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APPLICATION OF FOURIER METHOD TO THE SOLUTION OF INVERSE PROBLEM IN SCATTERING THEORY

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Introduction

As it is known the problem of reconstructing the potential function of Schrödinger equation using the spectral properties of its solutions is called an inverse problem of the scattering theory. Strict mathematical solution of this problem for the case of central symmetrical field was given in the well-known papers by I.M. Gelfand and B.M. Levitan |1,2|, M.G. Krein |3|, V.A. Marchenko |4| et al.

However, all these papers are based on the fact that the used spectral characteristic (spectral function or S-function) is known for all values of the parameter k, of the Schrödinger equation. Physically speaking it means that it is necessary to know the results of scattering experiments for all energies.

In the real physical situation we may know it only for a finite energy interval. At rather small energies Shrödinger equation becomes nonapplicable for the description of a physical phenomenon. This does not even allow to hope that this interval may be enlarged arbitrarily. Therefore, in order to apply the results of papers |1-4| it is necessary to extrapolate in some way the used spectral characteristic beyond the limits of the known energy interval. However, the known attempts to extrapolate analytically the limiting phase (or the S-function) are extremely unsatisfactory, since arbitrary small perturbations of the limiting phase (or S-function) at sufficiently high energies provide arbitrary large perturbations of the potential function. This fact makes us search for another way to solve the problem.

In the present paper based upon the application of the Fourier method is given a means for the reconstruction of the low frequency harmonics of the potential and wave functions by the limiting phase of the S-scattering known for the finite energy interval. The stability of this process is shown.

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I. The Problem

As is well-known the radial component of the wave function of the S-wave in the central symmetrical field satisfies the equation:

$$-\frac{\hbar^2}{2m}\frac{d^2R(r,E)}{dr^2} + U(r)R(r,E) = E\,R(r,E)$$

with the boundary condition in zero: $R(0,E) = 0$.

Substituting:

$r = x\hbar\left(\frac{\alpha}{2m}\right)^{1/2}$, $U(r)=\alpha^{-1}V(x)$, $E=\alpha^{-1}\kappa^2$, $R(r,E)=\varphi(x,\kappa)$ ($\alpha>0$ and arbitrary), and assuming that $R(r,E)$ are normalized so that $\varphi'(0,\kappa)=\kappa$. Then $\varphi(x,\kappa)$ satisfy the equation:

$$-\varphi''(x,\kappa)+V(x)\varphi(x,\kappa)=\kappa^2\varphi(x,\kappa) \tag{1}$$

with the initial conditions:

$$\varphi(0,\kappa)=0, \quad \varphi'(0,\kappa)=\kappa. \tag{1'}$$

As is shown in [2] $\varphi(x,\kappa)$ defined in this way permit the representation

$$\varphi(x,\kappa)=\sin\kappa x+\int_0^x K(x,t)\sin\kappa t\,dt, \tag{2}$$

where $K(x,t)$ with $0\leq t\leq x$ satisfies the equation

$$\frac{\partial^2 K(x,t)}{\partial x^2}-\frac{\partial^2 K(x,t)}{\partial t^2}=V(x)\,K(x,t) \tag{3}$$

with the conditions:

$$K(x,0)=0, \tag{4}$$

$$\frac{d}{dx}K(x,x)=\frac{1}{2}V(x). \tag{4'}$$

Let $V(x)$ be tending to zero sufficiently quickly at $x\to\infty$. Then for great x $\varphi(x,\kappa)$ have the asymptotic form:

$$\varphi(x,\kappa)\sim A(\kappa)\sin(\kappa x+\delta(\kappa)), \tag{5}$$

where $\delta(\kappa)$ are the so-called limiting phases, (A(k) may be calculated using $\delta(\kappa)$). Our purpose is to find $V(x)$ and $\varphi(x,\kappa)$ using $\delta(\kappa)$. Substituting (4') into (3) one obtains the equation

$$\frac{\partial^2 K(x,t)}{\partial x^2}-\frac{\partial^2 K(x,t)}{\partial t^2}=2K(x,t)\frac{d}{dx}K(x,x) \tag{6}$$

with the condition $K(x,0)=0$ which we use for the odd extension of $K(x,t)$ to the negative t $(-x\leq t\leq 0)$. Thus, we had to find a solution for the equation (6) such,that $\varphi(x,\kappa)$ defined by (2) would have the asymptotic form (5). Having found such $K(x,t)$ we find also $V(x)$ using the relation (4'). It is the solution of the problem we shall be concerned with. Everywhere further we shall consider V(x) to be continuous if the opposite is not especially mentioned.

