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Abstract:

In this paper, first of all, a metric space Π of certain complex-valued functions of a point in space-time is introduced with a symmetrical, nondegenerate, indefinite metrics determined by the Klein-Gordon operator. The equations of the quantum theory of a scalar field with selfinteraction are then formulated as tensor-equations in the space Π . A new equation which allows to consider the problem of computing the Green functions as an "initial value problem" is derived and solved formally by the method of continual integration.

Introduction:

By inspecting the equations of the quantum field theory one gets the impression that one has to do with a sort of tensor-equations in the function-space of the external potentials. For instance, it is hardly possible to pass over without observing that the Feynman [1] "diffusion equation"

$$\frac{dT}{d(x^2)} = \frac{i}{2} \iint \frac{\delta^2 T}{\delta B_\mu(1)\, \delta B_\mu(2)}\, \delta_+(1,2)\, d\tau_1\, d\tau_2$$

has the form of a tensor-equation in the space of external electromagnetic potentials $B_\mu(x)$ with a metrics determined by the "contra-variant metric tensor" $\delta_{\mu\nu}\, \delta_+(1,2)$. The covariant metric tensor is $\delta_{\mu\nu}\, \delta_+^{-1}(1,2)$ where the function $\delta_+^{-1}(1,2)$ is defined by the equation

$$\int \delta_+(1,2)\, \delta_+^{-1}(2,3)\, d\tau_2 = \delta(1,3).$$

Another example is given by the Dyson [2] equation for the Green function $G(x,y)$ i.e.

$$G(x,y) = S^c(x,y) + S^c(x,x')\, \Sigma^*(x',y')\, G(y',y)\, d^{(4)}x'\, d^{(4)}y'.$$

It suffices to interpret the quantities G, S^c and Σ^* as tensors (with continuous tensor-indices) and we have again a tensor-equation. It would be possible to mention a series of further equations of the quantum field theory showing a more or less striking similarity with tensor equations in a function-space. This situation raises the conjecture that the tensor-features which appear at various places in the quantum field theory are not accidental but are manifestations of some general principle which remained unobserved up to the present time.

In what follows we shall consider (as a rather academic example) such a tensorial reformulation of the quantum theory of a scalar field with selfinteraction.

1. The Function-Space Π

Let us consider the set of all complex-valued functions $\psi(x)$ in space-time which fulfil the following boundary conditions:

I) $\psi(\vec{x},t)=0$ for $|\vec{x}|\Rightarrow\infty$ and all values of t,

II)

$$\frac{1}{c}\frac{\partial\psi(\vec{x},t)}{\partial t}-i\int V(\vec{x}-\vec{x}\,')\,\psi(\vec{x}\,',t)\,d^{(3)}x'=0 \qquad (1.1)$$

for $t\Rightarrow-\infty$, and

III)

$$\frac{1}{c}\frac{\partial\psi(\vec{x},t)}{\partial t}+i\int V(\vec{x}-\vec{x}\,')\,\psi(\vec{x}\,',t)\,d^{(3)}x'=0 \qquad (1.2)$$

for $t\Rightarrow+\infty$.

The "nucleus" V of the integral operator[1)] in (1.1) and (1.2) is defined as follows:

$$V(\vec{x}) = \frac{1}{(2\pi)^3} \int \left| \sqrt{\vec{k}^2 + \varkappa^2} \right| e^{i\vec{k}\vec{x}} \, d^{(3)}k \quad . \tag{1.3}$$

1) This operator was introduced by Landau and Peierls [3] for $\varkappa=0$. The equation (1.1) admits as solutions only waves with negative frequencies and the equation (1.2) only waves with positive frequencies. (It is understood that $e^{-i|\omega|t}$ corresponds to positive frequency).

The set of all functions $\psi(x)$ satisfying the above-mentioned conditions I) - III) will be called the "space Π". In the space Π we introduce the scalar product (ψ_1, ψ_2) and the distance s^2 of two functions ψ_1, ψ_2 by the following formulas:

$$(\psi_1, \psi_2) = \int \psi_1(x) \left[-\Box + \varkappa^2 \right] \psi_2(x) \, d^{(4)}x \tag{1.4}$$

and

$$s^2 = (\psi_1 - \psi_2 \, , \, \psi_1 - \psi_2) \quad . \tag{1.5}$$

The equations (1.4) and (1.5) determine the metrics of the space Π. Notice that the metrics is "indefinite" (s^2 can even be a complex quantity) but symmetrical. The symmetry is a consequence of the conditions I) - III). In fact, by the Green theorem and I) we get

$$(\psi_1, \psi_2) - (\psi_2, \psi_1) = \frac{1}{c}\Big[\int\Big(\psi_1 \frac{\partial \psi_2}{\partial t} - \psi_2 \frac{\partial \psi_1}{\partial t}\Big) d^{(3)}x\,\Big]_{t=-\infty}^{t=+\infty} . \tag{1.6a}$$

Then in virtue of (1.1), (1.2) and from $V(\vec{x}) = V(-\vec{x})$ the integrals on the right hand side of (1.6a) vanish and we have

$$(\psi_1, \psi_2) = (\psi_2, \psi_1) . \tag{1.6b}$$

Since the equation

$$(-\Box + \varkappa^2)\psi = 0$$

with the boundary conditions I) - III) has the only solution $\psi(x) \equiv 0$, the space Π does not contain isotropic functions[2] and hence its metrics is nondegenerate.

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[2] Let us remark that isotropic functions are orthogonal (in the sense of the equation (1.4) to all functions of the space Π.

As the metrics is of the usual type we can take the advantage of the standard tensor notation. The fundamental metric tensor is

$$g(x,x') = g(x',x) = (-\Box + \varkappa^2)\delta(x-x') = (-\Box' + \varkappa^2)\delta(x-x'). \tag{1.7}$$

We shall agree to consider the components (1.7) of g as <u>covariant</u> and to emphasize this convention we shall write $g(\overline{xx'})$ instead of $g(xx')$ in (1.7). (The bar over the "indices" x and x' of g will remind us that this are the <u>lower</u> indices). In accordance with this convention we shall consider the "index" (x) of the ψ's in (1.4) as an <u>upper</u> index and shall write it like $(\underline{x})$. ($\psi(\underline{x})$ are <u>contravariant</u> components of the vector ψ). Similar notation will be used also for higher order tensors in the space Π. Besides this the summation over a pair of indices $\underline{x}$, $\bar{x}$ i.e. the integration as in (1.4) or (1.5) will not be depicted explicitly. (Einstein's summation-rule of the tensor calculus in the space Π).

