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Laboratory of Theoretical Physics

D.V.Shirkov

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ON THE COULOMB EFFECTS IN THE THEORY OF SUPERCONDUCTIVITY

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D.V.Shirkov

ON THE COULOMB EFFECTS IN THE THEORY OF SUPERCONDUCTIVITY [x)]

Объединенный институт
ядерных исследований
БИБЛИОТЕКА

x) This preprint is an exposition of the two paper [5,14].
Its fundamental results were reported at 5-th All Union Conference on Low Temperature Physics. (Tbilisi, October,1958).

A b s t r a c t

A new form of the connection between matrix elements of the functional derivatives of the Feynman scattering matrix and those of the energy operator is established. This form contains no infinite δ - factors or transitions to the limit. The kernel of the integral equation of compensation of "dangerous" electronic graphs in the theory of superconductivity is expressed by this way in terms of usual Green functions .

The summation of the Coulomb singularities in the kernel of the compensation equation is performed by means of the renormalization group method in the approximation of an electron gas at high density . The results of summation in the lowest approximation coincide with the formulae obtained before in a qualitative way.

Introductory remarks

In the paper[1] the effect of the Coulomb interaction between the electrons in the theory of superconductivity has been analysed. The structure of the kernel $Q(K, K')$ of the integral compensation equation for dangerous electron graphs has been investigated. The kernel Q is expressed in terms of matrix elements of the matrix

$$S^{\circ}_{-\infty} = T\exp\left(-i\int_{-\infty}^{0} H_c(t)dt\right)$$

and energy operator

$$R = H_c(0)\, S^{\circ}_{-\infty} \tag{1.1}$$

These matrix elements in the region of the infrared Coulomb singularity have been evaluated, however, in a qualitative way.

It is well known that the energy charaoteristios of the many body system can be expressed in terms of Feynman S -matrix[2,3]

$$S = S^{\infty}_{-\infty} = T\exp\left(-i\int_{-\infty}^{\infty} H_{int}(t)dt\right). \tag{1.2}$$

Using this idea we establish the connection between matrix elements of the funotional derivatives of the operators R and S and express the kernel Q in terms of usual Green functions. Then one formulates the renormalization group method for the problem of the electron system with Coulomb interaction and with its aid one determines the 4-vertex Green function in the infrared region in the approximation of an electron gas at high density. Using these expression for determination of the kernel Q we obtain in the lowest approximation the expressions established in[1] in a qualitative way.

2. Conneotion between the operators S and R and between their functional derivatives

The relation between the energy levels of the second-quantized system and the complete scattering matrix S was recently investigated by Sucher[2] and Rodberg[3]. The more oonvenient formula was derived by Rodberg. From his formulae it follows, in particular, the connection between matrix elements of the complete matrix S (1.2) and energy operator R (11), which can be represented in the form:

$$\langle \Phi^{*} S \Phi \rangle_c = -2\pi i \delta(E_n - E_n) \langle \Phi^{*}_n R \Phi_n \rangle_c . \tag{2.1}$$

The subscript "c" indicates here that when calculating matrix elements one takes into aocount only the connected graphs.

The formulae of the type (2.1) can be derived also for commutators of the matrices S, R with particle creation and annihilation operators, and, consequently, for the functional derivatives S and R with respect to these operators.

Let $a^{+}_{k\sigma}$ and $a_{k\sigma}$ be accordingly the operators of creation and annihilation of an electron with momentum k, energy $\varepsilon(k)$ and spin σ. Then considering the corresponding commutators by means of (2.1) we arrive to the formula (calculation details are given in [5])

$$i\int_{-\infty}^{\infty} \langle \Phi_1^* \frac{\delta^{n+m} S}{\delta a^{+}_{k_1\sigma_1}(0)\delta a^{+}_{k_2\sigma_2}(t_2)\dots \delta a_{\ell_m s_m}} \Phi_2 \rangle \times$$

$$\times \exp\left\{ i\sum_{2\le i\le n} \varepsilon(k_i)t_i - i\sum_j \varepsilon(\ell_j)\tau_j \right\} dt_2 \dots dt_n\, d\tau_1 \dots d\tau_m =$$

