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DISPERSION RELATIONS AND PERTURBATION THEORY

1.

An interesting result was obtained in a recent paper by Redmond[1] (see also[2]), which consists in the following: using the analiticity requirement it is possible to get the expressions for the propagators which do not contain the logarithmic poles.

In this way, we think, one may arrive to certain general conclusions concerning the perturbation theory method and, perhaps, concerning some general features of the modern quantum field theory.

The method of dispersion relations presents the most reasonable way for studying the analiticity properties of the propagators. Now it is the only approach to the problems of the quantum field theory, which is very likely free from inner difficulties. Therefore, it appears quite natural that further progress in the quantum field theory must be associated with the method of dispersion relations.

This method based on the most general principles of covariance, causality, unitarity and spectrality*) allows to obtain the expressions for the quantities of Green functions type and the matrix elements of the transitions in the form of the spectral expansions. Thereby the problem is reduced to the study of the properties of the corresponding spectral functions. They may be expressed through the Green functions for more complex processes by expansion in the complete set of physical states.

Here appears a possibility of obtaining the equation system for the determination of the Green functions. It should be noted that in contrast, e.g., to the Schwinger equation system no ultraviolet divergences arise here.

There are, however, some difficulties in the course of the realization of this program since the spectral representations for higher Green functions are not obtained yet.

2.

Here arises a palliative possibility of obtaining the missing information about the spectral functions by the data of the perturbation theory. It is this way which Symanzik followed in the paper[3]. Considering the n-th term of the perturbation theory for a certain vertex he showed that this term may be presented in a definite spectral form. Then, making use of the hypothesis about the possibility of summing the series for the spectral function Symanzik made a conclusion that the vertex under considera-

*) We mean by the spectrality principle the condition that the complete set of the physical states with positive energy exists.

tion may be presented as a whole in the given spectral form.

Symanzik used this method to obtain general theoretical conclusions leading to the proof of dispersion relations. The investigation of different possibilities for the approximations would be in our opinion of great interest. In this way we consider the Källen-Lehmann spectral formula as a simplest example. Instead of summing the whole series of the perturbation theory we restrict ourselves to the summation of only that class of diagramms the study of which leads from the standpoint of some authors[4] to the proof of the existence of a logarithmic pole.

The spectral representation of the boson propagator according to Källen-Lehmann theorem is of the form

$$\Delta_c(k^2)=\frac{1}{m^2-k^2}+\int_{m_0^2}^{\infty}\frac{I(z)\,dz}{(z-k^2-i\epsilon)} \tag{1}$$

The right hand side of this formula represents the function f analytical throughout the whole complex plane of the variable k^2 except the pole in the point $k^2=m^2$ and the cut along the real axis from $k^2=m_0^2$ to $k^2\longrightarrow\infty$. The function $\Delta_c(k^2)$ for the real $k^2\geq m_0^2$ equals the limiting value f along the upper edge of the cut, i.e., $\Delta_c(k^2)_{k^2\geq m_0^2}=\lim_{\epsilon\to 0} f(k^2+i\epsilon)=f_+(k^2)$

The spectral function $I(z)$ is real and defined by the discontinuity of the function f

$$2\pi i\, I(z)=f_+(z)-f_-(z)=\Delta_c(z)-\overset{*}{\Delta}_c(z) \tag{2}$$

where $f_-(z)$ is the value of the function f on the lower edge of the cut.

In case of electrodynamics we consider the class of those diagramms for the photon Green function D_c, which are represented as a photon line with an arbitrary number of simple insertions-electron-positron loops of the second order.

For the sake of brevity we shall call these graphs as accepted the main logarithmic diagramms. The contribution of the n-th term of this class in D_c is of the form

$$-\frac{1}{k^2}\left(e^2F(k^2,m^2)\right)^n \tag{3}$$

where $F(k^2,m^2)$ corresponds to the second order loop. The explicit expression for F is given, e.g., in § 32.1 of paper[5].

