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A PROOF OF SOME DISPERSION RELATIONS IN QUANTUM FIELD THEORY

D u b n a, 1959

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A b s t r a c t

Dispersion relations for some inelastic processes are proved by the Bogolubov's method using also the Jost-Lehmann-Dyson integral representation for causal commutators.

I n t r o d u c t i o n

The dispersion relations method was first suggested in quantum field theory by M.L.Goldberger [1] in 1955. Dispersion relations provide a certain connection between the real and imaginary parts of the scattering amplitude. This connection leads to the relation between the quantities which are directly measurable in the scattering experiments. An experimental check of dispersion relations provide a straightforward verification or indication on a possible violation of the most basic principles of the physical theory, if only it were clearly demonstrated that dispersion relations are unambiguos consequences of these principles. In this connection strict proofs of any possible dispersion relations are of great interest. The dispersion relations method is basically related to the analyticaly properties of the scattering amplitudes which are in general generalized functions*. It was quite difficult to explore the analiticity properties of the scattering amplitudes and for some time the dispersion relations had no strict mathematical foundation.

In 1956 Bogolubov** developed a method which allowed to prove the dispersion relations strictly basing upon the theory of generalized functions and that of the functions of several complex variables. In this way he proved (see [4], Mathematical Appendix) the π-nucleon dispersion relations for momentum transfer

$$\Delta^2 = \frac{1}{4}(p-p')^2 \le \frac{M}{M+\mu}\mu^2$$

where M and μ are nucleon and pion masses respectively.

At the same time K. Symanzik*** found out a proof of dispersion relations for $\Delta^2=0$. This case is importantly much simpler due to the absence of the so-called nonobservable region.

The Bogolubov's method was further improved and developed in paper [5] where the upper

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* We call generalized function any linear continuous functional over the S-space of L. Schwartz [2] or, that is the same, of the classes C(p,q,n) introduced by N.N.Bogolubov (see, for instance [3]).

** Report at the International conference of Phys.-theor. in Seattle, U.S.A. (September, 1956); see also the book by N.N. Bogolubov, B.V. Medvedev and M.K. Polivanov [4].

*** Report at the international conference of Phys.Theor. in Seattle, U.S.A. (September, 1956).

limit for Δ^2 in the case of meson-nucleon scattering was extended up to $2\mu^2$. Soon the same result was obtained in paper by H.J. Bremermann, R. Oehme and J.G. Tailor [6]. The method of [6] is based on the construction of the envelope of holomorphy (see, for instance, [7] ch. IV) for special domains called semitubes.

Using the results of [5] V.S. Vladimirov in [8] was able to increase the upper limit for Δ^2 up to $2{,}56\mu^2$ (assuming $M=7\mu$). At last recently using the Dyson integral representation for causal commutators H. Lehman [9] obtained the estimation

$$\Delta^2 < \Delta^2_{max} = \frac{8}{3}\,\frac{2M+\mu}{2M-\mu}\,\mu^2 \approx 3{,}10\mu^2 \qquad \left(M = \frac{940}{140}\mu\right). \tag{0.1}$$

The dispersion relations for the processes 1-6 (see below) were proved in papers [8,14,20].

We shall obtain here a proof of dispersion relations for the processes 1-6 using the Bogolubov's method [4,5] and the integral representation of Jost-Lehmann-Dyson [22,10]. The momentum transfer interval will be extended.

I. Nucleon Compton-effect $(\gamma+p\to\gamma'+p')$ (M. Gell-Mann, M.L. Goldberger and W.E.Thirring [11]; N.N. Bogolubov, D.V. Shirkov [12]; T. Akiba and I. Sato [13]). For this process the parameters γ_j and τ_j^0* have the values:

$$\gamma_1=\gamma_2=2\,,\quad \tau_1^0=\tau_2^0=0.$$

The upper limit for the momentum transfer Δ^2 is calculated by (1.17) and is equal to

$$\Delta^2_{max} = \frac{(2M+\mu)(6M^2+9M\mu+4\mu^2)}{4M(M+\mu)^2}\,\mu^2 \approx 3{,}02\mu^2. \tag{0.2}$$

2. The bremsstrahlung of γ-quanta in electron-nucleon scattering $(p+e\to p'+e'+\gamma')$ (A.A. Logunov [5]).

$$\gamma_1=\gamma_2=2\,,\quad \tau_1^0=\tau^0<0\,,\quad \tau_2^0=0\,;$$

$$\Delta^2_{max} = \frac{M}{M+\mu}\,\frac{2\mu^2-\tau^0}{4} + \frac{(\mu^2-\tau^0)\mu^2}{2(2M+2\mu)^2} + \frac{(2M+\mu)\mu}{4M}\sqrt{4\mu^2-\tau^0}\Bigg[1+ \tag{0.3}$$

$$+\sqrt{1+\frac{M(2M+\mu)}{(2M+2\mu)^2}}\sqrt{1+\frac{\mu^2-\tau^0}{4\mu^2-\tau^0}\,\frac{M}{(M+\mu)^2}\,\frac{(2M+\mu)^2-\tau^0}{2M+\mu}}\ \Bigg] \quad \text{if} \quad \tau^0\geq -3\mu^2;$$

when $\tau^0<-3\mu^2$ Δ^2_{max} is given by Eq. (1.12). The calculations lead to the following results:

τ^0/μ^2	-1	-2	-3	-4	-5	-6	-7	-8	-9	-10	-20
$\frac{\Delta^2_{max}}{\mu^2}$	3,69	4,30	4,87	5,41	5,91	6,40	6,88	7,33	7,77	8,19	12,23

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* As to the meaning of the parameters γ_j and τ_j^0 see the main theorem.