$$= \int_{-\infty}^{0} \langle \Phi_1^* \frac{\delta^{n+m} S}{\delta a^{+}_{k_1\sigma_1}(t_1)\dots \delta a_{\ell_m s_m}(\tau_m)} \Phi_2 \rangle \times \tag{2.2}$$

$$\times \exp\left\{ i\sum_{1\le i\le n} \varepsilon(k_i)t_i - i\sum_j \varepsilon(\ell_j)\tau_j \right\} dt_1 \dots dt_n\, d\tau_1 \dots d\tau_m$$

for

$$\sum_{1\le i\le n} \varepsilon(k_i) - \sum_j \varepsilon(\ell_j) = E_1 - E_2 . \tag{2.3}$$

In derivation of this formula one made use of the property of the translational invariance of matrix elements of functional derivatives of the S - matrix which allowed to carry out one integration over time in the left hand side and reduce δ - functions. The formula (2.2) is very convenient for further applications.

3. Transformation of the kernel Q of the compensation equation

Using Eq. (22) we proceed now to the transformation of the kernel Q of the integral compensation equation for electron graphs in the theory of superconductivity. It is defined by the formula

$$\int_{-\infty}^{0} \langle \frac{\delta^2 R}{\delta a^{+}_{k_1}(t)\delta a^{+}_{k_0}(t')} \rangle_c e^{i\tilde{\varepsilon}(k)(t+t')} dt\,dt' = 0 \tag{3.1}$$

(see Eq. 1. 5.20) , i.e. Eq. (5.20) in the paper [1].

According to (2.2) for small $\tilde{\varepsilon}(k)$ (i.e. in the small vicinity of the Fermi Surface) Eq. (3.1) may be replaced by

$$i\int_{-\infty}^{\infty} \langle \frac{\delta^2 S}{\delta a^{+}_{k_1}(0)\,\delta a^{+}_{k_0}(t)} \rangle_c e^{-i\tilde{\varepsilon}(k)|t|} dt = 0 . \tag{3.2}$$

Just as it has been made in [1] we reduce (3.2)' to the form:

$$2\xi(\kappa) u_\kappa v_\kappa = (u_\kappa^2 - v_\kappa^2) \sum_{\kappa'} u_{\kappa'} v_{\kappa'} Q(\kappa,\kappa') \tag{3.3}$$

where u_κ, v_κ are the parameters of the canonical transformation from a to α, $\xi(\kappa) = E - \lambda +$

$$+ i\int_{-\infty}^{\infty} e^{-i\tilde{\varepsilon}(\kappa)|t|} \left\{ \left\langle \frac{\delta^2 S}{\delta a_{\kappa,+}(0)\delta \hat{a}_{\kappa,+}(t)} \right\rangle_c - \left\langle \frac{\delta^2 S}{\delta \hat{a}_{-\kappa,-}(0)\delta a_{-\kappa,-}(t)} \right\rangle_c \right\} dt$$

and the kernel Q can be represented as a sum of the terms

$$Q(\kappa,\kappa') = Q_c(\kappa,\kappa') + Q_{ph}(\kappa,\kappa'). \tag{3.4}$$

The first term Q_c corresponds to pure Coulomb interaction. It equals

$$Q_c(\kappa,\kappa') = \begin{cases} q(\kappa,\kappa') & \text{for } \kappa > \kappa_F \\ q(\kappa',\kappa) & \text{for } \kappa < \kappa_F \end{cases} \tag{3.5}$$

$$Q(\kappa,\kappa') = i\int_{-\infty}^{\infty} d\tau\, dt\, d\tau'\, e^{-i\tilde{\varepsilon}(\kappa)|\tau-\tau'| - i\tilde{\varepsilon}(\kappa')|t|} \times \left\langle \frac{\delta^4 S}{\delta a_{\kappa',+}(0)\delta a_{-\kappa',-}(t)\delta \hat{a}_{\kappa,+}(\tau)\delta \hat{a}_{-\kappa,-}(\tau')} \right\rangle_c .$$

