-C^{-1}\gamma_\mu C = \gamma_\mu^T = \gamma_\mu^* \qquad (\gamma_\mu^\dagger = \gamma_\mu^{*T} = \gamma_\mu) \tag{4.22}$$
$$C^\dagger C = 1, \qquad C^T = -C$$

In terms of two-component spinors equation (4.20) may be written in the form

$$\begin{matrix} \varphi = -i\varkappa\sigma_2\chi^* & \chi = i\varkappa\sigma_2\varphi^* \\ \xi = -i\varkappa\sigma_2\eta^* & \eta = i\varkappa\sigma_2\xi^* \end{matrix} \tag{4.23}$$

Expressing with help of relations (4.23) in (4.18–19) the formulae for $I_{\mathfrak{r}}$ for spinors and the formulae for I_t for pseudospinors by means of χ and η respectively, we get

$$I_{\mathfrak{r}}: \quad \chi' = i\epsilon\varkappa\sigma_2\chi^* \qquad \chi'^* = i\epsilon\varkappa^*\sigma_2\chi \tag{4.24}$$

$$I_t: \quad \eta' = i\epsilon\varkappa\sigma_2\eta^* \qquad \eta'^* = i\epsilon\varkappa^*\sigma_2\eta \tag{4.25}$$

Introducing in the same way (4.23) into the equations for $I_{\mathfrak{r}}$ for pseudospinors and for I_t, for spinors we get four equations for two variables with are inconsistent. This shows once more that the notion of time inversion (space inversion) does not exist for two-component spinors (pseudospinors).

Expressing further by means of (4.23) φ through χ and ξ through η in equations (4.9) and (4.10), we get the following twocomponent equations for spinors and pseudospinors

$$(\sigma_i\partial_i + \sigma_0\partial_0)\chi = -im\varkappa\sigma_2\chi^* \tag{4.26}$$

$$(\sigma_i\partial_i + \sigma_0\partial_0)\eta = m\varkappa\sigma_2\eta^* \tag{4.27}$$

In the twocomponent theory of neutrino proposed by Landau, Lee and Yang and by Salam one proceeds in a different way. It is seen e.g. from (4.9) that the terms which mix the components φ and χ vanish if $m = 0$. In this case one may treat each equation of the pair separately. Neglecting e.g. φ we get only one (two-component) equation for χ and vice versa. It is easily seen that when one of the components φ or χ is neglected (put equal to zero) then it makes no sense of speaking about inversions since these transformations transform according to

(4.18-19) φ into χ and vice versa. The notion of inversions of space as well as of time does not exist in this kind of theory.

Of course we may also use another interpretation. Instead of neglecting one of the equations we may consider it as describing particles of an "antiworld". The world does not mix with the "antiworld" due to $m = 0$. The notion of inversions is then preserved and each inversion transforms the world into the "antiworld" and vice versa. We have then to do with an essentially four-component theory.

The difference between the theory of Landau, Salam and Lee and Yang and the Majorana type of theory consists, apart from the transformation properties, in the fact that the latter gives a two-component theory of particles with not necessarily vanishing rest mass, whereas in the first the vanishing of the rest mass is essential. In the Lee and Yang-Landau-Salam theory the spin of the particle is always paralell or antiparalell to its momentum, whereas in the theory described by equations (4.26) or (4.27) this is not the case. Of course if we put in equations (4.26-27) $m = 0$ we may obtain all the theoretical results of Lee and Yang, Salam and Landau by proper choice of the interaction. However we are free to introduce a certain amount of depolarization by allowing $m \neq 0$ which may be of importance when more accurate measurements of polarization phenomena are available.

5. Bilinear forms and the theorem of Lüders and Pauli.

The existence of bilinear invariant and covariant forms is of importance for the construction of such physical quantities as Lagrangean, current, momentum, angular momentum etc. In the usual four-component theory we have the following forms

$$\bar{\psi}\psi, \quad i\bar{\psi}\gamma_5\psi, \quad i\bar{\psi}\gamma_\mu\psi, \quad i\bar{\psi}\gamma_5\gamma_\mu\psi, \qquad i\bar{\psi}\sigma_{\mu\nu}\psi, \quad \bar{\psi}\gamma_5\sigma_{\mu\nu}\psi \tag{5.1}$$

where

$$\sigma_{\mu\nu} = \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \tag{5.2}$$

The transformation properties of these forms are easily obtained from (4.16-17) and the well known commutation properties of the γ_μ matrices. They are given in the following table 1.

