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# ON THE PRINCIPLE OF COMPENSATION AND METHOD OF THE SELF-CONSISTENT FIELD

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# ON THE PRINCIPLE OF COMPENSATION AND METHOD OF THE SELF-CONSISTENT FIELD

## 1. Principle of compensation.

We shall discuss in this paper possible generalizations of the compensation principle of the "dangerous" graphs in the cases of the spatially inhomogeneous states and determine its connection with the method of the self-consistent field too.

When we must investigate an effect of the dynamic system the external inhomogeneous field then the problem of the electrodynamics of the superconducting state may be here an important example.

Let $\vec{A}(\vec{r})$ be the vector-potential, depending on $\vec{r}$, then the Hamiltonian of one electron will have an additional term

$$-\frac{e}{2m}\{(pA)+(Ap)\}+\frac{e^2}{2m}A^2$$

which disturbs the spatial homogeneity.

Note that the presence of the terms of this type makes the compensation of the graphs corresponding to momenta $k,-k$ insufficient. In fact, determing $\vec{A}(\vec{r})$ as the superposition of the Fourier components

$$A(q)\,e^{-i(q,r)}$$

we see that in the same sense the graphs with arbitrary momenta $K_1K_2$ will be dangerous, in any case the graphs for which $q=k_1+k_2$ is sufficiently small. It is clear that it is impossible to exclude them by our ordinary canonical transformation, mixing up the amplitudes of creation and annihilation of the momenta $\pm k$, since it includes only one arbitrary function $u_k$ (or $v_k$).

In order to compensate graphs with any pair of momenta we must use the more general canonical transformation, formulated in the paper /1/

$$a_f=\sum_\nu (u_{f\nu}\,\alpha_\nu+v_{f\nu}\,\overset{+}{\alpha}_\nu) \qquad (1)$$

where $f=(p,\sigma)$ ; $\sigma$ – is spin index, $u_{f\nu}$ , $v_{f\nu}$ – are arbitrary functions, connected by orthonormality relations

$$\sum_\nu\{u_{f\nu}\overset{*}{u}_{f'\nu}+v_{f\nu}\overset{*}{v}_{f'\nu}\}=\delta(f-f') \qquad (2)$$

$$\sum_\nu\{u_{f\nu}v_{f'\nu}+u_{f'\nu}v_{f\nu}\}=0$$

Just these relations provide canonical character of the transformation considered (1). We study here for simplicity the generalized principle of compensation in for the Hamiltonian of the direct interaction between particles, since in such a case the first approximation leads to a nontrivial result.

As it was noted earlier [2] the insertion of the electron–photon interaction, for example, requires to use the second approximation. Meaning various applications let us take the expression of the total Hamiltonian in a sufficiently general form

$$H=\sum T(f,f')\overset{+}{a}_{f}a_{f'}+\frac{1}{2}\sum \mathcal{V}(f_1,f_2;f_2',f_1')\overset{+}{a}_{f_1}\overset{+}{a}_{f_2}a_{f_2'}a_{f_1'} \tag{3}$$

$$T(f,f')=I(f,f')-\lambda\delta(f-f')$$

where $\lambda$ is the chemical potential, $I$ – the Hamiltonian of particles, $\mathcal{V}$ – is the interaction energy.

We suppose here, that $I$ and $\mathcal{V}$ comply with the conditions of symmetry, Hermiticity e.g. The compensation principle of dangerous graphs in the considered first approximation is

$$<\alpha_{\nu_1}\alpha_{\nu_2}H>_0=0 \tag{4}$$

The expectation value in the state $C_0$ corresponding to vacuum in the $\alpha$ -representation

$$\alpha_{\nu}C_0=0\;;\qquad C_0^{*}\overset{+}{\alpha}_{\nu}=0 \tag{5}$$

One can expand the Eq. (4) substituting the expression (1) into it and calculating the expectation values.

By means of this we obtain explicit equations for determining the unknown quantities $u,v$ which are to be solved together with the conditions (2).

In a number of cases it should be more conveniently to write down these expressions in a slightly different form. Show for this, that from (4) it follows

$$\begin{aligned}\mathfrak{U}&\equiv<[a_{f_1}a_{f_2};H]>_0=0\\ \mathfrak{B}&\equiv<[\overset{+}{a}_{f_1}a_{f_2};H]>_0=0\end{aligned} \tag{6}$$

Indeed

$$\begin{aligned}\mathfrak{U}\equiv&\sum_{\nu_1,\nu_2}<[(u_{f_1\nu_1}\alpha_{\nu_1}+v_{f_1\nu_1}\overset{+}{\alpha}_{\nu_1})(u_{f_2\nu_2}\alpha_{\nu_2}+v_{f_2\nu_2}\overset{+}{\alpha}_{\nu_2}),H]>_0=\\ =&\sum_{\nu_1,\nu_2}u_{f_1\nu_1}u_{f_2\nu_2}<\alpha_{\nu_1}\alpha_{\nu_2}H-H\alpha_{\nu_1}\alpha_{\nu_2}>_0+\sum_{\nu_1,\nu_2}u_{f_1\nu_1}v_{f_2\nu_2}<\alpha_{\nu_1}\overset{+}{\alpha}_{\nu_2}H-H\alpha_{\nu_1}\overset{+}{\alpha}_{\nu_2}>_0+\\ +&\sum_{\nu_1,\nu_2}v_{f_1\nu_1}u_{f_2\nu_2}<\overset{+}{\alpha}_{\nu_1}\alpha_{\nu_2}H-H\overset{+}{\alpha}_{\nu_1}\alpha_{\nu_2}>_0+\sum_{\nu_1,\nu_2}v_{f_1\nu_1}v_{f_2\nu_2}<\overset{+}{\alpha}_{\nu_1}\overset{+}{\alpha}_{\nu_2}H-H\overset{+}{\alpha}_{\nu_1}\overset{+}{\alpha}_{\nu_2}>_0\end{aligned}$$

Due to (5)

$$<H\alpha_{\nu_1}\alpha_{\nu_2}>=<\overset{+}{\alpha}_{\nu_1}\overset{+}{\alpha}_{\nu_2}H>=<\overset{+}{\alpha}_{\nu_1}\alpha_{\nu_2}H>_0=<H\overset{+}{\alpha}_{\nu_1}\alpha_{\nu_2}>_0=0$$

and besides

$$\langle \alpha_{\nu_1} \overset{+}{\alpha}_{\nu_2} H - H\alpha_{\nu_1} \overset{+}{\alpha}_{\nu_2} \rangle_0 = \langle -\overset{+}{\alpha}_{\nu_2} \alpha_{\nu_1} H + H\overset{+}{\alpha}_{\nu_2} \alpha_{\nu_1} \rangle_0 = 0$$

It follows from (4)

$$\mathcal{U} = \sum_{\nu_1,\nu_2} U_{f_1\nu_1} U_{f_2\nu_2} \langle \alpha_{\nu_1} \alpha_{\nu_2} H \rangle_0 - \sum_{\nu_1,\nu_2} V_{f_1\nu_1} V_{f_2\nu_2} \langle \alpha_{\nu_2} \alpha_{\nu_1} H \rangle_0^* = 0$$

One proves similarly the second Eq. (6). It is not difficult to make sure, that (4) follows from (6). Thus both systems (4) and (6) are completely equivalent.

Now we show that $\mathcal{U}$ and $\mathcal{B}$ are not independent. Firstly we transform the orthonormality relations (2). Insert the combined indices

$$g = (f, \rho)\,; \qquad \rho = 0, 1$$
$$\omega = (\nu, \tau)\,; \qquad \tau = 0, 1 \tag{7}$$

and put

$$\varphi_{\nu,0}(f, 0) = \overset{*}{v}_{f\nu}\,, \qquad \varphi_{\nu,0}(f, 1) = u_{f\nu}$$
$$\varphi_{\nu,1}(f, 0) = \overset{*}{u}_{f\nu}\,, \qquad \varphi_{\nu,1}(f, 1) = v_{f\nu} \tag{8}$$

In such notations the considered relations take the usual form

$$\sum_{\omega} \overset{*}{\varphi}_{\omega}(g)\, \varphi_{\omega}(g') = \delta(g - g') \tag{9}$$

it follows from this

$$\sum_{g} \overset{*}{\varphi}_{\omega}(g)\, \varphi_{\omega'}(g) = \delta(\omega - \omega')$$

or in the old notations

$$\sum_f \{ \overset{*}{u}_{f\nu_1} u_{f\nu_2} + v_{f\nu_2}^* v_{f\nu_1} \} = \delta(\nu_1 - \nu_2)$$
$$\sum_f \{ \overset{*}{v}_{f\nu_1} u_{f\nu_2} + \overset{*}{v}_{f\nu_2} u_{f\nu_1} \} = 0 \tag{10}$$

By means of these relations it is not difficult to express the amplitudes $\alpha, \overset{+}{\alpha}$ in terms of $a, \overset{+}{a}$ ,

$$\alpha_\nu = \sum_f \{ u_{f\nu}^* a_f + v_{f\nu} \overset{+}{a}_f \} \tag{11}$$

Now we turn our attention to the identity

$$<[\overset{+}{\alpha}_{\nu_1}\alpha_{\nu_2}\,;\,H]>_0=0 \tag{12}$$

conditioned only by the properties (5). Substituting here the expressions (11), we find

$$\sum_{f_1,f_2}<[(u_{f_1\nu_1}\overset{+}{a}_{f_1}+\overset{*}{v}_{f_1\nu_1}a_{f_1})(\overset{*}{u}_{f_2\nu_2}a_{f_2}+v_{f_2\nu_2}\overset{+}{a}_{f_2});\,H]>_0=0$$

or expanding

$$\sum_{f_1,f_2}u_{f_1\nu_1}\overset{*}{u}_{f_2\nu_2}<[\overset{+}{a}_{f_1}a_{f_2}\,;H]>_0+\sum_{f_1,f_2}u_{f_1\nu_1}v_{f_2\nu_2}<[\overset{+}{a}_{f_1}\overset{+}{a}_{f_2}\,;\,H]>_0+$$
$$+\sum_{f_1,f_2}\overset{*}{v}_{f_1\nu_1}\overset{*}{u}_{f_2\nu_2}<[a_{f_1}a_{f_2}\,;\,H]>_0+\sum_{f_1,f_2}\overset{*}{v}_{f_1\nu_1}v_{f_2\nu_2}<[a_{f_1}\overset{+}{a}_{f_2}\,;\,H]>_0=0$$

Thus, the forms $\mathfrak{U}$ and $\mathfrak{B}$ are connected by the identities[x)]

$$\sum_{f_1,f_2}\{u_{f_1\nu_1}\overset{*}{u}_{f_2\nu_2}\mathfrak{B}(f_1,f_2)+u_{f_1\nu_1}v_{f_2\nu_2}\overset{*}{\mathfrak{U}}(f_1,f_2)+$$
$$+\overset{*}{v}_{f_1\nu_1}\overset{*}{u}_{f_2\nu_2}\mathfrak{U}(f_1,f_2)+\overset{*}{v}_{f_1\nu_1}v_{f_2\nu_2}\overset{*}{\mathfrak{B}}(f_1,f_2)\}=0 \tag{13}$$

Now we proceed to obtaining explicit expressions for $\mathfrak{U}$ and $\mathfrak{B}$. We have

$$\mathfrak{U}(f_1,f_2)=\sum_f\{T(f_1,f)<a_fa_{f_2}>_0+T(f_2,f)<a_{f_1}a_f>_0\}+$$
$$+\sum_{f_1',f_2'}V(f_1,f_2;f_2',f_1')<a_{f_1'}a_{f_2'}>_0+$$
$$+\sum_{f,f_1',f_2}V(f_1,f;f_2',f_1')<\overset{+}{a}_fa_{f_2}a_{f_2'}a_{f_1'}>_0+ \tag{14}$$
$$+\sum_{f,f_1',f_2'}V(f,f_2;f_2',f_1')<\overset{+}{a}_fa_{f_1}a_{f_2'}a_{f_1'}>_0$$

and also

$$\mathfrak{B}(f_1,f_2)=\sum_f\{T(f_2,f)<\overset{+}{a}_{f_1}a_f>_0-T(f,f_1)<\overset{+}{a}_fa_{f_2}>_0\}-$$
$$-\sum_{f,f_1',f_2'}\{V(f_1',f_2';f,f_1)<\overset{+}{a}_{f_1'}\overset{+}{a}_{f_2'}a_fa_{f_2}>_0-$$
$$-V(f_2,f;f_2',f_1')<\overset{+}{a}_{f_1}\overset{+}{a}_{f}a_{f_2'}a_{f_1'}>_0\} \tag{15}$$

The vacuum expectation values of the type

$$F(f,f')=<\overset{+}{a}_fa_{f'}>_0\;;\qquad \Phi(f_1,f_2)=<a_{f_1}a_{f_2}>_0. \tag{16}$$

$$F_2(f_1,f_2;f_2',f_1')=<\overset{+}{a}_{f_1}\overset{+}{a}_{f_2}a_{f_2'}a_{f_1'}>_0$$

$$\Phi_2(f_1,f_2,f_3,f_4)=<\overset{+}{a}_{f_1}a_{f_2}a_{f_3}a_{f_4}>_0 \tag{17}$$

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x) Note that if in (6) for $\mathfrak{U}$, $\mathfrak{B}$ we changed the averaging over the vacuum state $C_0$ by averaging

$$\frac{Sp\,(\ldots D)}{Sp\,D}$$

over some distribution $D$ diagonal in the representation $\ldots n_\nu \ldots\ (n_\nu=\overset{+}{\alpha}_\nu\alpha_\nu)$ then the identities (13) would take place for $\nu_1=\nu_2$. In fact due to the diagonality of $D$:

$$Sp\,(\overset{+}{\alpha}_\nu\alpha_\nu HD)=Sp\,(HD\overset{+}{\alpha}_\nu\alpha_\nu)=Sp\,(H\overset{+}{\alpha}_\nu\alpha_\nu D)$$

and consequently

$$\overline{[\overset{+}{\alpha}_\nu\alpha_\nu\,;\,H]}=0$$

are determined by means of (1), expressing the amplitudes $a, \overset{+}{a}$ in terms of $\alpha, \overset{+}{\alpha}$. In that way we find

$$F(f,f') = \sum_{\nu} \overset{*}{v}_{f\nu} v_{f'\nu}, \qquad \Phi(f_1,f_2) = \sum_{\nu} u_{f_1\nu} v_{f_2\nu}, \tag{18}$$

$$\begin{aligned} F_2(f_1,f_2;f_2',f_1') &= F(f_1,f_1')F(f_2,f_2') - \\ &- F(f_1,f_2')F(f_2,f_1') + \overset{*}{\Phi}(f_1,f_2)\Phi(f_1',f_2') \end{aligned} \tag{19}$$

$$\begin{aligned} \Phi_2(f_1,f_2,f_3,f_4) &= F(f_1,f_2)\Phi(f_3,f_4) - \\ &- F(f_1,f_3)\Phi(f_2,f_4) + F(f_1,f_4)\Phi(f_3,f_4) \end{aligned} \tag{20}$$

