JOINT INSTITUTE FOR NUCLEAR RESEARCH

Laboratory of Theoretical Physics

P-308

V.G. Ogievetsky, I.V. Polubarinov

ON WAVE EQUATIONS FOR ZERO AND NONZERO REST MASS

ЖЭТФ, 1959, т37, в2, с470-476.

D u b n a 1959.

P-308

V.G. Ogievetsky, I.V. Polubarinov

ON WAVE EQUATIONS
FOR ZERO AND NONZERO REST MASS

A b s t r a c t

It is proved that nonzero-rest-mass wave equations are invariant under the 15-parameter group of transformations. This group is a representation of the conformal group.

I n t r o d u c t i o n

Klein-Gordon and Dirac equations are invariant under the ten-parameter group which is a representation of the inhomogeneous Lorentz group L_{10}.

It was established in papers of E. Cunningham,|1| H. Bateman,|2| P.A.M. Dirac,|3|, H. Bhaba|4|, W. Pauli|5|, A. McLennan|6| et al that if the rest mass is zero the wave equations are invariant under a more extended 15-parameter group of transformations which forms a representation of the conformal group C_4, involving L_{10}. The Dirac equation for a neutrino is also invariant under the four-parameter Pauli group.

It is accepted to believe that these properties of invariance may be ascribed to the wave equations for zero rest mass only.

It will be proved in the present paper, however, that Klein-Gordon and Dirac equations for non-zero mass are also invariant under the 15-parameter group of transformations G_{15} which is the representation of the conformal group C_4. For the Dirac equation there is an analog of the Pauli group also. The operators of all these transformations depend on the mass m as a parameter and turn into the known operators in the limit $m=0$. In the group G_{15} it is necessary to adopt the form different from the usual one for some operators of the representation of the Lorentz group. This leads to the difficulty: under Lorentz rotations the particle momentum will not be transformed as a 4-vector.

The derivation of the transformations for $m \neq 0$ is essentially based upon the transformations for $m = 0$.

Therefore, Sec. 2 is concerned with the consideration of the conformal group of representations for zero rest mass.

The way for the derivation of the corresponding transformations for $m \neq 0$ is being established in Sec. 3.

In the following Section there are given the infinitesimal operators of the group G_{15} for the wave equations with $m \neq 0$ as well as the analog of the Pauli group.

2. Conformal Group and Zero Rest Mass Wave Equations

As it was pointed out in the introduction a number of papers [1-6] were devoted to the proof of the conformal invariance of the zero rest mass wave equations. In this Section is given a summary [3-8] of main results concerning both the conformal group and the invariance of the zero rest mass Klein-Gordon -Dirac equations.

The 15-parameter conformal group C_4 consists of the Lorentz group L_{10} (translations and rotations), dilatation transformations and of four proper conformal transformations (see the first row of Table 1). The proper conformal transformation is the product of the inversion in the unit hypersphere $x'_\mu = x_\mu / x^2$ then of a translation and again of an inversion. Three of them (spatial) connected with the transition into a uniform accelerated frame of reference. [9-13]

The laws of transformations for the solutions of zero-rest-mass of Klein-Gordon and Dirac equations

$$\Box^2 \varphi_0(x) = 0 \tag{1}$$

$$\gamma_\mu \frac{\partial}{\partial x_\mu} \psi_0(x) = 0 \tag{2}$$

are given in the 3^{d} and 4^{th} rows of Table 1.

It is seen from Table 1 that even the solution of Klein-Gordon equation is not a scalar under the dilatation and proper conformal transformations.

At infinitesimal transformations

$$\psi'(x) = \left(1 + i\, a_\mu P_\mu\right) \psi(x) \tag{3}$$

$$\psi'(x) = \left(1 + i\,\omega_{\mu\nu} M_{\mu\nu}\right) \psi(x) \tag{4}$$

$$\psi'(x) = \left(1 + \varepsilon I\right) \psi(x) \tag{5}$$

$$\psi'(x) = \left(1 + d_\mu L_\mu\right) \psi(x) \tag{6}$$

T a b l e I.