2. The Case of a Finite Potential

Let $V(x) \equiv 0$ if $x \geqslant x_0$. Then according to (3) $K(x,t)$ satisfies (if $x \geqslant x_0$) the equation

$$\frac{\partial^2 K(x,t)}{\partial x^2} - \frac{\partial^2 K(x,t)}{\partial t^2} = 0 \tag{7}$$

and, therefore, if $x \geqslant x_0$

$$K(x,t) = f_1(x-t) + f_2(x+t)$$

It follows from condition (4) that $f_2(x) = -f_1(x)$ i.e.,

$$K(x,t) = f(x-t) - f(x+t), \tag{8}$$

where $f(x) = f_1(x)$. It follows from (4') that by $\xi \geqslant 2x_0$ (i.e. $x+t \geqslant 2x_0$ or $x-t \geqslant 2x_0$) $f'(\xi) \equiv 0$. Making use of the latter remark we obtain by $x+t \geqslant 2x_0$

$$\frac{\partial K(x,t)}{\partial x} = -\frac{\partial K(x,t)}{\partial t}, \tag{9}$$

and by $x-t \geqslant 2x_0$

$$\frac{\partial K(x,t)}{\partial x} = \frac{\partial K(x,t)}{\partial t}. \tag{9'}$$

Thus, having determined $K(2x_0,t)$ we can easily obtain that

$$K(x_0,t) = K(2x_0, x_0+t) - K(2x_0, x_0-t), \tag{10}$$

$$\left.\frac{\partial K(x,t)}{\partial x}\right|_{x=x_0} = -\frac{d}{dt}\left[K(2x_0, x_0+t) + K(2x_0, x_0-t)\right], \tag{10'}$$

and our original problem reduces to the solution of Cauchy problem for Eq.(6) in a triangle limited by the straight lines $x = t$, $x = -t$, and $x = x_0$.

3. Determination of $K(2x_0,t)$ and Cauchy Problem

Let for $x \geqslant x_0$ $V(x) \equiv 0$. Then by $x \geqslant x_0$ the asymptotic equality (5) will pass into an exact one, i.e., by $x \geqslant x_0$

$$\sin kx + \int_0^x K(x,t)\sin kt\,dt = A(k)\sin(kx+\delta(k)). \tag{11}$$

Differentiating (2) over X and making use of (9) we can easily obtain that by $x \geqslant 2x_0$

$$\varphi'(x,k) = k\left(\cos kx + \int_0^x K(x,t)\cos kt\,dt\right). \tag{12}$$

From (5) we have by $x \geqslant x_0$

$$\varphi'(x,k) = k\,A(k)\cos(kx+\delta(k)),$$

i.e. by $x \geqslant 2x_0$

$$\cos kx + \int_0^x K(x,t)\cos kt\,dt = A(k)\cos(kx+\delta(k)). \tag{13}$$

Thus, formulae (11) and (13) allow, $\delta(k)$ being known, to determine the Fourier transform from $K(x,t)$ by $x \gg 2x_0$. Further, making use of relations (10) and (10') we find that

$$\int_{-x_0}^{x_0} K(x_0,t)\sin k_n t\,dt = (-1)^n\, 2A(k_n)\sin\delta(k_n), \tag{14}$$

$$\int_{-x_0}^{x_0} K(x_0,t)\cos k_n t\,dt = 0, \tag{14'}$$

$$\int_{-x_0}^{x_0} \left.\frac{\partial K(x,t)}{\partial x}\right|_{x=x_0} \sin k_n t\,dt = (-1)^{n+1}\, 2k_n\left(A(k_n)\cos\delta(k_n) - 1\right), \tag{15}$$

$$\int_{-x_0}^{x_0} \left.\frac{\partial K(x,t)}{\partial x}\right|_{x=x_0} \cos k_n t\,dt = 0 \tag{15'}$$

for $k_n = n\frac{\pi}{x_0}$, where n is an integer.