In order to obtain an explicit expression for the "phonon" term Q_{ph} using (1.5.3) we represent the right hand side of Eq. (3.5) for $\kappa < \kappa_F$ in the form

$$\sum_{\kappa'} u_{\kappa'} v_{\kappa'} Q(\kappa,\kappa') = i\int_{-\infty}^{\infty} \left\langle \frac{\delta^2 S}{\delta \hat{a}_{\kappa,+}(0)\delta \hat{a}_{-\kappa,-}(t)} \right\rangle_0 e^{-i\tilde{\varepsilon}(\kappa)|t|}\, dt . \tag{3.6}$$

The r.h.s. matrix element with due account of the causality property of the S - matrix (compare, for ex. Eq. (48.15)[4] can be represented in the form of a chronological product of the two "currents"

$$\left\langle \frac{\delta^2 S}{\delta \hat{a}_{\kappa,+}(0)\delta \hat{a}_{-\kappa,-}(t)} \right\rangle = \left\langle \frac{\delta^2 S}{\delta \hat{a}_{\kappa,+}(0)\delta \hat{a}_{-\kappa,-}(t)} \hat{S}^+ \right\rangle_{\alpha_0} =$$

$$= \left\langle T\left(\frac{\delta S}{\delta \hat{a}_{\kappa,+}(0)} \hat{S}^+ \right)\left(\frac{\delta S}{\delta \hat{a}_{-\kappa,-}(t)} \hat{S}^+ \right) \right\rangle_{\alpha_0} . \tag{3.7}$$

The subscript "α_0" denotes here the expectation value over the α - vacuum.

The obtained chronological product of the currents for each of the two cases ($t > 0$ or $t < 0$) is a usual product of these currents. This product can be expanded in the complete set of physical states. Restricting ourselves in this expansion to the states with one α - electron and one β - phonon we single out "main terms" containing the small denominator:

$$\tilde{\omega}(q) + \tilde{\varepsilon}(\kappa) + \tilde{\varepsilon}(\kappa') , \qquad (q = k - k').$$

Using the property of translational invariance we obtain, for example, for $t < 0$:

$$\left\langle \frac{\delta^2 S}{\delta \hat{a}^{+}_{K_1,+}(0)\,\delta \hat{a}^{+}_{-K_1,-}(t)} \right\rangle_c = \sum_{K',S} \left\langle \frac{\delta S}{\delta \hat{a}^{+}_{K,+}(0)} \hat{S}^{+} \hat{\alpha}^{+}_{K'S} \hat{\beta}^{+}_{K-K'} \right\rangle \left\langle \alpha_{K'S} \beta_{K-K'} \frac{\delta S}{\delta \hat{a}^{+}_{-K_1,-}(t)} \hat{S}^{+} \right\rangle_{a_0} =$$

$$= \sum_{K',S} e^{it[\tilde{\omega}(K-K') + \tilde{\varepsilon}(K')]} \left\langle \frac{\delta S}{\delta \hat{a}^{+}_{K,+}(0)} \hat{\alpha}^{+}_{K'S} \hat{\beta}^{+}_{K-K'} \right\rangle_{a_0} \left\langle \alpha_{K'S} \beta_{K-K'} \frac{\delta S}{\delta \hat{a}^{+}_{-K,-}(0)} \hat{S}^{+} \right\rangle_{a_0}. \tag{3.8}$$

Then performing the commutation of the operators $\hat{\alpha}^{+}, \alpha, \hat{\beta}^{+}, \beta$ with $\delta S / \delta \hat{a}^{+}$ we make the transition to the limit $\alpha \to a$. Adding the results for $t > 0$ and $t < 0$ taking into acoount the symmetry of H_{ph} with respeot to β and $\hat{\beta}^{+}$ and using (3.7), (3.8) we obtain from (3.6)

$$Q_{ph}(K,K') = -2 \frac{\Gamma(K,K',q)\,\Gamma(-K,-K',-q)}{\tilde{\omega}(q) + \tilde{\varepsilon}(K) + \tilde{\varepsilon}(K')}; \quad q = K - K' \tag{3.9}$$

where $\Gamma(K,K',q) =$

$$= \int_{-\infty}^{\infty} d\tau\, d\theta \left\langle \frac{\delta^3 S}{\delta a_{K',+}(\tau)\,\delta \hat{a}^{+}_{K,+}(0)\,\delta \beta_q(\theta)} \right\rangle_{a_c}. \tag{3.10}$$

