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V.G. Grishin, M.I. Podgoretsky

REMARKS ON THE OPTICAL MODEL OF A NUCLEUS

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Abstract

Some specifications of the optical model are considered.

§ 1. The problem of particle scattering on a force centre is known to reduce to the solution of Shrödinger equation:

$$\nabla^2 \psi + [k_0^2 - U(\vec{r})]\psi = 0. \tag{1}$$

In the notations shown in Fig. 1 the solution has the form[1]

$$\varphi(P) = \int_V L\, U(\vec{r}')\psi(\vec{r}')\, d\tau' - \int_S \left(\varphi \frac{\partial L}{\partial n} - L \frac{\partial \varphi}{\partial n}\right) ds' \tag{2}$$

where $\varphi(P) = \psi(P) - e^{ik_0 z}$, $L = -\frac{1}{4\pi} \frac{e^{ik_0|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}$ and

$\frac{\partial}{\partial n}$ denotes the differentiation over the external normal to the surface element. The surface S and the volume V confined in it may be chosen arbitrary.

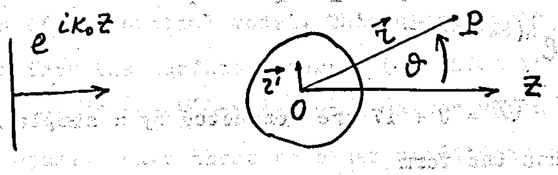


Fig. 1.

At great distances $\varphi \sim 1/r$. Therefore, the integral over the surface infinitely removed from a scatterer is tending to zero. If the surface S includes the points O and P and is tending to infinity, then the solution may be written as follows

$$\varphi(P) = \int_V L\, U \psi\, d\tau' \tag{3}$$

By another choice of the surface (see Fig.2) the integral over the infinite parts of the surface is tending to zero, the volume integral falls out, and the integral by the plane Σ remains. Thus, the same solution may be presented in the form

$$\varphi(P) = -\int_{\Sigma} \left(\varphi \frac{\partial L}{\partial n} - L \frac{\partial \varphi}{\partial n} \right) ds' \tag{4}$$

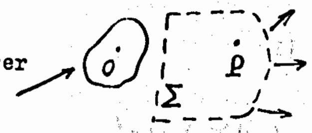
scatterer


Fig.2

The expression (3) is usually used in the Bohr approximation supposing that the incident wave does not distort, i.e. $\Psi \simeq e^{ik_0 z}$. An analogous approach may be used in other cases if the form of Ψ inside a scatterer is approximately known. In particular, in the analysis of fast particles interaction with nuclei and nucleons the function Ψ may be substituted in the approximation of the geometry optics and, thus, to obtain the expression for φ.

Let us consider such an approach taking as an example a homogeneous disk |Fig.3|:

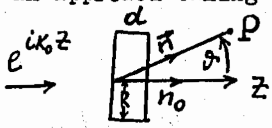


Fig.3

Here n and n_0 are the unit vectors in the direction of scattering and Z-axis respectively. Inside the disk we have $k = k_0 + k_1$ (k_1 may be complex), i.e.

$$\Psi = A e^{i(k_0 + k_1) z} \tag{5}$$

The quantities k_0, k and $\mathcal{V} = U + iV$ are connected by a simple relation. Indeed the equation (1) for our case has the form

$$\frac{d^2\Psi}{dz^2} + (k_0^2 - U + iV)\Psi = 0,$$

that after the substitution (5) yields

$$\mathcal{V} = U + iV = k_0^2 - k^2. \tag{5'}$$

Substituting (5) and (5') into (3) we find

$$\varphi(P) = -\frac{A(k_0^2 - k^2)}{4\pi} \int_V \frac{e^{ik_0|\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|} e^{i(k_0 + k_1) z'} d\tau'$$

at great distances

$$|\vec{r} - \vec{r}'| \simeq r - \vec{r}'\vec{n} \qquad \text{and} \qquad \frac{1}{|\vec{r} - \vec{r}'|} \simeq \frac{1}{r}.$$

Then
$$\varphi(P) = -\frac{A(k_0^2-k^2)e^{ik_0 r}}{4\pi r}\int_V e^{ik_0(\vec{n}_0-\vec{n})\vec{r}'}\cdot e^{ik_1 z'}d\tau'.$$

From here the scattering amplitude
$$f(\vartheta) = -\frac{A(k_0^2-k^2)}{4\pi}\int_V e^{ik_0(\vec{n}_0-\vec{n})\vec{r}'}\cdot e^{ik_1 z'}d\tau' \tag{6}$$

