JOINT INSTITUTE FOR NUCLEAR RESEARCH

Laboratory of Theoretical Physics

P. 272

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Nuovo Cim., 1959, v 12, N 3, c 264-285.

Dubna, 1959 г.

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PARTIAL WAVE ANALYSIS OF THE PRODUCTION OF BOSON PAIRS***

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Abstract

Partial wave analysis for boson pair production on nucleons is made. The corresponding angular operators, which completely characterize the spin and angular dependence of the S-matrix, are expressed with the help of the Legendre polynomials and tabulated.

Simultaneously, a general method of calculating the angular operators is given for processes containing more than four particles, in which cases the straightforward method leads to very lengthy calculations.

I. Introduction

Boson production processes as pion and photon pair production, rediative pion scattering etc. have been intensively investigated during the last years. The theoretical bibliography on this subject may be divided into two parts: one containing pure-theoretical treatises, which solve the problem by means of the Chew-Low method or dispersion relations, and the other consisting of works which use some semiempirical postulates concerning the dynamical character of the process, such as assumptions on the exixtence of resonant states.

For both these methods, it is useful to study the angular and charge dependence of the S-matrix. In the first case, one obtains in this way a set of independent equations for the energy-dependent coefficients; in the latter case, one can write directly the amplitude of the assumed resonant state as a function of angular and isotopic variables .

To perform this angular and isotopic analysis, it is sufficient to know the general laws of conservation, without assuming anything concerning the dynamics of the reaction. The result of such an analysis is a set of orthonormal polynomials, which can also be used for the phase analysis of the experimental data. This is especially important for processes with an interaction character which is not very well known.

In the present paper, the ordinary space structure of the S-matrix is studied for processes which may be described with the help of the following formula

$$b + N \longrightarrow \sum_{i=1}^{n} b_i + N \qquad (1)$$

where N denotes the nucleon and b , b_i bosons of any kind. For processes of the type $b + N \rightarrow b' + N$ this analysis has been made by Ritus [1]. However, the straightforward generalization of his method to the case of more particles leads to lengthy and cumbersome calculations. For instance, for $n = 2$ in (1) almost two hundred terms must be calculated and for higher values of n the calculations are practically impossible. We give a method for simplifying this procedure, namely by reducing the production of n bosons to that of $n-1$ bosons. We demonstrate it for $n = 2$, but it has general validity (n = 2 is chosen only to deal with simple formulae). For the same value of n we have calculated the explicit form of the angular operators. The resulting polynomials are given in the tables I, II and III for different coupling schemes of the final angular momenta.

Readers interested only in the practical use of these tables can omit sections 3 and 4, which deal with the calculation formalism of the angular operators in the boson-boson and boson-nucleon coupling schemes respectively.

2. The angular operators

Let the initial state of our system be characterized by the total angular momentum J, its z-component M and by the eigenvalues (i) of a system of quantities which together with J and M form a complete set of commuting observables. Then, the initial state will be described by the following ket-vector

$$|JM(i)>$$

Similarly, the final state will be described by

$$|J'M'(f)>$$

where the symbols have an analogical meaning to that of $J, M, (i)$ respectively. The operator which describes the transition $|JM(i)> \longrightarrow |J'M'(f)>$ is defined by

$$|J'M'(f)><JM(i)|$$

Consequently, the S-matrix can be expressed as follows

$$S = \sum a(J'M'(f); JM(i)) \; |J'M'(f)><JM(i)| \tag{2}$$

where $a(J'M'(f); JM(i))$ determines the "breadth" of the corresponding channel. Doing the partial wave analysis, we shall be concerned only with the angular momentum part of the channel labels (i), (f) and we shall omit any explicit reference to other quantities such as energy, isotopic spin etc., which would be included in the a-coefficients. Due to the three-dimensional rotational invariance of the S-matrix, the right-hand side of (2) may be written in the following form:

$$\sum_{J(f)(i)} a(J(f)(i)) \sum_{M=-J}^{+J} |JM(f)><JM(i)|$$

where

$$\sum_{M=-J}^{J} |JM(f)\rangle\langle JM(i)| = \hat{\mathcal{J}}(J(f)(i)) \tag{3}$$

are the so-called angular operators of the reaction (see [1]); they determine the angular and spin dependence of the S-matrix. They are orthogonal with one another and are normalized in the following way:

$$\underset{\substack{\text{over}\\ \text{spins}}}{tr} \int_{\substack{\text{over}\\ \text{all angles}}} \hat{\mathcal{J}}^{+}(J'(f')(i'))\,\hat{\mathcal{J}}(J(f)(i)) =$$

$$= \sum_{M'=-J'}^{+J'} \sum_{M=-J}^{+J} tr \int |J'M'(i')\rangle\langle J'M'(f)|\,|JM(f)\rangle\langle JM(i)| = (2J+1)\delta_{J'J}\,\delta_{(f')(f)}\,\delta_{(i')(i)}$$

if the initial and the final state eigenfunotions are normalized to 1.

Before calculating the explicit form of (3) for reactions belonging to the type (1), let us remark that if we know the angular operators for b, b_i (see (1)) having spin zero, then those for spin one may be obtained by operating with:

$\frac{1}{\sqrt{\ell_k(\ell_k+1)}} \frac{\partial}{\partial \vec{k}}$ for the electric multipole,

$\frac{-i}{\sqrt{\ell_k(\ell_k+1)}} \left[\vec{k} \frac{\partial}{\partial \vec{k}}\right]$ for the magnetic multipole and

$\vec{k}$ for the "longitudinal" multipole

on the corresponding angular operators, Therefore we calculate only the angular operators for b and b_i having spin zero, and we do not suppose any definite parity for them. (So, if some of the b, b_i are π-mesons, the corresponding angular operators are obtained by postulating the internal oddness of these particles). In this particular case, the set (i) of the initial eigenvalues consists of the spin $\sigma = 1/2$ of the nucleon and the orbital momentum ℓ of the ingoing boson. The symbol (f), in the two-boson case, represents by itself either

$$\sigma' = 1/2,\ \ell_1, \ell_2, L \quad \text{where: } \hat{\vec{L}} = \hat{\vec{\ell}}_1 + \hat{\vec{\ell}}_2 \tag{I}$$

or

$$\sigma' = 1/2,\ \ell_1, j, \ell_2 \quad \text{where: } \hat{\vec{j}} = \hat{\vec{\sigma}}' + \hat{\vec{\ell}}_1 \tag{II}$$

Both these coupling schemes must give the same information about the process, if transitions from all possible initial to all possible final states are considered i.e. if no one from the a -coefficients is neglected. Mathematically, this is an evident fact because of the completeness of the orthonormal set of the angular operators both in Case (I) and in Case (II). In practical calculations, however, we deal always only with certain non-complete subsets of these sets, having to restrict curselves to a finite number of terms in (2). Then, the two coupling schemes are no more equivalent and we must choose that of them which gives a better approximation to the ideal case. This depends, naturally, on the distribution of the a - coefficients in (2): if only few of them are large and the others small, then it is sufficient to consider only a small number of channels for obtaining a true picture of the reality. If, on the contrary, we deal with such a coupling in which all or many a 's are large, the calculations approache very slowly to the real case. Thus, this practical reason gives a criterion in choosing the coupling.

Besides this practical argument, from the physical point of view, the fact that in a given coupling scheme only a small number of channels are important, gives a deeper insight into the understanding of the nature of the process. Indeed, the existence of such resonant states cannot be predicted by means of general group-theoretical methods, but is in a direct connection with the dynamical properties of the interaction. For instance, it is to be expected that the isobaric state ($^3/_2$ $^3/_2$) of the nucleon will play an important role in processes in which at least one π -meson is present, such as the case for nucleon-nucleon collisions and for elastic scattering of pions*.

In general, we can say that the choice of the coupling scheme is determined by the interaction: if in a system of three particles, a certain pair interacts much stronger than other pairs, it may be considered approximately as a shortly-living subsystem with spin equal to the vectorial sum (according to the vector model) of angular momenta of these particles. Naturally, by specifying the spin eigenvalue of this subsystem we determine, simultaneously, the coupling scheme of the angular momenta. This situation is quite analogous to that in an atom, in which the angular momenta may be coupled either according to the (LS)-scheme or according to the (jj)-scheme.

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* We refer the reader to the clear discussion on this topic contained in the paper of R.F. Peierls (see [2]).