The expression (3.10) is written for small $\mathcal{E}$, ω. Performing the funotional differentiation with respect to β and making the transition to the limit $g = 0$ we get

$$Q_{ph}(K,K') = \frac{2\tilde{g}^2(q)\tilde{\omega}(q)}{\tilde{\omega}(q) + \tilde{\varepsilon}(K') + \tilde{\varepsilon}(K)} \Lambda(K,K',q)\Lambda(-K,-K',-q) \tag{3.11}$$

where

$$\Lambda(K,K',q) = \int_{-\infty}^{\infty} d\tau\, dt \left\langle \frac{\delta^2 [T(H_q(t) S_c)]}{\delta a_{K',+}(\tau)\,\delta \hat{a}^{+}_{K,+}(0)} \right\rangle_{a_c} \tag{3.12}$$

S_c being a pure Coulomb matrix, and

$$H_q(t) = \frac{1}{\sqrt{2V}} \sum_{\ell,s} \hat{a}^{+}_{\ell+q,s}(t)\, a_{\ell,s}(t). \tag{3.13}$$

Performing explioitly in (3.12) the functional differentiation and using "generalized Wiok theorm" (see Eq. (34.11)[4]) we obtain for Λ the representation in terms of one - eleotron Green functions and 4-vertex Green function of the pure Coulomb problem. This representation is given as Eq. (3.15) in the paper[5]. It will not be written down here. It is essential for us that in the infrared region for $q^2 \sim 0$ it has the form:

$$\Lambda(K,K',q) = \Lambda(q) = \frac{1}{\sqrt{2V}} \left\{ 1 - \frac{\gamma(q) F(q)}{2} \sum_{S} \int_{-\infty}^{\infty} dt\, \Psi_{Sq}(t) \right\} \tag{3.14}$$

where $\nu(q)$ is a kernel of the Coulomb interaction,

$$F(q) = \frac{4}{V} \sum_{\substack{\ell+q>\kappa_F \\ \ell<\kappa_F}} \frac{1}{\tilde{\varepsilon}(\ell+q)+\tilde{\varepsilon}(\ell)}$$

and $\Psi_{s,q}(t)$ is a coefficient for the main part of the 4-th order funotional derivatives

$$\left\langle \frac{\delta^4 S_c}{\delta \tilde{a}^+_{\ell,s}(\tau)\,\delta \tilde{a}^+_{\kappa,+}(\theta)\,\delta a_{\ell+q,s}(0)\,\delta a_{\kappa',+}(t)} \right\rangle = \frac{i\nu(q)}{V}\,\delta(\tau)\,\delta(t-\theta)\,\Psi_{s,q}(t) \qquad (3.15)$$

4. Problem of summation of the Coulomb singularities

The obtained expressions (3.5), (3.14) and (3.15) for the kernels Q and Q_{ph} of the integral oompensation equation ocntain the fourth order functional derivatives of the Coulomb matrix S_c. Their expressions oan be analysed completely in the approximation of an electron gas at high density when the effeotive parameter of the Coulomb interaction

$$\nu(\kappa_F)\frac{N}{V}$$

is small in comparison with eleotron energy on the Fermi surfaoe $\kappa_F^2/2m$, i.e. for

$$\frac{4\pi\hbar^2 e^2}{\kappa_F^2}\cdot\frac{N}{V}\Big/\frac{\kappa_F^2}{2m} = \frac{8}{\pi^3}\,r_s\,\alpha \ll 1. \qquad (4.1)$$

Here $\alpha = (4/9\pi)^{1/3}$, and r_s is a dimensionless intereleotron spaoing measured in Bohr radius units.

In this case the Coulomb energy is small in comparison with the kinetic energy everywhere except the region of small momentum transfers

$$q^2 \ll \kappa_F^2 \qquad (4.2)$$

this is so called "infrared region". Therefore, one can emplay the usual perturbation theory everywhere except the infrared region. In this region where the effective expansion parameter $r_s\,\kappa_F^2/q^2$ is not small it is necessary to summarize the infinite set of the "main" Coulomb terms proportional to the powers $(r_s\,\kappa_F^2/q^2)^n$.