Thus, by $\delta(k)$ we may calculate the Fourier coefficients of the initial data on a segment $-x_0 \leq t \leq x_0$ of the straight line $x = x_0$. Let us extend our data periodically over the whole straight line $x = x_0$ and apply the Fourier method to the solution of equation (6) with the initial data (14), (14'), (15) and (15').

Preliminarily let us make a substitution

$$\xi = \frac{\pi}{x_0}x, \quad \tau = \frac{\pi}{x_0}t, \quad K(\xi,\tau) = \frac{x_0}{\pi}K(x,t).$$

It is not easy to check that $K(\xi,\tau)$ satisfies the equation

$$\frac{\partial^2 K(\xi,\tau)}{\partial\xi^2} - \frac{\partial^2 K(\xi,\tau)}{\partial\tau^2} = 2K(\xi,\tau)\frac{d}{d\xi}K(\xi,\xi) \tag{16}$$

with the initial data on the straight line $\xi = \pi$:

$$K(\pi,\tau) = \frac{x_0}{\pi}K(x_0,t), \tag{17}$$

$$\left.\frac{\partial K(\xi,\tau)}{\partial\xi}\right|_{\xi=\pi} = \frac{x_0^2}{\pi^2}\left.\frac{\partial K(x,t)}{\partial x}\right|_{x=x_0}. \tag{17'}$$

We shall look for $K(\xi,\tau)$ in the form:

$$K(\xi,\tau) = \sum_{n=1}^{\infty} a_n(\xi)\sin n\tau. \tag{18}$$

Having differentiated (18) formally and substituting it into (16) we obtain the system of equation for the determination of $a_n(\xi)$:

$$a_n''(\xi) + n^2 a_n(\xi) = 2a_n(\xi)\frac{d}{d\xi}\sum_{p=1}^{\infty} a_p(\xi)\sin p\xi . \tag{19}$$

Making use of (14), (15), (17) and (17') we find that

$$a_n(\pi) = \frac{1}{\pi}\int_{-\pi}^{\pi} K(\pi,\tau)\sin n\tau\, d\tau =$$

$$= \frac{1}{\pi}\int_{-x_0}^{x_0} K(x_0,t)\sin \kappa_n t\, dt = (-1)^n \frac{2}{\pi} A(\kappa_n)\sin\delta(\kappa_n) .$$

and

$$a_n'(\pi) = \frac{1}{\pi}\int_{-\pi}^{\pi} \left.\frac{\partial K(\xi,\tau)}{\partial \xi}\right|_{\xi=\pi} \sin n\tau\, d\tau =$$

$$= \frac{x_0}{\pi^2}\int_{-x_0}^{x_0} \left.\frac{\partial K(x,t)}{\partial x}\right|_{x=x_0} \sin \kappa_n t\, dt = (-1)^{n+1}\frac{2\kappa_n x_0}{\pi^2}\left(A(\kappa_n)\cos\delta(\kappa_n) - 1\right).$$

Substituting $n\frac{\pi}{x_0}$ for κ_n we obtain

$$a_n(\pi) = (-1)^n \frac{2}{\pi} A\left(n\frac{\pi}{x_0}\right)\sin\delta\left(n\frac{\pi}{x_0}\right), \tag{20}$$

$$a_n'(\pi) = (-1)^{n+1}\frac{2n}{\pi}\left(A\left(n\frac{\pi}{x_0}\right)\cos\delta\left(n\frac{\pi}{x_0}\right) - 1\right), \tag{20'}$$

and our problem reduced to the solution of the system (19) with the initial data (20) and (20').

Note, that equality (9) by $x = 2x_0$ and $t > 0$ is also correct for the case when $V(x) \equiv 0$ by $x > x_0$ and in the point $x = x_0$ it has a discontinuity of the first kind (for instance, the rectiangular potential well). Therefore, equality (13) also holds for this case. It can be easily verified that Eqs. (10) and (10') also hold for this case. Therefore, we may also apply the Fourier method here and reduce the problem to the solution of system (19) with the initial data (20) and (20'). The specific features of this case must be reflected in the asymptotics for δ(k) for great k.