3. The boson-boson coupling

In the case of the boson-boson coupling, the final state may be written as follows

$$|JM\ell_1\ell_2 L\,{}^1\!/_2\rangle = \sum_{\mu'=-1/2}^{1/2} C_{JM}^{LM-\mu'\,{}^1\!/_2\mu'}\, \mathcal{Y}_{LM-\mu'}^{\ell_1\ell_2}\, |{}^1\!/_2\mu'\rangle$$

where

$$\mathcal{Y}_{LM-\mu'}^{\ell_1\ell_2}(\vec{q},\vec{\tau}) = \sum_{\lambda_1=-\ell_1}^{\ell_1} C_{LM-\mu'}^{\ell_1\lambda_1\,\ell_2 M-\mu'-\lambda_1}\, Y_{\ell_1\lambda_1}(\vec{q})\, Y_{\ell_2 M-\mu'-\lambda_1}(\vec{\tau})$$

is the orbital eigenfunction of the outgoing bosons and $|{}^1\!/_2\,\mu'\rangle$ is the spin eigenfunction of the nucleon. If we write the initial state also as a Clebsch-Gordan combination of the angular momenta of the ingoing particles, we find that the angular operator (3) has the following form

$$^{(L)}\hat{\mathcal{J}}(J\ell_1\ell_2 L\,{}^1\!/_2\,\ell\,{}^1\!/_2) = \sum_{\mu'=-1/2}^{1/2}\sum_{\mu=-1/2}^{1/2} \mathcal{L}_{\mu'\mu}\, |{}^1\!/_2\mu'\rangle\langle{}^1\!/_2\mu| \qquad (4)$$

where

$$\mathcal{L}_{\mu'\mu} = \sum_M C_{JM}^{LM-\mu'\,{}^1\!/_2\mu'}\, C_{JM}^{\ell M-\mu\,{}^1\!/_2\mu}\, \mathcal{Y}_{LM-\mu'}^{\ell_1\ell_2}(\vec{q},\vec{\tau})\, Y^*_{\ell M-\mu}(\vec{p}) \qquad (5)$$

will be called the orbital operator and

$$|{}^1\!/_2\mu'\rangle\langle{}^1\!/_2\mu| = \begin{cases} \dfrac{1\pm\sigma_z}{2} & \mu'=\mu=\pm{}^1\!/_2 \\[2mm] \dfrac{\sigma_x \pm i\sigma_y}{2} & \mu'=-\mu=\pm{}^1\!/_2 \end{cases} \qquad (6)$$

will be called the spin operator (σ_x, σ_y, σ_z are the Pauli matrices); $\vec{p}$, $\vec{q}$ and $\vec{\tau}$ are unit vectors parallel to the momenta of the b , b_1 and b_2 -particles respectively.

Comparing this with formula (5.II) of [1] we see that the angular operators (4) have the same form as those for the elastic scattering of (parity-less) pions on nucleons; the only difference is that the orbital eigenfunction of the final pion is replaced here by the $\mathcal{Y}$ -function. Therefore, all calculations are analogous to those for the elastic scattering, namely, the sum over $M=-J,\ldots\ldots+J$ in (5) can be reduced, by choosing the z -axis parallel to $\vec{p}$, to one term:

$$C_{J\mu}^{L\,\mu-\mu'\,{}^1\!/_2\mu'}\, C_{J\mu}^{\ell 0\,{}^1\!/_2\mu}\, \mathcal{Y}_{L\,\mu-\mu'}^{\ell_1\ell_2}(\vec{q},\vec{\tau})\, \frac{\sqrt{2\ell+1}}{\sqrt{4\pi}} \qquad (7)$$

(because $M-\mu = 0$ in this coordinate frame). By inserting this expression in (4) we obtain the form of $\hat{\mathcal{J}}$ in a special frame and, using the rotational invariance of $\hat{\mathcal{J}}$, we can write it in an invariant form.

Let us perform the calculations some more in detail. The first Clebsch-Gordan coefficient in (7) is different from zero only for $J=L\pm 1/2$ and the second one only for $J=\ell\pm 1/2$ Furthemore, $\mu=\pm 1/2$ and $\mu'=\pm 1/2$ so that there are sixteen values of $\mathcal{L}_{\mu'\mu}$ for given ℓ, L, ℓ_1 and ℓ_2, Let us remark now that the magnetic index $\mu-\mu'$ in (7) has only the values 1,0 and -1, so that the $\mathcal{Y}$-function may be expressed with the help of the operator

$$\hat{\vec{L}} = \frac{1}{i}\left[\vec{q}\frac{\partial}{\partial\vec{q}}\right] + \frac{1}{i}\left[\vec{\tau}\frac{\partial}{\partial\vec{\tau}}\right]$$

in the following way

$$\sqrt{L(L+1)}\,\mathcal{Y}_{L,\pm 1} = \hat{L}_{\pm}\mathcal{Y}_{L0} \qquad (8)$$

where $\hat{L}_{\pm} = \hat{L}_x \pm i\hat{L}_y$. Now, according to (6) and (7), the four terms on the right-hand side of (4) may be written as follows (we take, for example, the case $J=L+1/2=\ell+1/2$):

$$\frac{L+1}{\sqrt{2L+1}}\mathcal{Y}^{\ell,\ell_1}_{L0}(\vec{q},\vec{\tau})\frac{1\pm\sigma_z}{2} \qquad (9)$$

for $\mu'=\mu=\pm 1/2$, and

$$\frac{1}{\sqrt{2L+1}}\hat{L}_{\mp}\mathcal{Y}^{\ell,\ell_2}_{L0}(\vec{q}\vec{\tau})\frac{\sigma_x\pm i\sigma_y}{2}$$

for $\mu'=-\mu=\pm 1/2$. They depend, evidently, on the coordinate frame chosen. In order to write them invariantly, let us define the vectors

$$\vec{P}_x = \vec{\tau} - \vec{p}\;\vec{p}\cdot\vec{\tau}\,, \qquad \vec{P}_y = [\vec{p}\,\vec{\tau}]$$

which together with $\vec{p}$ form a carthesian system and are, in the $p\parallel z$-frame, parallel to the x, y, z-axis respectively (the x-axis in shosen in the plane of the vectors $\vec{p}, \vec{\tau}$). Their lengths are $|\vec{P}_x|=|\vec{P}_y|=\sqrt{1-(\vec{p}\cdot\vec{\tau})^2}$, $|\vec{p}|=1$. With the help of them, we can write (9) in the following invariant form:

$$\frac{L+1}{\sqrt{2L+1}}\mathcal{Y}_{L0}\,\frac{1\pm\vec{\sigma}\cdot\vec{p}}{2}$$

$$\frac{1}{2\vec{P}_x^2\sqrt{2L+1}}\vec{P}_{\mp}\cdot\hat{\vec{L}}\,\mathcal{Y}_{L0}\,\vec{P}_{\pm}\cdot\vec{\sigma}$$

where $\vec{P}_\pm = \vec{P}_x \pm i\vec{P}_y$. Taking into account that

$$\frac{\vec{P}_-\cdot\hat{\vec{L}}\;\vec{P}_+\cdot\vec{\sigma} + \vec{P}_+\cdot\hat{\vec{L}}\vec{P}_-\cdot\vec{\sigma}}{2P_x^2} = \sigma_x \hat{L}_x + \sigma_y \hat{L}_y$$

and that $\hat{L}_z \, Y_{L0} = 0$

we obtain the following form for the angular operator:

$$\hat{\mathcal{J}} = \frac{(L+1)}{\sqrt{2L+1}} \, Y_{L0} + \frac{1}{\sqrt{2L+1}} \, \vec{\sigma}\cdot\hat{\vec{L}} \, Y_{L0} \tag{10}$$

In this form, the angular operator is still independent of the number of outgoing particles (for more particles it is only necessary to write more arguments behind Y_{L0}). For the two-particle case, we can write

$$Y_{L0}^{\ell_1\ell_2} = \sum_{\lambda_1=-\ell_1}^{\ell_1} C_{L0}^{\ell_1\lambda_1,\ell_2,-\lambda_1} Y_{\ell_1\lambda_1} Y_{\ell_2,-\lambda_1} \tag{11}$$

For simplicity, we restrict ourselves to ℓ_1 = 0 and 1, ℓ_2 being arbitrary. We obtain

for $\ell_1 = 0$

$$Y_{\ell_2 0}^{0\ell_2} = \frac{1}{\sqrt{4\pi}} Y_{\ell_2 0} = \frac{\sqrt{2\ell_2+1}}{4\pi} \mathcal{P}_{\ell_2}(\vec{p}\cdot\vec{\tau})$$

and for $\ell_1 = 1$

$$Y_{\ell_2+1\,0}^{1\ell_2} = \frac{\sqrt{3}}{4\pi\sqrt{\ell_2+1}} \Big((\ell_2+1)\vec{p}\cdot\vec{q}\; \mathcal{P}_{\ell_2}(\vec{p}\cdot\vec{\tau}) - [\vec{p}\vec{q}]\cdot[\vec{p}\vec{\tau}]\,\mathcal{P}_{\ell_2}'(\vec{p}\cdot\vec{\tau}) \Big)$$

$$Y_{\ell_2 0}^{1\ell_2} = \frac{-i\sqrt{3}}{4\pi} \sqrt{\frac{2\ell_2+1}{\ell_2(\ell_2+1)}} \; \vec{p}\cdot[\vec{q}\vec{\tau}]\,\mathcal{P}_{\ell_2}'(\vec{p}\cdot\vec{\tau})$$

$$Y_{\ell_2-1\,0}^{1\ell_2} = \frac{\sqrt{3}}{4\pi\sqrt{\ell_2}} \big(-\ell_2\, \vec{p}\cdot\vec{q}\; \mathcal{P}_{\ell_2}(\vec{p}\cdot\vec{\tau}) - [\vec{p}\vec{q}]\cdot[\vec{p}\vec{\tau}]\,\mathcal{P}_{\ell_2}'(\vec{p}\cdot\vec{\tau}) \big)$$

where $\mathcal{P}_{\ell_2}(\vec{p}\cdot\vec{\tau})$ is the Legendre function of order ℓ_2 .