Substituting these expressions into (14), (15) we obtain explicit expressions

$$\mathcal{U}(f_1,f_2) = \mathcal{U}(f_1,f_2 \mid F,\Phi)$$
$$\mathcal{B}(f_1,f_2) = \mathcal{B}(f_1,f_2 \mid F,\Phi)$$

We have, for example

$$\begin{aligned} \mathcal{U}(f_1,f_2 \mid F,\Phi) &= \sum_f \{E(f_1,f)\Phi(f,f_2) + E(f_2,f)\Phi(f_1,f)\} + \\ &+ S(f_1,f_2) - \sum_f \{F(f,f_1)S(f,f_2) + F(f,f_2)S(f_1,f)\} \end{aligned} \tag{21}$$

where

$$\begin{aligned} E(f_1,f) &= T(f_1,f) + \\ &+ \sum_{f',f''} \{V(f_1,f'';f',f) - V(f_1,f'';f,f')\} F(f'',f') \\ S(f_1,f_2) &= \sum_{f_1'f_2'} V(f_1,f_2;f_2',f_1')\Phi(f_1',f_2') \end{aligned} \tag{22}$$

So, our generalized principle of the compensation leads in the first approximation to the equations

$$\left.\begin{aligned} \mathcal{U}(f_1,f_2 \mid F,\Phi) &= 0 \\ \mathcal{B}(f_1,f_2 \mid F,\Phi) &= 0 \end{aligned}\right\} \tag{23}$$

which have been obtained earlier [3,4] by means of generalization of the well known the Fock method [8]. Besides these expressions we have one subsidiary condition according to which the functions $F$, $\Phi$ can be represented in the form (18). It would be useful to formulate such subsidiary condition in form of a set of relations, for $F, \Phi$. Let us note that from (18) it follows

$$\overset{*}{F}(f,f') = F(f',f); \qquad \Phi(f_2,f_1) = -\Phi(f_1,f_2) \tag{24}$$

We introduce again the combined indices $g$, $\omega$ and consider the matrix

$$K(g,g') = \sum_{\omega} \overset{*}{\varphi}_{\omega}(g)\varphi_{\omega}(g')n_{\omega} \tag{25}$$

in which

$$n_{\nu,0} = 1 \; ; \qquad n_{\nu,1} = 0$$

Then

$$K(f,0;\, f',0) = \sum_{\nu} v_{f\nu}\overset{*}{v}_{f'\nu} \; ; \qquad K(f,0;\, f',1) = \sum_{\nu} v_{f\nu}u_{f'\nu}$$

$$K(f,1;\, f',0) = \sum_{\nu} \overset{*}{u}_{f\nu}\overset{*}{v}_{f'\nu} \; ; \qquad K(f,1;\, f',1) = \sum_{\nu} \overset{*}{u}_{f\nu}u_{f'\nu}$$

We obtain from here, according to (2)

$$K(g,g') = \begin{vmatrix} F(f',f), & -\Phi(f,f') \\ \Phi^*(f,f'), & \delta(f-f') - F(f,f') \end{vmatrix} \tag{26}$$

On the other hand we see from (25) that $\varphi_{\omega}(g)$, $n_{\omega}$ are accordingly eigenvectors and eigenvalues of the operator $K$. As these eigenvalues are equal to zeró or unity, $K$ is projection operator and therefore

$$K = K^2 \tag{27}$$

Expanding this relation we find subsidiary conditions which must be satisfied by the functions $F$ and $\Phi$ :

$$F(f_1,f_2) = \sum_{f}\left\{F(f_1,f)F(f,f_2) + \overset{*}{\Phi}(f,f_1)\Phi(f,f_2)\right\}$$

$$\sum_{f}\left\{F(f_1,f)\overset{*}{\Phi}(f,f_2) + F(f_2,f)\overset{*}{\Phi}(f,f_1)\right\} = 0 \tag{28}$$

Now we show that the conditions (24) and (28) are completely equivalent to the condition that the functions $F$, $\Phi$ can be represented in the form (18). So, we have to prove, that any $F$ and $\Phi$ complying with the conditions (24), (28) can in fact be represented in the form (18). First of all we profit by the conditions (24) and introduce the matrix $K(g,g')$ by the relation (26). Owing to (24) $K$ is Hermitian and therefore may be represented in form (25) where $\varphi_{\omega}(g)$ will represent the orthonormalized system of the eigenvectors of $K$. Let us make in the space of the points $\{g\}$ point transformation $T$, substituting $(f,0)$ into $(f,1)$ and vice versa. We have

$$TK \equiv K(Tg, Tg') =$$

$$= \begin{vmatrix} \delta(f-f') - F(f,f'), & \overset{*}{\Phi}(f,f') \\ & \\ -\Phi(f,f'), & F(f',f) \end{vmatrix}' = \delta(g-g') - \overset{*}{K}(g,g')$$

In view of this property it is not difficult to see, that if $\varphi(g)$ is any eigenvector of the operator $K$ and $n$ is the corresponding eigenvalue than $\overset{*}{\varphi}(Tg)$, $1-n$ are the eigenvector and eigenvalue of $K$ too.

Thus, the numeration $\{\omega\}$ of the eigenvectors and eigenvalues of the operator $\{K\}$ one may realised by system of the two indices $\{\nu,\tau\}$ $(\tau = 0,1)$ , putting

$$n_{\nu,0} = n_\nu \; ; \quad n_{\nu,1} = 1 - n_\nu \tag{29}$$

$$\varphi_{\nu,0}(g) = \varphi_\nu(g) \; ; \quad \varphi_{\nu,1}(g) = \overset{*}{\varphi}_\nu(Tg)$$

Now we use the conditions (28) from which it follows, that $K = K^2$ and hence $n_\omega = 0,1$. Let us attribute to $n_\nu$ unit value, and to $1-n_\nu$ zero value, eliminating by means of this the ambiguity of the index $\omega$ into $(\nu,0)$ and $(\nu,1)$ .

After determining $\varphi_{\nu,0}(g)$ , $\varphi_{\nu,1}(g)$ we can obtain the functions $u_\nu(f)$, $v_\nu(f)$ by means of relation (8). Since $\varphi_\omega(g)$ form the orthonormalized system we see, that the obtained functions $u$ , $v$ satisfy the relations (2). For the end of prove we have only to expand (25) and note that the representation[x)] (18) follows immediately from them.

So we must solve the equations (23) together with the subsidiary conditions (24), (28) . There are no functions $u$ , $v$ in them. After obtaining expressions for $F$ and $\Phi$ we can determine a system of the functions $\{u, v\}$ using the above mentioned method.

Let us stress here, that the determination of the system $\{u, v\}$ has large ambiguity. Indeed let $\varphi_{\nu,0}(g)$ be orthonormalized system of the eigenvectors of the operator $K$ corresponding to the eigenvalue equal to unity . If we subject it to arbitrary unitary transformation we obtain again the orthonormalized system of the eigenvectors of the operator $K$ , corresponding to the eigenvalue equal to unity. The same remark is true for $\varphi_{\nu,1}(g)$ .

We see that the systems $\{\varphi_{\nu,0}(g)\}$, $\{\varphi_{\nu,1}(g)\}$ are determined only with the

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x) It is interesting to note if we deal with functions satisfying only the conditions (24) then after making once again these considerations we should obtain instead of (18) representations of the form

$$F(f,f') = \Sigma_\nu \{\overset{*}{v}_{f\nu} v_{f'\nu} (1-n_\nu) + \overset{*}{u}_{f\nu} u_{f'\nu} n_\nu\}$$
$$\Phi(f,f') = \Sigma_\nu \{u_{f\nu} v_{f'\nu} (1-n_\nu) + v_{f\nu} u_{f'\nu} n_\nu\}$$

Let us note if $F$ and $\Phi$ are determined by means of averaging

$$F(f,f') = S_p\{\overset{+}{a}_f a_{f'} D\} \cdot (S_p D)^{-1}; \quad \Phi(f,f') = S_p\{a_f a_{f'} D\} \cdot (S_p D)^{-1}$$

over any positive statistical operator $D$ then the operators $K$, $I-K$ must be both non-negative and therefore, in the obtained representation $0 \le n_\nu \le 1$ .

accuracy of the arbitrary unitary transformations over the index $\nu$ . Therefore, the functions $\{u, v\}$ have the same degree of the ambiguity.

As we noted already the equations (23) are not independent, since the forms $\mathcal{U}$, $\mathcal{B}$ are connected by the identities (13). Therefore in a number of cases it is conveni - ent to consider one[x)] of them

$$\mathcal{U}(f_1, f_2 \mid F, \Phi) = 0$$

together with the subsidiary conditions (24), (28). The second of the Eq. (23) will be carry out automatically. Let us consider as an example the problem of the determination of the ground superconducting state in the theory of superconductivity. Let us put in our formulae $f=(p, \sigma)$ where $p$ - is momentum and $\sigma$ - is spin index and we shall denote the two values of the latter by symbols + and - .

We take as usually[xx)]

$$I(p, p') = E(p)\,\delta(p-p')$$

$$V(f_1, f_2; f_2', f_1') = \frac{J(p_1, p_2; p_2', p_1')}{V}\,\delta(p_1+p_2-p_1'-p_2')\,\delta(\sigma_1-\sigma_1')(\sigma_2-\sigma_2') \qquad (30)$$

where $V$ is the volume of the system.

$J$ is supposed as real function invariant with respect to the transformation of the reflection of momentum $p \to -p$ . It is not difficult to check then that we satisfy all the equations and subsidiary conditions by putting

$$F(f, f') = F(p)\,\delta(f-f'), \quad \Phi(f, f') = \delta(f+f')\,\Phi(f);$$
$$\Phi(p,+) = -\Phi(p), \quad \Phi(p,-) = \Phi(p) \qquad (31)$$

where $F(p)$, $\Phi(p)$ are real functions of $p$ , invariant with respect to the transformation of the momentum reflection. They are determined from the equations:

$$2\xi(p)\,\Phi(p) + \frac{1-2F(p)}{V}\sum_{p'} J(p,-p;-p',p')\,\Phi(p') = 0$$
$$F(p) = F^2(p) + \Phi^2(p) \qquad (32)$$

x) The case may arise when $\Phi \equiv 0$. Then, the equation $\mathcal{U}=0$ is carried out trivially and we must restrict ourselves to considering the equation $\mathcal{B}=0$

xx) Here we use a discrete delta-function, i.e. the Kronecker symbol

$$\delta(p) = 1 \qquad p = 0;$$
$$\delta(p) = 0 \qquad p \neq 0.$$

where

$$\xi(p) = E(p) - \lambda + \frac{1}{V}\sum_{p'}\{2J(p', p; p, p') - J(p, p'; p, p')\}F(p') \tag{33}$$

Put here

$$-\frac{1}{V}\sum_{p'} J(p, -p; -p', p')\Phi(p') = C(p)$$

Then from (32) we obtain

$$\Phi(p) = \frac{C(p)}{2\Omega(p)}, \quad \Omega(p) = \sqrt{\xi^2(p) + C^2(p)}$$

$$F(p) = \frac{1}{2}\left\{1 - \frac{\xi(p)}{\Omega(p)}\right\} \tag{34}$$

and make sure that $C(p)$ satisfies the equation

$$C(p) + \frac{1}{V}\sum_{p'} J(p, -p; -p', p')\frac{C(p')}{2\Omega(p')} = 0 \tag{35}$$

As one can see we oome to the usual formulae of the theory of superoonduotivity.