	$I_{\vec{r}}$	I_t	Transposition	I'_t	$I'_{\vec{r}t}$	C	$T=I'_tC$
$\frac{1}{2}(\bar{\psi}\psi - \psi\bar{\psi})$	+	−	−	+	+	+	+
$\frac{i}{2}(\bar{\psi}\gamma_5\psi - \psi\gamma_5^T\bar{\psi})$	−	+	−	−	+	+	−
$\frac{i}{2}(\bar{\psi}\gamma_i\psi - \psi\gamma_i^T\bar{\psi})$	−	−	−	+	−	−	−
$\frac{i}{2}(\bar{\psi}\gamma_4\psi - \psi\gamma_4^T\bar{\psi})$	+	+	−	−	−	−	+
$\frac{i}{2}(\bar{\psi}\gamma_5\gamma_i\psi - \psi\gamma_i^T\gamma_5^T\bar{\psi})$	+	+	−	−	−	+	−
$\frac{i}{2}(\bar{\psi}\gamma_5\gamma_4\psi - \psi\gamma_4^T\gamma_5^T\bar{\psi})$	−	−	−	+	−	+	+
$\frac{i}{2}(\bar{\psi}\sigma_{ik}\psi - \psi\sigma_{ik}^T\bar{\psi})$	+	−	−	+	+	−	−
$\frac{i}{2}(\bar{\psi}\sigma_{i4}\psi - \psi\sigma_{i4}^T\bar{\psi})$	−	+	−	−	+	−	+
$\frac{1}{2}(\bar{\psi}\sigma_{ik}\gamma_5\psi - \psi\gamma_5^T\sigma_{ik}^T\bar{\psi})$	−	+	−	−	+	−	+
$\frac{1}{2}(\bar{\psi}\sigma_{i4}\gamma_5\psi - \psi\gamma_5^T\sigma_{i4}^T\bar{\psi})$	+	−	−	+	+	−	−

Table 1.

In this table + denotes invariance and − denotes change of sign of the corresponding form. It is easily seen from column 1 that for space inversions $I_{\vec{r}}$ the forms containing γ_5 transform like pseudotensors, the forms without γ_5 like tensors. For time inversions I_t (second column) we have just the opposite transformation properties. The forms without γ_5 transform like pseudotensors and those with γ_5 like true tensors. This fact constitutes a difficulty when we wish to write invariant relations between forms of the type (5.1) and true tensors. This difficulty can be removed in frames of classical theory only be introducing instead of the true tensors, quantities which behave like (5.1). E.g. instead of the true vector A_μ (electromagnetic potential) we may introduce the vector $\epsilon(t)A_\mu$ where $\epsilon(t)$ changes sign when t is replaced by $-t$ ($\epsilon^2(t) = 1$).

In quantum theory there is another way out of the difficulty without changing the transformation properties of true tensors. This way out was proposed by Schwinger (1951) and has strong relations with the connection of spin and statistics. If we antisymmetrize in quantum theory the forms (5.1)

$$\bar{\psi} O \psi \rightarrow \frac{1}{2}(\bar{\psi} O \psi - \psi O^T \bar{\psi}) \tag{5.3}$$

(column 3)

we may obtain change of sing of these forms by the transposition

$$\bar{\psi} O \psi \rightarrow \psi O^T \bar{\psi} \tag{5.4}$$

If we now supply each time inversion I_t by a transposition (5.4) we get just the necessary change of sign. We shall denote time inversion followed by a transposition by I_t^0 (column 4). It is easily seen from column 5 that the product $I_{\vec{r}}\ I_t^0 = I_{\vec{r}\,t}'$ leaves the scalar and the pseudoscalar forms unchanged. This ~~ix~~ the content of Lüders Pauli theorem. It states simply that an even number of inversions does not change anything in a form which is invariant with respect to rotations (proper Lorentz group). This statement becomes for Bose fields still more trivial since in th~~i~~is case $I_t^0 = I_t$ due to the fact that Bose fields are quantized with commutators and symmetric forms must ~~by~~ be used.

It may be noted that one could even reverse the argument and use the demand of proper behaviour under I_t^0 of the forms (5.1) to conclude upon the quantization of the field by means of anticommutators for spin 1/2 fields. Similarly one could proceed for fields of integer spin.

For pseudospinors the situation is similar. In this case the forms with γ_5 (without γ_5) transform like pseudotensors (tensor~~x~~) for I_t and like tensors (pseudotensors) for $I_{\vec{r}}$. Using transposition together with $I_{\vec{r}}$ and denoting the resulting transformation by $I_{\vec{r}}^0$ we get again the necessary change of ~~the~~ sigh and the result that $I_{\vec{r}}'\ I_t = I_{\vec{r}\,t}^0$ leaves each form unchanged which is invariant with respect to the proper Lorentz group.

It may be noted that in the above formulation charge conjugation C does not enter explicitly into the theorem. The usual formulation with C is from the geometrical point of view rather unnatural. It may be obtained by multiplying $I_{\vec{r}}\ I_t^0$ by C^2. The operation $C^2 = C \circ C$ does not change anything and therefore the same results hold for $I_{\gamma} I_t^0$ as for $I_{\gamma} \circ C \circ C\, I_t^0$. Now $C\, I_t^0$ is just Wigners (1932) time reversal and thus the usual form follows in which Lüders Pauli theorem is applied. It may be noted however that from the geometrical point of view charge conjugation C appears in this theorem only apparently.

Let us go over now to the two-component theory described by equations (4.26-27). From the fact that in such a theory either space inversions (for pseudospinors) or time inversions (for spinors) do not exist we easily see that it makes no sense to speak about the Lüders Pauli theorem in the above formulation. Of course other invariance properties (time or space inversions excluded) may be derived from the invariance with respect to the proper Lorentz group. We do not want to discuss this point here.