By inserting them in (10) and in analogical expressions for $J = L \pm 1/2 = \ell \mp 1/2$, $J = L - 1/2 = \ell - 1/2$ we obtain sixteen different angular operators as given in Table I and II.

4. The boson-nucleon coupling

In the case of the boson-nucleon coupling, the angular operator is given by a formula which is quite analogical to (4):

$$^{(j)}\hat{\mathcal{Y}}(J\ell_1\ell_2 j\,{}^1\!/_2\,\ell\,{}^1\!/_2) = \sum_{\mu'=-1/2}^{1/2}\sum_{\mu=-1/2}^{1/2} \mathcal{L}_{\mu'\mu}\,|{}^1\!/_2\,\mu'\rangle\langle{}^1\!/_2\,\mu| \qquad (12)$$

However, we can easily see that this form of writing, in which the orbital and spin quantities are separated, is very disadvantageous in this case. Indeed, $\mathcal{L}_{\mu'\mu}$ in (12) have the following form:

$$\mathcal{L}_{\mu'\mu} = \sum_{M=-J}^{J}\sum_{m=-j}^{j} C_{JM}^{\ell_2 M-m\, jm}\, C_{jm}^{\ell_1 m-\mu'\, 1/2\mu'}\, C_{JM}^{\ell M-\mu\, 1/2\mu}\, Y_{\ell_2 M-m}(\vec{\tau})\, Y_{\ell_1 m-\mu'}(\vec{q})\, Y^{*}_{\ell\, M-\mu}(\vec{p}) \qquad (13)$$

If comparing it with (5) we see that (13) contains two summations (namely over M and over m), while (5) has only one, namely that over M . The second summation in (5) is involved implicitly, through $\mathcal{Y}$, which must be expressed through $Y(\vec{q})$ and $Y(\vec{\tau})$ according to (11). For this reason, even if the sum over M is eliminated in (13) by suitable choosing of the coordinate frame (as it was done in (5)), the sum over m remains. This circumstance, naturally, complicates considerably the calculations. Therefore it is more convenient to write (11) in another form:

$$^{(j)}\hat{\mathcal{Y}}(J\ell_1\ell_2 j\,{}^1\!/_2\,\ell\,{}^1\!/_2) = \sum_{m=-j}^{j}\sum_{\mu=-1/2}^{1/2} \mathcal{K}_{m\mu}\,|jm\,\ell_1\,{}^1\!/_2\rangle\langle{}^1\!/_2\,\mu| \qquad (14)$$

where

$$\mathcal{K}_{m\mu} = \sum_{M=-J}^{J} C_{JM}^{\ell_2 M-m\, jm}\, C_{JM}^{\ell M-\mu\, 1/2\mu}\, Y_{\ell_2 M-m}(\vec{\tau})\, Y^{*}_{\ell M-\mu}(\vec{p}) \qquad (15)$$

and

$$|jm\,\ell_1\,{}^1\!/_2\rangle = \sum_{\mu'=-1/2}^{1/2} C_{jm}^{\ell_1 m-\mu'\, 1/2\mu'}\, Y_{\ell_1 m-\mu'}(\vec{q})\,|{}^1\!/_2\,\mu'\rangle$$

It is clear that all the four summs will subsist in (14), namely the two "small" (over μ and μ')* and the two "great" ones (over m and M). The last two are responsible for the existance of a great amount of terms, which make all direct computations impossible. Nevertheless, (14) has the advantage of an explicit summation only over m , M being included in $\mathcal{K}_{m\mu}$ (see (15)).

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* One of them, (μ'), is contained implicitely through $|jm\,\ell_1\,{}^1\!/_2\rangle$

To get rid of the summation over m, we have to take the vector $\vec{q}$ parallel to the z-axis; on the other hand, the expression for $\mathcal{K}_{m\mu}$ reduces to one term only in the coordinate frame in which $\vec{p}$ is parallel to the z-axis. Since $\mathcal{K}_{m\mu}$ is not invariant we must, having found its form in the $\vec{p} \parallel z$-frame, transform it into the $\vec{q} \parallel z$-frame, in which (14) may easily be calculated. For this reason, we shall investigate the transformation properties of $\mathcal{K}_{m\mu}$. Let us consider the following angular operator

$$\hat{\mathcal{K}}(J\ell_2 j\,\ell\,{}^1\!/_2) = \sum_{m=-j}^{j} \sum_{\mu=-1/2}^{1/2} \mathcal{K}_{m\mu}\, |j\,m\rangle\langle {}^1\!/_2\,\mu| \tag{16}$$

The only difference between (16) and (14) is that $|j\,m\rangle$ in (16) is a pure spin function corresponding to spin j, while $|j\,m\,\ell_1\,{}^1\!/_2\rangle$ in (14) is a combination of a spherical harmonic $Y_{\ell\,m-\mu'}(\vec{q})$ and a one-half spin function $|{}^1\!/_2\,\mu'\rangle$. The operators

$$|j\,m\rangle\langle {}^1\!/_2\,\mu|$$

describe the conversion of the nucleon into a particle with spin j. Thus, the $\hat{\mathcal{K}}$ operator defined by (16) describes the elastic scattering on a fermion the spin of which changes from ${}^1\!/_2$ to j.

The method consists now in the following: from (15), we find the form of $\mathcal{K}_{m\mu}$ in the $\vec{p} \parallel z$-frame and inserting it in (16) we obtain the explicit form of $\hat{\mathcal{K}}$. Since $\hat{\mathcal{K}}$ is invariant, it has the same form in the $\vec{q} \parallel z$-system. The form of $\mathcal{K}_{m\mu}$ in this system may now be obtained by means of a trace operation on the product of $\mathcal{K}$ with the spin transition operators $(|j\,m\rangle\langle {}^1\!/_2\,\mu|)^+$, written in the $\vec{q} \parallel z$-system:

$$\hat{\mathcal{K}} = \sum_{m\mu} \mathcal{K}_{m\mu}(z \parallel \vec{p})\, |j\,m\,(z \parallel \vec{p})\rangle\langle {}^1\!/_2\,\mu\,(z \parallel \vec{p})|$$

$$\mathcal{K}_{m\mu}(z \parallel \vec{q}) = \mathrm{tr}\left(\hat{\mathcal{K}}\, |{}^1\!/_2\,\mu\,(z \parallel \vec{q})\rangle\langle j\,m\,(z \parallel \vec{q})|\right) \tag{17}$$

where $|{}^1\!/_2\,\mu\,(z \parallel \vec{q})\rangle$ and $\langle j\,m\,(z \parallel \vec{q})|$ are the spin functions in the $\vec{q} \parallel z$-frame, for spin ${}^1\!/_2$ and j respectively. In order to write easily these transition operators in different coordinate frames it is convenient to express them as linear combinations of the spin-tensors of Racah, which, as it is known provide a basis for the coresponding matrix algebra. Moreover, they have the advantage that their transformations properties are known, so that they can easily be written in the desired coordinate frame. The spin-tensors are defined as follows:

$$T^{I\xi}(j,{}^1\!/_2)=\sum_{\mu=-1/2}^{1/2}(-1)^{1/2-\mu}\,C^{j\,\xi-\mu\ 1/2\,\mu}_{I\xi}\,|j\,\xi-\mu\rangle\langle{}^1\!/_2\,\mu| \qquad (18)$$

and transform according to the irreducible representation of weight $2I+1$ of the three-dimensional rotation group (in our case $I=j+{}^1\!/_2$ or $I=j-{}^1\!/_2$

Let us perform this calculation for $j={}^1\!/_2$ and $j={}^3\!/_2$ for which values the Clebsch-Gordan coefficients are tabulated (see [4]). We obtain, for $j={}^1\!/_2$

$$T^{00}({}^1\!/_2\,{}^1\!/_2)=\frac{1}{\sqrt{2}}\begin{pmatrix}1&0\\0&1\end{pmatrix};\qquad T^{1,\pm1}({}^1\!/_2\,{}^1\!/_2)=\mp\frac{\sigma_x\pm i\sigma_y}{2};\ T^{1,0}({}^1\!/_2\,{}^1\!/_2)=\frac{1}{\sqrt{2}}\sigma_z \qquad (19a)$$

and for. $j={}^3\!/_2$

$$T^{11}=-\frac{1}{2}\begin{pmatrix}\sqrt{3}&0\\0&1\\0&0\\0&0\end{pmatrix}\qquad T^{10}=-\frac{1}{2}\begin{pmatrix}0&0\\\sqrt{2}&0\\0&\sqrt{2}\\0&0\end{pmatrix}\qquad T^{1\,-1}=-\frac{1}{2}\begin{pmatrix}0&0\\0&0\\1&0\\0&\sqrt{3}\end{pmatrix} \qquad (19b)$$

$$T^{22}=\begin{pmatrix}0&1\\0&0\\0&0\\0&0\end{pmatrix}\qquad T^{21}=-\frac{1}{2}\begin{pmatrix}1&0\\0&-\sqrt{3}\\0&0\\0&0\end{pmatrix}\qquad T^{20}=-\frac{1}{2}\begin{pmatrix}0&0\\\sqrt{2}&0\\0&-\sqrt{2}\\0&0\end{pmatrix}$$

$$T^{2\,-1}=-\frac{1}{2}\begin{pmatrix}0&0\\0&0\\\sqrt{3}&0\\0&-1\end{pmatrix}\qquad T^{2\,-2}=-\begin{pmatrix}0&0\\0&0\\0&0\\1&0\end{pmatrix}$$