With these conventions the equation (1.4) for instance can be written in the form

$$(\psi_1, \psi_2) = \psi_1(\underline{x})\, g(\overline{xx'})\, \psi_2(\underline{x}') \,. \tag{1.4}$$

The contravariant components $g(\underline{xx'})$ of the metric tensor g are determined by the equation

$$g(\overline{xx'})\, g(\underline{x''x'}) = g(\bar{x}\, \underline{x}'') \tag{1.8}$$

where $g(\bar{x}\,\underline{x}'') = \delta(x - x'')$. By the help of the tensors $g(\overline{xx'})$ and $g(\underline{xx'})$ we can raise and lower, the "indices" in the usual manner. Thus, for instance, the relations giving $\psi(\bar{x})$ in terms of $\psi(\underline{x})$ and conversely read

$$\psi(\overline{x}) = g(\overline{x x'})\,\psi(\underline{x}') \,, \tag{1.9}$$

$$\psi(\underline{x}) = g(\underline{x x'})\,\psi(\overline{x}') \,. \tag{1.9'}$$

The <u>contravariant</u> ténsor $g(\underline{x x'})$ is the Green function of the Klein-Gordon equation for a problem with the boundary conditions (1.1), (1.2) and is connected with the Feynman-Dyson causal function $\Delta^F(x-x')$ by the relation

$$g(\underline{x x'}) = \frac{i}{2}\,\Delta^F(x-x') \,. \tag{1.10}$$

2. <u>Creation and Destruction Operators</u>

Let us define the operators $b_c(\underline{x})$ and $b_a(\underline{x})$ by the commutation rules

$$\left[b_c(\underline{x});\, b_a(\underline{x}')\right] = i\,g(\underline{x x'}) \,,$$
$$\left[b_a(\underline{x});\, b_a(\underline{x}')\right] = \left[b_c(\underline{x});\, b_c(\underline{x}')\right] = 0 \,. \tag{2.1}$$

From these operators we can construct the operators

$$\underline{I}(\underline{x x'}) = i\left(b_c(\underline{x})\, b_a(\underline{x}') - b_c(\underline{x}')\, b_a(\underline{x})\right)$$

which fulfil the commutation relations

$$\left[I(\underline{x x'});\, b_a(\underline{x}'')\right] = g(\underline{x' x''})\, b_a(\underline{x}) - g(\underline{x x''})\, b_a(\underline{x}') \,,$$
$$\left[\underline{I}(\underline{x x'});\, b_c(\underline{x}'')\right] = g(\underline{x' x''})\, b_c(\underline{x}) - g(\underline{x x''})\, b_c(\underline{x}') \,. \tag{2.2}$$

Finally, using (2.2) we obtain the relations

$$[I(\underline{x}\underline{x}'); I(\underline{x}''\underline{x}''')] = g(\underline{x}\underline{x}''')I(\underline{x}'\underline{x}'') + g(\underline{x}'\underline{x}'')I(\underline{x}\underline{x}''') - \\ - g(\underline{x}\underline{x}'')I(\underline{x}'\underline{x}''') - g(\underline{x}'\underline{x}''')I(\underline{x}\underline{x}') \tag{2.3}$$

which show that $I(\underline{x}\underline{x}')$ are the operators of infinitesimal rotations in the space Π, determining a representation of the group of rotations[3].

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[3] Golfand in the paper [9] constructed the operators of infinitesimal rotations in an euclidean space with infinite number of dimensions starting from creation and destruction operators of the usual type belonging to Fermi particles. Our operators $I(\underline{x}\underline{x}')$ are constructed from the so called causal operators belonging to Bose particles. (See below footnote [6]).

This representation is reducible since the operator

$$N = i\, g(\overline{\underline{x}''\underline{x}'''})\, \underset{c}{b}(\underline{x}'')\, \underset{a}{b}(\underline{x}''') = i\, \underset{c}{b}(\underline{x}'')\, \underset{a}{b}(\bar{\underline{x}}'')$$

which is not a multiple of the unit operator[4], commutes with $I(\underline{x}\underline{x}')$:

$$[I(\underline{x}\underline{x}'); N] = 0 \; . \tag{2.4}$$

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[4] The fact that N is not a multiple of the unit operator is clear since N does not commute for instance with the

operators $b_c(\underline{x})$ and $b_a(\underline{x})$:

$$[N; b_c(\underline{x})] = b_c(\underline{x}), \quad [N; b_a(\underline{x})] = -b_a(\underline{x})$$

Hence N is a scalar operator which remains unchanged under rotations. To find its eigenvalues n and to construct the eigenvectors Ψ satisfying the equation

$$i\,g(\overline{x x'})\, b_c(\underline{x})\, b_a(\underline{x}')\, \Psi = n\Psi, \tag{2.5}$$

we can start with the vector Ψ_0 satisfying the equation

$$b_a(\underline{x})\Psi_0 = 0 . \tag{2.6}$$

The existence of such a Ψ_0 is clear in the "standard" representation in which the operation of $b_c(\underline{x})$ on Ψ means simply a multiplication whereas $b_a(\underline{x})$ is given by the help of a functional derivative, namely

$$b_a(\underline{x})\Psi = -i\,g(\underline{x}\underline{x}')\frac{\delta}{\delta b_c(\underline{x}')}\Psi .$$

Then Ψ_0 is simply a functional which is independent of $b_c(\underline{x})$.

The vector Ψ_0 belongs obviously to the eigenvalue n = 0 of the operator N . Further eigenvectors Ψ of N can now be obtained by applying the operators $b_c(\underline{x})$ on Ψ_0 :

$$\Psi(\underline{x}) = b_c(\underline{x})\,\Psi_0 ,$$

$$\Psi(\underline{x_1, x_2}) = b_c(\underline{x_1})\, b_c(\underline{x_2})\, \Psi_0 ,$$

generally[5]

$$\Psi(\underline{x_1 \dots x_n}) = b_c(\underline{x_1}) \dots b_c(\underline{x_n})\, \Psi_0 \quad . \tag{2.7}$$

[5] In the paper [8] , in which a formalism similar to ours is developed, the space of all Ψ -vectors is denoted by the symbol Ω .