One may determine the oorresponding functions $\{U, V\}$, putting

$$U_\nu(f) = U(p)\delta(\nu - f), \quad V_\nu(f) = V(f)\delta(\nu + f)$$

$$V(p, +) = V(p), \quad V(p, -) = -V(p) \tag{36}$$

where

$$V^2(p) = F(p), \quad U^2(p) = 1 - F(p)$$

2. Method of the self-oonsistent field.

We oonsidered up to now only the problem of determination of the ground state, independent on the time. It is not diffioult however to generalize the method of the self-oonsistent field for studying processes, depending on the timo. Let us introduoe for this funotions depending on the time.

$$F_t(f_1, f_2) = a^+_{f_1} a_{f_2}, \quad \Phi_t(f_1, f_2) = a_{f_1} a_{f_2} \tag{37}$$

and we shall oonsider the amplitudes a in Heisenberg-representation. The averaging

$$\bar{A} = \frac{Sp(A\mathcal{D})}{Sp\mathcal{D}}$$

performs here over some statistical operator $\mathcal{D}$ whioh does not depend out. Now we note that the exact relations

$$i\frac{\partial F(f_1, f_2)}{\partial t} = \overline{[a^+_{f_1} a_{f_2}; H]}; \quad i\frac{\partial F(f_1, f_2)}{\partial t} = \overline{[a_{f_1} a_{f_2}; H]}$$

yield from the motion equations or in more expanded form

$$i\frac{\partial F(f_1, f_2)}{\partial t} = \sum_f\{T(f_2, f)F(f_1, f) - T(f, f_1)F(f, f_2)\} -$$

$$-\sum_{f,f_1',f_2'}\{V(f_1',f_2';f,f_1)F_2(f_1',f_2';f,f_2)- \tag{38}$$
$$-V(f_2,f;f_2',f_1')F_2(f_1,f;f_2',f_1')\}$$

$$i\frac{\partial\Phi(f_1,f_2)}{\partial t}=\sum_f\{T(f_1,f)\Phi(f,f_2)+T(f_2,f)\Phi(f_1,f)\}+$$
$$+\sum_{f_1',f_2'}V(f_1,f_2;f_2',f_1')\Phi(f_1',f_2')+ \tag{39}$$
$$+\sum_{f,f_1',f_2'}\{V(f_1,f;f_2',f_1')\Phi_2(f;f_2,f_2',f_1')+$$
$$+V(f,f_2;f_2',f_1')\Phi_2(f;f_1,f_2'f_1')$$

where again

$$F_2(f_1,f_2;f_2',f_1')=\overline{a^+_{f_1}a^+_{f_2}a_{f_2'}a_{f_1'}} \tag{40}$$
$$\Phi_2(f_1;f_2,f_3,f_4)=\overline{a^+_{f_1}a_{f_2}a_{f_3}a_{f_4}}$$

According to principles of the theory of distribution function chains we should express again $\frac{\partial F_2}{\partial t}$, $\frac{\partial\Phi}{\partial t}$ in terms of the distribution functions of higher order and so on. The transition to the closed system of the approximate equations might be performed due to "unlinking" of one of these equations, for example, by means of some suitable approximation which expresses the highest correlation function of this equation in terms of the lowest ones. In the method of the self-consistent field we restrict ourselves only to the first equations (38),(39) obtained already and substitute approximately $F_2, \Phi_2$ into $F, \Phi$. Let us take these functions[x)]:

$$F(f_1,f_2)=\frac{S_p\{a^+_{f_1}(t)a_{f_2}(t)\mathcal{D}\}}{S_p\mathcal{D}} \qquad \Phi(f_1,f_2)=\frac{S_p\{a_{f_1}(t)a_{f_2}(t)\mathcal{D}\}}{S_p\mathcal{D}}$$
$$F_2(f_1,f_2;f_2',f_1')=\frac{S_p\{a^+_{f_1}(t)a^+_{f_2}(t)a_{f_2'}(t)a_{f_1'}(t)\mathcal{D}\}}{S_p\mathcal{D}} \tag{41}$$

and suppose, that the statistical operator $\mathcal{D}$ is diagonal in the representation of $\ldots n_\nu$, $\ldots$, in which $n_\nu=\overset{+}{\alpha}_\nu(t)\alpha_\nu(t)$. Strictly speaking one may make such an assumption only for one fixed time momentum since $\mathcal{D}$ remains constant and $\alpha(t)$, $\overset{+}{\alpha}(t)$ in the general case vary with time. Nevertheless one may consider our assumption as true one for the first approximation in cases when the main part of the Hamiltonian $H$ in amplitudes $\alpha$ have the following form: $\sum_\nu\Omega(\nu)\overset{+}{\alpha}_\nu\alpha_\nu$

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x) It follows from this that $F, \Phi$ satisfy always the conditions (24) then in "zero approximation" the equations of motion will be:

$$i\frac{\partial \alpha_\nu}{\partial t} = \Omega(\nu)\alpha_\nu$$

for them

$$\overset{+}{\alpha}_\nu(t)\,\alpha_\nu(t) = Const$$

Here the main part of the dependence $a_f(t)$, $\overset{+}{a}_f(t)$ from $t$, so to say, is oompensated by the time dependence of the functions $U$, $V$. Using the mentioned approximation we substitute the expressions (I) into (41) and perform the averaging with account of the diagonality of $\mathcal{D}$ in the representation ... $n_\nu$...

We get

$$F(f_1,f_2) = \sum_\nu \{ V_{f_1\nu} V_{f_2\nu}(1-\bar{n}_\nu) + \overset{*}{U}_{f_1\nu} U_{f_2\nu}\bar{n}_\nu \}$$
$$\Phi(f_1,f_2) = \sum_\nu \{ U_{f_1\nu} V_{f_2\nu}(1-\bar{n}_\nu) + V_{f_1\nu} V_{f_2\nu}\bar{n}_\nu \} \qquad (42)$$

where $\bar{n}_\nu$ is the mean value of

$$\overset{+}{\alpha}_\nu(t)\,\alpha_\nu(t)$$

We obtain also

$$F_2(f_1,f_2;f_2',f_1') = F(f_1,f_1')F(f_2,f_2') - F(f_1,f_2')F(f_2,f_2') + \overset{*}{\Phi}(f_1,f_2)\Phi(f_1',f_2') \qquad (43)$$

$$\Phi_2(f_1;f_2,f_3,f_4) = F(f_1,f_2)\Phi(f_3,f_4) + F(f_1,f_4)\Phi(f_2,f_3) - F(f_1,f_3)\Phi(f_2,f_4) \qquad (44)$$

Substituting these expressions (43),(44) into the equations (38), (39) we obtain time equations of the self-consistent field in the form:

$$\begin{cases} i\dfrac{\partial \Phi(f_1,f_2)}{\partial t} = \mathcal{U}(f_1,f_2|F,\Phi) \\ i\dfrac{\partial F(f_1,f_2)}{\partial t} = \mathcal{B}(f_1,f_2|F,\Phi) \end{cases} \qquad (45)$$

It is not diffioult to note, that the forms $\mathcal{U}$, $\mathcal{B}$ have the same expressions as before. This is conditioned by the ooinoidenoe of the r.h.s. of the Eq. (38), (39) with oorresponding expressions in (14), (15) and the coincidence of (43) with (19), (20). We may oonsequently use the properties of $\mathcal{U}$ and $\mathcal{B}$ which have been established before. Now let us turn our attention to the identity (13) whioh is true in the oonsidered case x) with $\nu_1 = \nu_2$. Basing on this identity, let us set up an important property of solution of the Eq. (45):

$$\frac{\partial \bar{n}_\nu}{\partial t} = 0 \qquad (46)$$

which holds for every solutior

x) As it was noted earlier the identity (13) is true with arbitrary $\nu_1$, $\nu_2$ if in the formulae (42) all the $\bar{n}_\nu = 0$.

In other words we show that the eigenvalue of the operators $K$ remains constant with time variation. But according to (25):

$$\varphi_\omega K = \bar{n}_\omega \varphi_\omega$$

Therefore, this assertion will be prove as soon as we shall make sure that with any $\omega$ :

$$\sum_{g,g'} \varphi_\omega(g) \frac{\partial K(g,g')}{dt} \varphi_\omega(g') = 0 \tag{47}$$

But since always:

$$n_{\nu 1} = 1 - n_{\nu 0}$$

we see that it is sufficient to prove the relation (47) only for $\omega = (\nu, 0)$ . Using (26) for the operator $K$ and the formulae (8) we obtain:

$$\sum_{g,g'} \varphi_{\nu 0}(g) \frac{dK(g,g')}{dt} \overset{*}{\varphi}_{\nu 0}(g') = -\sum_{f,f'} \overset{*}{V}_{\nu f} V_{\nu f'} \frac{d\overset{*}{F}(f,f')}{dt} + \sum_{f,f'} \overset{*}{V}_{\nu f} \overset{*}{U}_{\nu f'} \frac{d\Phi(f,f')}{dt} - \sum_{f,f'} U_{\nu f} V_{\nu f'} \frac{d\overset{*}{\Phi}(f,f')}{dt} + \sum_{f,f'} U_{\nu f} \overset{*}{U}_{\nu f'} \frac{dF(f,f')}{dt}$$

from which according to (45) and (13) we have:

$$i\sum_{g,g'} \varphi_{\nu 0}(g) \frac{dK(g,g')}{dt} \overset{*}{\varphi}_{\nu 0}(g') = \sum_{f,f'} \Big\{ \overset{*}{V}_{\nu f} V_{\nu f'} \mathcal{B}(f,f') + \overset{*}{V}_{\nu f} \overset{*}{U}_{\nu f'} \mathfrak{A}(f,f') + U_{\nu f} V_{\nu f'} \overset{*}{\mathfrak{A}}(f,f') + U_{\nu f} \overset{*}{U}_{\nu f'} \mathcal{B}(f,f') \Big\} = 0$$

this proves our assertion (46).

We have here a typical property of the method of the self-consistent field . One does not take into account relaxation effects.If any set $\ldots n_\nu \ldots$ is conserved then the particular system $n_\nu = 0$ is conserved also which corresponds to the ground state considered above. Therefore , the Eq. (45) are consistent with the subsidiary conditions (28). Let us write these equations and the subsidiary conditions in $r$ - representation for the case, when:

$$I = \frac{(P - e\mathcal{A})^2}{m}$$

and the interaction is described by the potential function $V(r_1, r_2)$ which does not depend on velocity and spins. We have x):

$$i\frac{\partial \Phi_{\sigma_1\sigma_2}(r_1,r_2)}{\partial t} = \Big\{ \frac{(i\frac{\partial}{\partial r_1} + e\vec{A}(r_1))^2 + (i\frac{\partial}{\partial r_2} + e\vec{A}(r_2))^2}{2m} - 2\lambda + \tag{48}$$

$$+\int V(r,r')\sum_\sigma F_{\sigma\sigma}(r',r')dr' + \int V(r_2,r')\sum_\sigma F_{\sigma\sigma}(r',r')dr' \Big\} \Phi_{\sigma_1\sigma_2}(r_1,r_2) + V(r_1,r_2)\Phi_{\sigma_1\sigma_2}(r_1,r_2) -$$

$$-\sum_\sigma \int dr' \Big\{ F_{\sigma\sigma'}(r',r_1) V(r',r_2) \Phi_{\sigma_1\sigma_{\sigma_2}}(r',r_2) + F_{\sigma\sigma_2}(r',r_2) V(r_1,r') \Phi_{\sigma_1\sigma}(r_1,r_2') \Big\} \tag{49}$$

$$i\frac{\partial F_{\sigma_1\sigma_2}(r_1,r_2)}{\partial t} = \Big\{ \frac{(i\frac{\partial}{\partial r_2} + e\vec{A}(r_2))^2 - (i\frac{\partial}{\partial r_1} - e\vec{A}(r_1))^2}{2m} \Big\} F_{\sigma_1\sigma_2}(r_1,r_2) +$$

---

x) Here $r$ is vector and $\vec{r}, dr$ is three-dimensional element of volume.

$$+\sum_{\sigma}\int d\tau\left\{\phi(\tau_2,\tau)-\phi(\tau_1,\tau)\right\}\left\{F_{\sigma\sigma}(\tau,\tau)F_{\sigma_1\sigma_2}(\tau_1,\tau_2)-F_{\sigma_1\sigma}(\tau_1,\tau)F_{\sigma,\sigma_2}(\tau,\tau_2)+\overset{*}{\phi}_{\sigma_1\sigma}(\tau_1,\tau)\phi_{\sigma\sigma_2}(\tau,\tau_2)\right\}$$

$$F_{\sigma_1\sigma_2}(\tau_1,\tau_2)=\sum_{\sigma}\int d\tau\left\{F_{\sigma_1\sigma}(\tau_1,\tau)F_{\sigma\sigma_2}(\tau,\tau_2)+\overset{*}{\phi}_{\sigma\sigma_1}(\tau,\tau_1)\phi_{\sigma\sigma_2}(\tau,\tau_2)\right\} \tag{50a}$$

$$\sum_{\sigma}\int d\tau\left\{F_{\sigma_1\sigma}(\tau_1,\tau)\overset{*}{\phi}_{\sigma\sigma_2}(\tau,\tau_2)+F_{\sigma_2\sigma}(\tau_2,\tau)\overset{*}{\phi}_{\sigma\sigma_1}(\tau,\tau_1)\right\}=0 \tag{50b}$$

As we see all this system of the equations is gauge invariant. The gauge transformation

$$e\vec{A}(\tau)\longrightarrow e\vec{A}(\tau)+\frac{\partial}{\partial\tau}\varphi(\tau) \tag{51}$$

is compensated by transformation of the function

$$\phi_{\sigma_1\sigma_2}(\tau_1,\tau_2)\longrightarrow\phi_{\sigma_1\sigma_2}(\tau_1,\tau_2)\,e^{i(\varphi(\tau_1)+\varphi(\tau_2))} \tag{52}$$

$$F_{\sigma_1\sigma_2}(\tau_1,\tau_2)\longrightarrow F_{\sigma_1\sigma_2}(\tau_1,\tau_2)\,e^{i(-\varphi(\tau_1)+\varphi(\tau_2))}$$

The gauge invariance is conditioned here by gauge invariance of the Hamiltonian. When considering here the problems of the theory of superconductivity in a model with direct interaction of electrons depending on velocities the corresponding Hamiltonian is already no longer exactly gauge invariant. This property is carried out only approximately, therefore the equations of the method of the self-consistent field will be gauge invariant with the same degree of approximation. It is essential to note, that the approximations themselves used by us do not distore the gauge invariance. This problem is discussed also in 4.

3. Representation with fixed number of particules.

Now independently on the above consideration we shall consider the correlation function

$$F_2(f_1, f_2; f_2', f_1')$$

taken in $\tau$ - representation. We put here $f = (\tau, \sigma)$ where $\sigma$ is some discrete, for example, spin index. Let this function may be represented in the form

$$F_2(f_1, f_2; f_2', f_1') = \sum_n \Psi_n^*(f_1, f_2)\Psi_n(f_1', f_2') + \tilde{F}_2 \tag{53}$$

so that

1) when the spacing between the pairs $(f_1, f_2)$ and $(f_1', f_2')$ tends to the infinity the additive term $\tilde{F}_2$ vanishes rapidly enough

2) when the spacing between the points $f_1$ and $f_2$ increases infinitely the function $\Psi_n(f_1, f_2)$ tends to zero and the integral

$$\int |\Psi_n(f_1, f_2)|^2 df_2 = \int |\Psi_n(f_2, f_1)|^2 df_2 \tag{54}$$

is convergent.

Then it is evident that we can interpret $\Psi_n(f_1, f_2)$ as the wave function of the pair of particles which is in one of the bound states, and the integral (54) interpret as proportional to the density of the number of these particles in the point $f_1$, which are connected by pairs in the state $\Psi_n$.