Having introduced the cylindrical coordinate system after the integration we obtain
$$f(\vartheta) = \frac{(1-e^{-ik_1 d})\cdot A(k_0^2-k^2)}{2ik_1}\cdot\frac{RJ_1(k_0R\sin\vartheta)}{k_0\sin\vartheta} \tag{7}$$

Here $I_1(k_0\ R\sin\vartheta)$ is the Bessel function of the first order. For simplicity we shall conduct further consideration only for two limiting cases.

a) $|k_1| << k_0$. Then the reflection does not practically take place on the boundaries of the division, i.e. $A \approx 1$.

Moreover, $U = (k_0-k)(k_0+k) \approx -2k_0k_1$

Therefore,
$$f(\vartheta) = i(1-e^{ik_1 d})\frac{RJ_1(k_0R\sin\vartheta)}{\sin\vartheta} \tag{8}$$

b) A very great absorption on the dist thickness.

In this case it is possible to neglect the wave reflection from the back wall of the disk while the reflection from the front one may be essential |if $k \gtrsim k_0$|. Taking into account the boundary conditions we obtain: $A = \frac{2k^0}{k_0+k}$, i.e. for $f(\vartheta)$ formula (8) is again valid.

The obtained result coincides with the expression for the scattering amplitude calculated on the basis of the optical model. This coincidence is not accidental since the optical model starts from the surface integral (4) which is equal to an initial volume integral (3) is instead of φ and Ψ their exact values would be substituted. Therefore, both approaches are equal. Under those or other conditions one is more convenient than the other from a methodical point of view.

§2. Suppose that the scatterer consists of independent elementary centres presented in Fig. 4 as hatched circles.

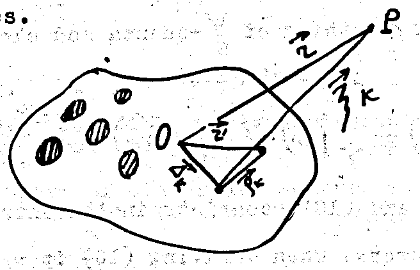


Fig. 4

Dut to additivity

$$V(\vec{r}') = \sum V_k(\vec{r}'), \qquad \text{where} \quad V_k(\vec{r}') = V_0(\rho_k)$$

|the function V_0 is equal for all the centres|. Then from (3) we obtain

$$\varphi(P) = \sum_k \int L V_k \psi \, d\tau' = \sum \varphi_k.$$

Since $|\vec{r} - \vec{r}'| = |\vec{\xi}_k - \vec{\rho}_k|$, then

$$\varphi_k = \int L(\vec{\xi}_k, \vec{\rho}_k) \cdot V_0(\vec{\rho}_k) \psi(\vec{\rho}_k) \, d\tau.$$

Here the integration is made over the variable vector $\vec{\rho}_k$. At sufficiently small absorption in the approximation of a geometry optics in the vicinity of each elementary scatterer one may write:

$$\psi(\vec{r}') = B \cdot e^{ik(\vec{n}_0 \vec{r}')} = B e^{ik(\vec{n}_0 \vec{\Delta}_k)} e^{ik(\vec{n}_0 \vec{\rho}_k)} = \psi(\vec{\Delta}_k) \cdot e^{ik(\vec{n}_0 \vec{\rho}_k)}$$

i.e.

$$\varphi_k(P) = \psi(\vec{\Delta}_k) \cdot \int L(\vec{\xi}_k, \vec{\rho}_k) V_0(\vec{\rho}_k) e^{ik(\vec{n}_0 \vec{\rho}_k)} \, d\tau \tag{9}$$

Further

$$L(\vec{\xi}_k, \vec{\rho}_k) = L(\vec{r}', \vec{r}) \simeq -\frac{e^{ik_0 r}}{4\pi r} e^{-ik_0(\vec{n}\vec{\Delta}_k + \vec{n}\vec{\rho}_k)}$$

Then

$$\varphi = \frac{\tilde{f}(\vartheta)}{r} e^{ik_0 r} \left(\sum \psi(\vec{\Delta}_k) e^{-ik_0 \vec{n} \vec{\Delta}_k} \right),$$

where

$$\tilde{f}(\vartheta) = -\frac{1}{4\pi} \int e^{i(k\vec{n}_0 \vec{\rho}_k - k_0 \vec{n} \vec{\rho}_k)} V_0(\rho_k) \, d\tau$$