From (2.1) and (2.6) it follows that the application of the operator $b_a(\underline{x})$ on an eigenvector Ψ of the type (2.7) gives

$$\begin{aligned} b_a(\underline{x})\, \Psi(\underline{x_1 \dots x_n}) = & -i\, g(\underline{x x_1})\, \Psi(\underline{x_2 \dots x_n}) - \\ & -i\, g(\underline{x x_2})\, \Psi(\underline{x_1, x_3 \dots x_n}) - \dots - \\ & -i\, g(\underline{x x_n})\, \Psi(\underline{x_1 \dots x_{n-1}}) \ . \end{aligned} \tag{2.8}$$

Using (2.7) and (2.8) it is easy to show that the vector $\Psi(\underline{x_1 \dots x_n})$ is an eigenvector of N belonging to the eigenvalue n where n = 0, 1, 2 ... .

Considering the analogy of our operators $b_c(\underline{x})$ and $b_a(\underline{x})$ with the usual creation and destruction operators of Bose particles we shall call our operators also creation and destruction operators[6].

6) They belong to the category of the so called <u>causal</u> operators as introduced for instance by Novozhilow [4]. See also [5] and [8].

Similarly the operator $\mathcal{N}$ will be called the operator of the total number of particles and the vector Ψ_0 the vacuum vector 7).

7) The problem of the connexion between the vectors Ψ of the space Ω and the usual state-vectors of the Hilbert space (as used in the customary formulation of the quantum field theory) is not at all trivial or simple and will be considered and solved in a subsequent paper.

Notice that the operators $b(\underline{x}) = b_a(\underline{x}) + b_c(\underline{x})$ are commutative

$$[b(\underline{x}) ; b(\underline{x}')] = 0 . \tag{2.9}$$

This circumstance will be of importance for our further consideration.

For the purpose of normalization of the Ψ-vectors we introduce a symmetrical matrix D (metric matrix in the Ω-space) which <u>in the interaction-free case</u> has to satisfy the following conditions: 8)

$$\begin{aligned} b_c^T(\underline{x}) D &= D\, b_a(\underline{x}) , \\ b_a^T(\underline{x}) D &= D\, b_c(\underline{x}) . \end{aligned} \tag{2.10}$$

($b_c^T(\underline{x})$ and $b_a^T(\underline{x})$ are the transposed of the matrices $b_c(\underline{x})$ and $b_a(\underline{x})$).

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8) The existence of the matrix D can be made clear as follows: From (2.1) we obtain the relations

$$[b_a^T(\underline{x}) ; b_c^T(\underline{x}')] = i g(\underline{x x'}), \qquad \text{etc.}$$

which are of the same form as (2.1).
Therefore a similarity transformation

$$b_a^T(\underline{x}) = D\, b_c(\underline{x})\, D^{-1}$$
$$b_c^T(\underline{x}') = D\, b_a(\underline{x}')\, D^{-1}$$

must exist which is equivalent to (2.10).

The vacuum vector Υ_0 can now be normalized by the equation

$$\Upsilon_0^T D \Upsilon_0 = 1 \quad . \tag{2.11}$$

The normalization of other Υ-vectors is given already by (2.7) and (2.11). Thus, for instance, it holds

$$\begin{aligned} \Upsilon^T(\underline{x}) D \Upsilon(\underline{x}') &= -i g(\underline{x x'}), \\ \Upsilon^T(\underline{x_1 x_2}) D \Upsilon(\underline{x_1' x_2'}) &= -g(\underline{x_1 x_1'})\, g(\underline{x_2 x_2'}) - \\ &\quad - g(\underline{x_1 x_2'})\, g(\underline{x_2 x_1'}), \quad \text{etc.} \end{aligned} \tag{2.12}$$

Besides (2.12), it is easy to show that the following orthogonality relations hold:

$$\underline{\Upsilon^T(x_1 \ldots x_n)} \, D \, \underline{\Upsilon(x_1' \ldots x_m')} = 0, \quad m \neq n. \tag{2.13}$$

3. The Green Tensors

In this paragraph we introduce the Green functions corresponding to a scalar field which is in a "quadratic"[9] interaction with itself. Even though such a field, apparently, does not exist in nature, its investigation has at least some heuristic value.[10] We shall show below that the Green functions transform as tensors under the rotations in the space Π . Therefore we shall call them Green tensors.

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[9] Under the "quadratic" self-interaction we understand an interaction in virtue of which in the simplest elementary act the particles A corresponding to the field are created or annihilated according to the scheme

$$A \rightleftarrows A' + A''.$$

In this sense a "linear" interaction would mean that the simplest process is $A \rightleftarrows B$.

10) Hurst and Thirring e.g. investigated this case in connection with the question of the convergence of the expansion series in the perturbation theory. [6,7].

The interaction is introduced by substituting for the conditions (2.10) the more general conditions

$$\begin{aligned}\left[b_c^T(\underline{x}) + \alpha\gamma(\underline{x}\,\overline{x'x''})\, b^T(\underline{x'})\, b^T(\underline{x''})\right]\Delta &= \Delta\, b_a(\underline{x}),\\ \left[b_a^T(\underline{x}) - \alpha\gamma(\underline{x}\,\overline{x'x''})\, b^T(\underline{x'})\, b^T(\underline{x''})\right]\Delta &= \Delta\, b_c(\underline{x})\end{aligned} \tag{3.1}$$

where $b(\underline{x}) = b_a(\underline{x}) + b_c(\underline{x})$, α is a coupling constant, $\gamma(\underline{x}\,\overline{x'x''}) = g(\underline{x x'''})\,\gamma(\overline{x''' x' x''})$ is some form-factor, $\gamma(\overline{x x' x''})$ being a symmetrical tensor in all its indices $\overline{x}, \overline{x}', \overline{x}''$ [11] and Δ is again a symmetrical matrix, the explicite form of which will be determined below. From (3.1) we obtain

$$b^T(\underline{x})\,\Delta = \Delta\, b(\underline{x}) \,. \tag{3.2}$$

The Green tensors are defined by the formula[12]

$$G(\underline{x_1 \dots x_n}) = \frac{1}{S_0}\, \Upsilon_0'^T \Delta\, b(\underline{x_1}) \dots b(\underline{x_n})\, \Upsilon_0' \tag{3.3}$$

where $S_0 = \Upsilon_0'^T \Delta\, \Upsilon_0$.

11) In the local theory $\gamma(\overline{xx'x''}) = \delta(x-x')\,\delta(x-x'')$.