Let us consider from this point of view our formula (43) and limit ourselves by the case of the theory of superconductivity. For the ground state we have :

$$\Phi_+(\tau_1, \tau_2) = \frac{1}{(2\pi)^3}\int e^{ik(\tau_1 - \tau_2)}\Phi(k)dk, \quad \Phi(k) = \frac{C(k)}{2\sqrt{\xi^2(k) + C^2(k)}}$$

$$F_{\sigma\sigma}(\tau_1, \tau_2) = \frac{1}{(2\pi)^3}\int e^{ik(\tau_1 - \tau_2)}F(k)dk, \quad F(k) = \frac{1}{2}\left\{1 - \frac{\xi(k)}{\sqrt{\xi^2(k) + C^2(k)}}\right\}$$

As we see the conditions (1), (2) hold here, therefore $\Phi(f_1, f_2) = \Phi_{-+}(\tau_1, \tau_2)$ may be considered as wave function of the bound pair of particles (with the opposite spins). In a given case there is only <u>one</u> state $\Phi(f_1, f_2)$ and we may say that all the bound quasi-molecules are in condensate. In view of the formula[x)] (43) one does not take into account the bound pairs dropped out of the condensate. Now turn our attention to the fact that in our considerations we have used essentially the canonical transformation (I). Due to this fact for the state $C_0$ and statistical operator $\mathcal{D}$ the total number of particles $\mathcal{N} = \sum \overset{+}{a}_f a_f$ ,

x) One can make such a calculation if we shall generalize the approximation of (43) according to the expression (53).

is not quantum number and has no fixed value. On the other hand $\mathcal{N}$ is always integral of motion for the Hamiltonian (3) considered. Therefore it is natural to require obtaining the same results with the representation in which $\mathcal{N}$ is quantum number.

Let us see however what would be in reality if we tried to make our consideration in such a representation. First of all we should not mixe up creation and annihilation amplitudes and hence we should be obliged to put in the formulae (1) that $\mathcal{V} \equiv 0$. But instead of (43) we should obtain the approximation

$$F(f_1,f_2;f_2',f_1') = F(f_1,f_1')F(f_2,f_2') - F(f_1,f_2')F(f_2,f_1') \tag{55}$$

of the Fock method not taking into account the possibility of appearance of bound states of pairs of particles. The state of affairs may show itself still worse, since independently of any approximations the equation

$$\overline{a_f a_f} = 0$$

takes place for any averaging procedure for which $\mathcal{N}$ is strictly fixed. It is no difficulties to find a way out of this paradox. If we want to operate with fixed $\mathcal{N}$ it is necessary to proceed further the chain of equations connecting the distribution functions and consider the correlation functions of the higher order. In order to exclude complex calculations we profit now by a simplified method.

Proceeding from the fact that in the dinamic system considered one has the bound pairs in the same state $\Phi(f_1,f_2)$ let us supplement the formula (55) of the usual Fock method with the term

$$\overset{*}{\Phi}(f_1,f_2)\Phi(f_1',f_2')$$

describing the contribution of such pairs. Substituting the obtained expression into the exact relation (38) we obtain immediately the second of the Eq.(45). In order to obtain the first from the Eq. (45) determining $\Phi$ we shall consider "two-time" correlation function in the form

$$\overline{a_{f_1}(t)a_{f_2}(t)\overset{+}{a}_{f_2''}(\tau)\overset{+}{a}_{f_1''}(\tau)}$$

and differentiate it over the time $t$. According to exact equations of motion we obtain[x)]:

$$i\frac{\partial}{\partial t}\langle a_{f_1}(t)a_{f_2}(t)\overset{+}{a}_{f_2''}\overset{+}{a}_{f_1''}(\tau)\rangle = \langle [a_{f_1}(t)a_{f_2}(t);H]\overset{+}{a}_{f_2''}(\tau)\overset{+}{a}_{f_1''}(\tau)\rangle =$$

$$= \sum_f \left\{ I(f_1,f)\langle a_f(t)a_{f_2}(t)\overset{+}{a}_{f_2''}(\tau)\overset{+}{a}_{f_1''}(\tau)\rangle + I(f_2,f)\langle a_{f_1}(t)a_f(t)\overset{+}{a}_{f_2''}(\tau)\overset{+}{a}_{f_1''}(\tau)\rangle \right\}$$

$$+\sum_{f_1'f_2'} \mathcal{V}(f_1,f_2;f_2',f_1')\langle a_{f_2'}(t)a_{f_1'}(t)\overset{+}{a}_{f_2''}(\tau)\overset{+}{a}_{f_1''}(\tau)\rangle +$$

x) The expectation value is denoted here by brackets $\langle \dots \rangle$ since it is more convenient for cumbersome expressions.

$$+\sum_{f,f_1',f_2'} V(f_1,f;f_2',f_1')<\overset{+}{a}_f(t)a_{f_2}(t)a_{f_2'}(t)a_{f_1'}(t)\overset{+}{a}_{f_2''}(\tau)\overset{+}{a}_{f_1''}(\tau)>+$$

$$+\sum_{f,f_1',f_2} V(f,f_2;f_2',f_1')<\overset{+}{a}_f(t)a_{f_1}(t)a_{f_2'}(t)a_{f_1'}(t)\overset{+}{a}_{f_2''}(\tau)\overset{+}{a}_{f_1''}(\tau)>$$

Let us note that the difference between this relation and (39) is that in the first we have on the right two operators $\overset{+}{a}$ compensating the variation of the number of particles. Perform here the transition to an approximate equation, expressing approximately the functions of the type:

$$<\overset{+}{a}_{f_1}(t)a_{f_2}(t)a_{f_2'}(t)a_{f_1'}(t)\overset{+}{a}_{f_2''}(\tau)\overset{+}{a}_{f_1'}(\tau)>$$

in terms of products of four and two operators. Note that here we must now take into account strict consration of the value $\mathcal{N}$. After this in the equation, obtained from (56) we displace the pair ( $f_2', f_1'$ ) to the infinity. Therefore, we use the following approximation:

$$<\overset{+}{a}_{f_1}(t)a_{f_2}(t)a_{f_2'}(t)a_{f_1'}(t)\overset{+}{a}_{f_2''}(\tau)\overset{+}{a}_{f_1'}(\tau)>=<\overset{+}{a}_{f_1}(t)a_{f_2}(t)><a_{f_2'}(t)a_{f_1'}(t)\overset{+}{a}_{f_2''}(\tau)\overset{+}{a}_{f_1'}(\tau)>- \tag{57}$$

$$-<\overset{+}{a}_{f_1}(t)a_{f_2'}(t)><a_{f_2}(t)a_{f_1'}(t)\overset{+}{a}_{f_2''}(\tau)\overset{+}{a}_{f_1'}(\tau)>+<\overset{+}{a}_{f_1}(t)a_{f_1'}(t)><a_{f_2}(t)a_{f_2'}(t)\overset{+}{a}_{f_2''}(\tau)\overset{+}{a}_{f_1'}(\tau)>+S'$$

where $S'$ is sum of the terms having the multpliers $<\overset{+}{a}_{f_1''}(\tau)a_f(t)$ or $<\overset{+}{a}_{f_2''}(\tau)a_f(t)$. We do not write down the explicite expression for $S'$ since such terms will vanish when displacing pair of the points $(f_2'', f_1'')$ to the infinity. We substitute (57) into (56) and displace this pair of points to the infinity. Then the expressions of the type

$$<a_{f_1}(t)a_{f_2}(t)\overset{+}{a}_{f_2'}(\tau)\overset{+}{a}_{f_1'}(\tau)>$$

will factorize into products

$$\Psi_t(f_1,f_2)\overset{*}{\Psi}_\tau(f_1',f_2')$$

in which $\Psi_t(f_1,f_2)$ is the wave function of the bound pairs and separating the common multiplier we obtain $\overset{*}{\Psi}_t(f_1',f_2')$,

$$i\frac{\partial\Psi_t(f_1,f_2)}{\partial t}=\sum_f\left\{I(f_1,f)\Psi_t(f,f_2)+I(f_2,f)\Psi(f_1,f)\right\}+\sum_{f_1',f_2'}V(f_1,f_2;f_2',f_1')\Psi_t(f_1',f_2')+$$

$$+\sum_{f,f_1',f_2'}V(f_1,f;f_2',f_1')\{F_t(f,f_2)\Psi_t(f_2',f_1')-F_t(f,f_2')\Psi_t(f,f_1')+F_t(f,f_1')\Psi_t(f,f_2')\}+ \tag{58}$$

$$+\sum_{f,f_1',f_2'}V(f_1,f_2;f_2',f_1')\left\{F_t(f,f_1)\Psi_t(f_2',f_1')-F_t(f,f_2')\Psi_t(f,f_1')+F_t(f,f_1')\Psi_t(f,f_2')\right\}$$

Note that in the ground stationary state $\Psi_t$ must be proportional to $e^{-iEt}$ where E is corresponding energy. Let us introduce the quantity x)

x) The sense of such a value $\lambda$ as chemical potential may be cleared from the following considerations. On the one hand the factor $exp(-iEt)$ must express time dependence of the wave function of the pair $<\overset{*}{C}_N a_{f_1}(t)a_{f_2}(t)C_{N+2}>$ where $C_N$ denotes the lowest state of the system in case when the number of particles equals N. On the other hand let the total energy of system in the state $C_N$ be E(N). Then the time dependence of the given form is determined by the multiplier $exp\{-i(E(N+2)-E(N))t\}$ Thus: $2\lambda \equiv E = E(N+2)-E(N)$

$$\lambda=\frac{\partial E(N)}{\partial N}$$

$$\lambda = \frac{E}{2}$$

and put in general non-equilibrium case:

$$\Psi_t(f_1, f_2) = e^{-2i\lambda t}\,\Phi_t(f_1, f_2)$$

so that:

$$i\frac{\partial \Psi_t}{\partial t} = e^{-2i\lambda t}\left\{ i\frac{\partial \Phi_t}{\partial t} + 2\lambda\Phi_t \right\}$$

Then the obtained equation (58) turns into the first of the equations (45).

These considerations may be given in more exaot form and with their aid one may obtain more exact equations, but this is not disoussed here. Now it is essential to stress, that the equations of the generalized method of the self-oonsistent field may be obtained in the scheme with the fixed number of partioles. For this one olears up the sense of the transformation (1). Namely by means of this transformation the results which would be obtained usually in higher approximation now are obtained in more lower approximation. This property is due to the fact that in terms of variables $\alpha$ the bound state drops out. For example, in our first approximation

$$\langle \alpha_{\nu_1}\alpha_{\nu_2}\overset{+}{\alpha}_{\nu_2'}\overset{+}{\alpha}_{\nu_1'}\rangle = (1-\bar{n}_{\nu_1})(1-\bar{n}_{\nu_2})\{\delta(\nu_1-\nu_1')\delta(\nu_2-\nu_2') - \delta(\nu_1-\nu_2')\delta(\nu_2-\nu_1')\} =$$

$$= \langle \alpha_{\nu_1}\overset{+}{\alpha}_{\nu_1'}\rangle\langle \alpha_{\nu_2}\overset{+}{\alpha}_{\nu_2'}\rangle - \langle \alpha_{\nu_1}\overset{+}{\alpha}_{\nu_2'}\rangle\langle \alpha_{\nu_2}\overset{+}{\alpha}_{\nu_1'}\rangle$$

The same situation may be obtained also in the highest approximations of the usual scheme. The principle of oompensation of the dangerous graphs gives us tool for the direot approach to these results. All the graphs compensated by this prinoiple just determine the bound state.

Thus, in cases when the possibility of appearance of the bound state of pairs of partioles (Bose - oondensate) prevents from application of the perturbation theory the prinoiple of oompensation when introduoing new variables $\alpha\alpha$ ( which leads to eliminate this state ) destroys the obstade in application of this usual theory.

## 4. Collective osoillations.

Now let us consider the problem of determining the speotrum of the elementary excitations of the ground state. From the point of view of the method of the self-oonsistent field one may solve this problem by the following way.

As it was already noted, the values $\bar{n}_\nu$ remain oonstant for the ground state they are equal to zero. Wishing to investigate small oscillations near suoh a state let us put that $\bar{n}_\nu = 0$ , i.e. with subsidiary conditions (28). Let $F_0, \Phi_0$ be $F, \Phi$ for the ground state. Consider the infinitesimal increments

$$F = F_0 + \delta F \quad , \quad \Phi = \Phi_0 + \delta\Phi$$

and write down for them linear equations in variations:

$$i\frac{\partial\delta\Phi(f_1,f_2)}{\partial t}=\delta\mathcal{U}(f_1,f_2|F,\Phi) \tag{59}$$

$$i\frac{\partial\delta F(f_1,f_2)}{\partial t}=\delta\mathcal{B}(f_1,f_2|F,\Phi)$$

Besides take into account, that $\delta F$ and $\delta\Phi$ must be connected by subsidiary conditions (28), and so

$$\delta\Big\{F(f_1,f_2)-\sum_f F(f_1,f)F(f,f_2)-\sum_f \overset{*}{\Phi}(f,f_1)\Phi(f,f_2)\Big\}=0 \tag{60}$$

$$\delta\Big\{\sum_f F(f_1,f)\overset{*}{\Phi}(f,f_2)+\sum_f F(f_2,f)\overset{*}{\Phi}(f,f_1)\Big\}=0$$

Note also that due to (24) $\delta\Phi$ must be antisymmetrical and $\delta F$ Hermitian. We shall solve the obtained homogeneous equations by superposition of the elementary solutions proportional to $exp(-iEt)$.