3. Electron-positron pair photoproduction on the nucleons ($p+\gamma \to p'+e^+ + e^-$) (A.A.Logunov [15]).

$$\gamma_1=\gamma_2=2, \quad \tau_1^\circ=0, \quad (2m_e)^2 \le \tau_2^\circ=\tau^\circ < 2{,}15\mu^2$$

where m_e is the electron mass. The upper limit for τ° follows from (1.5). Δ^2_{max} is given by (0.3):

τ°/μ^2	0	0,25	0,5	0,75	1,0	1,5	2,0
Δ^2_{max}/μ^2	3,02	2,84	2,66	2,46	2,25	1,79	1,15

4. Meson photoproduction on the nucleons ($p+\gamma \to p'+\pi$) (A.A. Logunov and B.M.Stepanov [16], A.A. Logunov, L.D. Solovyev and A.N. Tavkhelidze [17], E. Corinaldesi [18], G.Chew, M.L. Goldberger, F. Low and Y. Nambu [19]).

$$\gamma_1=2, \gamma_2=3, \quad \tau_1^\circ=0, \quad \tau_2^\circ=\mu^2;$$

$$\Delta^2_{max}=\frac{M}{M+\mu}\left[\frac{\mu^2}{4}+(2M+\mu)\mu^2\sqrt{\frac{2}{3M(2M-\mu)}}\right]\left[1+\sqrt{1+\frac{M(2M+\mu)}{4(M+\mu)^2}}\right]\approx 3{,}04\mu^2. \qquad (0.4)$$

5. π-meson production in electron-nucleon collisions ($p+e \to p'+e'+\pi$) (A.A.Logunov [15], S. Fubini, Y. Nambu and V. Wataghin [21]).

$$\gamma_1=2, \gamma_2=3, \quad \tau_1^\circ=\tau^\circ<0, \quad \tau_2^\circ=\mu^2;$$

$$\Delta^2_{max}=\frac{M}{M+\mu}\left[\frac{\mu^2-\tau^\circ}{4}+\frac{(2M+\mu)\mu}{2}\sqrt{\frac{2}{3}\frac{4\mu^2-\tau^\circ}{M(2M-\mu)}}\right]\left[1+\sqrt{1+\frac{\mu^2-\tau^\circ}{4\mu^2-\tau^\circ}\cdot\frac{M}{(M+\mu)^2}\cdot\frac{(2M+\mu)^2-\tau^\circ}{2M+\mu}}\right] \qquad (0.5)$$

if $\tau^\circ \ge -3\mu^2$. If $\tau^\circ < -3\mu^2$ Δ^2_{max} is given by Eq.(1.12). The calculations yield the following results:

τ°/μ^2	−1	−2	−3	−4	−5	−6	−7	−8	−9	−10	−20
Δ^2_{max}/μ^2	3,73	4,36	4,95	5,51	6,04	6,54	7,03	7,50	7,95	8,39	12,32

6. Electron-positron pair production by π-mesons ($p+\pi \to p'+e^+ + e^-$) (A.A. Logunov [15]).

$$\gamma_1=3, \gamma_2=2, \quad \tau_1^\circ=\mu^2, \quad (2m_e)^2 \le \tau^\circ=\tau_2^\circ < 2{,}15\mu^2.$$

Δ^2_{max} follows from (0.5):

τ°/μ^2	0	0,25	0,5	0,75	1,0	1,5	2,0
Δ^2_{max}/μ^2	3,04	2,85	2,65	2,45	2,23	1,83	1,09

Processes 2-6 are the simplest examples of inelastic processes. For meson-nucleon scattering ($\gamma_1=\gamma_2=3,\ \tau_1^\circ=\tau_2^\circ=\mu^2$) the obtained value of Δ^2_{max} coincides with that obtained by H. Lehmann in |9|. (Eq.(0.1)). In processes 1 and 4 the values of Δ^2_{max} obtained here coincide with the corresponding ones, calculated by R. Oheme and J.G.Taylor|24|. As to process 5 our results coincide with the corresponding ones of paper|24| only if

$$\tau^\circ \geq -4\mu^2 \frac{M+\mu}{M-\mu} = \tau'.$$

When $\tau^\circ < \tau'$ the results of|24| are not correct as by calculating Δ^2_{max} by (1.12) another possibility in Eq.(1.12) was not taken into account (see the proof of the auxiliary theorem).

As follows from the main theorem the anti-hermitian part of the scattering amplitude for any fixed $t \geq \frac{1}{2}(M+\mu)$ and $\tau < 0$ is a holomorphic function in Δ^2 inside the ellipses (17) with the boundary

$$\Delta^2 = A + B\cos\delta + i C\sin\delta, \quad 0 \leq \delta < 2\pi$$

and the foci at

$$A \pm \frac{1}{2}\sqrt{\varphi^2(t,\tau+\tau_1^\circ)\,\varphi^2(t,\tau+\tau_2^\circ)}$$

if $\tau_j^\circ \leq \mu^2$ *. The functions A, B, C and φ^2 are determined by Eq. (1.8) - (1.10). If $t > \frac{1}{2}(M+\mu)$ we may introduce the new variable $\cos\theta$ by the equation

$$\cos\theta = \frac{2(\Delta^2 - A)}{\sqrt{\varphi^2(t,\tau+\tau_1^\circ)\,\varphi^2(t,\tau+\tau_2^\circ)}}.$$

Note that for real values this cosine is that of the angle between the momenta of the initial and final nucleons.

