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In the paper [1] a new variational principle was proposed by one of the authors (N.N.B.). This principle is a generalization of the well known Fock method [2]. Consideration of the new variational principle was extended by S.V.Tyablikov [4,7]. In this paper we shall make further investigations of the variational principle in the many body problem, obtain stationary equations in a evident form and consider the application of this variational principle to two particular cases.

Let us consider a system interacting Fermi-particles with a Hamiltonian

$$H=\sum_{f,f'} E(f',f)a_f^+ a_{f'} + \frac{1}{2}\sum_{f_1,f_2,f_1',f_2'} K(f_2',f_1';f_1,f_2)a_{f_1}^+ a_{f_2}^+ a_{f_2'} a_{f_1'}, \tag{1}$$

here $E(f,'f)=T(f,'f)-\lambda\delta_{ff'}$, λ - is the chemical potential, f - set of indices characterising one particle state and the real function $K(f_2',f_1';f_1,f_2)$ has the following properties:

$$K(f_2',f_1';f_1,f_2)=K(f_2,f_1;f_1',f_2')=K(f_1',f_2';f_2,f_1) \tag{2}$$

Perform a canonical transformation of the Fermi-amplitudes:

$$a_f=\sum_\nu\{u_{f\nu}\alpha_\nu+v_{f\nu}\alpha_\nu^+\} \tag{3}$$

In order to preserve their commutation properties it is necessary to carry out the following conditions:

$$\xi_{f,f'}\equiv\sum_\nu\{u_{f\nu}u_{f'\nu}^*+v_{f\nu}v_{f'\nu}^*\}-\delta_{f,f'}=0. \tag{4}$$

$$\eta_{f,f'}\equiv\sum_\nu\{u_{f\nu}v_{f'\nu}+v_{f\nu}u_{f'\nu}\}=0.$$

We determine a new vacuum state C_o

$$\alpha_\nu C_o=0$$

and find a mean value of H by this state, e.i.

$$\bar{H}=\sum_{f,f'}E(f',f)F(f,f')+\sum_{f_1,f_2,f_1',f_2'}K(f_2',f_1';f_1,f_2)\left\{F(f_1,f_1')F(f_2,f_2')+\frac{1}{2}\Phi^*(f_1,f_2)\Phi(f_1',f_2')\right\}, \tag{5}$$

where

$$F(f,f')=\sum_\nu v_{f\nu}^* v_{f'\nu};\quad \Phi(f,f')=\sum_\nu u_{f\nu}v_{f'\nu}. \tag{6}$$

Let us determine u and v from the minimum $\bar{H}$ condition with the subsidiary conditions (4):

$$\delta\left\{\bar{H}(u,v)+\sum_{f_1,f_2}\left[\lambda(f_2,f_1)\xi_{f_1,f_2}+\mu^*(f_2,f_1)\eta_{f_1,f_2}+\mu(f_2,f_1)\eta^*_{f_1,f_2}\right]\right\} \tag{7}$$

here $\lambda(f,'f)$, $\mu(f,'f)$ are Lagrange factors. Considering the variations of $\delta u, \delta v, \delta u^*, \delta v^*$ as independent ones we find:

$$\frac{\delta\bar{H}}{\delta u_{f\omega}^*}+\sum_{f''}\left\{\lambda(f,f'')u_{f''\omega}+\mu(f,f'')v_{f''\omega}^*+\mu(f'',f)v_{f''\omega}^*\right\}=0 \tag{8}$$

$$\frac{\delta\bar{H}}{\delta v_{f\omega}^*}+\sum_{f''}\left\{\lambda(f,f'')v_{f''\omega}+\mu(f,f'')u_{f''\omega}^*+\mu(f'',f)u_{f''\omega}^*\right\}=0$$

and two equations complexly conjugated with these. We obtain:

$$A(f,f') \equiv \sum_{\omega} \left\{ v_{f'\omega} \frac{\delta \bar{H}}{\delta u^*_{f\omega}} + u_{f'\omega} \frac{\delta \bar{H}}{\delta u^*_{f\omega}} \right\} + \mu(f,f') + \mu(f',f) = 0,$$

$$B(f,f') \equiv \sum_{\omega} \left\{ u^*_{f'\omega} \frac{\delta \bar{H}}{\delta u^*_{f\omega}} + v^*_{f'\omega} \frac{\delta \bar{H}}{\delta v_{f\omega}} \right\} + \lambda(f,f') = 0. \qquad (9)$$