4. Relationship between V(x) and δ(k)

As is well-known the solution of equation (1) with the initial data (1') satisfies the following integral equation:

$$\varphi(x,\kappa) = \sin\kappa x + \frac{1}{\kappa}\int_0^x V(\xi)\sin\kappa(x-\xi)\,\varphi(\xi,\kappa)\,d\xi . \tag{21}$$

The solution of this equation have the form:

$$\varphi(x,\kappa) = \sin\kappa x + \sum_{n=1}^{\infty}\frac{1}{\kappa^n} Q_n(x,\kappa), \tag{22}$$

where

$$Q_n(x,k)=\int_0^x\int_0^{x_1}\dots\int_0^{x_{n-1}} V(x_1)\cdots V(x_n)\sin k(x-x_1)\cdots\sin k(x_{n-1}-x_n)\sin kx_n\,dx_1\dots dx_n.$$

It can be easily seen that

$$|Q_n(x,k)|\le\int_0^x\int_0^{x_1}\dots\int_0^{x_{n-1}}|V(x_1)\cdots V(x_n)|\,dx_1\dots dx_n=\frac{1}{n!}\Big(\int_0^x|V(\xi)|d\xi\Big)^n,$$

and, therefore, a series (22) by any $k\neq 0$ is convergent uniformly on any segment $[0,x]$. If in addition $\int_0^\infty|V(\xi)|d\xi<\infty$ then for any $k\neq 0$ series (22) is convergent uniformly along a half straight line $[0,\infty)$.

Let by $x>x_0$ $V(x)\equiv 0$. Then according to (22) by $x\geq x_0$

$$\varphi(x,k)=\Big(1+\sum_{n=1}^\infty\frac{1}{k^n}\Phi_n(k)\Big)\sin kx+\Big(\sum_{n=1}^\infty\frac{1}{k^n}\Psi_n(k)\Big)\cos kx \tag{23}$$

where

$$\Phi_n(k)=\int_0^{x_0}\int_0^{x_1}\dots\int_0^{x_{n-1}}V(x_1)\dots V(x_n)\cos kx_1\sin k(x_1-x_2)\cdots\sin k(x_{n-1}-x_n)\sin kx_n\,dx_1\dots dx_n, \tag{24}$$

whereas

$$\Psi_n(k)=-\int_0^{x_0}\int_0^{x_1}\dots\int_0^{x_{n-1}}V(x_1)\dots V(x_n)\sin kx_1\sin k(x_1-x_2)\cdots\sin k(x_{n-1}-x_n)\sin kx_n\,dx_1\dots dx_n. \tag{25}$$

Comparing (23) with (5) which in the case $V(x)\equiv 0$ by $x>x_0$ passes by $x\geq x_0$ into an exact equality we have

$$A(k)\cos\delta(k)=1+\sum_{n=1}^\infty\frac{1}{k^n}\Phi_n(k), \tag{26}$$

$$A(k)\sin\delta(k)=\sum_{n=1}^\infty\frac{1}{k^n}\Psi_n(k). \tag{27}$$

The latter formulae are also correct in the case of $x_0=\infty$ and $\int_0^\infty|V(\xi)|d\xi<\infty$. In this case $\varphi(x,k)$ allows the representation

$$\varphi(x,k)=A(k,x)\sin(kx+\delta(k,x)).$$

Put $A(k)=\lim_{x\to\infty}A(k,x)$ and $\delta(k)=\lim_{x\to\infty}\delta(k,x)$. If $\int_0^\infty|V(\xi)|d\xi<\infty$ then these limits exist. Supposing $x_0=\infty$ in formulas (24) and (25) we obtain such $\Phi_n(k)$ and $\Psi_n(k)$ that equalities (26) and (27) hold. If in addition we assume that $\int_0^\infty\xi|V(\xi)|d\xi<\infty$ then it can be easily shown that for small k

$$|\Phi_n(k)|\le\frac{k^n}{n!}C^n,\ \text{while}\ |\Psi_n(k)|\le\frac{k^n\,o(k)}{n!}C^n$$

where C is a certain constant, while $o(k)\to 0$ by $k\to 0$ and is independent of n. Consequently, the right-hand side of equality (27) is tending to zero by $k\to 0$, whereas the right-hand side of equality (26) is limited in the neighbourhood of the point $k=0$. It can be easily shown that by $k\to 0$ the right-hand side of (26) is tending to a certain limit. It follows easily from this that $A(k)$ and $\delta(k)$ are continuous function of k. If we suppose in addition that $\int\limits_0 \xi^{\ell} |V(\xi)| d\xi < \infty$ for $\ell = 0, 1, \dots, m+1$, then it is easy to show that by $\kappa > 0$ $A(\kappa)\cos\delta(\kappa)$ and $A(\kappa)\sin\delta(\kappa)$ have continuous derivatives up to the order m. The expressions for them are found by means of the corresponding limiting process in the right-hand sides of equalities (26) and (27).