12) In the interaction-free case ($\alpha = 0$, $\Delta = D$) we have in virtue (2.11), (2.6), (2.10) and (2.12)

$$iG^{(0)}(\underline{xx'}) = \Upsilon^T(\underline{x})\,D\,\Upsilon(\underline{x'}) = g(\underline{xx'})$$

which has been interpreted as the metric tensor in the Π -space. If we should introduce the base of vectors χ as follows

$$\chi_0 = \frac{1}{\sqrt{S_0}}\Upsilon_0\,,$$

$$\chi(\underline{x}) = \frac{1}{\sqrt{S_0}}\,b(\underline{x})\,\Upsilon_0\,,$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$\chi(\underline{x_1\ldots x_n}) = \frac{1}{\sqrt{S_0}}\,b(\underline{x_1})\ldots b(\underline{x_n})\,\Upsilon_0$$

and should define the scalar products of these vectors by means of the matrix Δ [i.e. as $\chi^T(\underline{x_1\ldots x_n})\,\Delta\,\chi(\underline{x'_1\ldots x'_m})$] we could look also upon the whole set of the Green functions (3.3) as components of the metric tensor in some other space. This point of view would lead to another "geometrical model" of the quantum field theory under consideration. However, we do not intend to develop it further in this paper.

Now we can derive an equation for the generating functional $Z[J(\bar{x})]$ of the Green tensors (3.3). Using (2.1), (3.1) and (2.6) we can perform $G(\underline{x' x_1 \cdots x_n})$ as follows

$$G(\underline{x' x_1 \ldots x_n}) = \frac{1}{S_0} \Upsilon_0'^T \Delta\, b_a(\underline{x'})\, b(\underline{x_1}) \ldots b(\underline{x_n})\, \Upsilon_0' + $$

$$+ \frac{1}{S_0} \Upsilon_0'^T \Delta\, b_c(\underline{x'})\, b(\underline{x_1}) \ldots b(\underline{x_n})\, \Upsilon_0' =$$

$$= -ig(\underline{x' x_1})\, G(\underline{x_2 \ldots x_n}) - ig(\underline{x' x_2})\, G(\underline{x_1 x_3 \cdots x_n}) - \cdots -$$

$$- ig(\underline{x' x_n})\, G(\underline{x_1 \ldots x_{n-1}}) - \alpha \gamma(\underline{x'} x'' x''')\, G(\underline{x'' x''' x_1 \ldots x_n}) .$$

Multiplying this equation by $g(\overline{x x'})$ we get

$$g(\overline{x x'})\, G(\underline{x' x_1 \cdots x_n}) + \alpha \gamma(\overline{x x' x''})\, G(\underline{x' x'' x_1 \cdots x_n}) =$$

$$= -ig(\bar{x}\, \underline{x_1})\, G(\underline{x_2 \cdots x_n}) - \cdots - ig(\bar{x}\, \underline{x_n})\, G(\underline{x_1 \ldots x_{n-1}}) . \qquad (3.4)$$

Note separately the special cases

$$g(\overline{x x'})\, G(\underline{x'}) + \alpha \gamma(\overline{x x' x''})\, G(\underline{x' x''}) = 0 , \qquad (3.5)$$

$$g(\overline{x x'})\, G(\underline{x' x_1}) + \alpha \gamma(\overline{x x' x''})\, G(\underline{x' x'' x_1}) = -ig(\bar{x}\, \underline{x_1}) \qquad (3.6)$$

which can also be deduced in similar way like (3.4).

Further multiplying (3.4) by the product $J(\bar{x}_1)\dots J(\bar{x}_n)$, then dividing by $n!$ and summing up over n from n=2 to $n=\infty$, we obtain after some rearrangement on the right hand side the equation

$$\sum_{n=2}^{\infty}\frac{1}{n!}g(\overline{xx'})\,G(\underline{x'x_1\dots x_n})\,J(\bar{x}_1)\dots J(\bar{x}_n)+$$

$$+\alpha\sum_{n=2}^{\infty}\frac{1}{n!}\gamma(\overline{xx'x''})\,G(\underline{x'x''x_1\dots x_n})\,J(\bar{x}_1)\dots J(\bar{x}_n)=$$

$$=-i\,J(\bar{x})\sum_{n=1}^{\infty}\frac{1}{n!}G(\underline{x_1\dots x_n})\,J(\bar{x}_1)\dots J(\bar{x}_n)\ .$$

To this equation we add the equation (3.5) and the equation (3.6) multiplied by $J(\bar{x}_1)$ and obtain

$$g(\overline{xx'})\Big[G(\underline{x'})+\sum_{n=1}^{\infty}\frac{1}{n!}G(\underline{x'x_1\dots x_n})\,J(\bar{x}_1)\dots J(\bar{x}_n)\Big]+$$

$$+\alpha\gamma(\overline{xx'x''})\Big[G(\underline{x'x''})+\sum_{n=1}^{\infty}\frac{1}{n!}G(\underline{x'x''x_1\dots x_n})\,J(\bar{x}_1)\dots J(\bar{x}_n)\Big]=$$

$$=-i\,J(\bar{x})\Big[1+\sum_{n=1}^{\infty}\frac{1}{n!}G(\underline{x_1\dots x_n})\,J(\bar{x}_1)\dots J(\bar{x}_n)\Big]\ .$$

This last equation can be written down in the form

$$g(\overline{xx'})\frac{\delta Z}{\delta J(\overline{x'})}+\alpha\gamma(\overline{xx'x''})\frac{\delta^2 Z}{\delta J(\overline{x'})\,\delta J(\overline{x''})}=-iJ(\overline{x})Z \qquad (3.7)$$

where

$$Z=1+\sum_{n=1}^{\infty}\frac{1}{n!}G(\underline{x_1\ldots x_n})\,J(\overline{x}_1)\ldots J(\overline{x}_n) \qquad (3.8)$$

The equation (3.7), considered as an equation determining the functional $Z[J]$, must be completed by the condition $Z[0]=1$.