Thus, we find[x)] secular equations for determining the spectrum of oscillations. Due to the conditions (60) $\delta F$ and $\delta\Phi$ are not independent and therefore it is practically convenient to represent them by the expressions in terms of new <u>independent</u> unknown variables which satisfy automatically the conditions (60). One may obtain such expressions immediately taking into account that due to (60) $n_\nu\equiv 0$ but and not $u_{f\nu}, u_{f\nu}$ suffers infinitesimal transformation. These transformations must be compatible with the orthonormality conditions (2). Instead of varying $u, v$ we may perform the infinitesimal transformation with $\alpha$.

$$\alpha_\nu \longrightarrow \alpha_\nu+\sum_{\nu'}\mu(\nu',\nu)\alpha_{\nu'}+\sum_{\nu'}\lambda(\nu',\nu)\overset{+}{\alpha}_{\nu'} \tag{61}$$

From the canonical conditions of this transformation it follows, then that

$$\lambda(\nu_1,\nu_2)+\lambda(\nu_2,\nu_1)=0 \tag{62}$$

$$\mu(\nu_1,\nu_2)+\overset{*}{\mu}(\nu_2,\nu_1)=0 \tag{63}$$

and

$$\langle\alpha_\nu\alpha_{\nu'}\rangle_0\rightarrow\lambda(\nu,\nu')$$

$\langle\overset{+}{\alpha}_\nu\alpha_{\nu'}\rangle_0$ remain equal to zero, and hence

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x) Stress that such a method of determination of the spectrum of elementary exitations is similar to that in the well known papers of A.A.Vlasov . One should note that these papers have a great influence on the development of the consption of collective oscillations.

$$F_0(f_1 f_2)+\delta F(f_1,f_2)=\sum_{\nu_1,\nu_2}\langle(\overset{*}{U}_{f_1\nu_1}\overset{+}{\alpha}_{\nu_1}+\overset{*}{V}_{f_1\nu_1}\alpha_{\nu_1})(U_{f_2\nu_2}\alpha_{\nu_2}+V_{f_2\nu_2}\overset{+}{\alpha}_{\nu_2})\rangle_0 =$$

$$=F_0(f_1,f_2)+\sum_{\nu_1,\nu_2}\{\overset{*}{V}_{f_1\nu_1}U_{f_2\nu_2}\lambda(\nu_1,\nu_2)+\overset{*}{U}_{f_2\nu_2}V_{f_2\nu_2}\overset{*}{\lambda}(\nu_2,\nu_1)\}$$

$$\Phi_0(f_1,f_2)+\delta\Phi(f_1,f_2)=\sum_{\nu_1,\nu_2}\langle(\overset{*}{U}_{f_1\nu_1}\alpha_{\nu_1}+V_{f_1\nu_1}\overset{+}{\alpha}_{\nu_1})(U_{f_2\nu_2}\alpha_{\nu_2}+V_{f_2\nu_2}\overset{+}{\alpha}_{\nu_2})\rangle_0 =$$

$$=\Phi_0(f_1,f_2)+\sum_{\nu_1,\nu_2}\{U_{f_1\nu_1}U_{f_2\nu_2}\lambda(\nu_1,\nu_2)+V_{f_1\nu_1}V_{f_2\nu_2}\overset{*}{\lambda}(\nu_2,\nu_1)\}$$

As one can see the coefficient $\mu$ did not enter our formulae. This is conditioned by the fact that in the considered case $\bar{n}_\nu\equiv 0$. Let us note that independently on the given consideration it is not difficult to check that the expressions:

$$\delta F(f_1,f_2)=\sum_{\nu_1,\nu_2}\{\overset{*}{V}_{f_1\nu_1}U_{f_2\nu_2}\lambda(\nu_1,\nu_2)+\overset{*}{U}_{f_1\nu_1}V_{f_2\nu_2}\overset{*}{\lambda}(\nu_2,\nu_1)\} \tag{64}$$

$$\delta\Phi(f_1,f_2)=\sum_{\nu_1,\nu_2}\{U_{f_1\nu_1}U_{f_2\nu_2}\lambda(\nu_1,\nu_2)+V_{f_1\nu_1}V_{f_2\nu_2}\overset{*}{\lambda}(\nu_2,\nu_1)\} \tag{65}$$

for arbitrary antisymmetrical function give the general solution of the subsidiary conditions (60), (24). In order to obtain equation for $\partial\lambda/\partial t$ it is advisable to express also $\lambda$ in terms of $\delta F$, $\delta\Phi$. Multiply (64) by $V_{f_1\gamma_1}$ and (65) by $\overset{*}{U}_{f_1\gamma_1}$ and take the sum. Then owing to the orthonormality conditions in the form (10) we obtain

$$\sum_{f_1}\{V_{f_1\gamma_1}\delta F(f_1,f_2)+\overset{*}{U}_{f_1\gamma_1}\delta\overset{*}{\Phi}(f_1,f_2)\}=\sum_{\nu_2}V_{f_2\nu_2}\overset{*}{\lambda}(\gamma_1,\gamma_2) \tag{66}$$

Multiply (64) by $U_{f_1\gamma_1}$ and (65) by $\overset{*}{V}_{f_1\gamma_1}$ and take the sum again. We get

$$\sum\{U_{f_1\gamma_1}\delta F(f_1,f_2)+\overset{*}{V}_{f_1\gamma_1}\delta\Phi(f_1,f_2)\}=\sum_{\nu_2}V_{f_2\nu_2}\overset{*}{\lambda}(\nu_2,\gamma_1)$$

or

$$\sum_{f_1}\{\overset{*}{U}_{f_1\gamma_1}\delta\overset{*}{F}(f_1,f_2)+V_{f_1\gamma_1}\delta\overset{*}{\Phi}(f_1,f_2)\}=-\sum_{\nu_2}\overset{*}{V}_{f_2\nu_2}\lambda(\gamma_1,\nu_2) \tag{67}$$

From (66) and (67) we obtain by the same way the unknown expression for

$$\lambda(\nu_1,\nu_2)=\sum\{\overset{*}{U}_{f_2\nu_2}V_{f_1\nu_1}\delta F(f_1,f_2)+\overset{*}{U}_{f_2\nu_2}\overset{*}{U}_{f_1\nu_1}\delta\Phi(f_1,f_2)- \tag{68}$$

$$-V_{f_2\nu_2}\overset{*}{U}_{f_1\nu_1}\delta\overset{*}{F}(f_1,f_2)-V_{f_2\nu_2}V_{f_1\nu_1}\delta\overset{*}{\Phi}(f_1,f_2)\}$$

After differentiating this expression over $t$ and taking into account (59) we obtain an equation for determining $\lambda$:

$$i\frac{\partial\lambda(\nu_1,\nu_2)}{\partial t}=\sum_{f_1,f_2}\{\overset{*}{U}_{f_2\nu_2}V_{f_1\nu_1}\delta\mathcal{B}(f_1,f_2)+\overset{*}{U}_{f_2\nu_2}\overset{*}{U}_{f_1\nu_1}\delta\mathcal{U}(f_1,f_2)+$$

$$+V_{f_2\nu_2}\overset{*}{U}_{f_1\nu_1}\delta\overset{*}{\mathcal{B}}(f_1,f_2)+V_{f_2\nu_2}V_{f_1\nu_1}\delta\overset{*}{\mathcal{U}}(f_1,f_2)\} \tag{69}$$

In order to expand completely this equation it is necessary to vary the forms $\mathcal{U}, \mathcal{V}$ and express $\delta F, \delta \Phi$ in terms of $\lambda$ by means of (64), (65). After calculations we have[x]:

$$i\frac{\partial \lambda(\nu_1,\nu_2)}{\partial t}=\sum_{\omega}\left\{\Omega(\nu_2,\omega)\lambda(\nu_1,\omega)-\Omega(\nu_1,\omega)\lambda(\nu_2,\omega)\right\}+ \tag{70}$$

$$+\sum\left\{X(\nu_1,\nu_2,\omega_1,\omega_2)\lambda(\omega_1,\omega_2)+Y(\nu_1,\nu_2;\omega_1,\omega_2)\lambda^*(\omega_2,\omega_1)\right\}$$

where

$$\Omega(\nu,\omega)=\sum_{f,f'}\mathcal{E}(f,f')(\overset{*}{u}_{f\nu}u_{f'\omega}-\overset{*}{v}_{f\omega}v_{f'\nu})+$$

$$+\sum_{f_1f_2f_1'f_2'}V(f_1,f_2;f_2',f_1')\Phi_0(f_2',f_1')\overset{*}{u}_{f_1\nu}\overset{*}{v}_{f_2\omega}+$$

$$+\sum_{f_1f_2f_1'f_2'}V(f_1,f_2;f_2',f_1')\overset{*}{\Phi}_0(f_2,f_1)v_{f_2'\nu}u_{f_1'\omega};$$

$$\mathcal{E}(f,f')=T(f,f')+\sum\left\{V(f_1,f;f',f_1')-V(f_1,f;f_1',f')\right\}F_0(f_1,f_1');$$

$$X(\nu_1,\nu_2;\omega_1,\omega_2)=\frac{1}{2}\sum V(f_1,f_2;f_2',f_1')(\overset{*}{u}_{f_2\nu_1}\overset{*}{u}_{f_1\nu_2}-\overset{*}{u}_{f_1\nu_1}\overset{*}{u}_{f_2\nu_2})u_{f_2'\omega_2}u_{f_1'\omega_1}+$$

$$+\frac{1}{2}\sum V(f_1,f_2;f_2',f_1')(v_{f_1'\nu_1}v_{f_2'\nu_2}-v_{f_2'\nu_1}v_{f_1'\nu_2})\overset{*}{v}_{f_1\omega_1}\overset{*}{v}_{f_2\omega_2}+$$

$$+\frac{1}{2}\sum\left\{V(f_1,f_2;f_2',f_1')-V(f_1,f_2;f_1',f_2')\right\}\times \tag{71}$$

$$\times(v_{f_1'\nu_2}\overset{*}{u}_{f_1\nu_2}-\overset{*}{u}_{f_1\nu_1}v_{f_1'\nu_2})(\overset{*}{v}_{f_2\omega_1}u_{f_2'\omega_2}-\overset{*}{v}_{f_2\omega_2}u_{f_2'\omega_1})$$

$$Y(\nu_1,\nu_2;\omega_1,\omega_2)=\frac{1}{2}\sum V(f_1,f_2;f_2',f_1')(\overset{*}{u}_{f_2\nu_1}\overset{*}{u}_{f_1\nu_2}-\overset{*}{u}_{f_1\nu_1}u_{f_2\nu_2})v_{f_2'\omega_1}v_{f_1'\omega_2}+$$

$$+\frac{1}{2}\sum V(f_1,f_2;f_2',f_1')(v_{f_1'\nu_1}v_{f_2'\nu_2}-v_{f_2'\nu_1}v_{f_1\nu_2})\overset{*}{u}_{f_2\omega_2}u_{f_1\omega_1}+$$

$$+\frac{1}{2}\sum\left\{V(f_1,f_2;f_2',f_1')-V(f_1,f_2;f_1',f_2')\right\}\times$$

$$\times(v_{f_1'\nu_1}\overset{*}{u}_{f_1\nu_2}-v_{f_1'\nu_2}\overset{*}{u}_{f_1\nu_1})(\overset{*}{u}_{f_2\omega_1}u_{f_2'\omega_2}-\overset{*}{u}_{f_2\omega_2}v_{f_2'\omega_1})$$

From (70) we obtain also:

$$-i\frac{\partial \lambda^*(\nu_1,\nu_2)}{\partial t}=\sum_{\omega}\left\{\overset{*}{\Omega}(\nu_2,\omega)\overset{*}{\lambda}(\nu,\omega)-\overset{*}{\Omega}(\nu,\omega)\overset{*}{\lambda}(\nu_2,\omega)\right\}+ \tag{72}$$

$$+\sum_{\omega}\left\{\overset{*}{X}(\nu_1,\nu_2;\omega_1,\omega_2)\overset{*}{\lambda}(\omega_1,\omega_2)+\overset{*}{Y}(\nu_1,\nu_2;\omega_1,\omega_2)\lambda(\omega_2,\omega_1)\right\}$$

We shall solve the system of the linear homogeneous equations (70), (72) by superposition of the normal oscillations

$$\lambda(\nu_1,\nu_2)=\sum_E e^{iEt}\xi_E(\nu_1,\nu_2) \tag{73}$$

$$\lambda^*(\nu_1,\nu_2)=\sum_E e^{-iEt}\eta_E(\nu_1,\nu_2); \qquad \overset{*}{\xi}_{-E}=\eta_E$$

x) Here $\omega$ is the index of summation over $\nu$ in contrast to the notations in 1.

Substituting (73) into (70) and (72) we obtain secular equations for determining the spectrum in form:

$$E\xi(\nu_1,\nu_2)=\sum\{\Omega(\nu_2,\omega)\xi(\nu_1,\omega)-\Omega(\nu_1,\omega)\xi(\nu_2,\omega)\}+$$

$$+\sum\{X(\nu_1,\nu_2;\omega_1,\omega_2)\xi(\omega_1,\omega_2)+Y(\nu_1,\nu_2;\omega_1,\omega_2)\eta(\omega_2,\omega_1)\};$$

$$-E\eta(\nu_1,\nu_2)=\sum\{\overset{*}{\Omega}(\nu_2,\omega)\eta(\nu_1,\omega)-\overset{*}{\Omega}(\nu_1,\omega)\eta(\nu_2,\omega)\}+ \tag{74}$$

$$+\sum\{\overset{*}{X}(\nu_1,\nu_2;\omega_1,\omega_2)\eta(\omega_1,\omega_2)+\overset{*}{Y}(\nu_1,\nu_2;\omega_1,\omega_2)\xi(\omega_2,\omega_1)\}.$$

Let us stress that we should obtain the same expressions if we took the method of the approximate second quantization instead of that of the self-consistent field. In this method we should introduce Bose-amplitudes $\beta_{\nu\mu}$ $(\beta_{\mu\nu}=-\beta_{\nu\mu})$, instead of the Fermi-amplitudes $\alpha_\nu\alpha_\mu$. Then, we should perform the diagonalization of the corresponding Hamiltonian which represents the quadratic form of the operators $\beta,\overset{+}{\beta}$ by means of canonical transformation

$$\beta_{\nu_1,\nu_2}=\sum_n\{\xi_n(\nu_1,\nu_2)\zeta_n+\eta_n(\nu_1,\nu_2)\overset{+}{\zeta}_n\} \tag{75}$$

with normalization condition

$$\sum_n\{|\xi_n|^2-|\eta_n|^2\}=1 \tag{76}$$

Here $\zeta_n$ are new Bose-amplitudes depending on the time througt the factor $exp(-iE_nt)$. Then it would turn out that $\xi$ and $\eta$ should just satisfy our equations (74). Let us note that the obtaining these equations by means of the method of approximate secondary quantization has some advantages over the above-mentioned one, since it leads in natural way to the normalization condition (76) determining the sign of $E$. In the method of the self-consistent field this sign is not fixed, it is easy to note that if $E,\xi,\eta$ is the solution of the system of secular equations (74) then the transformation

$$E\to -E,\ \xi\to\eta^*,\ \eta\to\xi^*$$

leads again to the solution of the same system.