In order to be able to write them in different coordinate frames, it is convenient to make use of the carthesian components T_i, T_{ik}, rather than the circularly polarized ones. They are connected by the following relations:

$$T^{1,\pm1}=\mp\frac{T_x\pm i\,T_y}{\sqrt{2}}\qquad T^{1,0}=T_z \qquad (20a)$$

$$T^{2,\pm2}=\frac{1}{2}(T_{xx}-T_{yy})\pm i\,T_{xy}$$

$$T^{2,\pm1}=\mp(T_{zx}\pm i\,T_{yz}) \qquad (20b)$$

$$T^{2,0}=\frac{1}{\sqrt{6}}(2T_{zz}-T_{xx}-T_{yy})$$

(see Appendix). Now, if we use the relation inverse to (18) we can express, with the help of (20), the spin transition operators $|jm\rangle\langle {}^1\!/_2\, \mu|$ in their tensorial form. We obtain (omitting the labels j and $^1\!/_2$)
for $j = {}^1\!/_2$

$$|\pm {}^1\!/_2\rangle\langle \pm {}^1\!/_2| = \tfrac{1}{\sqrt{2}}\,(T^{00} \pm T_z) \qquad (21a)$$

$$|\pm {}^1\!/_2\rangle\langle \mp {}^1\!/_2| = \tfrac{1}{\sqrt{2}}\,(T_x \pm i\, T_y)$$

and for $j = {}^3\!/_2$

$$|\pm {}^3\!/_2\rangle\langle \pm {}^1\!/_2| = \tfrac{1}{2}\,(T_{xz} \pm i T_{yz}) \pm \tfrac{1}{2}\sqrt{\tfrac{3}{2}}\,(T_x \pm i\, T_y)$$

$$|\pm {}^1\!/_2\rangle\langle \pm {}^1\!/_2| = \mp \tfrac{1}{2\sqrt{3}}\,(2\, T_{zz} - T_{xx} - T_{yy}) - \tfrac{1}{\sqrt{2}}\, T_z$$

$$|\mp {}^1\!/_2\rangle\langle \pm {}^1\!/_2| = -\tfrac{\sqrt{3}}{2}\,(T_{xz} \mp i\, T_{yz}) \mp \tfrac{1}{2\sqrt{2}}\,(T_x \mp i\, T_y) \qquad (21b)$$

$$|\mp {}^3\!/_2\rangle\langle \pm {}^1\!/_2| = \mp \tfrac{1}{2}\,(T_{xx} - T_{yy}) + i\, T_{xy}$$

Now, we must compute $\mathcal{K}_{m\mu}$ both for $j = {}^3\!/_2$ and $j = {}^1\!/_2$. For $j = {}^3\!/_2$, there are eight possibilities of combining ℓ_2 with j and ℓ with $\sigma = {}^1\!/_2$, namely:

$$J = \ell_2 + {}^3\!/_2 = \ell \pm {}^1\!/_2$$

$$J = \ell_2 + {}^1\!/_2 = \ell \pm {}^1\!/_2$$

$$J = \ell_2 - {}^1\!/_2 = \ell \pm {}^1\!/_2$$

$$J = \ell_2 - {}^3\!/_2 = \ell \pm {}^1\!/_2$$

For $j = {}^1\!/_2$, there are the following four possibilities:

$$J = \ell' + {}^1\!/_2 = \ell \pm {}^1\!/_2$$

$$J = \ell' - {}^1\!/_2 = \ell \pm {}^1\!/_2$$

Let us perform the calculations for an example, say $J = \ell_2 + 3/2 = \ell + 1/2$ i.e. $\ell = \ell_2 + 1$. In the coordinate frame in which $z \parallel \vec{p}$ and the x -axis is chosen in the plane of the vectors $\vec{p}, \vec{\tau}$, we have, using the Clebsch-Gordan coefficients:

$$\mathcal{K}_{3/2\,1/2} = \frac{1}{4\pi}\sqrt{\frac{\ell_2(\ell_2+1)(\ell_2+2)}{(2\ell_2+1)(2\ell_2+2)(2\ell_2+3)}}\sqrt{\frac{\ell+1}{2\ell+1}}\, Y_{\ell_2,-1}(\vec{\tau})\sqrt{2\ell+1}$$

$$= \frac{1}{4\pi}\,\frac{\ell_2+2}{\sqrt{(2\ell_2+2)(2\ell_2+3)}}\,\tau_-\,\mathcal{P}'_{\ell_2}(\vec{p}\cdot\vec{\tau})$$

This expression is to be multiplied by

$$\frac{1}{2}(T_{xz} + i\,T_{yz}) + \frac{1}{2}\sqrt{\frac{3}{2}}(T_x + i\,T_y)$$

Since, however,

$$\mathcal{K}_{-\frac{1}{2}\cdot\frac{1}{2}} = -\frac{1}{4\pi}\,\frac{\ell_2+2}{\sqrt{(2\ell_2+2)(2\ell_2+3)}}\,\tau_+\,\mathcal{P}'_{\ell_2}(\vec{p}\cdot\vec{\tau})$$

is to be multiplied by $\frac{1}{2}(T_{xz} - i\,T_{yz}) - \frac{1}{2}\sqrt{\frac{3}{2}}(T_x - i\,T_y)$ and since $\tau_- = \tau_+ = \tau_x$ in our system, the sum of these two terms gives

$$\frac{1}{4\pi}\,\frac{\ell_2+2}{\sqrt{(2\ell_2+2)(2\ell_2+3)}}\,\mathcal{P}'_{\ell_2}(\vec{p}\cdot\vec{\tau})\,\tau_x\,\frac{1}{2}\left(i\,T_{yz} + \sqrt{\frac{3}{2}}\,T_x\right)$$

This expression can be written in the following invariant form

$$\frac{\ell_2+2}{\sqrt{(2\ell_2+2)(2\ell_2+3)}}\,\mathcal{P}'_{\ell_2}(\vec{p}\cdot\vec{\tau})\,\frac{1}{2}\left(\sqrt{\frac{3}{2}}\,\vec{P}_x\cdot\vec{T} + i\,\vec{P}_y\cdot\vec{\vec{T}}\cdot\vec{p}\right)$$

In a similar way, terms containing $\mathcal{K}_{\pm 3/2,\mp 1/2}$, $\mathcal{K}_{\pm 1/2,\pm 1/2}$, $\mathcal{K}_{\pm 1/2,\mp 1/2}$ are calculated. Summing all these contributions, one obtains the following expression for $\hat{\mathcal{K}}$:

$$\hat{\mathcal{K}} = \frac{1}{\sqrt{(2\ell_2+2)(2\ell_2+3)}}\Big\{-\sqrt{6}(\ell_2+2)(\ell_2+1)\,\vec{p}\cdot\vec{T}\,\mathcal{P}_{\ell_2}(\vec{p}\cdot\vec{\tau}) +$$

$$+(\ell_2+2)(\sqrt{6}\,\vec{P}_x\cdot\vec{T} - 2i\,\vec{P}_y\cdot\vec{\vec{T}}\cdot\vec{p})\,\mathcal{P}'_{\ell_2}(\vec{p}\cdot\vec{\tau}) + 2i\,\vec{P}_y\cdot\vec{\vec{T}}\,\vec{P}_x\,\mathcal{P}''_{\ell_2}(\vec{p}\cdot\vec{\tau})\Big\}$$

Similar expressions were obtained also for the other six cases of $j = 3/2$

By the same method, also the $\hat{\mathcal{K}}$ - operators for $j = 1/2$ were obtained.

To find the $\hat{\mathcal{J}}$ -operators, we make use of (17). It is easy to see that the orbital operators $\mathcal{L}_{\mu'\mu}$ may be written as follows :

$$\mathcal{L}_{\mu'\mu} = \frac{1}{(4\pi)^{3/2}} C_{j\mu'}^{\ell_1 0\, 1/2 \mu'} \sqrt{2\ell_1+1}\ tr\,(\hat{\mathcal{K}}\,|\,1/2\,\mu><j\mu'|)$$

The trace operations can quickly be performed using the formulae

$$tr\ (T_i^+ T_j) = \delta_{ij}$$

$$tr\ (T_{ij}^+ T_{k\ell}) = \tfrac{1}{2}(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk})$$

$$tr\ (T_{ij}^+ T_k) = 0$$

These relations have only a restricted domain of validity, but for our purpose one can use them without supplementary precautions (see Appendix).