Finite Transformations

	Translation (4-parameter)	Rotations (6-parameter)	Dilatation 1-parameter	Proper conformal transformations (4-parameter)*
Transformation of coordinates	$x'_\mu = x_\mu - a_\mu$	$x'_\mu = a_{\mu\nu} x_\nu$ $(a_{\mu\nu} a_{\mu\lambda} = \delta_{\nu\lambda})$	$x'_\mu = \frac{x_\mu}{\alpha}$	$x'_\mu = \frac{x_\mu - \alpha_\mu x^2}{1 - 2(\alpha x) + \alpha^2 x^2}$
Transformation of a scalar function	$f'(x_\mu) = f(x_\mu + a_\mu)$	$f'(x_\mu) = f(a^{-1}_{\mu\nu} x_\nu)$	$f'(x_\mu) = f(\alpha x_\mu)$	$f'(x_\mu) = f\left(\frac{x_\mu + \alpha_\mu x^2}{1 + 2(\alpha x) + \alpha^2 x^2}\right)$
Transformation of the solutions of the Klein-Gordon equations with m = 0	$\varphi'_0(x_\mu) = \varphi_0(x_\mu + a_\mu)$	$\varphi'_0(x_\mu) = \varphi_0(a^{-1}_{\mu\nu} x_\nu)$	$\varphi'_0(x_\mu) = \alpha \varphi_0(\alpha x_\mu)$***	$\varphi'_0(x_\mu) = [1 + 2(\alpha x) + \alpha^2 x^2]^{-1} \varphi_0\left(\frac{x_\mu + \alpha_\mu x^2}{1 + 2(\alpha x) + \alpha^2 x^2}\right)$
Transformation of the solutions of Dirac equation with m = 0	$\psi'_0(x_\mu) = \psi_0(x_\mu + a_\mu)$	$\psi'_0(x_\mu) = S \psi_0(a^{-1}_{\mu\nu} x_\nu)$**	$\psi'_0(x_\mu) = \alpha^{3/2} \psi_0(\alpha x_\mu)$	$\psi'_0(x_\mu) = \frac{1 + 2(\alpha x) - (\alpha\gamma)(\gamma x)}{[1 + 2(\alpha x) + \alpha^2 x^2]^2} \psi_0\left(\frac{x_\mu + \alpha_\mu x^2}{1 + 2(\alpha x) + \alpha^2 x^2}\right)$

---

* $x^2 = x_\mu x_\mu$, $\alpha^2 = \alpha_\mu \alpha_\mu$, $(\alpha x) = \alpha_\mu x_\mu$, $\mu = 1,2,3,4$.

** S is the operator of the transformation of the Dirac spinor under rotations.

*** The numerical value of the exponent of α by $\varphi_0(\alpha x)$ and $\psi_0(\alpha x)$ is uniquelly defined by the requirement that φ_0 and ψ_0 would be transformed according to the representation of group C_4 as a whole and may be found from the relation of the structure (15).

where P_μ , $M_{\mu\nu}$, I and I_μ are infinitesimal operators of translation, rotation,* dilatation and proper-conformal transformations whereas $a_\mu, \omega_{\mu\nu}, \varepsilon$ and α_μ are corresponding infinitesimal parameters of transformations.

The infinitesimal operators of the conformal group and its representations satisfy the following structure relations

$$[P_\mu, P_\nu] = 0 \tag{7}$$

$$[P_\mu, M_{\nu\lambda}] = -i(\delta_{\mu\nu}P_\lambda - \delta_{\mu\lambda}P_\nu) \tag{8}$$

$$[M_{\mu\nu}, M_{\lambda\rho}] = i(\delta_{\mu\lambda}M_{\nu\rho} + \delta_{\nu\rho}M_{\mu\lambda} - \delta_{\mu\rho}M_{\nu\lambda} - \delta_{\nu\lambda}M_{\mu\rho}) \tag{9}$$

$$[I_\mu, M_{\nu\lambda}] = -i(\delta_{\mu\nu}I_\lambda - \delta_{\mu\lambda}I_\nu) \tag{10}$$