In the region $|k^2|\gg m^2$ the function F assumes the form

$$F(k^2,m^2)=\frac{1}{3\pi}\ln\frac{4m^2-k^2}{4m^2} \tag{4}$$

We introduced the term $4m^2$ under the logarithmic sign to give correctly the imaginary component of the function F

$$\operatorname{Im} F(k^2, m^2) = -\frac{1}{3}\theta(k^2-4m^2), \quad \theta(x) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases}$$

preserving at this its normalization

$$F(0, m^2) = 0$$

Note that the direot summation of the terms (4), made first in [6], leads to the expression

$$-\frac{1}{k^2}\left(1 - \frac{e^2}{3\pi}\ln\frac{4m^2-k^2}{4m^2}\right)^{-1} \qquad (5)$$

on the basis of which the conclusion was made [4] that there exists a logarithmic pole and, therefore, there is an internal contradiction in the theory. It can be easily seen that the n-th term (3) may be represented in the Källen-Lehmann spectral form.

Restricting ourselves by the approximation (4) we get

$$D_n = -\frac{1}{k^2}\left(\frac{e^2}{3\pi}\ln\frac{4m^2-k^2}{4m^2}\right)^n = \int_{4m^2}^{\infty}\frac{I_n(z)dz}{z-k^2}$$

where the function $I_n(z)$ is defined by the discontinuity of the function D_n by formula (2). Performing the summation for the spectral function I

$$I(z) = \sum_{n=1}^{\infty} I_n(z)$$

we make sure that $I(z)$ as a whole is a discontinuity of expression (5), Substituting (5) into (2) we find

$$I(z) = \begin{cases} \frac{e^2}{3\pi z}\left[\left(1 - \frac{e^2}{3\pi}\ln\frac{z-4m^2}{4m^2}\right)^2 + e^4/9\right]^{-1}, & z \geq 4m^2 \\ 0 & \end{cases} \qquad (6)$$

Thus, for the photon Green function be obtain

$$D_c(k^2) = -\frac{1}{k^2} + \frac{e^2}{3\pi}\int_{4m^2}^{\infty}\frac{dz}{z(z-k^2-i\epsilon)\left[\left(1-\frac{e^2}{3\pi}\ln\frac{z-4m^2}{4m^2}\right)^2+e^4/9\right]} \qquad (7)$$

The formulas of such a type are just discussed in paper [1].

As can be easily seen the expression (7) represents the function

$$D_c(k^2) = -\frac{1}{k^2}\left[1 - \frac{e^2}{3\pi}\ln\frac{4m^2-k^2}{4m^2}\right]^{-1} - \\ -\frac{3\pi}{e^2}\left[\left(1-e^{-3\pi/e^2}\right)\left(k^2-4m^2+4m^2e^{3\pi/e^2}\right)\right]^{-1}$$

which with account of the smallness e^2, for $|p^2|$ much more than m^2, may be written in the form

$$D_c(k^2) = -\frac{1}{k^2}\left[1 - \frac{e^2}{3\pi}\ln\left(-\frac{k^2}{4m^2}\right)\right]^{-1} - \frac{3\pi/e^2}{k^2+4m^2e^{3\pi/e^2}} \qquad (8)$$

The function (8) has the following remarkable properties: this function: 1) has no logarithmic pole; 2) in the neighbourhood of the point $e^2 = 0$ as a function of e^2 it has a singularity of "superconducting" type" exp $(-3\pi/e^2)$ 3) in the vicinity of the point $e^2 = 0$ it permits the asymptotic expansion coinciding with the expansion of the usual perturbation theory and representable in the form (5). It is also clear that the second term in the right hand side of (8) cannot be principally obtained in the perturbation theory due to the exponential order of smallness. Because of this it does not correspond to any Feynmann diagramms.