Here we meet with the situation well known in the relativistic quantum eleotrodynamios where in spite of the smallness of the dimensionless coupling constant (fine struoture parameter $e^2/4\pi = 1/137$) in the so called "ultraviolet" and the "infrared" regions of momentum variables the produot of $e^2/4\pi$ and "large" logarithm is an effeotive expansion parameter. In the quantum eleotrodynamios there is a number of well known methods,

allowing to summarize infinite sets of main terms. We mean here the method of summation of the main graphs by Landau, Abrikosov and Halatnikov [6] and renormalization group method [4].

In the quantum statistics there is a number of methods yielding analogous results. So, the procedure of summing the main Coulomb graphs suggested by Gell-Mann, Brueckner, and Sawada [7] is essentially equivalent to the method given in the paper [6]. The well known method of approximate second quantization [8] in the problem of an electron gas at high density leads also to the analogous results [1,9]. Finally, for summation of the Coulomb singularities of the Green functions one may use the renormalization group technique. For the first time this possibility was indicated in the paper [10] *).

An important advantage of the renormalization group method is its regularity. As it will be shown the first approximation of the renormalization group method leads to the formulae which coincide with the results of summation of the main graphs (just as in the quantum field theory) and with the formulae of the approximate second quantization method .

The results of the higher approximations of the renormalization group method may be of interest for investigating the problem of extension of the limit of applicability of the electron gas at high density approximation.

5. Renormalization group in the problem of the Coulomb interaction between electrons

The possibility of use of the renormalization group method in the problem of the Coulomb interaction between electrons is based (the more detailed consideration see in [10]) on the group character of the finite multiplicative transformation of the basic quantities : one - electron Green function G , two-electron Green function (4-vertex) Γ and dimensionless parameter of the Coulomb interaction r (see (4.1)

$$G \to G' = Z_2 G, \quad \Gamma \to \Gamma' = Z_1^{-1} \Gamma, \quad r \to r' = Z_1 Z_2^{-2} r \qquad (5.1)$$

The meaning of the transformation (5.1) resides in the fact that the sets (G', Γ', r') and (G, Γ, r) describe the same physical picture.

The tool of the renormalization group method is Lie differential equations. In order to obtain them it should write down at first functional group equations, corresponding to the transformation (5.1). As a first step it is necessary to choose the representation of the Green functions G and Γ .

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x) For application of the renormalization group method in quantum statistic problems see also [15] .

For this purpose we proceed to the symmetrical momentum - energy representation of the electron operators (see, for ex. [11])

$$a_s(p) = a_s(p_0, \bar{p}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ip_0\tau} a_{\bar{p},s}(\tau)\, d\tau \tag{5.2}$$

$$\dot{a}_s(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ip_0\tau} \dot{a}_{\bar{p},s}(\tau)\, d\tau .$$

In this representation the chronological pairing has the form:

$$\overbrace{a_s(p)\, \dot{a}_\sigma(k)} = i\,\delta(p-k)\,\delta_{s\sigma}\,G_0(p)$$

where

$$\delta(p-k) \equiv \delta_{\bar{p}\bar{k}}\,\delta(p_0 - k_0)$$

$$G_0(p) = \frac{1}{p_0 - |\tilde{\varepsilon}(\bar{p})| + i\delta\, \mathrm{sign}\,\tilde{\varepsilon}(\bar{p})} . \tag{5.3}$$

In the case under consideration due to the determination of the renormalized energy $\tilde{\varepsilon}$ for the total Green function

$$\frac{\langle T a_s(p) \dot{a}_\sigma(k) S\rangle_0}{S_0} = i\,\delta(p-k)\,\delta_{s\sigma}\,G(p)$$

we have also

$$G(p) = \frac{S(p)}{p_0 - |\tilde{\varepsilon}(\bar{p})| + i\delta\, \mathrm{sign}\,\tilde{\varepsilon}} . \tag{5.4}$$