For instance, for $j = 3/2$ ($\ell_1 = 1$), $J = \ell_2 + 3/2 = \ell + 1/2$ we obtain the following result, if we are working in the $\vec{q} \parallel z$-frame*:

$$\mathcal{L}_{1/2\,1/2} = \frac{1}{(4\pi)^{3/2}} \frac{\sqrt{6}}{\sqrt{(2\ell_2+2)(2\ell_2+3)}} \Big\{ (\ell_2+1)(\ell_2+2)\,\vec{p}\cdot\vec{q}\ \mathcal{P}_{\ell_2} +$$

$$+ (\ell_2+2)(i\,\vec{p}\cdot\vec{q}\ \vec{P}_Y\cdot\vec{q} - \vec{P}_X\vec{q})\,\mathcal{P}'_{\ell_2} - i\,\vec{P}_X\cdot\vec{q}\ \vec{P}_Y\cdot\vec{q}\ \mathcal{P}''_{\ell_2} \Big\}$$

The other orbital operators have a similar form or are somewhat more complicated. By multiplying $\mathcal{L}_{\mu'\mu}$ with $|1/2\mu'><1/2\mu|$ and summing over μ' and μ , we obtain the corresponding angular operator. The resulting angular operators are tabulated in Table I and Table III.

All practical calculations of Section 3 and Section 4 were checked. Moreover, the normalization and orthogonality of each angular operator was verified by direct integration.

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* The result is dependent on the coordinate frame.

5. Conclusion

We must mention that the calculation of the angular operators is only the first step in investigating the S-matrix of a given process. To do the second one means to perform the same analysis in the isotopic space. The method of obtaining the corresponding "isotopic angular operators" is quite analogous, the only difference being that if one or more photons are present, the S-matrix is no more a scalar but rather a sum of a scalar and the third component of a vector. This means that instead of each angular operator in the ordinary space, there are one scalar and there vectorial operators in the isotopic space. So, the number of operators is greater, but the calculations are not more complicated because the whole procedure is quite automatical.

The third step consists in constructing the theory of interaction. This problem must be solved by other means than the first two, for instance, with the help of dispersions or of the Chew-Low equation. However, it should be mentioned that for the complete analysis of the reaction the first two steps are necessary, because only after having done them, one can compare the theory with the experiment. Indeed, if the angular analysis gives the explicit form of the angular operator $\hat{\mathcal{J}}(J,(f),(i))$ and the form of the S-matrix is obtained by use of the interaction theory, the experimentally observable coefficient $a(J(f)(i))$ is defined by the theoretical formula

$$a(J(f)(i)) = tr\int \hat{\mathcal{J}}^{+}(J(f)(i))\, S$$

where $tr\int$ means integration over all continuous and sum over all discrete variables. In the isotopic space, the situation is quite analogous.

Acknowledgements

We would like to express our gratitude to Professor Ning Hu, Professor S. Ţiţeica and Professor Ya. A. Smorodinsky for many illuminating discussions. We thank also Mr. V.G. Efimov for helpful conversations during this work.

Appendix

Relation (18) is a unitary transformation connecting the spin transition operators $|jm\rangle\langle {}^1\!/\!_2\,\mu|$ with the circularly polarized spin-tensors $T^{j+1/2}$ and $T^{j-1/2}$. We see from it that the circularly polarized spin-tensors are orthonormal:

$$tr\,\left((T^{I'\xi'})^+ .\ T^{I\xi}\right) = \delta_{I'I}\,\delta_{\xi'\xi} \tag{A1}$$

Further, we can use (18) for recognizing the transformation character of the spin transition operators $|{}^3\!/\!_2\, m\rangle\langle {}^1\!/\!_2\,\mu|$. For this purpose, we express T^2 and T^1 through the corresponding carthesian components T_{ik} and T_i. The relation between T^1 and T_i is well-known:

$$\begin{aligned} T^{1,\pm 1} &= \mp \frac{1}{\sqrt{2}}\,(T_x \pm i\, T_y) \\ T^{1,0} &= T_z \end{aligned} \tag{A2}$$

To find the relation between T^2 and T_{ik}, we express T^2 as a Clebsch-Gordan combination of two independent vectors. We obtain

$$\begin{aligned} T^{2,\pm 2} &= \frac{1}{2}\,U_x V_x - \frac{1}{2}\,U_y V_y \pm \frac{i}{2}\,(U_x V_y + U_y V_x) \\ T^{2,\pm 1} &= \mp\frac{1}{2}\,(U_z V_x + U_x V_z) - \frac{i}{2}\,(U_y V_z + U_z V_y) \\ T^{2,0} &= -\frac{1}{\sqrt{6}}\,U_x V_x - \frac{1}{\sqrt{6}}\,U_y V_y + \sqrt{\frac{2}{3}}\,U_z V_z \end{aligned} \tag{A3}$$

and

$$T^{0,0} = \frac{1}{\sqrt{3}}\,(U_x V_x + U_y V_y + U_z V_z)$$

This is a unitary transformation between the circularly polarized tensors and the symmetrized pairs. The corresponding carthesian symmetrical tensor is defined by

$$\begin{aligned} T_{xx} &= U_x V_x\ , \ldots \\ T_{xy} &= T_{yx} = \frac{1}{2}\,(U_x V_y + U_y V_x),\ \ldots \end{aligned} \tag{A4}$$

To make the transformation (A4) unitary, we define also

$$\tilde{T}_{xx} = T_{xx} \, , \ldots$$
$$\tilde{T}_{xy} = \sqrt{2}\, T_{xy} \tag{A5}$$

In this way $\tilde{T}_{ik}$ will be expressed through T^2 and $T^{oo} \equiv T$ by a product U of two unitary transformations. However, being not contained in the relations (18), for $j = {}^3\!/\!_2$, T cannot be expressed through the spin transition operators $|{}^3\!/\!_2\, m><{}^1\!/\!_2\, \mu|$ Therefore, it must be constructed artificially, with the help of a supplementary vector, $\vec{p}$, say. For instance, let us take $T = \vec{p} \cdot \vec{T}$ Then

$$tr\,(T^+ T^{2,3}) = 0 \quad , \quad tr\,(T^+ T) = 1 \tag{A6}$$

Hence, according to (A1) and owing to the unitarity of U , $\tilde{T}_{ik}$ will be an orthonormal set of matrices:

$$tr\,(\tilde{T}^+_{xx}\ \tilde{T}_{xx}) = tr\,(\tilde{T}^+_{xy}\ \tilde{T}_{xy}) = \ldots = 1 \tag{A7}$$

i.e.

$$tr\,(T^+_{ij}\ T_{ke}) = \frac{1}{2}\,(\delta_{ik}\,\delta_{je} + \delta_{ie}\,\delta_{jk}) \tag{A7}$$

Combining now the three transformations (A3), (A4) and (A5) we obtain the explicit form of the matrix U

$$U = \begin{array}{c} \begin{array}{cccccc} (xx) & (yy) & (zz) & (xy) & (yz) & (zx) \end{array} \\ \begin{pmatrix} \frac{1}{2} & -1/2 & 0 & \frac{1}{\sqrt2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{i}{\sqrt2} & -\frac{1}{\sqrt2} \\ -\frac{1}{\sqrt6} & -\frac{1}{\sqrt6} & \frac{2}{\sqrt6} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{i}{\sqrt2} & \frac{1}{\sqrt2} \\ \frac{1}{2} & -1/2 & 0 & -\frac{1}{\sqrt2} & 0 & 0 \\ \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & 0 & 0 & 0 \end{pmatrix} \end{array} \tag{A8}$$

This is the transformation matrix between $T^{2\}}$, T and $\tilde{T}_{ik}$

Similarly, from (A2) it follows

$$tr\ (T_i^+ T_k) = \delta_{ik} \tag{A9}$$

However, T_i and T_{ik} will not be orthogonal one to another because T_{ik} contains T_i (through $T = \vec{\rho}\cdot\vec{T}$). By means of another choice of T , they can be made orthogonal, but in this way, either (A6) or (A7) or (A9) would be violated. Nevertheless, this arbitrariness in choosing T is not embarrassing: being a linear combination of elements of the product representation $\mathcal{D}^{1/2} \times \mathcal{D}^{1/2}$, $\hat{\mathcal{K}}$ will not contain elements of the $\mathcal{D}^0$ representation so that terms containing T will cancel in all our expressions.

This is also the reason which enables us to take

$$tr\ (T_i^+ T_{ke}) = 0 \tag{A10}$$

in our calculations. This relation is correct for all angular operators $\hat{\mathcal{K}}$ because all non-orthogonal terms between T_i and T_{ke} are due to T .

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This manuscript received by Publishing Department on December 17, 1958.

TABLES OF THE ANGULAR OPERATORS.