Note that the final expression (8) coordinates with the initial approximation (5). In the region where we used formula (5) for the calculation of the spectral function it differs from the final expression (8) by negligibly small terms of the order $exp(-3\pi/e^2)$

Thus, it is the formula (8) but not (5) represents the result of consistent summation of the main logarithmic terms which contain no paradoxes like a "zero-charge[4] difficulty".

Expression (6) for the spectral function was obtained from formula (5), which represents the result of the summation of the main logarithmic terms of the perturbation theory. We could start, however, not from formula (5) but from the expression (§43.I from[5]): $-\frac{1}{K^2}\left[1-e^2F(K^2,m^2)\right]^{-1}$ corresponding to the summation of the main logarithmic diagramms and passing into (5) in the limit $K^2 \gg m^2$

The corresponding spectral function is more cumbersome than formula (6), keeping at this all its essential properties.

3.

Similar correction of the formulas of the logarithmic summation may be fulfilled also for other propagators. Consider, e.g., a meson propagator in the symmetrical pseudoscalar theory of meson-nucleon interaction. The expression for this propagator obtained by improving the perturbation theory has the form[5,7]

$$\Delta_c(p^2) = \frac{1}{(m^2-p^2)\left[1-\frac{5g^2}{4\pi}\ln\left(\frac{-p^2}{m^2}\right)\right]^{\alpha}}$$

where $\alpha = 4/5$. The corresponding spectral function I_α, obtained by summing under the sign of the spectral representation is

$$I_1(z)=\frac{1}{m^2-z}\cdot\frac{\sin\left(\alpha\cdot \operatorname{arctg}\frac{5g^2/4}{1-t}\right)}{\pi\left[(1-t)^2+\frac{25g^4}{16}\right]^{\alpha/2}} \tag{9}$$

where $$t=-\frac{5g^2}{4\pi}\ln\frac{z}{m^2}$$

A general feature of the spectral functions (6) and (9) is their resonance character. As it was pointed out these formulas are obtained on the basis of the summation of the main logarithmic terms. It is interesting, therefore, to clear up whether the resonance character is conserved for the spectral functions which are obtained by summing the logarithmic terms of higher order of smallness. With this purpose we take as an initial expression for the photon propagator the formula (§43,2 from[5]) obtained by summing the terms of the form $(e^2\ln z)^n$ and $e^2(e^2\ln z)^m$

$$D_c^{T.B}(z)=-\frac{1}{z}\left\{1-t+\frac{ie^2}{3}\theta(z-4m^2)-\frac{3e^2}{4\pi}\ln\left[1-t+\frac{ie^2}{3}\theta(z-4m^2)\right]\right\}^{-1}$$

where $$t=\frac{e^2}{3\pi}\ln\frac{z-4m^2}{4m^2}$$

From this expression we get instead of (6) (at $z\geq 4m^2$)

$$I(z)=\frac{e^2}{3\pi z}\quad\frac{1+\frac{9}{4\pi}\operatorname{arctg}\frac{e^2/3\pi}{1-t}}{\left\{1-t+\frac{3e^2}{2\pi}\ln\left[(1-t)^2+\frac{e^4}{9}\right]\right\}^2+\frac{e^4}{9}\left\{1+\frac{9}{4\pi}\operatorname{arctg}\frac{e^2/3\pi}{1-t}\right\}^2} \tag{10}$$

It can be seen from formula (10) that the resonance character of the spectral function is conserved with account of higher logarithmic terms. The effect of these higher terms, however, upon the behaviour of the spectral function in the region of the resonance at $1-t\sim 0$ is not small. It is seen from the comparison of the expressions (6) and (10) that they differ by a factor $(1+9/8)$ independent of the degree of the smallness of the parameter e^2.