The scattering amplitude is of the form

$$A(\vartheta) = \tilde{f}(\vartheta) \cdot \left(\sum_k \psi(\vec{\Delta}_k) e^{-ik_0 \vec{n} \vec{\Delta}_k} \right) \tag{10}$$

If $|k_1| \Delta l \ll 1$, where Δl are the dimensions of an elementary center, then coincides practically with the scattering amplitude $f(\vartheta)$ on a free elementary centre. It is this case which occurs in quick π-meson and nucleon interaction with nuclei ($|k_1| \sim 10^{12}\,cm^{-1}$ and $\Delta l \sim 10^{-13}\,cm$) to say nothing of γ-quanta and electrons. Therefore, one may consider that $\tilde{f}(\vartheta) \approx f(\vartheta)$ that yields

$$A(\vartheta) = f(\vartheta) \sum \psi(\vec{\Delta}_k) e^{-ik_0 \vec{n} \vec{\Delta}_k} \tag{10'}$$

The physical meaning of (10) and (10') consists in the interference of secondary scattered waves from elementary scatterers. When deriving (10) it was supposed that inside each elementary centre the wave does not change /$e^{ikz'}$ both outside and inside the emitter/, i.e. for each emitter the Bohr approximation is used. It is clear from the physical meaning of (10) that it is not obligatory and that the same result will also hold for strong distortion of the field inside elementary scatterers.

As is known the differential cross section for elastic scattering $\sigma(\vartheta) = |A(\vartheta)|^2$. Since the elementary centres are moving we have for experimentally measured $\sigma_{exp}(\vartheta) = \overline{\sigma(\vartheta)} = \overline{|A(\vartheta)|^2} =$

$$= |f(\vartheta)|^2 \cdot \overline{|\sum_k \psi(\vec{\Delta}_k) e^{-ik_0 \vec{n}\vec{\Delta}_k}|^2}$$

As a result of averaging the term dependent only upon the mean number of elementary centres in different parts of the scatterer and the term dependent upon the fluctuation in the number of centres are obtained. Further we shall consider only the first term. Therefore, from the very beginning one may average A(ϑ), then calculate $|\bar{A}|^2$*. Let us introduce the mean density of elementary scatterers $\rho(\vec{\Delta})$ **, then instead of (10') we obtain:

$$A(\vartheta) = f(\vartheta) \cdot F(\vartheta), \tag{11}$$

where

$$F(\vartheta) = \int_\tau \rho(\vec{\Delta}) \psi(\vec{\Delta}) e^{-ik_0 \vec{n}\vec{\Delta}} \cdot d\tau. \tag{11'}$$

has the meaning of a generalized form-factor. If $\psi(\vec{\Delta}) \sim e^{ik_0 n_0}$, then the generalized form-factor passes into a usual one, used in Bohr approximation |2,3|.

From (11) and (11') follows that

$$\sigma(\vartheta) = |f(\vartheta)|^2 \cdot |F(\vartheta)|^2 \tag{11''}$$

§ 3. By a usual consideration in the optical model it is supposed that the approximation of geometry optics is valid inside a scatterer. Then in view of (3) and (4) being equal to each other one may write

$$A_{o\delta}(\vartheta) = -\frac{1}{4\pi} \int_V e^{-ik_0 \vec{n}\vec{\Delta}} \cdot V(\vec{\Delta}) \psi(\vec{\Delta}) d\tau$$

Further one may believe that

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* The investigation of the second term makes it possible to come to the problem about the character of intranuclear fluctuations.

** $\rho(\vec{\Delta})$ is the mean number of elementary scatterers the centres of which are situated in the volume under consideration. Further by $\rho(\vec{\Delta})$ we shall understand the density of nucleon number.

$$U(\vec{\Delta}) = -4\pi f(0)\rho(\vec{\Delta}) \tag{12}$$

i.e.
$$A_{o\delta}(\vartheta) = f(0)\cdot\int_V e^{-ik_0\vec{n}\vec{\Delta}}\rho(\vec{\Delta})\psi(\vec{\Delta})\,d\tau .$$

On the other hand we have from (11) $A(\vartheta) = f(\vartheta)\cdot\int_V e^{-ik_0\vec{n}\vec{\Delta}}\rho(\vec{\Delta})\psi(\vec{\Delta})\,d\tau$