Now let us return to the equations (3.1) and look for Δ satisfying these equations. Assume $\Delta = DS = S^T D$ where D is the matrix satisfying (2.10). Then S has to fulfil the equations

$$\begin{aligned}[b_c(\underline{x});S]&=\alpha\gamma(\underline{x}\overline{x'x''})\,b(\underline{x}')\,b(\underline{x}'')\,S\,,\\ [b_a(\underline{x});S]&=-\alpha\gamma(\underline{x}\overline{x'x''})\,b(\underline{x}')\,b(\underline{x}'')\,S\,.\end{aligned} \qquad (3.9)$$

Develop S in a series according to powers of α :

$$S=S^{(0)}+S^{(1)}+S^{(2)}+\ldots$$

Then from (3.9) we get a recurrent system of equations

$$\begin{aligned}[b_c(\underline{x});S^{(n)}]&=\alpha\gamma(\underline{x}\overline{x'x''})\,b(\underline{x}')\,b(\underline{x}'')\,S^{(n-1)}\,,\\ [b_a(\underline{x});S^{(n)}]&=-\alpha\gamma(\underline{x}\overline{x'x''})\,b(\underline{x}')\,b(\underline{x}'')\,S^{(n-1)}\end{aligned}$$

which can be satisfied by putting

$$S^{(n)} = \frac{(-i\alpha)^n}{n!} L^n$$

where

$$L = \frac{1}{3} g(\overline{x\, x'\, x''})\, b(\underline{x})\, b(\underline{x}')\, b(\underline{x}'')$$

Hence the matrix S can be expressed in the form

$$S = e^{i\alpha L} \quad . \tag{3.10}$$

4. Transformation Properties of the Green Functions

We shall investigate the transformation properties of the Green functions first of all in case of free field ($\alpha = 0$). Under the rotations in the Π-space the operators $b(\underline{x})$ transform as contravariant vectors, i.e.

$$b'(\underline{x}') = c(\underline{x}'\, \overline{x})\, b(\underline{x}) \tag{4.1}$$

where

$$c(\underline{x}\, \overline{x}')\, c(\underline{x}''\, \overline{x}''')\, g(\underline{x}'\, x''') = g(\underline{x\, x''}) \quad . \tag{4.2}$$

The operators $b_a(\underline{x})$ and $b_c(\underline{x})$ transform in the same manner as $b(\underline{x})$. Since $b'_a(\underline{x})$ and $b'_c(\underline{x})$ fulfil (in virtue of (4.2)) the same commutation relations as $b_a(\underline{x})$ and $b_c(\underline{x})$, there exists a matrix R for which

$$b'(\underline{x}') = R^{-1}\, b(\underline{x}')\, R = c\,(\underline{x}'\, \overline{x})\, b(\underline{x}) \quad . \tag{4.3}$$

Clearly R in a matrix representation of the rotation (4.1). For infinitesimal transformations (4.1) we have[13]

$$C(\underline{x}'\bar{x}) = g(\underline{x}'\bar{x}) + \varepsilon(\underline{x}'\bar{x})$$

and R is given by

$$R = I + \frac{1}{2}\, I(\underline{x}\underline{x}')\,\varepsilon(\bar{x}\bar{x}') \quad .$$

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[13] Let us remark that the group of rotations (4.1) contains as a <u>subgroup</u> also the group of inhomogeneous Lorentz transformations. Indeed, under an infinitesimal transformation

$$x'_\mu = (\delta_{\mu\nu} + \varepsilon_{\mu\nu})x_\nu + \varepsilon_\mu$$

or

$$x_\nu = x'_\nu - (\varepsilon_{\nu\rho}x'_\rho + \varepsilon_\nu)$$

a Lorentz scalar $\psi(\underline{x})$ transforms as follows

$$\psi'(\underline{x}') = \psi(\underline{x}) = [g(\underline{x}'\bar{x}'') + \varepsilon(\underline{x}'\bar{x}'')]\,\psi(\underline{x}'')$$

where

$$\varepsilon(\underline{x}'\bar{x}'') = -\varepsilon(\bar{x}''\underline{x}') =$$

$$= \left[\frac{1}{2}\varepsilon_{\nu\rho}\left(x'_\nu \frac{\partial}{\partial x'_\rho} - x'_\rho \frac{\partial}{\partial x'_\nu}\right) - \varepsilon_\nu \frac{\partial}{\partial x'_\nu}\right] g(\underline{x}'\bar{x}'') \, .$$

Inserting for R and $C(\underline{x}'\bar{x})$ into (4.3) we obtain the condition

$$[I(\underline{x}\underline{x}');\, b(\underline{x}'')] = g(\underline{x}'\underline{x}'')\,b(\underline{x}) - g(\underline{x}\underline{x}'')\,b(\underline{x}') \tag{4.4}$$

which is consistent with (2.2). From (2.10) we have

$$I^T(\underline{xx'})\, D = - D I(\underline{xx'}) \tag{4.5}$$

so that

$$R^T D = D R^{-1} \, . \tag{4.6}$$

Now we can investigate the transformations of the Green functions $G^{(0)}(\underline{x_1 \ldots x_n})$ (corresponding to free particles) under the substitution

$$\Psi_0 \to \Psi_0' = R \Psi_0 \, . \tag{4.7}$$

First of all, it follows from (4.6) that the quantity $S_0^{(0)}$ is scalar since due to (4.6)

$$S_0^{(0)\prime} = \Psi_0'^T D \Psi_0' = \Psi_0^T D \Psi_0 = S_0^{(0)} = 1 \, .$$

Therefore, using (4.6), (4.3) and (4.1) we can write

$$G^{(0)\prime}(\underline{x_1' \ldots x_n'}) \equiv \frac{1}{S_0^{(0)\prime}} \Psi_0'^T D\, b(\underline{x_1'}) \ldots b(\underline{x_n'}) \Psi_0' =$$

$$= \frac{1}{S_0^{(0)}} \Psi_0'^T R^T D\, b(\underline{x_1'}) \ldots b(\underline{x_n'}) R \Psi_0 =$$

$$= c(\underline{x_1'} \, \overline{x}_1) \ldots c(\underline{x_n'} \, \overline{x}_n)\, G^{(0)}(\underline{x_1 \ldots x_n}) \, . \tag{4.8}$$

From this we see that the Green functions $G^{(0)}(\underline{x_1 \dots x_n})$ transform as tensors under rotations which entitled us to give them the name Green tensors. Of course, from (2.6) we have

$$\Psi_0' = R\Psi_0 = \Psi_0 \tag{4.7a}$$

so that also

$$G^{(0)\prime}(\underline{x_1 \dots x_n}) = G^{(0)}(\underline{x_1 \dots x_n}), \tag{4.9}$$

i.e. the Green tensors in case of $\alpha = 0$ are reproduced by (4.8). Indeed, in this case, they can be expressed in terms of the metric tensor $g(\underline{x x'})$.