Then the anti-hermitian part of the amplitude will be an analytical function of $\cos\theta$ holomorphic inside the elipse with the foci at ± 1 and with the boundary

$$\cos\theta = D\cos\delta + i\sqrt{D^2-1}\,\sin\delta, \quad 0 \leq \delta < 2\pi \tag{0.6}$$

where

$$D = \frac{2B}{\sqrt{\varphi^2(t,\tau+\tau_1^\circ)\,\varphi^2(t,\tau+\tau_2^\circ)}}.$$

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* If one of $\tau_j^\circ > \mu^2$, the foci may lie on the emaginary axis.

Therefore it can be expanded in terms of Legendre polynomials [9,24,25] and this expansion converges inside the ellipse (0.6). The coefficients of this expansion are determined by the values of the anti-hermitian part by $-1 \leq \cos\theta \leq 1$. The obtained expansion, in particular, converges also for all real Δ^2 from the interval:

$$\Delta^2_{min} < \Delta^2 < \Delta^2_{max} .$$

This circumstance allows to express the anti-hermitian part in the non-observable region contained in the dispersion relations by the values from the physical region.

§ 1. Main Theorem

In this section we shall state the main theorem which is an extension and generalization of the corresponding Bogolubov's theorem ([14] Mathematical appendix, see also [5.6.8]).

This theorem provides the proof of dispersion relations for all mentioned above processes (see papers [14,20]) and leads, at the same time, to some analytical properties of the scattering amplitude as a function of energy and momentum transfer. The obtained results are given in the introduction. It is necessary to remark that the analytical properties of the scattering amplitude are proved according to the general principles of the quantum field theory such as causality, spectral conditions and covariance.

Main theorem. Let the four generalized function of four 4-vectors

$$\mathcal{F}_{ij}(x_1, x_2, x_3, x_4) , \quad i,j = r,a$$

be given which are invariant under the transformations of inhomogeneous orthochronous Lorentz group. Suppose that these generalized functions satisfy the following conditions:

$$\begin{aligned} &\mathcal{F}_{rr}=0 \text{ if } x_1 \lesssim x_3 \text{ or } x_2 \lesssim x_4 ; \\ &\mathcal{F}_{ra}=0 \text{ if } x_1 \lesssim x_3 \text{ or } x_2 \gtrsim x_4 ; \\ &\mathcal{F}_{ar}=0 \text{ if } x_1 \gtrsim x_3 \text{ or } x_2 \lesssim x_4 ; \\ &\mathcal{F}_{aa}=0 \text{ if } x_1 \gtrsim x_3 \text{ or } x_2 \gtrsim x_4 . \end{aligned} \tag{1.1}$$

Assume further that their Fourier-transforms $\widetilde{\mathcal{F}}_{ij}(p_1,\dots,p_4)$,

$$\int \mathcal{F}_{ij}(x_1,\dots,x_4) \exp i(p_1x_1+\dots+p_4x_4)\, dx_1\dots dx_4 = (2\pi)^4 \delta(p_1+\dots+p_4) \widetilde{\mathcal{F}}_{ij}(p_1,\dots,p_4)$$

defined, evidently, on a manifold

$$p_1+p_2+p_3+p_4=0 \tag{1.2}$$

satisfy the conditions

$$\widetilde{\mathcal{F}}_{rj} = \widetilde{\mathcal{F}}_{aj} \quad \text{if} \quad p_1^2<(\mathcal{M}+\mu)^2 \quad \text{and} \quad p_3^2<\gamma_1^2\mu^2, \; j=r,a;$$

$$\widetilde{\mathcal{F}}_{ir} = \widetilde{\mathcal{F}}_{ia} \quad \text{if} \quad p_2^2<(\mathcal{M}+\mu)^2 \quad \text{and} \quad p_4^2<\gamma_2^2\mu^2; \; i=r,a; \tag{1.3}$$

$$\widetilde{\mathcal{F}}_{ij}=0 \quad \text{if} \quad (p_1+p_3)^2<(\mathcal{M}+\mu)^2 \quad \text{or} \quad p_{10}+p_{30}<0, \; i,j=r,a. \tag{1.4}$$

We assume that $\gamma_j>1$, $\mathcal{M}+\mu\geq\gamma_j\mu>0$, $j=1,2$.

Let τ_j^0 be any fixed numbers with the property

$$\min_{t\geq\frac{1}{2}(\mathcal{M}+\mu)}\left[\left(t+\frac{\mathcal{M}^2-\tau_j^0}{4t}\right)^2-\mathcal{M}^2+\frac{(2\mathcal{M}+\mu)(\gamma_j^2\mu^2-\tau_j^0)}{4t^2-(\mathcal{M}+\mu-\gamma_j\mu)^2}\mu\right]>0, \; j=1,2. \tag{1.5}$$

Then it is possible to construct the generalized function $\varphi(z_1,z_2,\ldots,z_5;t)$ of the real variable t with the properties:

1) $\varphi(z_1,\ldots,z_5;t)$ is holomorphio with respect to the variables $z=(z_1,z_2,\ldots,z_5)$ in a certain domain D_t. The domain $\mathcal{D}_t$, $t\geq\frac{1}{2}(\mathcal{M}+\mu)$, contains all the points z of the form

$$z_1=\mathcal{M}^2, \; z_2=\mathcal{M}^2, \; z_3=\tau+\tau_1^0, \; z_4=\tau+\tau_2^0, \; z_5=-4\Delta^2 \tag{1.6}$$

where τ is any number less or equal to 0 and Δ^2 takes any complex numbers from the ellipse

$$\mathcal{A}(t,\tau)+\mathcal{B}(t,\tau)\cos\delta+i\,C(t,\tau)\sin\delta, \quad 0\leq\delta<2\pi \tag{1.7}$$

with

$$\left.\begin{aligned}
\mathcal{A}(t,\tau)&=\tfrac{1}{4}\varphi^2(t,\tau+\tau_1^0)+\tfrac{1}{4}\varphi^2(t,\tau+\tau_2^0)-\left(\frac{\tau_1^0-\tau_2^0}{8\tau}\right)^2,\\
\mathcal{B}(t,\tau)&=\tfrac{1}{2}\psi(t,\gamma_1,\tau+\tau_1^0)\psi(t,\gamma_2,\tau+\tau_2^0)+\\
&\quad+\tfrac{1}{2}\sqrt{\psi^2(t,\gamma_1,\tau+\tau_1^0)-\varphi^2(t,\tau+\tau_1^0)}\sqrt{\psi^2(t,\gamma_2,\tau+\tau_2^0)-\varphi^2(t,\tau+\tau_2^0)},\\
C(t,\tau)&=\tfrac{1}{2}\psi(t,\gamma_1,\tau+\tau_1^0)\sqrt{\psi^2(t,\gamma_2,\tau+\tau_2^0)-\varphi^2(t,\tau+\tau_2^0)}+\\
&\quad+\tfrac{1}{2}\psi(t,\gamma_2,\tau+\tau_2^0)\sqrt{\psi^2(t,\gamma_1,\tau+\tau_1^0)-\varphi^2(t,\tau+\tau_1^0)};
\end{aligned}\right\} \tag{1.8}$$

$$\varphi^2(t,\tau)=\left(t+\frac{\mathcal{M}^2-\tau}{4t}\right)^2-\mathcal{M}^2; \tag{1.9}$$

$$\psi(t,\tau,\gamma)=\begin{cases}\sqrt{\varphi^2(t,\tau)+\dfrac{(2\mathcal{M}+\mu)^2(\gamma^2\mu^2-\tau)}{4t^2-(\mathcal{M}+\mu-\gamma\mu)^2}} & \text{if} \quad \tau\geq\gamma\mu\left(\mathcal{M}+\mu-\dfrac{4t^2}{\mathcal{M}+\mu-\gamma\mu}\right),\\[2ex]
\dfrac{(2\mathcal{M}+\mu)\mu}{4t}+\dfrac{1}{4t}\sqrt{(2t+\mathcal{M}+\mu)^2-\tau}\sqrt{(2t-\mathcal{M}-\mu)^2-\tau} & \text{if} \quad \tau\leq\gamma\mu\left(\mathcal{M}+\mu-\dfrac{4t^2}{\mathcal{M}+\mu-\gamma\mu}\right).\end{cases} \tag{1.10}$$

In particular, if the inequality $\Delta^2_{min} < \Delta^2_{max}$* is fulfilled Δ^2 from the interval

$$\Delta^2_{min} < \Delta^2 < \Delta^2_{max} \tag{1.11}$$

belongs to all the domains $\mathcal{D}_t$. In Eq. (1.11)

$$\Delta^2_{max} = \min_{t \geq \frac{1}{2}(\mathcal{M}+\mu)} \left[\mathcal{A}(t,0) + \mathcal{B}(t,0) \right], \tag{1.12}$$

$$\Delta^2_{min} = \max_{\substack{t \geq \frac{1}{2}(\mathcal{M}+\mu) \\ \tau \leq 0}} \left[\mathcal{A}(t,\tau) - \mathcal{B}(t,\tau) \right]. \tag{1.13}$$

2) $\varphi(z_1, \dots, z_5; t) = 0$ if $t < \frac{1}{2}(\mathcal{M}+\mu)$.

3) For real ($P_1, P_2, P_3 P_4$) from the manifold (1.2) for which the values

$$z_1 = p_1^2,\ z_2 = p_2^2,\ z_3 = p_3^2,\ z_4 = p_4^2,\ z_5 = (p_1+p_2)^2$$

belong to the domain $\mathcal{D}_t$ where $t = \frac{1}{2}\sqrt{(p_1+p_3)^2}$ the representation

$$\mathcal{F}_{ij}(p_1, \dots, p_4) = \varphi\left[p_1^2, p_2^2, p_3^2, p_4^2, (p_1+p_2)^2; \tfrac{1}{2}\sqrt{(p_1+p_3)^2}\,\right]$$

takes place if

$$p_{10} + p_{30} \geq 0\ ,\ (p_1+p_3)^2 \geq (\mathcal{M}+\mu)^2.$$

Let us note some special cases of this theorem important for the applications.

1. Let

$$2 \leq \gamma_j \leq 3\ ,\ -\gamma_j^2 \mu^2 \frac{\mathcal{M}+\mu}{\mathcal{M}+\mu-\gamma_j\mu} \leq \tau_j^0 \leq \mu^2 \tag{1.14}$$

and $\mathcal{M}/\mu$ = the experimental radio of the nucleon mass to that π-meson (M = 6.71μ).