Comparing $A_{o\delta}(\vartheta)$ and $A(\vartheta)$ we obtain:

$$A(\vartheta) = \frac{f(\vartheta)}{f(0)}A_{o\delta}(\vartheta) \tag{13}$$

It is seen from (13) that $A(\vartheta)$ by $\vartheta \neq 0$ is different from $A_{o\delta}(\vartheta)$ since the usual optical consideration does not take into account secondary waves radiated (emitted) by elementary scatterers at the angles $\vartheta \neq 0$. In forward scattering from (13) we have:

$A(0) = A_{o\delta.}(0)$ that is clear from the physical point of view, since the introduction of K_1 just takes into account the superposition of all secondary waves radiated forward. The difference between $A(\vartheta)$ and $A_{o\delta}(\vartheta)$ vanishes if the scattering on an elementary centre is isotropic, but may be essential if there is an anisotropy.

From (13)

$$\sigma(\vartheta) = \frac{\sigma_0(\vartheta)}{\sigma_0(0)}\,\sigma_{o\delta}(\vartheta) \tag{14}$$

Here

$\sigma_0(\vartheta) = |f(\vartheta)|^2$ and $\sigma_{o\delta}(\vartheta) = |A_{o\delta.}(\vartheta)|^2$

As usual $\frac{\sigma_0(\vartheta)}{\sigma_0(0)} < 1$, i.e. $\sigma(\vartheta) < \sigma_{o\delta}(\vartheta)$ and the total elastic cross section σ_{el} will be less than an analogous usual one $\sigma_{el.o\delta.}$ It follows from $A(0) = A_{o\delta.}(0)$ and from the well-known optical theorem $\sigma_t = 4\pi\lambda\, \mathrm{Im}\, A(0)$ that the total cross sections σ_t are identical. From here the total inelastic cross section $\sigma_{in} = \sigma_t - \sigma_{el}$ is greater than an analogous usual one $\sigma_{in.o\delta.}$

$$\sigma_{in} = \sigma_t - \sigma_{el} = \int\sigma_{o\delta}(\vartheta)\,d\Omega + \sigma_{in.o\delta.} - \int\frac{\sigma_0(\vartheta)}{\sigma_0(0)}\sigma_{o\delta}(\vartheta)\,d\Omega \tag{15}$$

i.e., $\sigma_{in} = \sigma_{in.o\delta.} + \int\sigma_{o\delta.}(\vartheta)\left[1 - \frac{\sigma_0(\vartheta)}{\sigma_0(0)}\right]d\Omega .$

The previous consideration was conducted in the nonrelativistic case. From the standpoint of physical meaning all what has been said but a concrete form of an integral determining $f(\vartheta)$ and $\tilde{f}(\vartheta)$ is referred also to a relativistic case.

Formula (14) is referred to the scattering on heavy elementary centres, i.e. is correct in the lab.system. It can be easily seen that in case of small scattering angles it has the same form in any coordinate system. Indeed, it is known that the quantity $\sigma^*(\vartheta)d\Omega^*$ is an invariant*

Then $$\sigma(\vartheta) = \sigma^*(\vartheta^*)\frac{d\Omega^*}{d\Omega} = \frac{\sigma_0^*(\vartheta)}{\sigma_0^*(0)}\sigma_{o\delta}^*(\vartheta^*)\frac{d\Omega^*}{d\Omega}$$

but $$\sigma_{o\delta}^*(\vartheta^*)\frac{d\Omega^*}{d\Omega} = \sigma_{o\delta}(\vartheta), \quad \sigma_0^*(0) = \sigma_0(0)\cdot\left(\frac{d\Omega}{d\Omega^*}\right)_0$$

and $$\sigma_0^*(\vartheta^*) = \sigma_0(\vartheta)\left(\frac{d\Omega}{d\Omega^*}\right)_\vartheta$$

Therefore, $$\sigma(\vartheta) = \frac{\sigma_0(\vartheta)}{\sigma_0(0)}\sigma_{o\delta}(\vartheta)\cdot\left(\frac{d\Omega^*}{d\Omega}\right)_0 \Big/ \left(\frac{d\Omega^*}{d\Omega}\right)_\vartheta$$

If the angles are small, then $\left(\frac{d\Omega^*}{d\Omega}\right)_0 \Big/ \left(\frac{d\Omega^*}{d\Omega}\right)_\vartheta \approx 1 - \alpha\vartheta^2$,

where α is equal to 1/2 if the mass of the scattered particles is equal to a mass of elementary scatterers and $\alpha \ll 1$ in light particle scattering on heavy ones.

Thus, finally we again obtain the form of (14).