Before we proceed to investigate the transformation properties of the Green function in case of $\alpha \neq 0$, we derive some relations which will be useful later. First of all, we rewrite the equations (3.1) in the form

$$\begin{aligned} a_c^T(\underline{x})\,\Delta &= \Delta\, a_a(\underline{x}) \\ a_a^T(\underline{x})\,\Delta &= \Delta\, a_c(\underline{x}) \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} a_c(\underline{x}) &= b_c(\underline{x}) + \tfrac{1}{2}\alpha\, g(\underline{x\, x' x''})\, b(\underline{x'})\, b(\underline{x''}), \\ a_a(\underline{x}) &= b_a(\underline{x}) - \tfrac{1}{2}\alpha\, g(\underline{x\, x' x''})\, b(\underline{x'})\, b(\underline{x''}). \end{aligned} \tag{4.11}$$

It is easy to show that $a_c(\underline{x})$ and $a_a(\underline{x})$ fulfil the same commutation rules as $b_c(\underline{x})$ and $b_a(\underline{x})$. Therefore there exists a similarity transformation

$$a_a(\underline{x}) = U^{-1} b_a(\underline{x})\, U$$
$$a_c(\underline{x}) = U^{-1} b_c(\underline{x})\, U \; . \tag{4.12}$$

Define the operators

$$\mathcal{J}(\underline{x x'}) = i\left(a_c(\underline{x})\, a_a(\underline{x'}) - a_c(\underline{x'})\, a_a(\underline{x})\right) = U^{-1} \underline{T}(\underline{x x'})\, U . \tag{4.13}$$

From (4.10) we have

$$\mathcal{J}^T(\underline{x x'})\, \Delta = - \Delta\, \mathcal{J}(\underline{x x'})$$

and

$$\mathcal{R}^T \Delta = \Delta\, \mathcal{R}^{-1} \tag{4.14}$$

if

$$\mathcal{R} = I + \tfrac{1}{2}\, \mathcal{J}(\underline{x x'})\, \varepsilon(\overline{x x'}) = U^{-1} R\, U .$$

From (4.11) we get $a(\underline{x}) = a_a(\underline{x}) + a_c(\underline{x}) = b(\underline{x})$
and then from (4.12)

$$[\, b(\underline{x}),\; U] = 0 \tag{4.15}$$

and therefore from (4.4)

$$[\mathcal{J}(\underline{x x'}),\, b(\underline{x''})] = g(\underline{x' x''})\, b(\underline{x}) - g(\underline{x x''})\, b(\underline{x'}) \tag{4.16}$$

so that also

$$\mathcal{R}^{-1} b(\underline{x}') \mathcal{R} = c(\underline{x}' \bar{x}) b(\underline{x}) . \tag{4.17}$$

Now we can already prove that the Green functions (3.3) transform as tensors under the substitution

$$\Upsilon_0 \to \Upsilon_0' = \mathcal{R} \Upsilon_0 .$$

Again from (4.14) we can see that S_0 is invariant, and from (4.14), (4.17) we find at once that

$$G'(\underline{x_1' \dots x_n'}) = c(\underline{x_1'} \bar{x}_1) \dots c(\underline{x_n'} \bar{x}_n) G(\underline{x_1 \dots x_n}) . \tag{4.18}$$

Notice that in case of $\alpha \neq 0$ the Green tensors are not reproduced by (4.18) since $\mathcal{R}\Upsilon_0 \neq \Upsilon_0$.

Inserting from (4.12) into (4.10) and using (2.10) we find

$$\Delta = U^T D U . \tag{4.19}$$

Therefore the Green tensors can be written also in the form

$$G(\underline{x_1 \dots x_n}) = \frac{1}{S_0} \phi^T D b(x_1) \dots b(x_n) \phi \tag{4.20}$$

where

$$\phi = U \Upsilon_0 . \tag{4.21}$$

Explicite expression for U can be obtained from the equations (4.11) and (4.12) by the same method which was used at the end of § 3. We find that

$$U = e^{-\frac{i\alpha}{6} \gamma(\overline{x x' x''}) b(\underline{x}) b(\underline{x}') b(\underline{x}'')} \tag{4.22}$$

Comparing (4.22) with (3.10) we see that

$$S = U^2 \tag{4.23}$$

and therefore also $\Delta = DS = DU^2$. This is consistent with (4.19) since from (4.22) and from $b^T(\underline{x})D = D\,b(\underline{x})$ we have $U^T D = DU$.

Notice finally that under the rotations ϕ transforms according to

$$\phi' = R\phi . \tag{4.24}$$

5. Equation with Functional Derivatives for $\Xi(\alpha,\psi)$

For determining the Green tensors from the general formula (4.20) it is necessary to know the vector ϕ as given by (4.21). Evaluating of $U\Psi_0$ meets with difficulties connected with the disentanglement of the operator U. Usually such an operator is expanded in powers of α and individual terms are rearranged into the normal products of creation and destruction operators. This method is convenient if the coupling parameter α is small enough. If α is large, it is necessary to look for other methods.

In the following we shall derive an equation, by the help of which, as we hope, it will be possible to evaluate ϕ without using expansions in powers of α . For this purpose we introduce the functional

$$\Xi(\alpha,\psi) = e^{\alpha b(\underline{x})\frac{\delta}{\delta\psi(\underline{x})}}\; e^{-\frac{i}{6}\gamma(\overline{x_1 x_2 x_3})\, b(\underline{x}_1)\, b(\underline{x}_2)\,\psi(\underline{x}_3)}\,\Psi_0 \tag{5.1}$$

where $\psi(\underline{x})$ are functions of the space Π. Then it holds

$$\Xi(\alpha, 0) = \phi \tag{5.2}$$

where ϕ is given by the formula (4.21).

Proof: Let us develop the operator $e^{\alpha b(\underline{x}) \frac{\delta}{\delta\psi(\underline{x})}}$ in powers of α:

$$e^{\alpha b(\underline{x}) \delta/\delta\psi(\underline{x})} = 1 + \alpha b(\underline{x}) \frac{\delta}{\delta\psi(\underline{x})} + \frac{1}{2!}\left(\alpha b(\underline{x}) \frac{\delta}{\delta\psi(\underline{x})}\right)^2 + \cdots$$

Since it holds

$$\left(\alpha b(\underline{x}) \frac{\delta}{\delta\psi(\underline{x})}\right)^n e^{-\frac{i}{6}\gamma(\overline{x_1 x_2 x_3}) b(\underline{x}_1) b(\underline{x}_2) \psi(\underline{x}_3)} \Upsilon_0 =$$

$$= \left(-\frac{i\alpha}{6}\gamma(\overline{x x' x''}) b(\underline{x}) b(\underline{x}') b(\underline{x}'')\right)^n e^{-\frac{i}{6}\gamma(\overline{x_1 x_2 x_3}) b(\underline{x}_1) b(\underline{x}_2) \psi(\underline{x}_3)} \Upsilon_0$$

we obtain from (5.1)

$$\Xi(\alpha, \psi) = e^{-\frac{i\alpha}{6}\gamma(\overline{x x' x''}) b(\underline{x}) b(\underline{x}') b(\underline{x}'')} \cdot e^{-\frac{i}{6}\gamma(\overline{x_1 x_2 x_3}) b(\underline{x}_1) b(\underline{x}_2) \psi(\underline{x}_3)} \Upsilon_0$$

and putting $\psi(\underline{x}_3) \equiv 0$ we get (5.2).