$$[I_\mu, I_\nu] = 0 \tag{11}$$

$$[P_\mu, I] = P_\mu \tag{12}$$

$$[M_{\mu\nu}, I] = 0 \tag{13}$$

$$[I_\mu, I] = -I_\mu \tag{14}$$

$$[P_\mu, I_\nu] = 2(M_{\mu\nu} + i\delta_{\mu\nu}I) \tag{15}$$

In Table 2 are given the infinitesimal operators for the solutions of the Klein-Gordon and Dirac equations with m = 0 in x - and p-representations/ $\psi(p) = (2\pi)^{-2}\int \exp(-ipx)\psi(x)\,d^4x$ /.

While the operators P_μ , $M_{\mu\nu}$ commute with the operators of the wave equations, the corresponding commutators for the operators I and I_μ are equal /in the p-representation/: for Klein-Gordon equation

$$[I, p^2] = -2p^2 \tag{16}$$

$$[I_\mu, p^2] = 4i\frac{\partial}{\partial p_\mu}p^2 \tag{17}$$

x/ An imaginary unit i appeared because the physical operators of the momentum P_μ and angular momentum $M_{\mu\nu}$ are taken as infinitesimal operators of the translation and rotation.

for Dirac equation

$$[I, i\gamma p] = -i\gamma p \tag{18}$$

$$[I_\mu, i\gamma p] = 2i\frac{\partial}{\partial p_\mu} i\gamma p \tag{19}$$

Thus, these commutators are equal to zero if applied to the solutions of the wave equations. Therefore, the transformed functions will be also the solutions of these equations.

If we introduce the operators

$$M_{\mu 5} = \frac{1}{2}(I_\mu + iP_\mu) \ , \ M_{\mu 6} = \frac{1}{2}(P_\mu + iI_\mu) \ , \ M_{56} = -I \quad /\mu \neq 5,6/ \tag{20}$$

instead of I_μ , P_μ and I, then (7)-(15) may be written in the form of the structure relations of the rotation group in the 6-dimensional space, i.e., in the form (19) with μ, ν, λ, ρ = 1,2,3,4,5,6. The conformal group and invariance of the massless wave equations with respect to it by passing into 6-dimensional space were studied by several authors[3,4,14].

Table 2.

Infinitesimal Operators in X and P-Representations

Infinitesimal operators / For transformation	Translation P_μ	Rotations $M_{\mu\nu}$	Dilatation I	Proper conformal transformations I_μ
of the scalar function	$\frac{1}{i}\frac{\partial}{\partial x_\mu}$	$\frac{1}{i}\left(x_\mu\frac{\partial}{\partial x_\nu}-x_\nu\frac{\partial}{\partial x_\mu}\right)$	$x_\mu\frac{\partial}{\partial x_\mu}$	$x^2\frac{\partial}{\partial x_\mu}-2x_\mu x_\nu\frac{\partial}{\partial x_\nu}$
	p_μ	$\frac{1}{i}\left(p_\mu\frac{\partial}{\partial p_\nu}-p_\nu\frac{\partial}{\partial p_\mu}\right)$	$-p_\mu\frac{\partial}{\partial p_\mu}-4$	$-i\left\{p_\mu\frac{\partial^2}{\partial p_\nu^2}-2\left[4+p_\nu\frac{\partial}{\partial p_\nu}\right]\frac{\partial}{\partial p_\mu}\right\}$
of the solution of the Klein-Gordon equation with m=0	$\frac{1}{i}\frac{\partial}{\partial x_\mu}$	$\frac{1}{i}\left(x_\mu\frac{\partial}{\partial x_\nu}-x_\nu\frac{\partial}{\partial x_\mu}\right)$	$x_\mu\frac{\partial}{\partial x_\mu}+1$	$x^2\frac{\partial}{\partial x_\mu}-2x_\mu x_\nu\frac{\partial}{\partial x_\nu}-2x_\mu$
	p_μ	$\frac{1}{i}\left(p_\mu\frac{\partial}{\partial p_\nu}-p_\nu\frac{\partial}{\partial p_\mu}\right)$	$-p_\mu\frac{\partial}{\partial p_\mu}-3$	$-i\left\{p_\mu\frac{\partial^2}{\partial p_\nu^2}-2\left[3+p_\nu\frac{\partial}{\partial p_\nu}\right]\frac{\partial}{\partial p_\mu}\right\}$
of the solution of the Dirac equation with m = 0	$\frac{1}{i}\frac{\partial}{\partial x_\mu}$	$\frac{1}{i}\left(x_\mu\frac{\partial}{\partial x_\nu}-x_\nu\frac{\partial}{\partial x_\mu}\right)+\frac{1}{2}\sigma_{\mu\nu}$ x)	$x_\mu\frac{\partial}{\partial x_\mu}+\frac{3}{2}$	$x^2\frac{\partial}{\partial x_\mu}-2x_\mu x_\nu\frac{\partial}{\partial x_\nu}-2x_\mu-\gamma_\mu(\gamma x)$
	p_μ	$\frac{1}{i}\left(p_\mu\frac{\partial}{\partial p_\nu}-p_\nu\frac{\partial}{\partial p_\mu}\right)+\frac{1}{2}\sigma_{\mu\nu}$ x)	$-p_\mu\frac{\partial}{\partial p_\mu}-\frac{5}{2}$	$-i\left\{p_\mu\frac{\partial^2}{\partial p_\nu^2}-2\left[3+p_\nu\frac{\partial}{\partial p_\nu}\right]\frac{\partial}{\partial p_\mu}+\gamma_\mu\gamma_\nu\frac{\partial}{\partial p_\nu}\right\}$