§ 4. The experiments on quick electron scattering on nuclei [4] have been performed. The analysis of experimental data was made by formula (11) by substituting the Coulomb scattering amplitude on a point charge for $f(\vartheta)$. Thus, the obtained function $\rho_e(\vec{\Delta})$ is characteristic for the charge density distribution in a nucleus.

More direct and suitable characteristic valid for comparison with the results of other experiments is the nucleon density in a nucleus.

Then by formula (11") we have

$$\sigma_{exp}^{\text{я}} = \sigma_{exp}^{p}\cdot F_{\text{я}}^{2} \tag{16}$$

where $\sigma_{exp}^{\text{я}}$ is the experimental cross section for electron scattering on a nucleus, σ_{exp}^{p} is the experimental cross section for electron scattering on a proton

$$F_{\text{я}} = \int \rho(\vec{\Delta})\gamma(\vec{\Delta})e^{-ik_0\vec{n}\vec{\Delta}}d\tau$$

and $\rho_H(\vec{\Delta})$ is the nucleon density in a nucleus. On the other hand, the authors of the

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* The asterisk denotes the quantities referring to the moving coordinate system.

paper[4] believed

$$\sigma_{exp}^{Я} = \sigma_{мот.} F^{2}_{Я.ноф.} \tag{16'}$$

where σ_{mot} is the cross section for electron scattering by a point charge ("Mott scattering") and $F_{Я.ноф.}$ is the form-factor obtained from (16'). Making the right-hand sides of(16) and (16') equal we obtain

$$F_{Я}^{2} = \frac{\sigma_{мот.} F^{2}_{ноф.Я.}}{\sigma^{p}_{exp.}}$$

If the magnetic scattering is absent then $\sigma^{p}_{exp} = \sigma_{мот} F_{p}^{2}$, where F_p is the form-factor of a proton. Finally one gets

$$F_{Я} = \frac{F_{Я.ноф.}}{F_{p}.}$$

For small angles in the Bohr approximation we have[4] $F = 1 - \frac{q^2 \overline{r^2}}{6}$, where $\hbar \vec{q}$ is a transferred momentum $(|q| = K_0 \vartheta)$ and $\overline{r^2}$ is the r.m.s. radius. Expanding all the form-factors in a series we get from (17)

$$\overline{r^2_{Я}} = \overline{r^2_{Я.ноф.}} - \overline{r^2_{p}} \tag{18}$$

The difference in the root-mean-square radii $\sqrt{\overline{r^2_{Я}}}$ and $\sqrt{\overline{r^2_{Я.ноф.}}}$ is 5.5% for carbon.

Analogous considerations ought to be taken into account also in the analysis of π -meson and nucleon[5-8] scattering. It is necessary to start from the density distribution in a nucleus and to take into account the cross section $\sigma_0(\vartheta)$ of elementary interaction treating the experimental data in accordance with (11") or (14) and not simply by a usual optical model. In this connection the paper[5] should be mentioned in which there is a conclusion about a possible difference between the quantities $\overline{r^2}$ in scattering of electrons and π -mesons on nuclei. The existence of the above-mentioned effect gives rise to some doubts from an experimental point of view.

At any rate when discussing this effect it is necessary to take into account the considerations stated above. Unfortunately, this was not done by the authors in an explicit form.

To introduce the correction it is necessary to expand the generalized form-factor (11') in a series by q^2. In the limiting case of small absorptions the expansion has the form

$$1 - \frac{q^2 \overline{r^2}}{6}$$

in the opposite, limiting case of very strong absorption (over the whole nucleus) there practically occurs a diffraction on a black sphere and

$$\sigma(\vartheta) = 2\,\sigma_{os.}(0)\,\frac{J_1^2(K_0 R \sin\vartheta)}{K_0^2 R^2 (\sin\vartheta)^2}$$

Then the expansion for small angles is of the form $1-\frac{q^2\overline{r^2}}{6}$. Thus, in the general case the coefficient at $q^2\overline{r^2}$ is likely to confine between 1/6 and 1/8.

To obtain the corrections it is also necessary to expand the elementary cross sections $\pi-p$ and p-p interactions in a series by q^2.

These expansions are as follows

$$\sigma_0(\vartheta) \approx \sigma_0(0)\left(1-\frac{q^2R^2}{6}\right), \qquad \text{at } E_{\pi,p} \approx (1\div 3)\,\text{Bev and } R=1{,}2\cdot 10^{-13}\,\text{cm.}$$

When making the planned calculations for carbon nuclei the correction in $\overline{r^2}$ of about 6.5% is obtained.

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