Let us exclude the Lagrange factors and find the equations:

$$\mathfrak{A}(f,f') \equiv A(f,f') - A(f',f) = 0, \qquad (10)$$

$$\mathfrak{B}(f,f') \equiv B(f,f') - B^*(f',f) = 0,$$

which have the following form:

$$\mathfrak{A}(f,f') = \sum_{f''} \left\{ \Phi(f,f'') \xi(f'',f') - \xi(f,f'') \Phi(f'',f') - \right.$$

$$- \sum_{f_1,f_2,f} \left\{ K(f_2,f_1;f,f'') \Phi(f_1,f_2) F(f'',f') - K(f_2,f_1;f',f'') \Phi(f_1,f_2) F(f'',f) \right\} + \qquad (11)$$

$$+ \sum_{f_1,f_2} K(f_2,f_1;f,f') \Phi(f_1,f_2) = 0,$$

$$\mathfrak{B}(f,f') = \sum_{f''} \left\{ F(f,f'') \xi(f'',f') - \xi(f,f'') F(f'',f') \right\} + \qquad (12)$$

where $$+ \sum_{f_1,f_2,f''} \left\{ K(f_2,f_1;f',f'') \Phi(f_1,f_2) \Phi^*(f,f'') - K(f_2,f_1;f,f'') \Phi^*(f_1,f_2) \Phi(f',f'') \right\} = 0,$$

$$\xi(f,f') = E(f,f') + 2 \sum_{f_1,f_2} K(f_2,f;f',f_1) F(f_1,f_2). \qquad (13)$$

(11) and (12) coicide with the equations obtained in /3,4/.

It should be noted that the functions $\mathfrak{A}(f,f')$ and $\mathfrak{B}(f,f')$ are connected by the following relation:

$$\sum_{f,f'} \left\{ v^*_{f\nu} u^*_{f'\nu'} \mathfrak{A}(f,f') + u_{f\nu} v_{f'\nu'} \mathfrak{A}^*(f',f) \right\} +$$

$$+ \sum_{f,f'} \left\{ u_{f\nu} u^*_{f'\nu'} - v^*_{f'\nu} v_{f\nu'} \right\} \mathfrak{B}(f,f') = 0. \qquad (14)$$

Therefore, if $\mathfrak{A}(f,f')=0$ then it follows from (14) that $\mathfrak{B}(f,f')=0$ and one may consider only one of the equations (11), (12). As it was shown in the paper /3/ the functions $F(f,f')$ and $\Phi(f,f')$ are connected by the following means:

$$F(f,f') = \sum_{f''} \left\{ \Phi^*(f'',f) \Phi(f'',f') + F(f,f'') F(f'',f') \right\}, \qquad (15)$$

$$\sum_{f''} \left\{ \Phi(f,f'') F(f',f'') + \Phi(f',f'') F(f'',f) \right\} = 0.$$

One may proof these relations by a direct substitution of the expressions $\Phi(f,f')$ and $F(f,f')$, if to take the conditions (4) and those connected with them:

$$\sum_{f} \left\{ u^*_{f\omega} u_{f\nu} + v^*_{f\nu} v_{f\omega} \right\} = \delta_{\nu\omega}, \qquad (16)$$

$$\sum_{f} \left\{ v^*_{f\omega} u_{f\nu} + v^*_{f\nu} u_{f\omega} \right\} = 0.$$

Thus, one may operate with the functions F and Φ instead of the functions u and v, if the conditions (15) carry out, then F and Φ will be expressed in terms of u and v by (6).

We shall consider an application of the new variational principle in the many body problem to two particular cases.

First case. Let us choose $v_{f\nu}=v_f\delta(f-\nu),\ u_{f\nu}=u_f\delta(f+\nu)$, $u_{f}=u_{f},\ v_{f}=-v_{f}$,

then $\alpha_f=u_f\alpha_{-f}+v_f\alpha_f^+$, $F(f,f')=\delta(f-f')v_f^2,\quad \Phi(f,f')=-\delta(f+f')u_fv_{f'}$.

In this case the equation (11) obtains the form:

$$\xi(f,f)u_fv_f+(u_f^2-v_f^2)\sum_{f'>0}K(-f',f';f,-f)u_{f'}v_{f'}=0, \tag{18}$$

$$\xi(f,f)=E(f,f)-2\sum_{f'>0}\left\{K(f',f;f',f)+K(-f',f;-f',f)\right\}v_{f'}^2. \tag{19}$$

If $f=(\kappa,\sigma)$, where K – is a momentum and σ – is a spin, then (18) coincides with an equation of the compensation of "dangerous" graphs in the theory of superconductivity [5] and if f is quantum number of the momentum projection along the symmetry axis of the nucleus , then (18) transforms into an equation [6], taking into account an interaction of nucleons, locating on the outer shell of a heavy nucleus.