The functional $\Xi(\alpha, \psi)$ as defined by (5.1) represents the solution of the equation

$$\frac{d\Xi}{d\alpha} = b(\underline{x}) \frac{\delta \Xi}{\delta\psi(\underline{x})} \tag{5.3}$$

satisfying also the initial condition

$$\Xi(0, \psi) = e^{-\frac{i}{6}\gamma(\overline{x_1 x_2 x_3}) b(\underline{x}_1) b(\underline{x}_2) \psi(\underline{x}_3)} \Upsilon_0 . \tag{5.4}$$

This condition can be put into another form if we apply the operator $b_a(\underline{x})$. Using (2.1) and (2.6) we obtain

$$b_a(\underline{x}) \Xi(0, \psi) = -\frac{1}{3}\gamma(\overline{\underline{x}\, x' x''}) b(\underline{x}') \psi(\underline{x}'') \Xi(0, \psi) . \tag{5.5}$$

This last equation can be solved directly in the "standard" representation $b_a(\underline{x}) = -i g(\underline{x}\underline{x}')\, \delta/\delta b_c(\underline{x}')$.

The solution reads[14)]

$$\Xi(0,\psi,b_c) = e^{-\frac{1}{2}A^{-1}(\overline{x''x})\,B(\underline{x}\,\overline{x}')\,b_c(\underline{x}')\,b_c(\underline{x}'')}\,\chi_0'$$

where

$$B(\underline{x}\,\overline{x}') = \frac{i}{3}\,\gamma(\underline{x}\,\overline{x}'\overline{x}_1)\,\psi(\underline{x}_1)$$

and $A^{-1}(\overline{x''x})$ is defined by the equation

$$A^{-1}(\overline{x''x})\,A(\underline{x}'\underline{x}) = g(\overline{x}''\underline{x}')$$

with

$$A(\underline{x}\underline{x}') = -i g(\underline{x}\underline{x}') - \frac{i}{3}\,\gamma(\underline{x}\underline{x}'\overline{x}_2)\,\psi(\underline{x}_2)$$

---

14) This remark is owing to V. Votruba.

The equation (5.3) reminds of the Dirac equation. The analogy is useful for investigating the properties of the equation (5.3). For instance, quite similarly as in case of the Dirac equation, one can prove the covariance of the equation (5.3) under the rotations in the space Π. It is easily found that under the rotation

$$\psi'(\underline{x}') = C(\underline{x}'\,\overline{x})\,\psi(\underline{x})$$

the functional Ξ transforms in the same way as ϕ, i.e.

$$\Xi'(\alpha,\psi') = R\,\Xi(\alpha,\psi)\,.$$

Thus, to show the covariance of (5.3), we multiply this equation by R and obtain

$$\frac{d\Xi'}{d\alpha} = R\,b(\underline{x})\,R^{-1}\frac{\delta\Xi'}{\delta\psi(\underline{x})} = R\,b(\underline{x})\,R^{-1}\frac{\delta\psi'(\underline{x}')}{\delta\psi(\underline{x})}\cdot\frac{\delta\Xi'}{\delta\psi'(\underline{x}')} =$$

$$= R\, b(\underline{x})\, R^{-1} c(\underline{x}'\bar{x}) \frac{\delta \Xi'}{\delta \psi'(\underline{x}')} = b(\underline{x}') \frac{\delta \Xi'}{\delta \psi'(\underline{x}')} =$$

$$= b(\underline{x}) \frac{\delta \Xi'}{\delta \psi'(\underline{x})} .$$

Let us remark that the initial condition (5.5) is not covariant under the whole group of rotations since the tensor $\gamma(\underline{x}\,\overline{x'x''})$ is, generally, not reproduced by all transformations of the group. It is, of course, in any case reproduced by the subgroup of the inhomogeneous Lorentz transformations but one can expect that there exist distinguished formfactors which, in addition, are reproduced also by more general transformations.

In complete analogy with the Dirac equation we can derive also an equation of the type of continuity equation. Define

$$\varrho = \Xi^T D\, \Xi ,$$
$$j(\underline{x}) = - \Xi^T D\, b(\underline{x})\, \Xi .$$

Then it holds

$$\frac{d\varrho}{d\alpha} + \frac{\delta j(\underline{x})}{\delta \psi(\underline{x})} = 0 . \tag{5.6}$$

Besides the transposed equation of (5.3), i.e.

$$\frac{d\Xi^T}{d\alpha} = \frac{\delta \Xi^T}{\delta \psi(\underline{x})} b^T(\underline{x}) \tag{5.7}$$

also the relation $D\, b(\underline{x}) = b^T(\underline{x}) D$ was used.

Generalizing (5.6) it is possible to obtain a new system of coupled equations for the Green tensors. For the sake of simplicity we shall write out these equations for the modified Green tensors

$$\mathcal{Y}(\underline{x}_1 \dots \underline{x}_n) = \Xi^T D\, b(\underline{x}_1) \dots b(\underline{x}_n)\, \Xi \; . \tag{5.8}$$

Taking the α -derivative and using (5.3) and (5.7) we obtain

$$\frac{d\,\mathcal{Y}(\underline{x}_1 \dots \underline{x}_n)}{d\alpha} = \frac{\delta \mathcal{Y}(\underline{x}\,\underline{x}_1 \dots \underline{x}_n)}{\delta \psi(\underline{x})} \; . \tag{5.9}$$

The system (5.9) must be supplemented by the initial conditions (obtained by putting $\alpha = 0$ in (5.8) and using (5.4)):

$$\mathcal{Y}^{(0)}(\underline{x}_1 \dots \underline{x}_n) = \Upsilon_0^{\prime T} D\, e^{-\frac{i}{3}\gamma(\overline{x x' x''})\, b(\underline{x})\, b(\underline{x}')\, \psi(\underline{x}'')}\, b(\underline{x}_1) \dots b(\underline{x}_n)\, \Upsilon_0' \; .$$