Then Δ^2_{min} = 0 (see the interpretation at the end of Sect. 3) and, consequently, it is possible to change the interval for Δ^2 in (1.11) into

$$0 < \Delta^2 < \Delta^2_{max}\ . \tag{1.15}$$

2. If the minimum in Eq. (1.12) is realized at t $= \frac{1}{2}(\mathcal{M}+\mu)$ and

$$\tau_j^0 \geq -\gamma_j^2 \mu^2 \frac{\mathcal{M}+\mu}{\mathcal{M}+\mu-\gamma_j\mu} \tag{1.16}$$

the expression for Δ^2_{max} may be simplified and becomes:

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* It leads again to some restriotions on the numbers $\gamma_j,\ \tau_j^0,\ \frac{\mathcal{M}}{\mu}$.

$$\Delta^2_{max} = \frac{M}{M+\mu}\,\frac{2\mu^2-\tau_1^0-\tau_2^0}{4} + \frac{(\mu^2-\tau_1^0)(\mu^2-\tau_2^0)}{8(M+\mu)^2} + \frac{2M+\mu}{2}\sqrt{\frac{(\gamma_1^2\mu^2-\tau_1^0)(\gamma_2^2\mu^2-\tau_2^0)}{\gamma_1\gamma_2(2M+2\mu-\gamma_1\mu)(2M+2\mu-\gamma_2\mu)}} \times$$

$$\times\left[1+\sqrt{1+\frac{(\mu^2-\tau_1^0)}{(\gamma_1^2\mu^2-\tau_1^0)}\,\frac{\gamma_1(2M+2\mu-\gamma_1\mu)}{4(M+\mu)^2}\cdot\frac{(2M+\mu)^2-\tau_1^0}{2M+\mu}}\,\sqrt{1+\frac{(\mu^2-\tau_2^0)\gamma_2(2M+2\mu-\gamma_2\mu)}{(\gamma_2^2\mu^2-\tau_2^0)\,4(M+\mu)^2}\cdot\frac{(2M+\mu)^2-\tau_2^0}{2M+\mu}}\right]. \tag{1.17}$$

For processes 1-6 when $\tau^0 \geq -3\mu^2$ the minimum in (1.11) is realized in fact at the point $t = 1/2\,(M+\mu)$ and, consequently, Δ^2_{max} is calculated according to Eq. (1.17). This leads us to Eqs. (0.1)-(0.5).

3. If

$\gamma_1 = \gamma_2 = \gamma$, $\tau_1^0 = \tau_2^0 = \tau^0$, $\tau^0 \geq -\gamma^2\mu^2\frac{M+\mu}{M+\mu-\gamma\mu}$

it is easy to see that $\Delta^2_{min} = 0$. In this case Eq. (1.12) is considerably simplified

$$\Delta^2_{max} = \min_{t\geq\frac{1}{2}(M+\mu)} \psi^2(t,\gamma,\tau^0) = \min_{t\geq\frac{1}{2}(M+\mu)}\left[\left(t+\frac{M^2-\tau^0}{4t}\right)^2 - M^2 + \frac{(2M+\mu)\mu(\gamma^2\mu^2-\tau^0)}{4t^2-(M+\mu-\gamma\mu)^2}\right]. \tag{1.18}$$

In case of meson-nucleon scattering and Compton effect the minimum in (1.18) is realized at the point $t=\frac{1}{2}(M+\mu)$ that leads us to Eqs. (0.1) and (0.2). In case of equal masses ($M=\mu$, $\gamma=2$, $\tau^0=\mu^2$) the minimum in (1.18) is realized at the point $t=\sqrt{\frac{3}{2}}\,M$; it gives $\Delta^2_{max} = 2M^2$ pointed out in [6].

§ 2. Auxiliary Theorem

The proof of the main theorem is based on a theorem about the analytical properties of the retarded and advanced amplitudes. In this section an auxiliary theorem is proved. This theorem is of interest by itself.

Theorem. Let two generalized functions $F_r(x)$ and $F_a(x)$, $x = (x_0, x_1, x_2, x_3)$ be given which vanish outside the advanced and retarded light cones respectively

$$F_r(x) = 0 \quad \text{if} \quad x \lesssim 0 \quad ; \quad F_a(x) = 0 \quad \text{if} \quad x \gtrsim 0. \tag{2.1}$$

Let besides their Fourier transforms $\tilde{F}_j(p)$, $p=(p_0,p_1,p_2,p_3)$, $j = r, a$ coincide in the domain $G_t^0(\gamma)$,

$$G_t^0(\gamma): \quad t-\sqrt{\vec{p}^{\,2}+\gamma^2\mu^2} < p_0 < -t+\sqrt{\vec{p}^{\,2}+(M+\mu)^2}. \tag{2.2}$$

Assume further that

$$2t \geq \mathcal{M}+\mu-\gamma\mu \geq 0 \;, \gamma>1 \,, \mu>0 . \tag{2.3}$$

Then there exists an analytical function $\tilde{\varphi}(k)$ of the four complex variables

$$k=(k_0,k_1,k_2,k_3)=p+iq=(p_0+iq_0,\vec{p}+i\vec{q})$$

which is holomorphic in a domain $G_t(\gamma)$. The domain $G_t(\gamma)$ is a set of the points k satisfying the following conditions: either

$$q^2>0 \quad \text{or by} \quad q^2 \leq 0$$

$$(p_0+t)^2<\vec{p}^{\,2}+(\mathcal{M}+\mu)^2, \quad (p_0-t)^2<\vec{p}^{\,2}+\gamma^2\mu^2, \tag{2.4}$$

$$v_1(p_0,|\vec{p}|)\equiv-\frac{\alpha^2\beta^2}{4}-t^2+p^2+\sqrt{\vec{p}^{\,2}(4t^2-\beta^2)+(t\alpha-p_0\beta)^2}<q^2 \quad \text{if} \quad p_0+|\vec{p}|\leq t\frac{\alpha}{\beta}, \tag{2.5}$$