The main problem in the process of quantization is the construction of a Lagrangean. One easily verifies that in the classical theory a Lagrangean which does not vanish identically and is constructed out of spinors must necessarily be either invarinat with respect to $I_{\vec{\gamma}}$ but not hermitean or hermitean but not invariant with respect to $I_{\vec{\gamma}}$. For a Lagrangean constructed out of pseudospinors we get the result that it may be either hermitean but change sign in time inversions or be invariant with respect to I_t but not hermitean.

In quantum theory it is possible to write a Lagrangean which (due to the anticommutation rules) does not vanish identically and is hermitean and invariant with respect to $I_{\vec{\gamma}}$ for spinors and with respect to I_t for pseudospinors.

The easiest way to get such a Lagrangean is to start for spinors with the usual antisymmetrized Lagrangean for a Dirac spin 1/2 field

$$L = \frac{1}{2}\bar{\psi}\,(\gamma_\mu \partial_\mu + m)\,\psi + \frac{1}{2}\,\psi\,(\gamma_\mu^T \partial_\mu - m)\,\bar{\psi} \qquad (5.7)$$

In terms of the two-component spinors φ and χ this may be written

$$\begin{aligned} L = \frac{i}{2}\{&\varphi^*\sigma_i\partial_i\varphi + \varphi\sigma_i^T\partial_i\varphi^* - \chi^*\sigma_i\partial_i\chi - \chi\sigma_i^T\chi^*\partial_i\chi^* \\ &-\varphi^*\partial_0\varphi - \varphi\partial_0\varphi^* - \chi^*\partial_0\chi - \chi\partial_0\chi^* \\ &-m\varphi^*\chi + m\chi^*\varphi - m\varphi\chi^* + m\chi\varphi^*\} \end{aligned} \qquad (\cancel{6}^{5}.8)$$

Expressing here by means of relations (3.9-10) $\cancel{\chi}$ or (4.23) φ through χ we get

$$\begin{aligned} L = &-i\{\chi^*\sigma_i\partial_i\chi + \chi\sigma_i^T\partial_i\chi^* + \chi\partial_0\chi^* + \chi^*\partial_0\chi\} \\ &+m\frac{1}{2}(\varkappa+\varkappa^*)\{\chi\sigma_2\chi + \chi^*\sigma_2\chi^*\} \end{aligned} \qquad (5.9)$$

One easily verifies the hermicity of (5.9). Inversion of space (4.24) does not change the first term (independent of the mass constant m). The second term, containing the mass factor m transforms under a space inversion into

$$M\,\frac{1}{2}(\varkappa+\varkappa^*)\{\varkappa^2\chi^*\sigma_2\chi^* + \varkappa^{*2}\chi\sigma_2\chi\} \qquad (5.10)$$

We thus infer that for vanishing rest mass ($m = 0$) Lagrangean (5.9) is invariant with respect to space inversions for arbitrary $\varkappa$ satisfying condition (4.21). For non-vanishing rest mass ($m \neq 0$) we must put upon $\varkappa$ the further condition $\varkappa = \bar{\varkappa}$ This together with $|\varkappa|^2 = 1$ determines $\varkappa$ up to a sign factor $\varkappa = \pm 1$. Equantions (4.26) follow from (5.9) with $\varkappa = +1$

$$\begin{aligned} L = &-i\{\chi^*\sigma_i\partial_i\chi + \chi\sigma_i^T\partial_i\chi^* + \chi\partial_0\chi^* + \chi^*\partial_0\chi\} \\ &+m\{\chi\sigma_2\chi + \chi^*\sigma_2\chi^*\}. \end{aligned} \qquad (5.11)$$

An analogous situation presents itself for pseudospinors. Here to obtain invariance we have to assume (for $m \neq 0$) an imaginary $\varkappa$ ($\varkappa = \pm i$). This means that equation (4.27) gets just the same form as (4.26). In the two-component theory there is no difference between equations (4.9) and (4.10) apart from the transformation properties of the fields occurring in these equations (in (4.26) inversions of time are forbidden in (4.27) inversions of space are forbidden.

The above considerations reveal another deep-lying difference between the two-component theory described by equations (4.26-27) and the theory proposed by Lee and Yang., Landau and Salam. In the former the Lagrangean of the free fields is invariant (with proper choice of $\varkappa$) with respect to the allowed inversions (of space for spinors and of time for pseudospinors). In the latter an inversion brings the fields occuring in the Lagrangean into non-existing fields or (in an alternative interpretation) into fields describing the "antiworld". Thus in the theory of Lee and Yang, Landau and Salam non-invariance with respect to space and time inversions is an intrinsic property of the free particles. On the contrary equations (4.26) are invariant with respect to space inversions. The assymetry in the angular distribution results here from the non-invariance of the interaction Hamiltonian. For equations (4.27) the situation is just reversed with respect to space and time.

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Additional note. I received a preprint of Mr. K.M. Casá s paper in which similar problems are treated. There are essential differences as well in the approach to the subject as in some of the results especially those concerning the inversions of space and time in the two-component theory of Majorama.

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