Let us find an asymptotic expression for $\delta(\kappa)$ for great k, supposing that $\int\limits_0^\infty |V(\xi)| d\xi < \infty$. It is seen form (24) and (25) that $\phi_n(\kappa)$ and $\psi_n(\kappa)$ for a fixed n are the limited functions of k. Therefore, the first terms of the asymptotics δ(k) will be determined by the first ψ_n(k) and $\phi_n(\kappa)$. Let us find an explicit expression for them. We shall consider that $V(x)$ is continuous together with their derivatives up to the second order inclusive, $V'(x) \to 0$ by $x \to \infty$ and $\int\limits_0^\infty |V'(x)| dx < \infty$ Having divided (25) by (26) we find that

$$tg\,\delta(\kappa) = \frac{\sum\limits_{n=1}^{\infty} \frac{1}{\kappa^n} \psi_n(\kappa)}{1 + \sum\limits_{n=1}^{\infty} \frac{1}{\kappa^n} \phi_n(\kappa)} \tag{28}$$

From this we find that for great k

$$\delta(\kappa) = -m\pi + \frac{C_1}{\kappa} + \frac{C_2}{\kappa^3} + \frac{o(\frac{1}{\kappa})}{\kappa^3}, \tag{29}$$

where $C_1 = -\frac{1}{2} \int\limits_0^\infty V(x)\,dx$, $C_2 = -\frac{V'(0)}{8}$, m is integer, whereas $o(\frac{1}{\kappa}) \to 0$ by k→∞. If $V(x)$ or $V'(x)$ has a finite number of discontinuities of the first kind, then in the asymptotics for $\delta(\kappa)$ there will appear additionally a finite sum of the terms of type $\frac{A_s}{\kappa^2}\sin 2\kappa x_s$ or $\frac{B_s}{\kappa^3}\cos 2\kappa x_s'$, where x_s is the point of a discontinuity $V(x)$, whereas x_s' — $V'(x)$. The coefficient A_s is proportional to $V(x_s+0)-V(x_s-0)$, whereas B_s — $V'(x_s+0) - V'(x_s-0)$.

5. Calculation of C_1

Making use of relation (4') we find that $C_1 = -\lim\limits_{x\to\infty} K(x,x)$. Let us take an arbitrary $x_0 > 0$. Then we may write that

$$K(2x_0, t) = \frac{a_0(x_0)}{2x_0} + \frac{1}{x_0} \sum_{n=1}^{\infty} \left(a_n(x_0)\cos n\frac{\pi}{x_0} t + b_n(x_0) \sin n\frac{\pi}{x_0} t \right),$$

where

$$a_n(x_0) = \int\limits_0^{2x_0} K(2x_0, t) \cos n\frac{\pi}{x_0} t\,dt, \quad b_n(x_0) = \int\limits_0^{2x_0} K(2x_0, t) \sin n\frac{\pi}{x_0} t\,dt.$$

It is known that at the point of the discontinuity of the first kind the Fourier series is convergent to a half-sum of the values of the function to the right and left from the point of discontinuity. In our case

$$K(2X_0, 2X_0) = \frac{2}{X_0} \sum_{n=0}^{\infty} a_n(X_0) - \frac{a_0(X_0)}{X_0}, \tag{30}$$

since $K(2X_0, 0) = 0$.

Tending X_0 to infinity we can easily obtain that the sum in the right-hand side of (30) is formally tending to the expression

$$\frac{2}{\pi}\int_0^{\infty}\{A(\kappa)\cos\delta(\kappa) - 1\}\,d\kappa .$$

It is not easy to show that this limiting process is correct. Thus, we obtain the following expression for C_1:

$$C_1 = -\frac{2}{\pi}\int_0^{\infty}\{A(\kappa)\cos\delta(\kappa) - 1\}\,d\kappa . \tag{31}$$

Making use of (26) we easily establish that the integral in the right-hand side of (31) is convergent well enough.