$$v_2(p_0,|\vec{p}|)\equiv\frac{t-p_0-|\vec{p}|}{t+p_0+|\vec{p}|}\left[-(p_0+t)^2+\vec{p}^{\,2}+(\mathcal{M}+\mu)^2\right]<q^2 \quad \text{if} \quad p_0+|\vec{p}|\geq t\frac{\alpha}{\beta} \tag{2.6}$$

where

$$\alpha=\mathcal{M}+\mu+\gamma\mu, \quad \beta=\mathcal{M}+\mu-\gamma\mu .$$

The function $\tilde{\varphi}(k)$ is such that for all real k = p from the domain (2.2)

$$\tilde{\varphi}(p)=\tilde{\mathcal{F}}_r(p)=\tilde{\mathcal{F}}_a(p) .$$

<u>Proof.</u> In papers by R. Jost and H. Lehmann |22|, and F.J. Dyson |10| an integral representation of causal commutators was found. Applying the Dyson theorem to our case we get

$$\tilde{\mathcal{F}}_r(p)-\tilde{\mathcal{F}}_a(p)=\int\varepsilon(p_0-u_0)\,\varphi\left[u,(p-u)^2\right]du \tag{2.7}$$

where a generalized function $\varphi(u,\lambda^2)$ vanishes outside the domain

$$|u_0|+|\vec{u}|\leq t \,, \quad \lambda\geq \max\left[0,\mathcal{M}+\mu-\sqrt{(t+u_0)^2-\vec{u}^{\,2}},\ \gamma\mu-\sqrt{(t-u_0)^2-\vec{u}^{\,2}}\right]. \tag{2.8}$$

By (2.1) the functions $\tilde{\mathcal{F}}_r(k)$ and $\tilde{\mathcal{F}}_a(k)$ are holomorphic in the domains $q_0>|\vec{q}|$ and $q_0<-|\vec{q}|$ respectively and the weak limits

$$\underset{q\to0,\ q_0>|\vec{q}|}{\tilde{\mathcal{F}}_r(p+iq)\to\tilde{\mathcal{F}}_r(p)} \quad , \quad \underset{q\to0,\ q_0<-|\vec{q}|}{\tilde{\mathcal{F}}_a(p+iq)\to\tilde{\mathcal{F}}_a(p)} \tag{2.9}$$

hold. Let us fix a real vector $\vec{p}$ and consider the functions $\tilde{\mathcal{F}}_r(k_0,\vec{p})$ and $\tilde{\mathcal{F}}_a(k_0,\vec{p})$ which are holomorphic with respect to $k_0=p_0+iq_0$ in the upper and lower planes respectively. These functions are bounded polynomially in the domains $q_0\geq\delta$ and $q_0\leq-\delta$ respectively (by each $\delta>0$). From the proof of lemma I in [23] it follows that the degree m of the majorant polynomial does not depend on $\vec{p}$ and δ but only on a class $C(\tau,s;4)$ to which the functions $\tilde{\mathcal{F}}_j$ belong. Taking into account the properties of the functions $\tilde{\mathcal{F}}_j(k_0,\vec{p})$ and applying the Cauchy theorem by $q_0>0$ we get the expression

$$\tilde{\mathcal{F}}_r(k_0,\vec{p})=\frac{(k_0-iN)^{n+1}}{2\pi i}\int_{-\infty}^{\infty}\frac{\tilde{\mathcal{F}}_r(\xi,\vec{p})-\tilde{\mathcal{F}}_a(\xi,\vec{p})}{(\xi-k_0)(\xi-iN)^{n+1}}d\xi+\sum_{s=0}^{n}\frac{(k_0-iN)^s}{s!}\frac{\partial^s\tilde{\mathcal{F}}_r(\xi,\vec{p})}{\partial\xi^s}\Big|_{\xi=iN} \tag{2.10}$$

where n is a sufficiently large natural number (in any case not less than m) and N is an arbitrary positive number.

Taking into account the representation (2.7) and making simple calculations one can get from (2.10)

$$\tilde{\mathcal{F}}_r(k_0,\vec{p})=\sum_{s=0}^{n}\frac{(k_0-iN)^s}{s!}\tilde{\mathcal{F}}^{(s)}(iN,\vec{p})+\frac{(k_0-iN)^{n+1}}{2\pi i}\times \tag{2.11}$$
$$\times\int\frac{\left[k_0-u_0-\sqrt{\lambda^2+(\vec{p}-\vec{u})^2}\right]\left[u_0-iN+\sqrt{\lambda^2+(\vec{p}-\vec{u})^2}\right]^{n+1}-\left[k_0-u_0+\sqrt{\lambda^2+(\vec{p}-\vec{u})^2}\right]\left[u_0-iN-\sqrt{\lambda^2+(\vec{p}-\vec{u})^2}\right]^{n+1}}{2\sqrt{\lambda^2+(\vec{p}-\vec{u})^2}\left[(k_0-u_0)^2-(\vec{p}-\vec{u})^2-\lambda^2\right]\left[(u_0-iN)^2-(\vec{p}-\vec{u})^2-\lambda^2\right]^{n+1}}\varphi(u,\lambda^2)\,du\,d\lambda^2.$$

The integral in (2.11) may be treated as a result of action of the funotional (generalized function) φ on the corresponding function from a suitable class $C(\tau,s;4)$ (if n is sufficiently large).