We write down now the equations for the eigenoscillations. Let us consider the question of the forced oscillations, caused by small external fields, giving rise to the variation $I(f,f')$. (The interaction function $U$ is supposed as independent on external fields). Then repeating the above considerations we get instead of homogeneous equations (70), (72) inhomogeneous ones of the type

$$i\frac{\partial\lambda(\nu_1,\nu_2)}{\partial t}=\sum_{\omega}\left\{\Omega(\nu_2,\omega)\lambda(\nu_1,\omega)-\Omega(\nu_1,\omega)\lambda(\nu_2,\omega)\right\}+\sum_{\omega_1,\omega_2}\left\{X(\nu_1,\nu_2;\omega_1,\omega_2)\lambda(\omega_1,\omega_2)+\right.$$

$$\left.+Y(\nu_1,\nu_2;\omega_1,\omega_2)\lambda^*(\omega_2,\omega_1)\right\}+\sum_{f,f'}\left\{V_{f'\nu_1}\overset{*}{U}_{f\nu_2}-\overset{*}{U}_{f\nu_1}V_{f'\nu_2}\right\}\delta I(f,f') \tag{77}$$

$$-i\frac{\partial\overset{*}{\lambda}(\nu_1,\nu_2)}{\partial t}=\sum_{\omega}\left\{\overset{*}{\Omega}(\nu_2,\omega)\overset{*}{\lambda}(\nu_1,\omega)-\overset{*}{\Omega}(\nu_1,\omega)\overset{*}{\lambda}(\nu_2,\omega)\right\}+\sum_{\omega_1,\omega_2}\left\{\overset{*}{X}(\nu_1,\nu_2;\omega_1,\omega_2)\overset{*}{\lambda}(\omega_1,\omega_2)+\right.$$

$$\left.+\overset{*}{Y}(\nu_1,\nu_2;\omega_1,\omega_2)\lambda(\omega_2,\omega_1)\right\}+\sum_{f,f'}\left\{\overset{*}{V}_{f'\nu_1}U_{f\nu_2}-U_{f\nu_1}\overset{*}{V}_{f'\nu_2}\right\}\delta I^*(f,f')$$

Let us use the just obtained general equations in the case of the dinamic system, considered in 1 in connection with the theory of superconductivity.

Let us substitute the formulae (30), (31), (36) from 1 into the expressions (71) and expand by means of this the equations (74). Let us note, that the spectrum is divided into two branches . For one of them

$$\lambda_{\sigma\sigma}=0$$

and oscillations take place for pairs of particles having opposite spins. For the other branch $\lambda_{-+}=\lambda_{+-}=0$ and oscillations take place pairs having identical spins. Let us consider here the first branch and put :

$$\lambda_{-+}(p_1,p_2)=\lambda(p_1,p_2)$$

$$\xi_{-+}(p_1,p_2)=\xi(p_1,p_2)$$

$$\eta_{-+}(p_1,p_2)=\eta(p_1,p_2)$$

Then the system of Eq.(74) takes the form:

$$E\xi(p_1,p_2)=\left\{\Omega(p_1)+\Omega(p_2)\right\}\xi(p_1,p_2)+$$

$$+\sum_{p_1',p_2'}\frac{\delta(p_1+p_2-p_1'-p_2')}{V}\left\{X(p_1,p_2;p_1',p_2')\xi(p_1',p_2')+Y(p_1,p_2;p_1',p_2')\eta(-p_2',-p_1')\right\} \tag{78}$$

$$-E\eta(-p_2,-p_1)=\left\{\Omega(p_1)+\Omega(p_2)\right\}\eta(-p_2,-p_1)+$$

$$+\sum_{p_1',p_2'}\frac{\delta(p_1+p_2-p_1'-p_2')}{V}\left\{X(p_1,p_2;p_1',p_2')\eta(-p_2',-p_1')+Y(p_1,p_2;p_1',p_2')\eta(p_1',p_2')\right\}$$

where $\Omega(p)$ has the same expression as in 1 and where

$$X(p_1,p_2;p_1',p_2')=J(p_1,p_2;p_2',p_1')\left\{U(p_1)U(p_2)U(p_1')U(p_2')+V(p_1)V(p_2)V(p_1')V(p_2')\right\}+$$

$$+\mathcal{J}(p_1,-p_1',p_2',-p)\{\mathcal{U}(p_1)\mathcal{V}(p_2)\mathcal{V}(p_1')\mathcal{U}(p_2')+\mathcal{V}(p_1)\mathcal{U}(p_2)\mathcal{U}(p_1')\mathcal{V}(p_2')\}+$$

$$+[\mathcal{J}(-p_1,p_2';-p_1',p_2)-\mathcal{J}(p_2',-p_1;-p_1',p_2)]\times$$

$$\times\{\mathcal{V}(p_1)\mathcal{U}(p_2)\mathcal{V}(p_1')\mathcal{U}(p_2')+\mathcal{U}(p_1)\mathcal{V}(p_2)\mathcal{U}(p_1')\mathcal{V}(p_2')\} \tag{79a}$$

$$Y(p_1,p_2;p_1',p_2')=-\mathcal{J}(p_1,p_2;p_2',p_1')\{\mathcal{U}(p_1)\mathcal{U}(p_2)\mathcal{V}(p_1')\mathcal{V}(p_2')+\mathcal{V}(p_1)\mathcal{V}(p_2)\mathcal{U}(p_1')\mathcal{U}(p_2')\}+$$

$$+\mathcal{J}(p_1,-p_1';p_2',-p_2)\{\mathcal{V}(p_1)\mathcal{U}(p_2)\mathcal{V}(p_1')\mathcal{U}(p_2')+\mathcal{U}(p_1)\mathcal{V}(p_2)\mathcal{U}(p_1')\mathcal{V}(p_2')\}+$$

$$+[\mathcal{J}(-p_1,p_2';-p_1',p_2)-\mathcal{J}(p_2',-p_1;-p_1',p_2)]\times$$

$$\times\{\mathcal{U}(p_1)\mathcal{V}(p_2)\mathcal{V}(p_1')\mathcal{U}(p_2')+\mathcal{V}(p_1)\mathcal{U}(p_2)\mathcal{U}(p_1')\mathcal{V}(p_2')\} \tag{79b}$$

As one can see the obtained equations connect the functions

$$\xi(p_1,p_2),\quad \eta(-p_2;-p_1)$$

only with fixed $p_1+p_2$. Note also that the coefficients $X,Y$ are identical in both equations (78). Therefore we may put

$$p_1=p\quad,\quad p_2=-p+q$$

$$\xi(p_1,p_2)-\eta(-p_2,-p_1)=\theta_q(p)$$

$$\xi(p_1,p_2)+\eta(-p_2,-p_1)=\vartheta_q(p)$$

Transform the Eq.(78)

$$L_q(\theta)=E\vartheta;\quad \mathcal{M}_q(\vartheta)=E\theta \tag{80}$$

where

$$L_q(\theta)=\{\Omega(p)+\Omega(p-q)\}\theta(p)+\frac{1}{V}\sum_{p'}Q_p(p,p')\theta(p')$$

$$\mathcal{M}_q(\vartheta)=\{\Omega(p)+\Omega(p-q)\}\vartheta(p)+\frac{1}{V}\sum_{p'}R_q(p,p')\vartheta(p') \tag{81}$$

and where $Q_q(p,p')=$

$$=\mathcal{J}_q(p,p')\{\mathcal{U}(p)\mathcal{U}(p-q)+\mathcal{V}(p)\mathcal{V}(p-q)\}\{\mathcal{U}(p')\mathcal{U}(p'-q)+\mathcal{V}(p')\mathcal{V}(p'-q)\}+$$

$$+I_q(p,p')\{\mathcal{V}(p)\mathcal{U}(p-q)-\mathcal{U}(p)\mathcal{V}(p-q)\}\{\mathcal{V}(p')\mathcal{U}(p'-q)-\mathcal{U}(p')\mathcal{V}(p'-q)\}$$

$R_q(p,p')=$

$$=\mathcal{J}_q(p,p')[\mathcal{U}(p)\mathcal{U}(p-q)-\mathcal{V}(p)\mathcal{V}(p-q)][\mathcal{U}(p')\mathcal{U}(p'-q)-\mathcal{V}(p')\mathcal{V}(p'-q)]+$$

$$+G_q(p,p')[\mathcal{V}(p)\mathcal{U}(p-q)+\mathcal{U}(p)\mathcal{V}(p-q)][\mathcal{V}(p')\mathcal{U}(p'-q)+\mathcal{U}(p')\mathcal{V}(p'-q)]$$

where $$\mathcal{J}_q(p,p') = \mathcal{J}(p,-p+q;-p'+q,p')$$

$$I_q(p,p') = \mathcal{J}(p,p'-q;p',p-q) - \mathcal{J}(p,p'-q;p-q,p') - \mathcal{J}(p,-p';-p'+q,p-q)$$
$$G_q(p,p') = \mathcal{J}(p,p'-q;p',p-q) - \mathcal{J}(p,p'-q;p-q,p') + \mathcal{J}(p,-p';-p'+q,p-q) \tag{82}$$

Let us explain now physical sense of functions $\theta$ and $\vartheta$. Let us consider with this aim the expressions for the density of the number of particles $\rho(r)$ and density of the momentum $\vec{P}(r)$. We have

$$\rho(r) = \langle \sum_\sigma \overset{+}{\psi}_\sigma(r)\psi_\sigma(r)\rangle = \frac{1}{V}\sum_{p_1 p_2 \sigma} \langle \overset{+}{a}_{p_1\sigma} a_{p_2\sigma}\rangle e^{i(p_2-p_1)r} =$$
$$= \frac{1}{V}\sum_{p_1,p_2\sigma} F_{\sigma\sigma}(p_1,p_2) e^{i(p_2-p_1)r} \tag{83}$$

and

$$\vec{P}(r) = \frac{1}{2}\langle \sum_\sigma \{\overset{+}{\psi}_\sigma(r)(-i\frac{\partial}{\partial \vec{r}}\psi_\sigma(r) + i\frac{\partial \overset{+}{\psi}_\sigma(r)}{\partial \vec{r}}\psi_\sigma(r)\}\rangle =$$

$$= \frac{1}{V}\sum_{p_1,p_2\sigma} \langle \overset{+}{a}_{p_1\sigma} a_{p_2\sigma}\rangle(\bar{p}_1+\bar{p}_2) e^{i(p_2-p_1)r} = \frac{1}{V}\sum_{p_1 p_2 \sigma} F_{\sigma\sigma}(p_1,p_2)(\bar{p}_1+\bar{p}_2)e^{i(p_2-p_1)r}$$

Let us introduce the Fourier components for these densities

$$\rho(r) = \sum_q \rho_q e^{i(qr)} \qquad \bar{P}(r) = \sum_q \bar{P}_q e^{i(qr)}$$

and note, that our ground state is spatially homogeneous and non current-carrying. According to (82) we have

$$\rho_0 = \frac{2}{V}\sum_p v_p^2, \qquad \rho_q = \frac{1}{V}\sum_{p_2-p_1=q}\{\delta F_{++}(p_1,p_2) - \delta F_{--}(p_1,p_2)\}; \quad q \neq 0$$
$$\bar{P}_q = \frac{1}{2V}\sum_{p_2-p_1=q}(\bar{p}_1+\bar{p}_2)\{\delta F_{++}(p_1,p_2) + \delta F_{--}(p_1,p_2)\} \tag{84}$$

On the other hand expanding the formulae (64) we obtain:

$$\delta F_{--}(p_1,p_2) = v(p_1)u(p_2)\lambda(p_2,-p_1) + u(p_1)v(p_2)\lambda^*(p_1,-p_2)$$
$$\delta F_{++}(p_1,p_2) = v(p_1)u(p_2)\lambda(-p_1,p_2) + u(p_1)v(p_2)\lambda^*(-p_2,p_1) \tag{85}$$

Substituting these expressions into (84) we get

$$\rho_q = \frac{1}{V}\sum_{p_1+p_2=q}\{v(p_2)u(p_1) + v(p_1)u(p_2)\}\{\lambda(p_1,p_2) + \lambda^*(-p_2,-p_1)\}$$
$$\vec{P}_q = \frac{1}{2V}\sum_{p_1+p_2=q}(\bar{p}_1-\bar{p}_2)\{v(p_2)u(p_1) - v(p_1)u(p_2)\}\{\lambda(p_1,p_2) - \lambda^*(-p_2,-p_1)\} \tag{86}$$

From which according to (73)

$$\rho_q = \sum_E \rho_q^{(E)} e^{iEt}, \quad \vec{P}_q = \sum_E \vec{P}_q^{(E)} e^{-iEt}$$

$$\rho_q^{(E)} = \frac{1}{V} \sum \{ \mathcal{U}(p) \mathcal{V}(p-q) + \mathcal{V}(p) \mathcal{U}(p-q) \} \vartheta_q(p) \tag{87}$$

$$\vec{P}_q^{(E)} = \frac{1}{2V} \sum_p (2\bar{p} - \bar{q}) \{ \mathcal{U}(p) \mathcal{V}(p-q) - \mathcal{V}(p) \mathcal{U}(p-q) \} \theta_q(p)$$