x/ $\sigma_{\mu\nu}=-i(\gamma_\mu\gamma_\nu-\delta_{\mu\nu})$

3. Connection between Wave Equations with $m \neq 0$ and $m = 0$

The invariance of the Klein-Gordon and Dirac equations for the nonzero mass under the 15-parametric group will be proved by establishing the connection between the equations with the mass and without it. All the considerations will be carried on in the momentum space.

If in the Klein-Gordon equation

$$\left[\vec{p}^{\,2} - p_0^2 + m^2\right]\varphi(\vec{p}, p_0) = 0 \tag{21}$$

the substitution

$$\vec{q} = \vec{p} \quad , \quad q_0 = \varepsilon(p_0)\sqrt{p_0^2 - m^2} \tag{22}$$

$$\varphi_0(\vec{q}, q_0) = \varphi(\vec{p}, p_0) \tag{23}$$

is made, then equation (21) takes the form of the Klein-Gordon equation without a mass

$$(\vec{q}^{\,2} - q_0^2)\,\varphi_0(\vec{q}, q_0) = 0 \tag{24}$$

Similarly the Dirac equation

$$(i\gamma p + m)\,\psi(\vec{p}, p_0) = 0 \tag{25}$$

by means of substitution (22) and the transformation of the function

$$\psi_0(\vec{q}, q_0) = S\,\psi(\vec{p}, p_0) \tag{26}$$

where

$$S = \operatorname{Ch}\frac{\chi}{2} - \gamma_4 \operatorname{Sh}\frac{\chi}{2} \quad , \quad \chi = \operatorname{arth}\frac{m}{p_0} \tag{27}$$

is reduced to the form of the Dirac equation without a mass

$$i\gamma q\,\psi_0(\vec{q},q_0)=0 \tag{28}$$

since

$$S^{-1}(i\gamma p+m)S^{-1}=i\gamma q \tag{29}$$

One may point out the wide class of such transformations of Klein-Gordon and Dirac equations to the mass m = 0. All of them are not covariant. One of them with the unitary S is used in |15|.

Equations (25) and (28) are invariant under the 15-parameter conformal group. The transformations laws of the solutions of these euqations are given in §2. The 15-parameter transformation group G_{15} leaving the equation with nonzero mass invariant may be obtained in the following way:

1) Klein-Gordon and Dirac equations reduce to zero mass form by (22), (23) and (22), (26), respectively.