We determine the two-electron (4 -vertex) Green function Γ as follows:

$$i\left\langle \frac{\delta^4 S}{\delta a_{s_1}(p_1)\,\delta a_{s_2}(p_2)\,\delta \dot{a}_{\sigma_1}(k_1)\,\delta \dot{a}_{\sigma_2}(k_2)} \right\rangle_c = \delta(p_1+p_2-k_1-k_2)\,\Gamma_{s_1 s_2 \sigma_1 \sigma_2}(p_1, p_2, k_1, k_2)$$

where

$$\Gamma_{s_1 s_2 \sigma_1 \sigma_2}(p_1, p_2, k_1, k_2) = \frac{i}{2\pi} \int_{-\infty}^{\infty} d\tau\, dt_1\, dt_2\, e^{-ip_2^\circ \tau + ik_1^\circ t_1 + ik_2^\circ t_2} \times \tag{5.5}$$

$$\left\langle \frac{\delta^4 S}{\delta a_{s_1 p_1}(0)\,\delta a_{s_2 p_2}(\tau)\,\delta \dot{a}_{\sigma_1 k_1}(t_1)\,\delta \dot{a}_{\sigma_2 k_2}(t_2)} \right\rangle_c$$

for

$$p_1^\circ + p_2^\circ = k_1^\circ + k_2^\circ , \qquad \bar{p}_1 + \bar{p}_2 = \bar{k}_1 + \bar{k}_2 .$$

The function Γ has a simple matrix structure.

$$\Gamma_{s_1 s_2 \sigma_1 \sigma_2}(p_1, p_2, k_1, k_2) = \delta_{s_2 \sigma_1}\,\delta_{s_1 \sigma_2}\,\Gamma(k_1, k_2, p_1, p_2) - \delta_{s_1 \sigma_1}\,\delta_{s_2 \sigma_2}\,\Gamma(k_1, k_2, p_2, p_1)$$

In the case (we are interested in) of the infrared Coulomb asymptotic behaviour the one-electron Green function G has no singularity and , hence, one may put $S(p) = 1$. Taking into account the fact that for the compensation equation the vicinity of the Fermi

surfaoe is only essential we can restrict ourselves to the examination of the function Γ for $K_1^0 = K_2^0 = P_1^0 = P_2^0 = 0$

$$\Gamma(\bar{K}_1, \bar{K}_2, \bar{P}_1, \bar{P}_2) =$$

$$= \frac{i}{2\pi} \int_{-\infty}^{\infty} d\tau\, dt_1 dt_2 \left\langle \frac{\delta^4 S_c}{\delta a_{P_1,+}(0)\, \delta a_{P_2,-}(\tau)\, \delta \bar{a}^+_{K_1,-}(t_1)\, \delta \bar{a}^+_{K_2,+}(t_2)} \right\rangle_c \tag{5.6}$$

for $\bar{K}_1 + \bar{K}_2 = \bar{P}_1 + \bar{P}_2$.

Indeed, from (3.5) we get

$$Q(K, K') = -2\pi\, \Gamma(-\bar{K}, \bar{K}, \bar{K} - \bar{Q}, \bar{Q} - \bar{K}) . \tag{5.7}$$

On the other hand from (3.14) and (3.15) it follows

$$\Lambda(Q) = \frac{1}{\sqrt{2V}} \left\{ 1 - 2\pi V F(\bar{Q})\, \Gamma(\bar{Q}) \right\} \tag{5.8}$$

where $\Gamma(\bar{Q})$ is a "main" part of the function $\Gamma(\bar{e} + \bar{Q}, \bar{K} - \bar{Q}, \bar{K}, \bar{e})$ in the limit of small $\bar{Q}$ which does not depend on $\bar{K}$ and $\bar{e}$.

In the lowest perturbation order

$$\Gamma_0(K, e, e + Q, K - Q) = \frac{V(Q)}{2\pi V} .$$

Therefore it is convenient to introduce a new function g , defined by the relation

$$\Gamma(K, \ell, \ell + Q, K - Q) = \frac{V(Q)}{2\pi V}\, g(K, \ell, \ell + Q, K - Q) . \tag{5.9}$$

This function has the following important properties:

a) Under transformation (5.1) undergoes the transformation

$$g \longrightarrow g' = Z_1^{-1} g \tag{5.10}$$

b) in the limit of switching off the Coulomb interaction it tends to the oonstant value

c) being represented by the perturbation series it contains the terms proportional to powers of the ratio $(r_s K_F^2 / Q^2)$

In such a situation it is not difficult to write down the functional group equation for the vertex function g in the infrared region.