Let $A(\kappa)$ and $\delta(\kappa)$ are known up to some k_{max}. Putting $C_1 = -\frac{2}{\pi}\int_0^{K_{max}}\{A(\kappa)\cos\delta(\kappa)-1\}d\kappa$ we shall make an error in the definition of C_1 equal to $-\frac{2}{\pi}\int_{K_{max}}^{\infty}\{A(\kappa)\cos\delta(\kappa)-1\}d\kappa$. It will be seen later that the error in the determination of $V(x)$ and $\varphi(x,\kappa)$ will be of the order $\frac{2}{\pi}\int_{K_{max}}^{\infty}\{A(\kappa)\cos\delta(\kappa)-1\}d\kappa$. Therefore, in order the error in the determination of $V(x)$ and $\varphi(x,\kappa)$ to be small it is necessary that $\frac{2}{\pi}\int_{K_{max}}^{\infty}\{A(\kappa)\cos\delta(\kappa)-1\}d\kappa$ would be small.

6. Relation between $A(\kappa)$, $\delta(\kappa)$ and conjugated states*

Consider first the case of a finite potential. Multiply (11) by i and add it with (13). We obtain that for $x \gg 2X_0$ and k real

$$e^{i\kappa x} + \int_0^x K(x,t)\,e^{i\kappa t}dt = A(\kappa)\,e^{i\kappa x + i\delta(\kappa)} .$$

We have thus

$$1 + \int_0^x K(x,t)\,e^{i\kappa(t-x)}dt = A(\kappa)\,e^{i\delta(\kappa)} \tag{32}$$

for $x \gg 2X_0$ and k real.

Let $G(\kappa) = 1 + \int_0^{2X_0} K(2X_0,t)\,e^{i\kappa(t-2X_0)}dt$. It can be easily seen that $G(\kappa)$ is analytical in the lower half plane of the complex variable. It easily follows from

$$G(\kappa) = \left\{ i\varphi(2X_0,\kappa) + \frac{\varphi'(2X_0,\kappa)}{\kappa} \right\} e^{-i2\kappa X_0}$$

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* While writing this Section Chudov's paper [5] was an essential help for me.

that $G(k)$ has only a finite number of zeros in the lower half plane.

Indeed, the eigenfunction of (1) having by $x > x_0$ the form

$$\varphi(x,k) = C e^{-ik_0 x}$$

and belonging, therefore, to $L_2(0,\infty)$ corresponds to any zero k_0 of the function $G(k)$ in the lower half plane. As in the case of a finite potential the equation (1) has only a finite number of eigenfunctions belonging to $L_2(0,\infty)$, then our assertion is proved. It follows from the selfconjugation of Shrödinger equation that k_0^2 is a real number. Since by $k_0^2 > 0$ the eigenfunctions do not possess an integrated square then $k_0^2 \leq 0$. The case when $k_0 = 0$ is a zero of the function G(k) requires a special consideration since in this case one may obtain a new discrete level by whatever small change of a potential. The limiting phase in this oase also behaves in a particular way. Thus, excluding the oase when G(o) = Q we establish that in the considered case G(K) has a finite number of purely imaginary zeros in the lower halfplane and does not turn into zero along a real axis. Therefore, A(k) does not turn into zero along the real axis as well. From here, making use of formula (27) we find that $\sin\delta(0)=0$ i.e. $\delta(0)$ is multiple to π. By changing the sign of A(k) is necessary we may always obtain that $\delta(0)$ may be chosen equal to zero. Then, applying (26) and (27) we find that $\delta(k)$ is an odd function along a real axis, whereas A(k) is an even function.

A(k) is real along the real axis. Therefore, along the real axis either $A(k)=|G(k)|$ or $A(k)=-|G(k)|$. Let us clear up which of these possiblities really takes place. We have from (26) that $A(k)\cos\delta(k) \to 1$ by $k \to \infty$.
Since according to (29) $\delta(k) \to -m\pi$ by $k \to \infty$,
then $A(k) \to (-1)^m$ by $k \to \infty$ i.e., the relation