These Tables contain angular operators for processes of the type $b+N\to N+b_1+b_2$ (where b_1, b_2 and b are zero-spin bosons without definite parity) for $l_1=0,1$ and l, l_2 being arbitrarily large. l, l_1 and l_2 are orbital quantum numbers of b, b_1 and b_2 respectively. L is the intermediate quantum number in the case of boson-boson coupling in the final state ($\vec{L}=\vec{l}_1+\vec{l}_2$), j is the intermediate quantum number in the case of boson-nucleon coupling in the final state ($\vec{j}=\vec{l}_1+\vec{\sigma}'$). $\vec{p}, \vec{q}, \vec{k}$ are unit vectors in the directions of the momenta of b, b_1, b_2 respectively. For practical reasons, we make use of an orthogonal set of (not normalized) vectors $\vec{P}_x=\vec{k}-\vec{p}\,\vec{p}\cdot\vec{k}$, $\vec{P}_y=[\vec{p}\vec{k}]$, $\vec{p}$.

$\mathcal{P}_{l_2}$ denotes the Legendre function depending on $\vec{p}\cdot\vec{k}=\cos\theta$; $\mathcal{P}'_{l_2}$ and $\mathcal{P}''_{l_2}$ are derivatives with respect to $(\vec{p}\cdot\vec{k})$.

If one of the bosons is a vectorial particle, the corresponding angular operators can be derived from ours by means of the following operations:

$$\frac{1}{\sqrt{l_\kappa(l_\kappa+1)}}\frac{\partial}{\partial\vec{k}} \quad \text{for the electric multipole,}$$

$$\frac{-i}{\sqrt{l_\kappa(l_\kappa+1)}}\left[\vec{k}\frac{\partial}{\partial\vec{k}}\right] \quad \text{for the magnetic multipole,}$$

$$\vec{k} \quad \text{for the „longitudinal" multipole.}$$

Here $\vec{k}$ is the unitary momentum of the corresponding boson.

TABLE I.

$$\begin{cases} J=L+\tfrac12=l+\tfrac12 & l_1=0 \quad L=l_2 \quad l=l_2 \\ J=l_2+\tfrac12=l+\tfrac12 & l_1=0 \quad j=\tfrac12 \quad l=l_2 \end{cases}$$

$$\hat{\mathcal{Y}}=(4\pi)^{-3/2}\left\{(l_2+1)\mathcal{P}_{l_2}+i\,\vec{\sigma}\cdot\vec{P}_y\,\mathcal{P}'_{l_2}\right\}$$

$$\begin{cases} J=L+\tfrac12=l-\tfrac12 & l_1=0 \quad L=l_2 \quad l=l_2+1 \\ J=l_2+\tfrac12=l-\tfrac12 & l_1=0 \quad j=\tfrac12 \quad l=l_2+1 \end{cases}$$

$$\hat{\mathcal{Y}}=(4\pi)^{-3/2}\left\{-(l_2+1)\vec{\sigma}\cdot\vec{p}\,\mathcal{P}_{l_2}+\vec{\sigma}\cdot\vec{P}_x\,\mathcal{P}'_{l_2}\right\}$$

$$\begin{cases} J=L-\tfrac12=l+\tfrac12 & l_1=0 \quad L=l_2 \quad l=l_2-1 \\ J=l_2-\tfrac12=l+\tfrac12 & l_1=0 \quad j=\tfrac12 \quad l=l_2-1 \end{cases}$$

$$\hat{\mathcal{Y}}=(4\pi)^{-3/2}\left\{-l_2\vec{\sigma}\cdot\vec{p}\,\mathcal{P}_{l_2}-\vec{\sigma}\cdot\vec{P}_x\,\mathcal{P}'_{l_2}\right\}$$

$$\begin{cases} J=L-\tfrac12=l-\tfrac12 & l_1=0 \quad L=l_2 \quad l=l_2 \\ J=l_2-\tfrac12=l-\tfrac12 & l_1=0 \quad j=\tfrac12 \quad l=l_2 \end{cases}$$

$$\hat{\mathcal{Y}}=(4\pi)^{-3/2}\left\{l_2\,\mathcal{P}_{l_2}-i\,\vec{\sigma}\cdot\vec{P}_y\,\mathcal{P}'_{l_2}\right\}$$

TABLE II.

$J = L + 1/2 = l + 1/2 \quad l_1 = 1 \quad L = l_2 + 1 \quad l = l_2 + 1$

$$\hat{\mathcal{Y}} = (4\pi)^{-3/2} \frac{\sqrt{3}}{\sqrt{(l_2+1)(2l_2+3)}} \Big\{ (l_2+1)\big((l_2+2)\vec{p}\cdot\vec{q} + i\,\vec{\sigma}\cdot[\vec{p}\vec{q}]\big)\mathcal{P}_{l_2} - \\ - \big((l_2+2)(\vec{P}_x\cdot\vec{q} - i\,\vec{\sigma}\cdot\vec{P}_y\,\vec{p}\cdot\vec{q}) - i\,\vec{\sigma}\cdot[\vec{p}\vec{q}]\,\vec{p}\cdot\vec{\pi}\big)\mathcal{P}'_{l_2} - i\,\vec{\sigma}\cdot\vec{P}_y\,\vec{P}_x\cdot\vec{q}\,\mathcal{P}''_{l_2}\Big\}$$

$J = L + 1/2 = l - 1/2 \quad l_1 = 1 \quad L = l_2 + 1 \quad l = l_2 + 2$

$$\hat{\mathcal{Y}} = (4\pi)^{-3/2} \frac{\sqrt{3}}{\sqrt{(l_2+1)(2l_2+3)}} \Big\{ -(l_2+1)\big((l_2+3)\vec{\sigma}\cdot\vec{p}\,\vec{p}\cdot\vec{q} - \vec{\sigma}\cdot\vec{q}\big)\mathcal{P}_{l_2} + \\ + \big((l_2+2)(\vec{\sigma}\cdot\vec{p}\,\vec{P}_x\cdot\vec{q} + \vec{\sigma}\cdot\vec{P}_x\,\vec{p}\cdot\vec{q}) + (\vec{\sigma}\cdot\vec{q} - \vec{\sigma}\cdot\vec{p}\,\vec{p}\cdot\vec{q})\,\vec{p}\cdot\vec{\pi}\big)\mathcal{P}'_{l_2} - \vec{\sigma}\cdot\vec{P}_x\,\vec{P}_x\cdot\vec{q}\,\mathcal{P}''_{l_2}\Big\}$$

$J = L - 1/2 = l + 1/2 \quad l_1 = 1 \quad L = l_2 + 1 \quad l = l_2$

$$\hat{\mathcal{Y}} = (4\pi)^{-3/2} \frac{\sqrt{3}}{\sqrt{(l_2+1)(2l_2+3)}} \Big\{ -(l_2+1)\big(l_2\,\vec{\sigma}\cdot\vec{p}\,\vec{p}\cdot\vec{q} + \vec{\sigma}\cdot\vec{q}\big)\mathcal{P}_{l_2} + \\ + \big((l_2+1)\vec{\sigma}\cdot\vec{p}\,\vec{P}_x\cdot\vec{q} - (l_2+2)\vec{\sigma}\cdot\vec{P}_x\,\vec{p}\cdot\vec{q} - (\vec{\sigma}\cdot\vec{q} - \vec{\sigma}\cdot\vec{p}\,\vec{p}\cdot\vec{q})\,\vec{p}\cdot\vec{\pi}\big)\mathcal{P}'_{l_2} + \vec{\sigma}\cdot\vec{P}_x\,\vec{P}_x\cdot\vec{q}\,\mathcal{P}''_{l_2}\Big\}$$

$J = L - 1/2 = l - 1/2 \quad l_1 = 1 \quad L = l_2 + 1 \quad l = l_2 + 1$

$$\hat{\mathcal{Y}} = (4\pi)^{-3/2} \frac{\sqrt{3}}{\sqrt{(l_2+1)(2l_2+3)}} \Big\{ (l_2+1)\big((l_2+1)\vec{p}\cdot\vec{q} - i\,\vec{\sigma}\cdot[\vec{p}\vec{q}]\big)\mathcal{P}_{l_2} - \\ - \big((l_2+1)\vec{P}_x\cdot\vec{q} + i(l_2+2)\vec{\sigma}\cdot\vec{P}_y\,\vec{p}\cdot\vec{q} + i\,\vec{\sigma}\cdot[\vec{p}\vec{q}]\,\vec{p}\cdot\vec{\pi}\big)\mathcal{P}'_{l_2} + i\,\vec{\sigma}\cdot\vec{P}_y\,\vec{P}_x\cdot\vec{q}\,\mathcal{P}''_{l_2}\Big\}$$