It follows from (2.11) that the function $\tilde{\mathcal{F}}_r(k_0,\vec{p})$ may be analytically extended on complex values of $\vec{p}$. The corresponding function $\tilde{\mathcal{F}}_r(k)$ is holomorphic for those k whioh satisfy inequalities: $\vec{q}^{\,2}<N^2$ and

$$\left.\begin{aligned}&(k_0-u_0)^2-(\vec{k}-\vec{u})^2-\lambda^2\neq0\\&(iN-u_0)^2-(\vec{k}-\vec{u})^2-\lambda^2\neq0\end{aligned}\right\}\quad\text{for all } (u,\lambda^2) \text{ from the domain (2.8).} \tag{2.12}$$

As N is an arbitrary positive number and domain (2.12) monotonously decreases with the decrease of N one can pass to the limit in (2.12) as $N\to\infty$. Thus the points k for which inequality

$$(k_0-u_0)^2-(\vec{k}-\vec{u})^2-\lambda^2\neq0 \tag{2.13}$$

holds for all (u,λ^2) from (2.8) belong to the domain of holomorphy of the function $\tilde{\mathcal{F}}_r(k)$. From oondition (2.13) it follows immediately that the points k for which $q^2>0$ belong to the domain of holomorphy. Therefore we can restrict our consideration only by those points k for which $q^2 \leq 0$. The latter are contained owing to (2.13) in a domain determined by the following condition:

$$\mathrm{Re}\left[(k_0-u_0)^2-(\vec{k}-\vec{u})^2-\lambda^2\right] = (p-u)^2-q^2-\lambda^2 < 0 \tag{2.14}$$

for all (u,λ^2) from (2. 8).

In order to simplify the condition (2.14) we assume, for example, that $2t \geq M+\mu+\gamma\mu$. Taking into account (2.8) we can write (2.14) in the form:

$$\max_{|u_0|+|\vec{u}|\leq t} f(u_0,|\vec{u}|) < 0 \tag{2.15}$$

where f is a continous funotion of its arguments (see Fig. 1),

$$f(u_0,|\vec{u}|) = \begin{cases} (p_0-u_0)^2-(|\vec{p}|-|\vec{u}|)^2-q^2 & \text{if } (u_0,|\vec{u}|)\in \mathrm{I}, \\ (p_0-u_0)^2-(|\vec{p}|-|\vec{u}|)^2-q^2-\left[\gamma\mu-\sqrt{(t-u_0)^2-\vec{u}^2}\right]^2 & \text{if } (u_0,|\vec{u}|)\in \mathrm{II}, \\ (p_0-u_0)^2-(|\vec{p}|-|\vec{u}|)^2-q^2-\left[M+\mu-\sqrt{(t+u_0)^2-\vec{u}^2}\right]^2 & \text{if } (u_0,|\vec{u}|)\in \mathrm{III}. \end{cases}$$

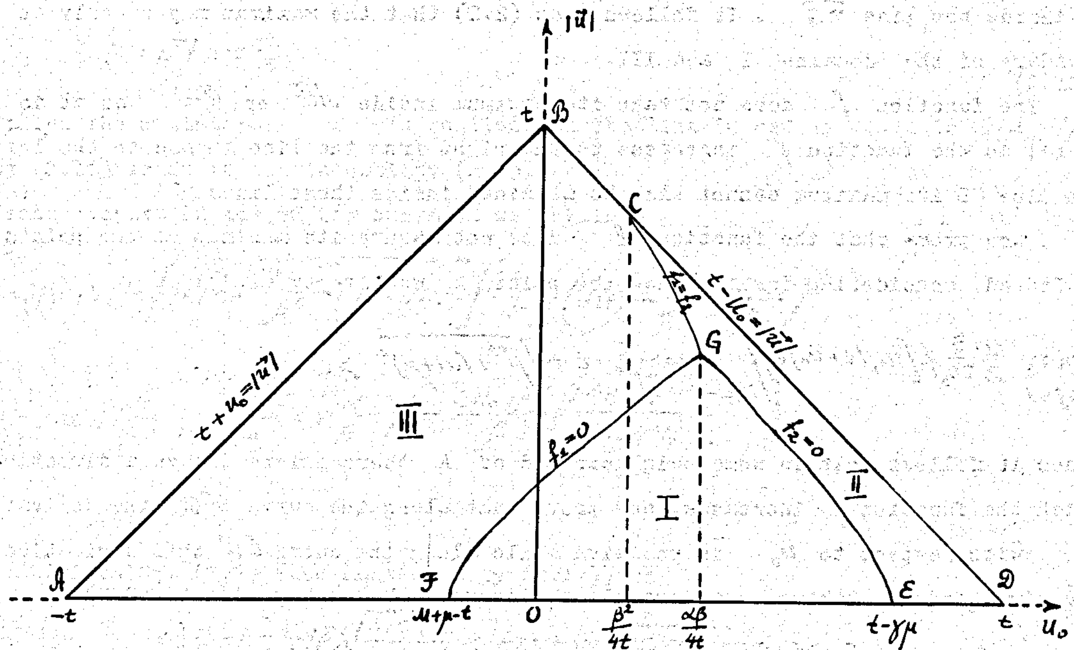

Fig. 1.

The equations of the curves FG, EG and CG are

$$f_1(u_0,|\vec{u}|)\equiv \mathcal{M}+\mu-\sqrt{(t+u_0)^2-\vec{u}^2}=0\,,$$

$$f_2(u_0,|\vec{u}|)\equiv \gamma\mu-\sqrt{(t-u_0)^2-\vec{u}^2}=0\,,$$

$$f_3(u_0,|\vec{u}|)\equiv f_1(u_0,|\vec{u}|)-f_2(u_0,|\vec{u}|)=0$$

respectively. The equation of the curve $\mathcal{C}\mathcal{G}$ takes a form

$$2\beta|\vec{u}|=\sqrt{(4t^2-\beta^2)(\beta^2-4u_0^2)}\quad,\ \frac{\beta^2}{4t}\leq u_0\leq\frac{\alpha\beta}{4t}\,. \tag{2.16}$$

Noticing that

$$f(-t,0)=(p_0+t)^2-\vec{p}^2-q^2-(\mathcal{M}+\mu)^2\,,\ f(t,0)=(p_0-t)^2-\vec{p}^2-q^2-\gamma^2\mu^2$$

one can see from (2.15) that in case $q^2\leq 0$ the domain of holomorphy contains only those points k which fulfil the inequalities (2.4) or, that is the same, the inequalities (2.2).