Thus, the contribution to the variations of the density of number of particles due to the elementary excitation is determined by the function $\vartheta$ and the corresponding contribution in variations of the density of momentum by the function $\theta$. Let us turn now to the equations (81) putting

$$\theta(p) = S_1 \delta(p - p_0)$$
$$\vartheta(p) = S_2 \delta(p - p_0) \tag{88}$$

where $S_1$ and $S_2$ are constant and $p_0$ is arbitrary fixed momentum. Omitting the terms of the order $V^{-1}$ vanishing after limit transition $V \to \infty$ and giving rise only to local changes in the wave function we see that (88) will be an admissible solution system if $S_1$ and $S_2$ are connected by the relation

$$S_1 \{ \Omega(p_0) + \Omega(p_0 - q) \} = E S_2$$
$$S_2 \{ \Omega(p_0) + \Omega(p_0 - q) \} = E S_1 \tag{89}$$

it follows from this

$$E^2 = \{ \Omega(p_0) + \Omega(p_0 - q) \}^2$$

Thus, we make sure of the presence of a continuous spectrum x)

$$E = \Omega(p_0) + \Omega(p_0 - q) \tag{90}$$

spaced by the gap. With the given $q$ the energy $E$ depends continuously on the momentum $p_0$. Let us write down an asymptotic part of the wave function for the elementary excitation of this type expanding the formula (65). We find

$$\delta\Phi_{-+}(p_1, p_2) = \mathcal{U}(p_1) \mathcal{U}(p_2) \lambda(p_1, p_2) - \mathcal{V}(p_1) \mathcal{V}(p_2) \lambda^*(-p_2, -p_1)$$

and hence in the considered case for

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x) We choose the positive sign according to the general normalization condition (76) which in our case is

$$\sum_p \theta(p) \vartheta(p) > 0 \tag{76}$$

Substituting into this condition the solution (88) we see that $S_1$ and $S_2$ must have the same signs. Therefore, the equation (89) leads to the positive sign of $E$.

$$\delta\Phi_{-+}(p_1,p_2)=\delta(p_1-p_0)\delta(p_2+p_0-q)S\exp\{-i(\Omega(p_0)+\Omega(p_0-q)t\}$$

where

$$S=U(p_0)U(p_0-q)\frac{S_1+S_2}{2}+V(p_0)V(p_0-q)\frac{S_1-S_2}{2}$$

We have consequently in $r$ - representation

$$\delta\Phi_{-+}(r_1,r_2)|_{|r_1-r_2|\to\infty}\simeq$$

$$\simeq const\ \exp\{-i[\Omega(p_0)+\Omega(p_0-q)]t+i[p_0r_1+(q-p_0)r_2]\}$$

Compare this expression with the wave function of the pair ( - , + ) in the ground state

$$\Phi^0_{-+}(r_1,r_2)=const\int e^{ip(r_1-r_2)}U(p)V(p)dp$$

It is clear that $\Phi^0_{-+}$ corresponds to the bound state of the pair of particles in particular for $|r_1-r_2|\to\infty$ this function tends to zero. The expression $\delta\Phi_{-+}$ factorize into the product of the two plane waves and corresponds to independent motion of the two particles with momenta $p_0, q-p_0$ .

Thus, one may interpret physically the elementary excitations from the continuous spectrum as those performing dissociation of the quasi-molecule into separate particles. Let us turn now to studying the spectrum of collective oscillations which is determined by means of the equations (81) corresponding to the discrete values of $E$ (for fixed $q$ ).

At first let us consider the case when the particles are not charged . In this case owing to absence of the Coulomb interaction we consider all the kernels $I, J, G$ as finite. Let us make a number of remarks. It follows from the expression (35)

$$L_0(\theta)=0 \quad for \quad \theta=U(p)V(p).$$

Therefore an inhomogeneous equation

$$L_0(\theta)=f(p)$$

may be solved only if

$$\sum_p f(p)U(p)V(p)=0 \tag{91}$$

Now we see that the system of the equations (81) for $q$ = 0 has the following solution

$$\theta=U(p)V(p)\ ,\quad \vartheta=0\ ,\ E=0 \tag{92}$$

and therefore we shall try to solve it for small $|q|$ by means of expanding in powers $|q|$

$$\theta = \mathcal{U}(p)\mathcal{V}(p) + |q|\theta_1(p,e) + |q|^2\theta_2(p,e) + \dots$$

$$\vartheta = |q|\vartheta_1(p,e) + \dots \tag{93}$$

$$E = |q|E_1 + \dots$$

where

$$\vec{e} = \vec{q}/|\vec{q}|$$

Substituting them into the Eq. (81) we obtain

$$L_0(\theta_1) = -\sum_{1\le\alpha\le3} e_\alpha \left\{ \frac{\partial L_q(\mathcal{U}\mathcal{V})}{\partial q_\alpha} \right\}_{q=0} \tag{94}$$

$$\mathcal{M}_0(\vartheta_1) = E_1 \mathcal{U}(p)\mathcal{V}(p) \tag{95}$$

$$L_0(\vartheta_2) = E_1\vartheta_1 - \sum_\alpha e_\alpha \left\{ \frac{\partial L_q(\theta_1)}{\partial q^\alpha} \right\}_{q=0} - \frac{1}{2}\sum_{\alpha,\beta} e_\alpha e_\beta \left\{ \frac{\partial^2 L_q(\mathcal{U}\mathcal{V})}{\partial q^\alpha \partial q^\beta} \right\}_{q=0} \tag{96}$$

One may resolve the Eq.(94) since the function $f(p)$ in its right part has a property:

$$f(-p) = -f(p)$$

due to which the condition (91) is carried out trivially. In order to resolve the equation (96), we must require according to (91) that

$$E_1 \sum_p \vartheta_1(p,e)\mathcal{U}(p)\mathcal{V}(p) =$$

$$= \sum_p \mathcal{U}(p)\mathcal{V}(p) \left\{ \sum_\alpha e_\alpha \left\{ \frac{\partial L_q(\theta_1)}{\partial q^\alpha} \right\}_{q=0} + \frac{1}{2}\sum_{\alpha,\beta} e_\alpha e_\beta \left\{ \frac{\partial^2 L_q(\mathcal{U}\mathcal{V})}{\partial q^\alpha \partial q^\beta} \right\}_{q=0} \right\} \tag{97}$$

From the Eq. (95) we see that $\vartheta_1$ is proportional to $E_1$. Therefore the condition (97) makes it possible to determine $E_1^2$ and so on. After making some calculations in the case of spherical symmetry for small $|q|$ we have

$$E = \frac{|q|\,S}{\sqrt{3}}$$

where without account of interaction corrections $S$ is equal to the velocity of the particles on the Fermi surface.

Thus, we obtain collective oscillations of the quasi- acoustic character. The region of their existence is limited by the momenta $q$ for which the corresponding $E$ is below the threshold of excitation of the continuous spectrum.

Let us see now what will occur with the oscillations of the dinamic system of electrons considered in the theory of superconductivity. Let us note that the presence of the Coulomb interaction leads to an essential singularity of the kernel $G_q$:

$$G_q = \frac{8\pi e^2}{|q|^2} + G'_q$$

Therefore it is advisible now to represent the operator $\mathcal{M}_q$ in the form

$$\mathcal{M}_q(\vartheta) = \mathcal{M}'_q |\vartheta| + \frac{8\pi e^2}{|q|^2}\left\{V(p)U(p-q)+U(p)V(p-q)\right\}\frac{1}{V}\sum_{p'}\vartheta(p')\left\{V(p')U(p'-q)+U(p')V(p'-q)\right\} \tag{98}$$

choosing obviously a part with singularity [x] with $q = 0$. In order to make regularization of the equations (81) let us introduce a new unknown value $\psi$, putting

$$\frac{1}{V}\sum_{p'}\vartheta(p')\left\{V(p')U(p'-q)+U(p')V(p'-q)\right\} = \frac{|q|^2}{16\pi e^2}\psi$$

Then, our system of equations may be written down in the forms

$$\begin{gathered} L_q(\theta) = E\vartheta \\ \mathcal{M}'_q(\vartheta) + \left\{V(p)U(p-q)+U(p)V(p-q)\right\}\frac{\psi}{2} = E\theta \\ \frac{1}{V}\sum_{p'}\vartheta(p')\left\{V(p')U(p'-q)+U(p')V(p'-q)\right\} = \frac{|q|^2}{16\pi e^2}\psi \end{gathered} \tag{99}$$

It has solution for $q = 0$ and when $E$ is arbitrary

$$\theta = U(p)V(p), \quad \vartheta = 0, \quad \psi = E$$

Therefore we try to solve it for small

$$\begin{gathered} \theta = U(p)V(p) + |q|\theta_1(p,e) + |q|^2\theta_2(p,e) + \dots \\ \vartheta = |q|\vartheta_1(p,e) + \dots; \quad \psi = E_0 + |q|\psi_1 + \dots \\ E = E_0 + |q|E_1 + \dots \end{gathered} \tag{100}$$

---

[x] It should not consider that with more exact interpretation we would obtain the screening effect in the multiplier $8\pi e^2/|q|^2$ and by means of this eliminate the singularity. The reason is that we deal here with variations of the density of the electrical charge, it is evident even if from the fact that the oscillation amplitude enter just the Eq. (98) (see (87)). When investigating the inhomogenety in distribution of the charge one takes into account the long-range Coulomb forces and hence the singularity for $q = 0$ must always take place in $p$ - representation.

Substituting them into (99) we obtain

$$L_o(\theta_1)=E_o\vartheta_1-\sum_\alpha e_\alpha\left\{\frac{\partial L_q(UV)}{\partial q_\alpha}\right\}_{q=0} \tag{101}$$

$$\mathcal{M}_o'(\vartheta_1)=U(p)V(p)(E_1-\Psi_1)+E_o\theta_1+\frac{E_o}{2}\sum_\alpha e_\alpha\left(U(p)\frac{\partial V(p)}{\partial p_\alpha}+V(p)\frac{\partial U(p)}{\partial p_\alpha}\right) \tag{102}$$

$$\frac{1}{V}\sum_{p'}\vartheta_1(p')U(p')V(p')=0 \tag{103}$$

$$L_o(\theta_2)=E_o\vartheta_2+E_1\vartheta_1-\sum_\alpha e_\alpha\left\{\frac{\partial L_q(\theta_1)}{\partial q_\alpha}\right\}_{q=0}-\frac{1}{2}\sum_{\alpha,\beta}e_\alpha e_\beta\left\{\frac{\partial^2 L_q(UV)}{\partial q_\alpha\partial q_\beta}\right\}_{q=0} \tag{104}$$

$$\frac{2}{V}\sum_p\vartheta_2(p)U(p)V(p)=\frac{E_o}{16\pi e^2}+\frac{1}{V}\sum_{\alpha,p}\vartheta_1(p)e_\alpha\left\{U(p)\frac{\partial V(p)}{\partial p_\alpha}+V(p)\frac{\partial U(p)}{\partial p_\alpha}\right\} \tag{105}$$

We take $E_1-\Psi_1=0$ in the Eq. (102). Then due to (100) and (102) one can note that $\theta_1$ and $\vartheta_1$ are asymmetrical when changing the sign $p$ and hence the condition (103) will be satisfied automaticly. In order to resolve (104) we write down our usual condition

$$E_o\sum\vartheta_2(p)U(p)V(p)=\sum U(p)V(p)\left\{\sum_\alpha e_\alpha\left(\frac{\partial L_q(\theta_1)}{\partial q^\alpha}\right)_{q=0}+\frac{1}{2}\sum_{\alpha,\beta}e_\alpha e_\beta\left[\frac{\partial^2 L_q(UV)}{\partial q_\alpha\partial q_\beta}\right]_{q=0}\right\} \tag{106}$$

The left part of this equation is

$$\frac{VE_o^2}{32\pi e^2}+\sum_{\alpha,\beta}\vartheta_1(p)e_\alpha\left\{U(p)\frac{\partial V(p)}{\gamma p_\alpha}+V(p)\frac{\partial U(p)}{\partial p_\alpha}\right\} \tag{107}$$

according to (105). Now we see that the Eq. (106) determining $E_o$ has no root equal to zero. Indeed it follows from (101) and (102) that the part of (106) which is represented by the expression (107) turns into zero for $E_o=0$. The right part of (106) for $E_o=0$ coincides with that of (97) and hence is not equal to zero.

Let us caloulate now $E_o$ for spherical symmetric case. We take

$$E(p)=\frac{p^2}{2m}$$

and suppose that

$$J(p_1,p_2;p_2',p_1')=J(p_1-p_1')\qquad p_1+p_2=p_1'+p_2' \tag{108}$$

Then one may oheck the identity

$$L_q(f_q)=\left\{\frac{(p-q)^2-p^2}{2m}\right\}\left(V(p)U(p-q)-U(p)V(p-q)\right) \tag{109}$$

in whioh

$$f_q(p)=U(p)V(p-q)+U(p-q)V(p) \tag{110}$$

Let us note that the case (108) is realized if the interaction does not depend on velocities and is determined by the potential $V(r_1-r_2)$ . In this case

$$J(p)=-J(-p)=\int V(r)\,e^{i(rp)}dr$$

In the theory of superconductivity it is necessary to take into account Fröhlich interaction due to the exchange of photons. For the Coulomb forces the condition (108) is fulfiled of course Fröhlich. Such interaction is effective only in narrow layer near the Fermi surface and in this region its contribution to $J$ is

$$J_{ph}(p_1,p_2;p_2',p_1')=-g^2(p_1-p_1') \qquad p_1+p_2=p_1'+p_2' \tag{111}$$

where $g(q)$ is the value characterizing the connection between electrons and phonons. Therefore we may use the relation (109). Strictly speaking in order this relation will take place exactly it is necessary to deform the expression (110). We should observe then deviations of the order $c/\omega$ , where $\omega$ is the mean energy of phonon, i.e. deviations of the order of the value of retarded effects of the electron-phonon interaction.

Owing to this sircumstance it is not advisable to make such more precise consideration in the model in which the electron - phonon interaction is substituted into direct interaction of electrons since this substitution itself is availble only with the accuracy of neglect of the retarded effect. Let us use now the relations (109), (110) for determing the value $E_0$ . As the operator $L_q$ is Hermitian we have

$$\sum_p\{L_q(\theta)\chi_q-L_q(\chi_q)\theta\}=0 \tag{112}$$

from where

$$E\frac{1}{V}\sum_p\vartheta(p)\{u(p)v(p-q)+v(p)u(p-q)\}= \tag{113}$$

$$=\frac{1}{V}\sum_p\theta(p)\left\{\frac{(p-q)^2}{2m}-\frac{p^2}{2m}\right\}(v(p)u(p-q)-u(p)v(p-q))$$

Let us calculate this equation with the accuracy of the value of the order $|q|^2$ __inclusive__. From (100) we see that

$$\theta(p)=u(p)v(p)+|q|\theta_1+\dots$$

From (99) and (100) we have

$$\frac{1}{V}\sum_p\vartheta(p)\{u(p)v(p-q)+v(p)u(p-q)\}=\frac{|q|^2}{16\pi e^2}\{E_0+|q|\psi_1+\dots\}$$

Therefore from (113) we obtain

$$E_0^2 = \frac{16\pi e^2}{V}\sum_p u(p)v(p)\frac{(\bar{p}\bar{e})}{m}\left\{v(p)\left(\bar{e}\,\frac{\partial u(p)}{\partial \bar{p}}\right) - u(p)\left(\bar{e}\frac{\partial v(p)}{\partial \bar{p}}\right)\right\} \qquad (114)$$

where $\bar{e} = \bar{q}/|q|$ . Substituting the expressions $u, v$ from (36) into (114) we find

$$E_0 = \sqrt{\frac{4e^2}{3\pi}\,\frac{p_F^3}{m}}\,, \qquad (115)$$

$p_F$ is the Fermi momentum.