These tensors $\mathcal{Y}^{(0)}$ satisfy the recurrent system of equations

$$[g(\overline{x x'}) + \tfrac{2}{3}\gamma(\overline{x x' x''})\psi(\underline{x}'')]\,\mathcal{Y}^{(0)}(\underline{x}'\underline{x}_1 \dots \underline{x}_n) =$$
$$= -i g(\overline{x}\,\underline{x}_1)\,\mathcal{Y}^{(0)}(\underline{x}_2 \dots \underline{x}_n) - \dots - i g(\overline{x}\,\underline{x}_n)\,\mathcal{Y}^{(0)}(\underline{x}_1 \dots \underline{x}_{n-1}) , \tag{5.10}$$

$$[g(\overline{x x'}) + \tfrac{2}{3}\gamma(\overline{x x' x''})\psi(\underline{x}'')]\,\mathcal{Y}^{(0)}(\underline{x}') = 0 \; . \tag{5.11}$$

The equations (5.9) - (5.11) are equivalent to the equations (5.3) and (5.5) and mean, in fact, the record of these equations in a certain representation of the matrices $b(\underline{x})$. Notice that the equations (5.10) are in some respect simpler than (3.4). Of course, in addition to (5.10) we have now also the equations (5.9).

Let us make finally a look on the problem of solving the equation (5.3). One can exploit also in this instance the analogy with the Dirac equation. Thus e.g. one can use the method of the development of $\Xi(\alpha, \psi)$ into a superposition of "monochromatic plane waves" which makes it possible avoid the expansions in powers of α . In the present paper, however, we shall not discuss this method but instead we shall be satisfied by shortly showing how one can deal with the problem of determining $\Xi(\alpha,\psi)$ as an "initial value problem," i.e. how one can express $\Xi(\alpha,\psi)$ explicitly in terms of $\Xi(0,\psi)$.

For this purpose we define the functional $F(\alpha,\psi)$ by the equation (5.3) with the initial condition

$$F(0,\psi) = \delta(\psi) \tag{5.12}$$

where $\delta(\psi)$ is the Dirac delta-functional in the space Π. In analogy with the formula (5.1) the functional $F(\alpha,\psi)$ can be written in the form

$$F(\alpha,\psi) = e^{\alpha b(\underline{x})\delta/\delta\psi(\underline{x})}\,\delta(\psi) = e^{\alpha b(\underline{x})\frac{\delta}{\delta\psi(\underline{x})}}\int e^{ip(\bar{x}')\psi(\underline{x}')}\,d\left(\frac{p}{2\pi}\right) =$$

$$= \int e^{i\alpha p(\bar{x})b(\underline{x})}\cdot e^{ip(\bar{x}')\psi(\underline{x}')}\,d\left(\frac{p}{2\pi}\right).$$

The first factor can be expressed by the help of normal products of creation and destruction operators in the form

$$e^{i\alpha p(\bar{x})b(\underline{x})} = e^{\frac{i}{2}\alpha^2 g(\underline{x}\underline{x}')p(\bar{x})p(\bar{x}')}\,\mathcal{N}\,e^{i\alpha p(\bar{x})b(\underline{x})}. \tag{5.13}$$

To prove this equation (5.13) we first of all remark that the left hand side, which will be denoted by $H(\alpha)$, satisfies the equation

$$dH/d\alpha = ip(\bar{x})b(\underline{x})H$$

with the initial condition $H(0)=1$. Taking the α-derivative of the right hand side of (5.13), which will be denoted by $\tilde{H}(\alpha)$, we obtain

$$d\tilde{H}/d\alpha = i\alpha g(\underline{x}\underline{x}')p(\bar{x})p(\bar{x}')\tilde{H} + ip(\bar{x})b_c(\underline{x})\tilde{H} + \tilde{H}ip(\bar{x})b_a(\underline{x}).$$

Since it holds

$$[b_a(\underline{x}) ; \tilde{H}] = \alpha\, g(\underline{x x'})\, p(\bar{x}')\, \tilde{H},$$

we find that also

$$d\tilde{H}/d\alpha = i\, p(\bar{x})\, b(\underline{x})\, \tilde{H} .$$

Besides this we have also $\tilde{H}(0) = 1$. Thus $H = \tilde{H}$ and the equation (5.13) is proved.

By the help of (5.13) the expression for $F(\alpha, \psi)$ can be cast in the form

$$F(\alpha,\psi) = \mathcal{N} e^{\alpha b(\underline{x}) \frac{\delta}{\delta\psi(\underline{x})}} \int e^{\frac{i}{2}\alpha^2 g(\underline{x x'}) p(\bar{x}) p(\bar{x}') + i p(\bar{x}) \psi(\underline{x})} d\left(\frac{p}{2\pi}\right).$$

If, in addition, we perform the substitution

$$p(\bar{x}) = \lambda(\bar{x}) - \frac{1}{\alpha^2}\, g(\overline{x x'})\, \psi(\underline{x}'),$$

we obtain finally

$$F(\alpha,\psi) = C \mathcal{N} e^{\alpha b(\underline{x}) \frac{\delta}{\delta\psi(\underline{x})}}\; e^{-\frac{i}{2\alpha^2} g(\overline{x' x''}) \psi(\underline{x}') \psi(\underline{x}'')} \tag{5.14}$$

where the constant C is given by

$$C = \int e^{\frac{i\alpha^2}{2} g(\underline{x x'}) \lambda(\bar{x}) \lambda(\bar{x}')} d\left(\frac{\lambda}{2\pi}\right) .$$

By the help of the functional $F(\alpha, \psi)$ the solution of the equation (5.3) can now be expressed in terms of $\Xi(0,\psi)$ by the formula

$$\Xi(\alpha,\psi) = \int F(\alpha, \psi - \varphi)\, \Xi(0,\varphi)\, d\varphi \tag{5.15}$$

Concluding remarks

The considerations of the last paragraph show that the quantized scalar field with selfinteraction can be characterized by the "wave functional" $\Xi(\alpha, \psi)$ from which all the Green tensors can be determined. The changement of this functional with the coupling parameter α is determined by the equation (5.3). This equation is so general that it does not depend of the character of the interaction (i.e. of the order of the interaction and whether local or nonlocal). One can therefore expect that this equation will remain valid in the Quantum field theory of any future form. The character of the interaction is specified only by the "initial condition" (5.5).

The purpose of this paper was not so much to provide effective methods for solving the equation (5.3) with the initial condition (5.5). The aim was rather to show that the fundamental equations of the quantum theory of a field can be cast in the form of tensor equations in the function space Π and to exhibit examples of the tensor algebra as well as tensor analysis in this space.

The author is grateful to Prof. V. Votruba for his interest in this work and for many valuable suggestions which contributed to improve the text of this paper.

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Received by Publishing Department
on May, II 1959.