$$A(k) = (-1)^m \, | G(k) | \tag{33}$$

is correct, where $m = -\lim_{k\to\infty} \frac{\delta(k)}{\pi}$, whereas $\delta(k)$ is chosen so that $\delta(0) = 0$. Let us determine the meaning of m in formula (29). We make use of the theorem about the connection between the number of zeros inside a closed contour of the analytical function G(k) and the change of its argument along thise contour (see [6], §23). As an integration contour G_R we assume a curve consisting of the segment $[-R, R]$ of the real axis and the lower half of the semicircle $|k| = R$. Let us take R so great that all zeros of the function G(k) lying in the lower halfplane would be inside our contour and proceed to the limit by $R\to\infty$. Then by means of elementary calculations we obtain that the number of zeros N of the function G(k) lying in the lower half plane is connected with $\delta(k)$ by the following relation:

$$2\pi N = \delta(-\infty) - \delta(\infty) = -2\delta(\infty).$$

Comparing the obtained equality with (29) we get that $m = N$. Since $N \geq 0$, then $m \geq 0$

as well.

Let us denote by $-i\varkappa_s$ $(\varkappa_s>0,\ s=1,\dots,m)$ the zeros of the function G(k) lying in the lower half-plane. Then $\kappa^2=-\varkappa_s^2$ are discrete eigenvalues of equation (1). Let

$$\widetilde{G}(\kappa) = G(\kappa)\prod_{s=1}^{m}\left(\frac{i\varkappa_s-\kappa}{i\varkappa_s+\kappa}\right). \tag{34}$$

It is not easy to see that $\widetilde{G}(\kappa)$ is analytical in the lower half plane and does not vanish in it.

Therefore, the function

$$\mathcal{F}(\kappa) = \ln\widetilde{G}(\kappa) = \ln|\widetilde{G}(\kappa)| + i\arg\widetilde{G}(\kappa)$$

will be also analytical everywhere in the lower half plane. It follows from the definition of the funotion F(k) that for great k $|\mathcal{F}(\kappa)|<\frac{C}{\kappa}$, where C is constant. Therefore, the so-oalled disperson relation may be written for the function F(k) |see, |7|, § 46). This dispersion relation connects the values of the real and imaginary parts of the funotion F(k) along the real axis:

$$\ln|\widetilde{G}(\kappa)| = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\arg\widetilde{G}(\kappa')}{\kappa-\kappa'}d\kappa',$$

i.e.

$$|\widetilde{G}(\kappa)| = \exp\left\{\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\arg\widetilde{G}(\kappa')}{\kappa-\kappa'}d\kappa'\right\}.$$

Using (32), (33), (34) and the definition of the function G(k) we obtain that along the real axis

$$A(\kappa) = (-1)^m\,|\widetilde{G}(\kappa)|$$

and

$$\arg\widetilde{G}(\kappa) = \delta(\kappa) + \sum_{s=1}^{m}\arg\left(\frac{i\varkappa_s-\kappa}{i\varkappa_s+\kappa}\right).$$

Finally we obtain that

$$A(\kappa) = (-1)^m\exp\left\{\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\delta(\kappa')+\sum_{s=1}^{m}\operatorname{arctg}\frac{2\varkappa_s\kappa'}{\varkappa_s^2-\kappa'^2}}{\kappa-\kappa'}d\kappa'\right\} \tag{35}$$

It is the connection of A(k) and δ (k) which is necessary. In order the obtained formula to be correct it is not neoessary to require the potential being finite. In this case, introducing

$$V_{x_0}(x) = \begin{cases} V(x) & \text{by } x<x_0 \\ 0 & \text{by } x>x_0 \end{cases}$$

we may write relation (35) for $A_{x_0}(\kappa)$ and $\delta_{x_0}(\kappa)$. If $\int_0^\infty|V(\xi)|d\xi<\infty$ and $\int_0^\infty\xi|V(\xi)|d\xi<\infty$, then in the obtained equality we may pass to the limit by $x_0=\infty$. The obtained limiting relation will be of the same form as (35). It establishes the connection between $\mathcal{A}(\kappa) = \mathcal{A}_\infty(\kappa)$ and $\delta(\kappa) = \delta_\infty(\kappa)$. When deducing (35) we excluded the case when $G(0)=0$. However, by means of considerations applying in the derivation of (35) one may obtain an analogous formula. The only difference is that it is necessary in this case to switch on a factor analytical in the lower half plane everywhere except the point k = 0 at which it has a pole of the necessary order and behaves sufficiently well by $|K| \to \infty$.