As one can see we obtained here the energy value for known plasma oscillations. The specificity of the superconducting state is absent completely[x)]. Since $E_0$ is greater considerably then the energy of the continuous spectrum(for small $q$ ) the obtained stationary solution will be in more exact interpretation only quasi-stationary.

Let us note however one interest fact. In spite of the obtained result in the system of equations (81) one may consider $E = 0$ as approximate eigenvalue.

Indeed taking into account (109) it is not difficult to observe that using

$$\theta_q(p) = \chi_q(p)\,, \qquad \vartheta_q(p) = 0\,, \quad E = 0$$

we satisfy the system (81) with the accuracy of the quantities of the order $|q|^2$ . We shall observe later that this fact is essential to ensure the gauge invariance of the theory. As the plasma oscillations with their great value of $E$ are not specify for the superconducting state then the following question may arise : are there the collective oscillations typical for such a state.

As we see now they are among the oscillations which do not change the density of the electrical charge distribution. In other words we must find solutions of the system (81) in which the expression

$$\frac{1}{V}\sum_p \vartheta(p)\{u(p)v(p-q) + v(p)u(p-q)\}$$

vanishes. This expression leads to the appearance of the singularity for $q = 0$ (see Eq. (98)). Let us consider a spherical symmetric case. Let us set the axis in direction of the vector $\vec{q}$ and introduce cylindrical coordinates. Let this solution have a form:

$$\theta_q(p) = e^{in\varphi}\,\theta(p^2, p_z)$$
$$\vartheta_q(p) = e^{in\varphi}\,\vartheta(p^2, p_z) \qquad n \neq 0$$

These solutions exist formally and for them the mentioned expression is equal to identically ze-

x) This result has been obtained earlier by Anderson [6] . An idea about the importance of the the superconducting state was not confirmed.(see 7 paper [1]).

ro. The question is if the corresponding values of $E$ will be below the threshold of excitation of the continuous spectrum.

We should have analysed also the oscillations of the spectrum branch which is not considered here and for which:

$$\lambda_{-+} = \lambda_{+-} = 0$$

## 5. The problems of the electrodynamics of superconducting state.

Let us consider here the problem of change of the ground superconducting state due to the effect of the external constant field $\mathcal{A}(r)$. In order to operate in a linear approximation let us consider $\mathcal{A}$ as infinitesimal value of the first order and use the general equations (77). Then not taking into account the presence of the paramagnetic term x) we obtain :

$$\lambda_{-+}(p_1, p_2) = \frac{1}{2}\theta_q(p); \qquad \lambda^*_{-+}(-p_2, -p_1) = -\frac{1}{2}\theta_q(p) \tag{116}$$

and

$$L_q(\theta_q) = -\frac{e}{m}(2\vec{p} - \vec{q})\vec{\mathcal{A}}(q)\{v(p)u(p-q) - u(p)v(p-q)\} \tag{117}$$

Now we investigate properties of this equation. Let us take

$$e\vec{\mathcal{A}}(q) = i\vec{q}\varphi(q) \tag{118}$$

Then in $r$ - representation with the presence of the gauge invariance we have

$$F(r_1, r_2) = e^{i(\varphi(r_2) - \varphi(r_1))} F_0(r_1, r_2)$$

or since in our case $\varphi$ is infinitesimal

$$\delta F(r_1, r_2) = i[\varphi(r_2) - \varphi(r_1)] F_0(r_1, r_2)$$

Transforming to the $p$ -representation and using (85) we obtain

$$\lambda(p_1, p_2) = i\varphi(p_1 + p_2)\{u_{p_1}v_{p_2} + v_{p_1}u_{p_2}\}$$

and

$$\theta_q(p) = 2i\varphi(q)\{u(p)v(p-q) + v(p)u(p-q)\} = 2i\varphi(q)\chi_q(p)$$

On the other hand the expression $\theta_q(p)$ obtained must satisfy Eq. (117) in the case of (118) and therefore

$$2i\varphi(q)L_q\{\chi_q\} = \frac{1}{im}\{(2\bar{p} - \bar{q}\}\varphi(q)[u(p)u(p-q) - u(p)v(p-q)]$$

x) In a linear approximation we may consider this effect independently.

But this is the relation (109).

Thus, the property of the gauge invariance takes place with the same degree of accuracy as the relation (109) i.e. with the accuracy of the retarded effects of electron-phonon interaction.

Let us represent ourselves the situation which will take place if we shall act by the following means. Let us consider firstly the Hamiltonian of the system without the external field and perform the canonical transformation

$$a_{K+} = U_K \alpha_{K0} + V_K \overset{+}{\alpha}_{K1}$$

$$a_{-K,-} = U_K \alpha_{K1} - V_K \overset{+}{\alpha}_{K0}$$

and determine $U, V$ from the condition of compensation of the dangerous graphs with the momenta $K, -K$.

Let us insert the small external field into Hamiltonian, transform the expression to the amplitudes $\alpha, \overset{+}{\alpha}$ after this let us apply the usual perturbation theory not taking care of the compensation of new dangereous graphs (arising due to the external field) with momenta $\sqrt{^{K,-K+q}}$.
Then instead of (117) we should obtain

$$\{\Omega(p) + \Omega(p-q)\}\,\theta_q(p) =$$

$$= -\frac{e}{m}(2\bar{p} - \bar{q})\bar{A}(q)\{V(p)U(p-q) - U(p)V(p-q)\}$$

from where

$$\theta_q(p) = \frac{-\frac{e}{m}(2\bar{p}-\bar{q})\bar{A}(\bar{q})}{\Omega(p)+\Omega(p-q)}\{V(p)U(p-q) - U(p)V(p-q)\} \qquad (119)$$

This result will not obviously gauge invariant already in any reasonable approximation. Substituting $L_q(\theta)$ into

$$\{\Omega(p) + \Omega(p-q)\}\,\theta$$

we distroyed by this way the property of this operator namely that zero is its eigenvalue for $q = 0$. Let us proceed now to studying the dependence of the current density on the vector-potential. We have according to (84)

$$m\vec{j}_q = e\vec{p}_q - e^2\vec{A}(q)\frac{2}{V}\sum V_p^2$$

and hence due to (87):

$$n\vec{j}_q = \frac{1}{V}\sum_p e(\bar{p}-\frac{\bar{q}}{2})\theta_q(p)[U(p)V(p-q)-V(p)U(p-q)] - e^2\bar{A}(q)\frac{2}{V}\sum_p V_p^2 \quad . \tag{120}$$

Let us denote through $T_\alpha(p,q)$ the solution of the equation

$$L_q(T_\alpha)_{\alpha=1,2,3} = -\frac{2p_\alpha - q_\alpha}{m}\left[V(p)U(p-q) - U(p)V(p-q)\right] \tag{121}$$

Then according to (117) and (120) we obtain

$$\theta_q(p) = e\sum_\alpha T_\alpha(p,q) A_\alpha(q)$$

and

$$\vec{j}_q = \frac{e^2\rho_0}{m}\sum_\beta\left\{S_{\alpha\beta}(q) - \delta(\alpha-\beta)\right\} A_\beta(q) \tag{122}$$

where

$$\rho_0 = \frac{2}{V}\sum_p V_p^2$$

$$S_{\alpha\beta}(q) = \frac{1}{V}\sum_p \frac{(2p_\alpha - q_\alpha)}{2\rho_0}\left[U(p)V(p-q) - V(p)U(p-q)\right] T_\beta(p,q) \tag{123}$$

In view of (121) we can write down also

$$S_{\alpha\beta}(q) = \frac{m}{V\rho_0}\sum_p L_q(T_\alpha)T_\beta \quad .$$

We make sure that $S_{\alpha\beta}$ is symmetrical

$$S_{\alpha\beta}(q) = S_{\beta\alpha}(q) \tag{124}$$

From (123) we have also

$$\sum_\alpha q_\alpha S_{\alpha\beta}(q) = \frac{m}{V\rho_0}\sum_p L_q(\chi_q)T_\beta = \frac{m}{V\rho_0}\sum_p \chi_q L_q(T_\beta) =$$

$$= \frac{1}{V\rho_0}\sum_p (2p_\beta - q_\beta)[U(p)V(p-q) - V(p)U(p-q)][U(p)V(p-q) + V(p)U(p-q)] =$$

$$= \frac{1}{V\rho_0}\sum_p (2p_\beta - q_\beta)\left\{U^2(p)V^2(p-q) - V^2(p)U^2(p-q)\right\} =$$

$$= \frac{1}{V\rho_0}\sum_p (2p_\beta - q_\beta)\left\{V^2(p-q) - V^2(p)\right\} =$$

$$= \frac{1}{V\rho_0}\sum_p (2p_\beta + q_\beta)V^2(p) - \frac{1}{V\rho_0}\sum_p (2p_\beta - q_\beta)V^2(p) \quad .$$

We obtain by this way the Buckingham relations [7]

$$\sum_{\alpha} q_\alpha S_{\alpha\beta}(q) = q_\beta \qquad (125)$$

Owing to these relations and (122) we make sure that the conservation low is fulfield

$$\vec{q}\,\vec{J}_q = 0$$

We see also that $\vec{J}_q$ depends only on the transverse part of the veotor-potential $\mathcal{A}$ :

$$\vec{J}_q = \frac{e^2\rho_0}{m}\sum_{\beta}\{S_{\alpha\beta}(q) - \delta(\alpha-\beta)\}\mathcal{U}_\beta(q)$$

$$\mathcal{U}_\alpha(q) = \mathcal{A}_\alpha(q) - \frac{(\vec{q}\,\vec{\mathcal{A}}(q))}{q^2}$$

Let us investigate now the dependence $\vec{J}_q$ on $\vec{\mathcal{U}}(q)$ for small $q$

As now

$$\vec{q}\,\vec{\mathcal{U}}(q) = 0$$

than the equation (117) may be written down in the form:

$$L_q(\Theta_q) = \frac{2e}{m}(\vec{p}_\perp\vec{\mathcal{U}})[\mathcal{U}(p)\mathcal{V}(p-q) - \mathcal{V}(p)\mathcal{U}(p-q)]$$

where $\vec{p}_\perp$ is componenta of $\bar{p}$ perpendicular to the vector $\bar{q}$ . After establishing in the space of the momenta the axis $z$ in the direction of $\vec{\mathcal{U}}_q$ and the axis $x$ in the direction $\vec{q}$ we obtain then

$$\Theta_q(p) = e\,\mathcal{U}(q)\,\tau(p,q) \qquad (126)$$

where

$$L_q(\tau) = f(p,q) = \frac{2}{m}p_z[\mathcal{U}(p)\mathcal{V}(p-q) - \mathcal{V}(p)\mathcal{U}(p-q)] \qquad (127)$$

As we see here $f(p,q)$ is the asymmetrio function $p_z$ .

$$f(p_x,p_y,-p_z;q) + f(p_x,p_y,p_z;q) = 0 \qquad (128)$$

Such a function will be orthogonal to $\mathcal{U}(p)\mathcal{V}(p)$ . Hence we may always x) try to solve the Eq. (127) in the form:

$$\tau(p,q) = q\tau_1(p) + q^2\tau_2(p)$$

where $\tau_1, \tau_2 \ldots$ are antisymmetrical functions of $p_z$ in sense of (128). On the other hand substituting (126) into (120) we obtain

$$\vec{I}_q = \frac{e^2\rho_0}{m}\{\vec{S}(q) - \vec{e}_z\}\mathcal{U}(q)$$

where $\vec{e}_z$ is unit vector in the direotion of the axis $z$ and

x) From the mathematical point of view the following case is possible : when the Eq. $L_0(\Theta) = 0$ besides the symmetrical solution $\Theta = \mathcal{U}(p)\mathcal{V}(p)$ has some more other solution antisymmetrical with regard to $p_z$ . Physically however the consideration of suoh a case has no reasons and we shall not take into account it.

$$\vec{S}(q)=\sum \frac{2\vec{p}-\vec{q}}{2p_0}\tau(p,q)[\mathcal{U}(p)\mathcal{V}(p-q)-\mathcal{V}(p)\mathcal{U}(p-q)]$$

But for $q \to 0$ the function $\tau$ is infinitesimal of the first order and hence $\vec{S}(q)$ will vanish as $q^2$. So, for sufficiently small $q$:

$$\vec{J}_q=-\frac{e^2 p_0}{m}\vec{\mathcal{A}}(q) \tag{129}$$

and we have Meissner effect [9], [10].

As we seen when considering the effect of vector-potential the operator $L_q(\theta)$ only was found essential. If we shall wished to consider the effect of the external scalar potential $V$, then we should obtain in a linear approximation the equation

$$\mathcal{M}_q(\vartheta)=-2eV(q)[\mathcal{U}(p)\mathcal{V}(p-q)+\mathcal{V}(p)\mathcal{U}(p-q)]$$

with the operator $\mathcal{M}_q$. Since this operator contains a singular turn due to the deformation of the charge density it is not difficult to make sure that the specifity of the superconducting state vanishes here and the screening effect will perform in the same way as in the normal state.

Let us note at last that if we shall investigate the effect of the term proportional to $\vec{H}\times\vec{\sigma}$ then we shall get a new operator which enters the equations of oscillations for the branch of spectrum where $\lambda_{-+} = 0$.

In conclusion I consider it my pleasant duty to thank to Prof. G.Wentzel and Prof. M.R.Shafroth for their useful discussion of questions on-electrodynamics of the superconducting state (Geneva, July 1958) which drew interest to the problems considered here.

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