It should be emphazed that the improvement of the perturbation theory by summing the logarithmic terms of different order of smallness is not a consistent operation. As is wellknown (see § 42.4 from [5]) the formulas obtained in such a way may be correct only in the region where the quantity

$$e^2 d(\kappa^2) = -e^2\kappa^2 D_c(\kappa^2)$$

is small in comparison with a unit.

It follows from here that the obtained expressions for the spectral functions may be believed to be correct only in the region up to a "resonance". In the resonance region and higher one cannot consider these formulas correct since we use the initial approximation outside the range of its application departing from the weak coupling formulation. Therefore only a hypothesis about the resonance character of the spectral functions may be suggested.

The nonapplicability of the obtained expressions for the spectral functions in the region of great $\varkappa$ may be also seen from the comparison with the general expression for the spectral functions [8]. It is not at all surprising since the determination of the correct asymptotics even for only one photon propagator requires a simultaneous consideration of the asymptotics of other higher verteces.

4.

It is not difficult to make sure that the expressions for the Green function obtained above are not renormalization invariant. In this section we intend to show how they may be transformed into an invariant form, the photon propagator is being taken as an example. As an initial consider formula (8) and write it as follows

$$\frac{e^2 d}{3\pi} = -\frac{e^2\kappa^2 D_c(\kappa^2)}{3\pi} = \tag{11}$$

$$= \frac{1}{\frac{3\pi}{e^2} - \ln\frac{\kappa^2}{m^2}} + \frac{1}{1-\exp\left[\frac{3\pi}{e^2} - \ln\frac{\kappa^2}{m^2}\right]}$$

The function $e^2 d$ which is called an invariant charge must be an invariant of the renormalization group transformation (see § 42 in [5]). However it is quite clear that expression (11) does not satisfy this requirement.

A usual technique of reducing the expressions to the renormalization invariant form employes the apparatus of the Lie differential equations, the correspondence with the usual perturbation theory is being taken into account. Since the expressions of type (11) cannot be expanded in powers of e^2 it will be more convenient from the technical point of view to start not from the Lie differential equations but from the functional equations of the renormalization group.

With this aim we shall look for an alalogue of the usual function d normalized to a unit at $k^2=\lambda^2$ as in the form

$$\frac{e_\lambda^2 d\left(\frac{k^2}{\lambda^2}, e_\lambda^2\right)}{3\pi} = \frac{1}{\Phi\left(\frac{e_\lambda^2}{3\pi}\right) - \ln\frac{k^2}{\lambda^2}} + \frac{1}{1-\exp\left[\Phi\left(\frac{e_\lambda^2}{3\pi}\right) - \ln\frac{k^2}{\lambda^2}\right]} \tag{12}$$

having in view that, (as it was first shown by Gell-Mann and Low [9] on the basis of the group considerations), the invariant function may depend upon e_λ^2 and λ^2 only through the argument

$$\varphi(e_\lambda^2) + \ln\lambda^2$$

The invariance requirement on the function $e^2 d$ is of the form:

$$e_\lambda^2 d\left(\frac{k^2}{\lambda^2}, e_\lambda^2\right) = e_0^2 d\left(\frac{k^2}{m_0^2}, e_0^2\right) = e_{\lambda_1}^2 d\left(\frac{k^2}{\lambda_1^2}, e_{\lambda_1}^2\right) \tag{13}$$

where m_0 is the quantity of the order of an electron mass, whereas e_0 - the corresponding value of the charge. This requirement establishes the relation between the transformation laws of the charge and normalization momentum in terms of the function Φ

$$\Phi\left(\frac{e_\lambda^2}{3\pi}\right) - \ln\frac{k^2}{\lambda^2} = \Phi\left(\frac{e_0^2}{3\pi}\right) - \ln\frac{k^2}{m_0^2} = \Phi\left(\frac{e_{\lambda_1}^2}{3\pi}\right) - \ln\frac{k^2}{\lambda_1^2}$$

from where, in particular, it follows

$$\Phi\left(\frac{e_\lambda^2}{3\pi}\right) - \Phi\left(\frac{e_0^2}{3\pi}\right) = -\ln\frac{\lambda^2}{m_0^2} \tag{14}$$