$J = L + 1/2 = l + 1/2 \quad l_1 = 1 \quad L = l_2 \quad l = l_2$

$$\hat{\mathcal{Y}} = (4\pi)^{-3/2} \frac{\sqrt{3}}{\sqrt{l_2(l_2+1)}} \Big\{ \big(i(l_2+1)\vec{P}_y\cdot\vec{q} - \vec{\sigma}\cdot\vec{\pi}\,\vec{p}\cdot\vec{q} + \vec{\sigma}\cdot\vec{q}\,\vec{p}\cdot\vec{\pi}\big)\mathcal{P}'_{l_2} - \\ - \vec{\sigma}\cdot\vec{P}_y\,\vec{P}_y\cdot\vec{q}\,\mathcal{P}''_{l_2}\Big\}$$

$J = L + 1/2 = l - 1/2 \quad l_1 = 1 \quad L = l_2 \quad l = l_2 + 1$

$$\hat{\mathcal{Y}} = (4\pi)^{-3/2} \frac{-i\sqrt{3}}{\sqrt{l_2(l_2+1)}} \Big\{ \big((l_2+2)\vec{\sigma}\cdot\vec{p}\,\vec{P}_y\cdot\vec{q} + \vec{\sigma}\cdot[\vec{q}\vec{\pi}]\big)\mathcal{P}'_{l_2} - \\ - \vec{\sigma}\cdot\vec{P}_x\,\vec{P}_y\cdot\vec{q}\,\mathcal{P}''_{l_2}\Big\}$$

$J = L - 1/2 = l + 1/2 \quad l_1 = 1 \quad L = l_2 \quad l = l_2 - 1$

$$\hat{y} = (4\pi)^{-3/2} \frac{-i\sqrt{3}}{\sqrt{l_2(l_2+1)}} \Big\{ \Big((l_2-1)\, \vec{\sigma}\cdot\vec{p}\; \vec{P}_y\cdot\vec{q} - \vec{\sigma}\cdot[\vec{q}\vec{\pi}] \Big) \mathcal{P}'_{l_2} + \\ + \vec{\sigma}\cdot\vec{P}_x\; \vec{P}_y\cdot\vec{q}\; \mathcal{P}''_{l_2} \Big\}$$

$J = L - 1/2 = l - 1/2 \quad l_1 = 1 \quad L = l_2 \quad l = l_2$

$$\hat{y} = (4\pi)^{-3/2} \frac{\sqrt{3}}{\sqrt{l_2(l_2+1)}} \Big\{ \Big(i l_2\, \vec{P}_y\cdot\vec{q} + \vec{\sigma}\cdot\vec{\pi}\; \vec{p}\cdot\vec{q} - \vec{\sigma}\cdot\vec{q}\; \vec{p}\cdot\vec{\pi} \Big) \mathcal{P}'_{l_2} + \\ + \vec{\sigma}\cdot\vec{P}_y\; \vec{P}_y\cdot\vec{q}\; \mathcal{P}''_{l_2} \Big\}$$

$J = L + 1/2 = l + 1/2 \quad l_1 = 1 \quad L = l_2 - 1 \quad l = l_2 - 1$

$$\hat{y} = (4\pi)^{-3/2} \frac{\sqrt{3}}{\sqrt{l_2(2l_2-1)}} \Big\{ -l_2 \Big(l_2\, \vec{p}\cdot\vec{q} + i\vec{\sigma}\cdot[\vec{p}\vec{q}] \Big) \mathcal{P}_{l_2} - \\ - \Big(l_2\, \vec{P}_x\cdot\vec{q} + i(l_2-1)\,\vec{\sigma}\cdot\vec{P}_y\; \vec{p}\cdot\vec{q} - i\,\vec{\sigma}\cdot[\vec{p}\vec{q}]\; \vec{p}\cdot\vec{\pi} \Big) \mathcal{P}'_{l_2} - i\,\vec{\sigma}\cdot\vec{P}_y\; \vec{P}_x\cdot\vec{q}\; \mathcal{P}''_{l_2} \Big\}$$

$J = L + 1/2 = l - 1/2 \quad l_1 = 1 \quad L = l_2 - 1 \quad l = l_2$

$$\hat{y} = (4\pi)^{-3/2} \frac{\sqrt{3}}{\sqrt{l_2(2l_2-1)}} \Big\{ l_2 \Big((l_2+1)\, \vec{\sigma}\cdot\vec{p}\; \vec{p}\cdot\vec{q} - \vec{\sigma}\cdot\vec{q} \Big) \mathcal{P}_{l_2} + \\ + \Big(l_2\, \vec{\sigma}\cdot\vec{p}\; \vec{P}_x\cdot\vec{q} - (l_2-1)\,\vec{\sigma}\cdot\vec{P}_x\; \vec{p}\cdot\vec{q} + (\vec{\sigma}\cdot\vec{q} - \vec{\sigma}\cdot\vec{p}\; \vec{p}\cdot\vec{q})\, \vec{p}\cdot\vec{\pi} \Big) \mathcal{P}'_{l_2} - \vec{\sigma}\cdot\vec{P}_x\; \vec{P}_x\cdot\vec{q}\; \mathcal{P}''_{l_2} \Big\}$$

$J = L - 1/2 = l + 1/2 \quad l_1 = 1 \quad L = l_2 - 1 \quad l = l_2 - 2$

$$\hat{y} = (4\pi)^{-3/2} \frac{\sqrt{3}}{\sqrt{l_2(2l_2-1)}} \Big\{ l_2 \Big((l_2-2)\, \vec{\sigma}\cdot\vec{p}\; \vec{p}\cdot\vec{q} + \vec{\sigma}\cdot\vec{q} \Big) \mathcal{P}_{l_2} + \\ + \Big((l_2-1)(\vec{\sigma}\cdot\vec{p}\; \vec{P}_x\cdot\vec{q} + \vec{\sigma}\cdot\vec{P}_x\; \vec{p}\cdot\vec{q}) - (\vec{\sigma}\cdot\vec{q} - \vec{\sigma}\cdot\vec{p}\; \vec{p}\cdot\vec{q})\, \vec{p}\cdot\vec{\pi} \Big) \mathcal{P}'_{l_2} + \vec{\sigma}\cdot\vec{P}_x\; \vec{P}_x\cdot\vec{q}\; \mathcal{P}''_{l_2} \Big\}$$

$J = L - 1/2 = l - 1/2 \quad l_1 = 1 \quad L = l_2 - 1 \quad l = l_2 - 1$

$$\hat{y} = (4\pi)^{-3/2} \frac{\sqrt{3}}{\sqrt{l_2(2l_2-1)}} \Big\{ -l_2 \Big((l_2-1)\, \vec{p}\cdot\vec{q} - i\,\vec{\sigma}\cdot[\vec{p}\vec{q}] \Big) \mathcal{P}_{l_2} - \\ - \Big((l_2-1)(\vec{P}_x\cdot\vec{q} - i\,\vec{\sigma}\cdot\vec{P}_y\; \vec{p}\cdot\vec{q}) + i\,\vec{\sigma}\cdot[\vec{p}\vec{q}]\; \vec{p}\cdot\vec{\pi} \Big) \mathcal{P}'_{l_2} + i\,\vec{\sigma}\cdot\vec{P}_y\; \vec{P}_x\cdot\vec{q}\; \mathcal{P}''_{l_2} \Big\}$$

TABLE III.

$J = l_2 + 1/2 = l + 1/2 \quad l_1 = 1 \quad j = 1/2 \quad l = l_2$

$$\hat{\mathcal{Y}} = -(4\pi)^{-3/2} \Big\{ (l_2+1)\,\vec{\sigma}\cdot\vec{q}\,\mathcal{P}_{l_2} + i(\vec{P}_y\cdot\vec{q} - i\vec{\sigma}\cdot[\vec{P}_y\vec{q}])\mathcal{P}'_{l_2} \Big\}$$

$J = l_2 + 1/2 = l - 1/2 \quad l_1 = 1 \quad j = 1/2 \quad l = l_2 + 1$

$$\hat{\mathcal{Y}} = (4\pi)^{-3/2} \Big\{ (l_2+1)(\vec{p}\cdot\vec{q} - i\vec{\sigma}\cdot[\vec{p}\vec{q}])\mathcal{P}_{l_2} - (\vec{P}_x\cdot\vec{q} - i\vec{\sigma}\cdot[\vec{P}_x\vec{q}])\mathcal{P}'_{l_2} \Big\}$$

$J = l_2 - 1/2 = l + 1/2 \quad l_1 = 1 \quad j = 1/2 \quad l = l_2 - 1$

$$\hat{\mathcal{Y}} = (4\pi)^{-3/2} \Big\{ l_2(\vec{p}\cdot\vec{q} - i\vec{\sigma}\cdot[\vec{p}\vec{q}])\mathcal{P}_{l_2} + (\vec{P}_x\cdot\vec{q} - i\vec{\sigma}\cdot[\vec{P}_x\vec{q}])\mathcal{P}'_{l_2} \Big\}$$

$J = l_2 - 1/2 = l - 1/2 \quad l_1 = 1 \quad j = 1/2 \quad l = l_2$

$$\hat{\mathcal{Y}} = (4\pi)^{-3/2} \Big\{ -l_2\,\vec{\sigma}\cdot\vec{q}\,\mathcal{P}_{l_2} + i(\vec{P}_y\cdot\vec{q} - i\vec{\sigma}\cdot[\vec{P}_y\vec{q}])\mathcal{P}'_{l_2} \Big\}$$