Introducing the "normalization" momentum λ let us consider the dimensionless variables:

$$g(Q^2, K_F^2, \ldots, r) = f\left(\frac{Q^2}{\lambda^2}, \frac{K_F^2}{\lambda^2}, r\right) .$$

The dots denote here uneseential momentum variables $K^2, \ell^2, (K+\ell)^2$ (as well as energy variables $K_0, Q_0, \ell_0, \ldots$) . We do not write down explicitly all these variable considering them as fixed parameters .

With due account of (5.1) and (5.10) we obtain

$$f\left(\frac{q^2}{\lambda_1^2}, \frac{\kappa_F^2}{\lambda_1^2}, r_1\right) = Z_1^{-1} f\left(\frac{q^2}{\lambda^2}, \frac{\kappa_F^2}{\lambda^2}, r\right)_{r_1 = Z_1 r}. \tag{5.11}$$

Choosing the normalization momentum λ so as

$$f(1, y, r) = 1 \tag{5.12}$$

we are led to the functional group equation

$$f(x, y, r) = f(t, y, r) f\left(\frac{x}{t}, \frac{y}{t}, r f(t, y, r)\right). \tag{5.13}$$

Differentiating (5.12) with respect to x and setting $t = x$ we get the Lie equation

$$\frac{\partial f(x, y, r)}{\partial x} = \frac{f(x, y, r)}{x} \Phi\left(\frac{y}{x}, r f(x, y, r)\right) \tag{5.14}$$

where the funotion

$$\Phi(y, r) = \left.\frac{\partial f(x, y, r)}{\partial x}\right|_{x=1} \tag{5.15}$$

is to be computed by means of the perturbation theory.

6. Infrared asymptotic behaviour of the vertex function

The calculation of the function f entering the right hand side of (5.15) must be performed with account of the multiplicative ambiquity (5.11) providing the fulfilment of the normalization condition (5.12).

In this way with the accuracy of the terms of the third order we get

$$f_{T,B}(x, y, r) = 1 - ar\left(\frac{y}{x} - 1\right) + br^2\left(\frac{y}{x} - 1\right)^2 - cr^2\left(\frac{y}{x} - 1\right) + \cdots \tag{6.1}$$

a, b, c being certain oonstants (which can depend on above-mentioned parameters). Substituting (6.1) into 5.15 and 5.14 and carrying out the integration we obtain consequently:

$$\Phi(y, z) = ary + cr^2 y$$

$$\frac{df}{f^2\left(1 + r\frac{c}{a} f\right)} = ary \frac{dx}{x^2} \tag{6.2}$$

$$r\frac{c}{a} \ln\left|\frac{a + rcf}{(a + rc) f}\right| + 1 - \frac{1}{f} = ary\left(1 - \frac{1}{x}\right).$$

We got here a transcendental equation for f which can be solved by method of successive approximations taking into account the smallness of r . This yields

$$f(x,y,r) = \left\{ 1 + a r y \left(\frac{1}{x} - 1\right) + r \frac{c}{a} \ln \left| \frac{1 + a r y \left(\frac{1}{x} - 1\right) + p \frac{c}{a}}{1 + r \frac{c}{a}} \right| \right\}^{-1} . \tag{6.3}$$

Let us note that the constant b of Eq. (6.1) did not enter the formulae (6.2),(6.3). The fact is that b really is not idependent . Expanding (6.3) in a power series in and comparing with (6.1) we find

$$b = a^2 . \tag{6.4}$$

Now we have only to pass on to the usual non-normalized function f_0 , depending on the observable value of the coupling constant $r = r_s$ and not containing the normali - zation momentum λ .

The perturbation theory yields for f_0 :

$$f_0^{T.B}(q^2, r_s) = 1 - a r_s \left(\frac{K_F^2}{q^2} + d \right) + a^2 r_s^2 \left(\frac{K_F^2}{q^2} + d \right)^2 - c r_s^2 \frac{K_F^2}{q^2} + \dots . \tag{6.5}$$

Here d is a constant like a, c .