Second case. Choose $f=(r,\sigma)$ where r - radius-vector, and let us give functions $K(f_2',f_1';f_1,f_2)$, $E(f',f)$ in the form:

$$\tfrac{1}{2}U(r_1-r_2;\sigma_1,\sigma_2)\left\{\delta_{\sigma_1\sigma_1'}\delta_{\sigma_2\sigma_2'}\,\delta(r_1-r_1')\delta(r_2-r_2')-\delta_{\sigma_1\sigma_2'}\delta_{\sigma_2\sigma_1'}\,\delta(r_1-r_2')\delta(r_2-r_1')\right\},$$

$$E(r)\delta_{\sigma\sigma'}\,\delta(r-r')=\left\{T(r)-\lambda\right\}\delta_{\sigma\sigma'}\,\delta(r-r').$$

In this case

$$\begin{aligned}\bar{H}=&\sum_{r,\sigma}\left\{T(r)-\lambda\right\}F(r,\sigma;r,\sigma)+\\&+\frac{1}{2}\sum_{r_1,\sigma_1,r_2,\sigma_2}U(r_1-r_2;\sigma_1,\sigma_2)\Big\{F(r_1,\sigma_1;r_1,\sigma_1)F(r_2,\sigma_2;r_2,\sigma_2)-\\&-F(r_1,\sigma_1;r_2\sigma_2)F(r_2,\sigma_2;r_1,\sigma_1)+\Phi^*(r_1,\sigma_1;r_2,\sigma_2)\Phi(r_1,\sigma_1;r_2,\sigma_2)\Big\}\end{aligned} \tag{20}$$

and the equation (11) will be :

$$\begin{aligned}&\Big\{E(r_1)+E(r_2)+\sum_{r'\sigma'}U(r_1-r';\sigma_1\sigma')F(r',\sigma';r',\sigma')+\\&+\sum_{r'\sigma'}U(r_2-r';\sigma_2,\sigma')F(r',\sigma';r',\sigma')+U(r_1-r_2;\sigma_1,\sigma_2)\Big\}\Phi(r_1,\sigma_1;r_2,\sigma_2)+\\&+\sum_{r'\sigma'}\Big\{U(r_1-r';\sigma_1,\sigma')F(r',\sigma';r_1,\sigma_1)\Phi(r_2,\sigma_2;r',\sigma')+U(r_2-r';\sigma_2,\sigma')F(r',\sigma';r_2\sigma_2).\\&\Phi(r',\sigma';r,\sigma_1)\Big\}-\sum_{r',\sigma'}\Big\{U(r_1-r';\sigma_1,\sigma')F(r',\sigma';r_2,\sigma_2)\Phi(r',\sigma';r_1,\sigma_1)+\\&+U(r_2-r';\sigma_2,\sigma')F(r',\sigma';r_1,\sigma_1)\Phi(r_2,\sigma_2;r',\sigma')\Big\}=0,\end{aligned} \tag{21}$$

Let us write it in the form:

$$\left\{E_1+E_2+V_1+V_2\right\}\Phi_{12}+\sum_K\left\{\mathcal{E}_{1K}\Phi_{2K}+\mathcal{E}_{2K}\Phi_{K1}\right\}+Z_{12}-\sum_K\left\{Z_{2K}F_{K1}+Z_{K1}F_{K2}\right\}=0. \tag{22}$$

Here $V_i=\sum_{r'\sigma'}U(r_i-r';\sigma_i,\sigma')F(r',\sigma';r',\sigma')$ is a potential, created by all particles in the point r_i , $\mathcal{E}_{iK}=U(r_i-r_K;\sigma_i,\sigma_K)F(r_K,\sigma_K;r_i,\sigma_i)$ is an addition to the kinetic energy of the i-particle; the term: $Z_{12}=U(r_1-r_2;\sigma_1,\sigma_2)\Phi(r_1,\sigma_1;r_2,\sigma_2)$ describes an interaction of the two particles, and the expression $-\sum\left\{Z_{2K}F_{K1}+Z_{K1}F_{K2}\right\}$ takes into account the influence of the others particles, distributed with the density $F(r,\sigma;r,\sigma)$.

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