2) Then some transformation of the 15-parameter group for m = 0 is being performed.

3) Having made the transitions inverse to (22), (23) and (22), (26) we find the operators of the 15-parameter group for nonzero mass.

4. Infinitesimal Operators of the 15-Parameter Group for Klein-Gordon and Dirac equations with $m \neq 0$.

The finite transformations of the 15-parameter group G_{15} by $m \neq 0$ are extremely complicated. But since they are entirely defined by their infinitesimal operators then there will be given only the latter ones calculated by the abovementioned way.

The infinitesimal operators have the following form:*

a) In case of Klein-Gordon equation

$$P_r^K = p_r \quad /r=1,2,3/\,, \quad P_0^K=\sqrt{p_0^2-m^2} \tag{30}$$

$$\left.\begin{aligned} M_{rn}^K &= \frac{1}{i}\left(p_r\frac{\partial}{\partial p_n}-p_n\frac{\partial}{\partial p_r}\right)\\ M_{r4}^K &= \frac{1}{i}\,\frac{\sqrt{p_0^2-m^2}}{p_0}\left(p_r\frac{\partial}{\partial p_4}-p_4\frac{\partial}{\partial p_r}\right) \qquad /p_4=ip_0/ \end{aligned}\right\} \tag{31}$$

---

* For simplicity only the case of the positive frequencies $p_0 \geqslant m$ is considered

$$I^K = -\left[3 + p_\mu \frac{\partial}{\partial p_\mu} - \frac{m^2}{p_0}\frac{\partial}{\partial p_0}\right] \tag{32}$$

$$\left.\begin{aligned} I_r^K &= -i p_r\left[\frac{\partial^2}{\partial p_\mu^2} + \frac{m^2}{p_0^2}\frac{\partial^2}{\partial p_0^2} - \frac{m^2}{p_0^3}\frac{\partial}{\partial p_0}\right] - 2i I^K \frac{\partial}{\partial p_r} \\ I_0^K &= -i\frac{\sqrt{p_0^2-m^2}}{p_0}\left\{p_0\left[\frac{\partial^2}{\partial p_\mu^2} + \frac{m^2}{p_0^2}\frac{\partial^2}{\partial p_0^2} + \frac{m^2}{p_0^3}\frac{\partial}{\partial p_0}\right] - 2I^K\frac{\partial}{\partial p_0}\right\}\end{aligned}\right\} \tag{33}$$

In case of Dirac equation

$$P_r^D = p_r \quad /r=1,2,3/, \quad P_0^D = \sqrt{p_0^2 - m^2} \tag{34}$$

$$\left.\begin{aligned} M_{rn}^D &= M_{rn}^K + \frac{1}{2}\sigma_{rn} \\ M_{r4}^D &= M_{r4}^K + \frac{1}{2}\frac{\sqrt{p_0^2-m^2}}{p_0}\sigma_{r4} + \frac{m}{2p_0\sqrt{p_0^2-m^2}}\left(\gamma_r p_4 - \gamma_4 p_r + m\sigma_{r4}\right)\end{aligned}\right\} \tag{35}$$

$$I^D = -\left[\frac{5}{2} + p_\mu\frac{\partial}{\partial p_\mu} - \frac{m^2}{p_0}\frac{\partial}{\partial p_0} + \gamma_4\frac{m}{2p_0}\right] \tag{36}$$

$$\left.\begin{aligned} I_r^D &= I_r^K + \frac{im}{4p_0^3(p_0^2-m^2)}\left[mp_0p_r - 2p_r\gamma_4(2p_0^2-m^2) + 2i\gamma_r p_0^2(p_0-\gamma_4 m)\right] - \\ &\quad - i\gamma_r\left(\gamma_\mu\frac{\partial}{\partial p_\mu} + \frac{im}{p_0}\frac{\partial}{\partial p_0}\right) + \frac{im\gamma_4}{p_0^2}\left(p_r\frac{\partial}{\partial p_0} + p_0\frac{\partial}{\partial p_r}\right) \\ I_0^D &= I_0^K + \frac{im}{4p_0^3\sqrt{p_0^2-m^2}}\left[2\gamma_4(2p_0^2-m^2) + mp_0\right] + \\ &\quad + \frac{im\gamma_4}{p_0\sqrt{p_0^2-m^2}} I^D - \frac{m+\gamma_4 p_0}{\sqrt{p_0^2-m^2}}\left(\gamma_\mu\frac{\partial}{\partial p_\mu} + \frac{im}{p_0}\frac{\partial}{\partial p_0}\right) \end{aligned}\right\} \quad (37)$$