$J = l_2 + 3/2 = l + 1/2 \quad l_1 = 1 \quad j = 3/2 \quad l = l_2 + 1$

$$\hat{\mathcal{Y}} = (4\pi)^{-3/2} \frac{\sqrt{3}}{2\sqrt{(l_2+1)(2l_2+3)}} \Big\{ (l_2+2)(l_2+1)(2\vec{p}\cdot\vec{q} + i\vec{\sigma}\cdot[\vec{p}\vec{q}])\mathcal{P}_{l_2} + (l_2+2)\Big(i(\vec{\sigma}\cdot\vec{P}_y\,\vec{p}\cdot\vec{q} + \vec{\sigma}\cdot\vec{p}\,\vec{P}_y\cdot\vec{q}) - (2\vec{P}_x\cdot\vec{q} + i\vec{\sigma}\cdot[\vec{P}_x\vec{q}])\Big)\mathcal{P}'_{l_2} - i(\vec{\sigma}\cdot\vec{P}_x\,\vec{P}_y\cdot\vec{q} + \vec{\sigma}\cdot\vec{P}_y\,\vec{P}_x\cdot\vec{q})\mathcal{P}''_{l_2} \Big\}$$

$J = l_2 + 3/2 = l - 1/2 \quad l_1 = 1 \quad j = 3/2 \quad l = l_2 + 2$

$$\hat{\mathcal{Y}} = (4\pi)^{-3/2} \frac{\sqrt{3}}{2\sqrt{(l_2+1)(2l_2+3)}} \Big\{ -(l_2+2)(l_2+1)(3\vec{\sigma}\cdot\vec{p}\,\vec{p}\cdot\vec{q} - \vec{\sigma}\cdot\vec{q})\mathcal{P}_{l_2} + 2(l_2+2)(\vec{\sigma}\cdot\vec{P}_x\,\vec{p}\cdot\vec{q} + \vec{\sigma}\cdot\vec{p}\,\vec{P}_x\cdot\vec{q})\mathcal{P}'_{l_2} - (\vec{\sigma}\cdot\vec{P}_x\,\vec{P}_x\cdot\vec{q} - \vec{\sigma}\cdot\vec{P}_y\,\vec{P}_y\cdot\vec{q})\mathcal{P}''_{l_2} \Big\}$$

$J = l_2 + 1/2 = l + 1/2 \quad l_1 = 1 \quad j = 3/2 \quad l = l_2$

$$\begin{aligned}\hat{\mathcal{Y}} = (4\pi)^{-3/2}\frac{1}{2\sqrt{l_2(2l_2+3)}}\Big\{&-l_2(l_2+1)(3\,\vec\sigma\cdot\vec p\;\vec p\cdot\vec q - \vec\sigma\cdot\vec q)\,\mathcal{P}_{l_2} - \\ &-\Big(3(\vec\sigma\cdot\vec P_x\;\vec p\cdot\vec q + \vec\sigma\cdot\vec p\;\vec P_x\cdot\vec q) - i(2l_2+3)(2\vec P_y\cdot\vec q + i\,\vec\sigma\cdot[\vec P_y\vec q])\Big)\mathcal{P}'_{l_2} + 3(\vec\sigma\cdot\vec P_x\;\vec P_x\cdot\vec q - \vec\sigma\cdot\vec P_y\;\vec P_y\cdot\vec q)\,\mathcal{P}''_{l_2}\Big\}\end{aligned}$$

$J = l_2 + 1/2 = l - 1/2 \quad l_1 = 1 \quad j = 3/2 \quad l = l_2 + 1$

$$\begin{aligned}\hat{\mathcal{Y}} = (4\pi)^{-3/2}\frac{1}{2\sqrt{l_2(2l_2+3)}}\Big\{&l_2(l_2+1)(2\vec p\cdot\vec q + i\,\vec\sigma\cdot[\vec p\vec q])\,\mathcal{P}_{l_2} - \\ &-\Big(3i(l_2+2)(\vec\sigma\cdot\vec P_y\;\vec p\cdot\vec q + \vec\sigma\cdot\vec p\;\vec P_y\cdot\vec q) + l_2(2\vec P_x\cdot\vec q + i\,\vec\sigma\cdot[\vec P_x\vec q])\Big)\mathcal{P}'_{l_2} + 3i(\vec\sigma\cdot\vec P_x\;\vec P_y\cdot\vec q + \vec\sigma\cdot\vec P_y\;\vec P_x\cdot\vec q)\,\mathcal{P}''_{l_2}\Big\}\end{aligned}$$

$J = l_2 - 1/2 = l + 1/2 \quad l_1 = 1 \quad j = 3/2 \quad l = l_2 - 1$

$$\begin{aligned}\hat{\mathcal{Y}} = (4\pi)^{-3/2}\frac{1}{2\sqrt{(l_2+1)(2l_2-1)}}\Big\{&-l_2(l_2+1)(2\vec p\cdot\vec q + i\,\vec\sigma\cdot[\vec p\vec q])\,\mathcal{P}_{l_2} - \\ &-\Big(3i(l_2-1)(\vec\sigma\cdot\vec p\;\vec P_y\cdot\vec q + \vec\sigma\cdot\vec P_y\;\vec p\cdot\vec q) + (l_2+1)(2\vec P_x\cdot\vec q + i\,\vec\sigma\cdot[\vec P_x\vec q])\Big)\mathcal{P}'_{l_2} - 3i(\vec\sigma\cdot\vec P_x\;\vec P_y\cdot\vec q + \vec\sigma\cdot\vec P_y\;\vec P_x\cdot\vec q)\,\mathcal{P}''_{l_2}\Big\}\end{aligned}$$

$J = l_2 - 1/2 = l - 1/2 \quad l_1 = 1 \quad j = 3/2 \quad l = l_2$

$$\begin{aligned}\hat{\mathcal{Y}} = (4\pi)^{-3/2}\frac{1}{2\sqrt{(l_2+1)(2l_2-1)}}\Big\{&l_2(l_2+1)(3\,\vec\sigma\cdot\vec p\;\vec p\cdot\vec q - \vec\sigma\cdot\vec q)\,\mathcal{P}_{l_2} + \\ &+\Big(3(\vec\sigma\cdot\vec P_x\;\vec p\cdot\vec q + \vec\sigma\cdot\vec p\;\vec P_x\cdot\vec q) + i(2l_2-1)(2\vec P_y\cdot\vec q + i\,\vec\sigma\cdot[\vec P_y\vec q])\Big)\mathcal{P}'_{l_2} - 3(\vec\sigma\cdot\vec P_x\;\vec P_x\cdot\vec q - \vec\sigma\cdot\vec P_y\;\vec P_y\cdot\vec q)\,\mathcal{P}''_{l_2}\Big\}\end{aligned}$$

$J = l_2 - 3/2 = l + 1/2 \quad l_1 = 1 \quad j = 3/2 \quad l = l_2 - 2$

$$\begin{aligned}\hat{\mathcal{Y}} = (4\pi)^{-3/2}\frac{\sqrt{3}}{2\sqrt{l_2(2l_2-1)}}\Big\{&l_2(l_2-1)(3\,\vec\sigma\cdot\vec p\;\vec p\cdot\vec q - \vec\sigma\cdot\vec q)\,\mathcal{P}_{l_2} + \\ &+ 2(l_2-1)(\vec\sigma\cdot\vec P_x\;\vec p\cdot\vec q + \vec\sigma\cdot\vec p\;\vec P_x\cdot\vec q)\,\mathcal{P}'_{l_2} + (\vec\sigma\cdot\vec P_x\;\vec P_x\cdot\vec q - \vec\sigma\cdot\vec P_y\;\vec P_y\cdot\vec q)\,\mathcal{P}''_{l_2}\Big\}\end{aligned}$$

$J = l_2 - 3/2 = l - 1/2 \quad l_1 = 1 \quad j = 3/2 \quad l = l_2 - 1$

$$\begin{aligned}\hat{\mathcal{Y}} = (4\pi)^{-3/2}\frac{\sqrt{3}}{2\sqrt{l_2(2l_2-1)}}\Big\{&-l_2(l_2-1)(2\vec p\cdot\vec q + i\,\vec\sigma\cdot[\vec p\vec q])\,\mathcal{P}_{l_2} + \\ &+(l_2-1)\Big(i(\vec\sigma\cdot\vec P_y\;\vec p\cdot\vec q + \vec\sigma\cdot\vec p\;\vec P_y\cdot\vec q) - (2\vec P_x\cdot\vec q + i\,\vec\sigma\cdot[\vec P_x\vec q])\Big)\mathcal{P}'_{l_2} + i(\vec\sigma\cdot\vec P_x\;\vec P_y\cdot\vec q + \vec\sigma\cdot\vec P_y\;\vec P_x\cdot\vec q)\,\mathcal{P}''_{l_2}\Big\}\end{aligned}$$