Formula (14) yields qualitative limitation upon the function Φ. For $\lambda^2 \longrightarrow \infty$ the function $\Phi\left(\frac{e_\lambda^2}{3\pi}\right)$ must monotonously tend to $-\infty$. The explicit form of this function may now be defined from the normalization condition on d. Putting in (12) $k^2=\lambda^2$ we obtain

$$\frac{e_\lambda^2}{3\pi} = \frac{1}{\Phi\left(\frac{e_\lambda^2}{3\pi}\right)} + \frac{1}{1-\exp\left[\Phi\left(\frac{e_\lambda^2}{3\pi}\right)\right]}$$

i.e.

$$x = \frac{1}{\Phi(x)} - \frac{1}{e^{\Phi(x)} - 1}$$

NOTE ADDED IN PROOF OF THE ENGLISH VARIANT

The above considerations with some modifications may be applied to the nonrenormalizable theories. We take as an example the nonlinear fermion theory with 4-fermion interaction Lagrangian and investigate the four-fermion vertex function $\Gamma(p', q', p, q)$. One may assert if the renormalization group considerations are taken into account that the main contributions to the symmetrical momentum asymptotic behaviour of $\Gamma(p, p, p, p)$ are due to the graphs with an arbitrary number of the simplest insertions-fermion loops of the second order. The contribution of the n-th term will be of the type

$$\Gamma_n(p^2) = g^n\left(\frac{p^2}{m^2}\ln\frac{p^2}{m^2}\right)^n \tag{A1}$$

Eaoh term (A1) may be presented in the following spectral form

$$\Gamma_n(p^2) = \int_{m_0^2}^{\infty} \frac{\rho_n(z)}{z - p^2 - i\varepsilon}\, dz$$

Taking into account the branching of the logarithm we made the summation over n for the spectral function ρ in accordance with the above mentioned programme. It yields

$$\Gamma(p^2) = 1 + \frac{g}{m_0^2}\int_{m_0^2}^{\infty} \frac{z.\,dz}{z - p^2 - i\varepsilon}\cdot\frac{1}{\left[1 - g\frac{z}{m^2}\ln\frac{z}{m^2}\right]^2 + \pi^2 g^2\frac{z^2}{m^4}} \tag{A2}$$

Eq. (A2) reminds Eq. (3) from paper [2]. The main difference is that formula (A2) contains the logarithm in the dominator of the spectral function.

The righthand side of (A2) represents the function of the type

$$\frac{1}{1 - g\frac{p^2}{m^2}\ln\frac{p^2}{m^2}} + \frac{m^2}{g(p^2 - p_0^2)\left(1 + \ln\frac{p_0^2}{m^2}\right)} \tag{A 3}$$

where p_0^2 is the root of the equation $\quad 1 - g\frac{p_0^2}{m^2}\ln\frac{p_0^2}{m^2} = 0$

Within the limit of small g $\quad p_0^2 = m^2/g$

Therefore, the second term in (A3) takes the form

$$\frac{m^2}{(gp^2 - m^2)(1 - \ln g)} \tag{A 4}$$

For $p^2 \ll p_0^2$ this expression reduces to the function

$$+f(g) = -\frac{1}{1-\ln g}$$

possessing rather interesting properties

$$f(0) = 0\,, \quad f'(0) = \infty\,, \quad f''(0) = \infty\,, \ldots$$

Thus, term (A4) being small in the vicinity of the point $g = 0$ has no asymptotic expansion in powers of g. Therefore, the function (A2) as a whole cannot be expanded in such a series. (This is in agreement with the remark of paper [2]).

It is worth stressing once more that no final meaning should be assigned to the expression of the type (A3) since the consistent solution of the problem requires a simultaneous analysis of all Green functions.

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