The infinitesimal operators (30)-(33) and (34)-(37) satisfy the relations of the structure (7-15). Therefore, the transformations determined by them form the representation of the conformal group C_4. In the limit $m = 0$ they pass into corresponding operators for equations without mass, written in Table 2.

All the infinitesimal operators either commute with their wave equations or commute if applied to the solutions of these equations similarly to the case $m = 0$ /see formulas (16)-(19)/, for instance,

$$[I^D, i\gamma p+m] = -\left(1+\gamma_4\frac{m}{p_0}\right)(i\gamma p+m) \quad (38)$$

Note, that all the commutators equal to zero for $m = 0$ are equal to zero also for $m \neq 0$ except

$$\left[M_{r4}^D, i\gamma p+m\right] = \frac{m}{p_0\sqrt{p_0^2-m^2}}(p_r\gamma_4 - p_4\gamma_r)(i\gamma p+m) \quad (39)$$

So, the Klein-Gordon and Dirac equations with the mass are invariant under the 15-parameter group G_{15}.

However, in order to obtain the representation of the conformal group C_4 as a whole we had to change the representation of the Lorentz group L_{10} /the operators P_0 and M_{r4} have changed/. In view of this the law of transformations for 4-momentum of the particle p_μ /which does not coincide, generally speaking, with the operator of the infinitesimal translation P_μ/ is different from the transformation law of a 4-vector*. This is a difficulty. Analogous ones may take place also for other physical quantities.

---

* P_μ is transformed as a 4-vector.

In conclusion we point out that it is possible to find the analog of Pauli group for nonzero mass Dirac equation using the method given in Sec.3. For instance, the one-parameter group is

$$\psi'(p) = \exp(i\alpha\Gamma_5)\psi(p) \tag{40}$$

where

$$\Gamma_5 = \varepsilon(p_0)\frac{p_0+\gamma_4 m}{\sqrt{p_0^2-m^2}}\gamma_5 \ , \ \Gamma_5^2 = 1 \tag{41}$$

Γ_5 is a Lorentz pseudoscalar in the representation of the group C_4 considered above.

The authors are grateful to Prof. M.A.Markov for the interest to this work. We wish to thank L.G. Zastavenko for useful discussions.

Literature

I. E.Cunningham, Proc.Lond.Math.Soc. 8, 77, I9IO.
2. H.Bateman, Proc.Lond.Math.Soc. 8, 223, I9IO.
3. P.A.M. Dirac, Annals of Math. 37, 429, I936.
4. H. Bahba, Proc. Cambr. Phil. Soc. 32, 622, I936.
5. W. Pauli, Helv.Phys.Acta, I3, 204, I940.
6. A.McLennan Nuovo Cim. 3, I360, I956.
7. R.L. Ingraham, Nuovo Cim. 9, 886, I952; I2, 825, I954.
8. S. Bludmen, Phys.Rev. I07, II63, I957.
9. L. Page, Phys.Rev. 49, 254, I936.
IO. H.Robertson, Phys.Rev. 49, 755, I936.
II. H. T.E.Engström, M. Zorn, Phys.Rev. 49, 70I, I936.
I2. J. Haantjes, Proc.Ned.Akad.Wet. 43, I288, I940.
I3. E.L. Hill, Phys.Pev. 67, 358, I945; 72, I43, I947; 84, II65, I95I.
I4. J. Murai, Prog. Theor. Phys. 9, I47, I953; II, 44I, I954.
I5. M.Cini, B.Touschek, Nuovo Cim. 7, 422, I958.