7. Asymptotics A(k)

We have noticed earlier that by $K \to \infty$ $\mathcal{A}(\kappa) \to (-1)^m$. Let us find the following term of the asymptotics. For this there are two possibilities: equality (27) and equality (35). Making use of equality (27) we easily obtain that for great k

$$\mathcal{A}(\kappa) = (-1)^m \left(1 + \frac{\alpha_1}{K^2} + O\left(\frac{1}{K^4}\right)\right). \tag{36}$$

Thus, the problem is to find α_1. It may be done by means of equality (35).

8. Solution of System (19)

Using (20), (20'), (29) and (36) we may write the first terms of the asymptotics $a_n(\pi)$ and $a_n'(\pi)$ for great n. We have for great n:

$$a_n(\pi) = (-1)^n \frac{2X_0}{\pi^2} \frac{C_1}{n} + O\left(\frac{1}{n^3}\right), \tag{37}$$

$$a_n'(\pi) = (-1)^n \frac{X_0^2}{\pi^3} \frac{C_1^2 - 2\alpha_1}{n} + O\left(\frac{1}{n^3}\right), \tag{37'}$$

where $\alpha_1 = \frac{V(0)}{Y}$.

We shall make the substitution:

$$a_n(\xi) = b_n(\xi) + \frac{2C_1 X_0 n}{X_0^2 + \pi^2 n^2} \cos n\xi + \frac{1}{\pi} \frac{X_0^2 (C_1^2 - 2\alpha_1)}{X_0^2 + \pi^2 n^2} \sin n\xi .$$

Substituting into (19), for the determination of $b_n(\xi)$ we obtain the equation system:

$$b_n''(\xi) + n^2 b_n(\xi) = 2\left\{b_n(\xi) + \frac{2C_1 X_0 n}{X_0^2+\pi^2 n^2} \cos n\xi + \frac{1}{\pi} \frac{X_0^2 (C_1^2 - 2\alpha_1)}{X_0^2 + \pi^2 n^2} \sin n\xi\right\} \cdot$$

$$\cdot \frac{d}{d\xi} \sum_{p=1}^{\infty} \left(b_p(\xi) \sin p\xi + \frac{C_1 X_0 p}{X_0^2+\pi^2 p^2} \sin 2p\xi + \frac{1}{\pi} \frac{X_0^2 (C_1^2 - 2\alpha_1)}{X_0^2 + \pi^2 p^2} \sin^2 p\xi \right\}. \tag{38}$$

The following should noted about system (38). When differentiating a series $\sum_{p=1}^{\infty} \frac{C_1 X_0 p}{X_0^2+\pi^2 p^2} \sin 2p\xi$ by each term a series arises which have two δ-shaped particularities, one at the point $\xi = \pi$ the other - at the point $\xi = 0$. This is due to the fact that a series

$$\sum_{p=1}^{\infty} \frac{C_1 X_0 p}{X_0^2 + \pi^2 p^2} \sin 2p\xi$$

is a Fourier series of a certain smooth periodical function having discontinuities of the first kind at the points $\xi = n\pi$ (n is an integer). A discontinuity $a_n(\xi)$ and $a_n'(\xi)$ at the point $\xi=\pi$ and $\xi = 0$ requires that these particularities would be switched off. This shows that a formal approach to the derivation of system (19) is insufficient. However, this derivation has no other defects. It can be seen from (37) and (37') that for great n

$$b_n(\pi) = O\left(\frac{1}{n^3}\right) \quad and \quad b_n'(\pi) = O\left(\frac{1}{n^3}\right).$$

For the first $M = \left[\frac{X_0 K_{max}}{\pi}\right]$ values n we may determine $b_n(\pi)$ and $b_n'(\pi)$ and assuming $b_n(\xi) = b_n'(\xi) \equiv 0$ by $n > M$ to solve a cuf-off system. For a fixed X_0 an error in the determination of $b_n(\xi)$ will depend upon M and by $M \to \infty$ (i.e. $K_{max} \to \infty$) will tend to zero. It is clear from the form of the system (38) that the solution of this system depends continuously upon C_1 and α_1. The system (38) allows to estimate the mistake in the determination of $b_n(\xi)$ and, therefore, in the determination of K(x,t) due to an error in the determination of the phase.

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