To proceed to r_s we profit by the usual method (see 42.3 [4]) based on the invariant charge rf property :

$$rf\left(\frac{q^2}{\lambda^2}, \frac{K_F^2}{\lambda^2}, r \right) = r_s f_0 (q^2, K_F^2, r_s) . \tag{6.6}$$

Putting in (6.6) $q^2 = \lambda^2$ we obtain for r an explicit expression

$$r = r_s f_0^{T.B}(\lambda^2, K_F^2, r_s)$$

Substituting this value into the left hand side of (6.6) with account of (6.3) and after a simple calculation we are led to the formula :

$$f_0(q^2, r_s) =$$

$$= \left[1 + a r_s \left(\frac{K_F^2}{q^2} + d \right) + r_s \frac{c}{a} \ln \left(1 + a r_s \frac{K_F^2}{q^2} \right) \right]^{-1} . \tag{6.7}$$

It is not difficult to make sure that (6.5) is an expansion of (6.7) in powers of small parameter r_s .

7. Discussion of the results.

Let us discuss the obtained formula (6.7) desoribing the behaviour of the Coulomb 4-vertex in the infrared region.

Recalling that according to (5.9)

$$\Gamma(q) = \frac{V(q)}{2\pi V} f_0(q^2, r_s) = \frac{2e^2}{V(q)^2} f_0(q^2, r_s) \tag{7.1}$$

we consider the expression

$$Q^{-2} = \frac{1}{q^2} f_0(q^2, r_s) = \frac{1}{q^2 + a r_s \kappa_F^2 + q^2 a r_s d + r_s \frac{c}{a} q^2 \ln\left(1 + a r_s \frac{\kappa_F^2}{q^2}\right)} . \tag{7.2}$$

In the limit of the small r_s and q^2 we get

$$Q^{-2} = \frac{1}{q^2 + a r_s \kappa_F^2} = \frac{1}{q^2 + q_s^2} . \tag{7.3}$$

$q_s = 0{,}814\, r_s^{1/2}\, K_F$ being the inverse length of the Tomas-Fermi screening.

Thus, in the first approximation (in powers r_s) the formula (7.2) leads to the well known Coulomb screening formula.

This shows that the results of summation of the main Coulomb graphs by Gell-Mann, Brueckner and Sawada [7] can be obtained in a much simpler way, i.e. by the renormalization group method.

Let us note also that after substitution of (7.3) into (7.1) and (5.8) with due account of the fact that

$$\lim_{q^2 \to 0} q^2 F(q) V(q) = q_s^2$$

we obtain x)

$$\Lambda(q) = \frac{1}{\sqrt{2V}} \left\{ 1 - \frac{q_s^2}{q^2 + q_s^2} \right\} = \frac{1}{\sqrt{2V}} \frac{q^2}{q^2 + q_s^2} . \tag{7.4}$$

The formulae (7.3) and (7.4) coincide with the results obtained in the qualitative way in 6.2 [1].

However, the signification of the formulae (7.2) and (6.7) and the possibilities of the renormalization group method in the problem of interacting electrons is not exhausted by this.

The renormalization group technique gives a regular method of improvement of the approximation properties of the usual perturbation expansions. This fact is well known in the quantum field theory. The present investigation illustrates this fact for the quantum statistics.

The formulae (6.7) and (7.2) generalized to the oase of the non-zero energy arguments present the second approximation to the results of the paper [7].

The further generalization of these formulae to higher orders can be performed simply

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x) This formula agrees with the result given in the paper [12] .

taking into acoount in (6.1) the terms of the next orders with respect to the powers r_s .

Such a generalization may be of interest, for instance, in order to specify the superconductivity criterion (see § 6.3 [1]) and expansions for the correlation energy in the region of not very small r_s .

In this connection note that recently it became known a thesis of Dubois (13), which attempted to improve the formulae of the paper [7] by summing the main graphs of the second order.

A brief oomparison of Eq. (6.7) with the results of the paper [13] shows that the structure of the expressions obtained there (see Eqs. (2.7), (2.5) and (A.3) [13]) corresponds to the resulto of the substitution in Eq. (6.7)

$$r_s \frac{c}{a} \ln\left(1 + a r_s \frac{k_F^2}{q^2}\right) \to r_s^2 c \frac{k_F^2}{q^2}$$

consistent only for small r_s and not small q^2 .

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Объединенный институт ядерных исследований
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x) In the English translation the numbers of sections and equations are changed so, for example, Russian edition §6.3. §6.2. (5.20) , (5.34)
English translation §5.4 §5.2 (4.6) , (4.19).