It is clear that the function f does not take its maximum inside the domain I or inside the line $\mathcal{F}\mathcal{E}$. It follows from (2.2) that the maximum may be only at the boundary of the domains I and III.

The function f does not take its maximum inside $\mathcal{A}\mathcal{F}$ or $\mathcal{E}\mathcal{D}$ as it is linear there; as the function f increases to the right from the line AB and to the left from the line CD its maximum cannot also be attained inside these lines.

Now prove that the function f does not assume its maximum at the points A or D. Indeed, considering for instance the point A we have by (2.2)

$$\max_{0\leq\rho\leq 1}\frac{1}{2}\frac{\partial}{\partial u_0}f\big[u_0,(t+u_0)\rho\big]\Big|_{u_0=-t}=-p_0-t+\sqrt{\vec{p}^2+(\mathcal{M}+\mu)^2}>0\,.$$

Hence it follows that in some neighbourhood of A there exists always a direction along which the function f increases. Now prove that along the curve $\mathcal{F}\mathcal{G}$ the derivative of f with respect to u_0 is positive while along the curve $\mathcal{E}\mathcal{G}$ this derivative is negative. Indeed, taking as an example the curve $\mathcal{F}\mathcal{G}$ we have due to (2.2)

$$f=(p_0-u_0)^2-q^2-\Big[|\vec{p}|-\sqrt{(t+u_0)^2-(\mathcal{M}+\mu)^2}\Big]^2,\ \mathcal{M}+\mu-t\leq u_0\leq\frac{\alpha\beta}{4t}\ ;$$

$$\frac{1}{2}\frac{\partial f}{\partial u_0}=-p_0-t+\frac{(t+u_0)|\vec{p}|}{\sqrt{(t+u_0)^2-(\mathcal{M}+\mu)^2}}\geq -p_0-t+\frac{(4t^2+\alpha\beta)|\vec{p}|}{\sqrt{(4t^2+\alpha\beta)^2-16t^2(\mathcal{M}+\mu)^2}}\geq$$

$$\geq -p_0-t+\sqrt{\vec{p}^{\,2}+(\mathcal{M}+\mu)^2}>0 .$$

Here we took into account only those $\vec{p}^{\,2}$, which due to Eq. (2.2) satisfy the inequality

$$\vec{p}^{\,2}>\frac{(4t^2-\alpha^2)(4t^2-\beta^2)}{16t^2}. \tag{2.17}$$

From the above statements one can conclude that the function f may take its maximum either on the curve BC or on the line CG only. On the curve BC we have

$$f=(p_0-u_0)^2-(|\vec{p}|-t+u_0)^2-q^2-(\mathcal{M}+\mu-2\sqrt{tu_0})^2, \quad 0\leq u_0\leq\frac{\beta^2}{4t};$$

$$\frac{1}{2}\frac{\partial f}{\partial u_0}=-t-p_0-|\vec{p}|+(\mathcal{M}+\mu)\sqrt{\frac{t}{u_0}} .$$

Hence it follows that if the inequality

$$p_0+|\vec{p}|\geq t\frac{\alpha}{\beta} \tag{2.18}$$

is fulfilled the maximum of f is realised on the line BC and is equal to $\varphi_2(p_0,|\vec{p}|)-q^2$ which by (2.15) leads to the inequality (2.6).

Taking into account (2.16) on the curve CG we obtain

$$f=(p_0-u_0)^2-\left[|\vec{p}|-\sqrt{\frac{4t^2-\beta^2}{\beta^2}(\beta^2-4u_0^2)}\right]^2-q^2-\left(\mathcal{M}+\mu-\frac{\beta}{2}-\frac{2u_0t}{\beta}\right)^2;$$

$$\frac{\beta^2}{4t}\leq u_0\leq\frac{\alpha\beta}{4t}\ ;\quad \frac{1}{2}\frac{\partial f}{\partial u_0}=t\frac{\alpha}{\beta}-p_0-\frac{2u_0|\vec{p}|}{\beta}\sqrt{\frac{4t^2-\beta^2}{\beta^2-4u_0^2}} .$$

If p satisfies Eq. (2.2) (and consequently (2.17)) so

$$\left.\frac{1}{2}\frac{\partial f}{\partial u_0}\right|_{u_0=\frac{\alpha\beta}{4t}-0}=t\frac{\alpha}{\beta}-p_0-\frac{\alpha}{\beta}|\vec{p}|\sqrt{\frac{4t^2-\beta^2}{4t^2-\alpha^2}}<t-p_0-\sqrt{\vec{p}^{\,2}+\gamma^2\mu^2}<0.$$

Therefore if Eq. (2.18) is not fulfilled the maximum of f is certainly obtained on the curve CG and equals $v_1^2(p_0,|\vec{p}|)-q^2$. Due to (2.15) this leads to inequality (2.5).

Now prove that when $q=0$ inequalities (2.5)-(2.6) follow from inequalities (2.4)or, that is the same, from those (2.2).

To see this it is sufficient to notice that inequality (2.5) when $q=0$ may be written in the form

$$[(p_0+t)^2-\vec{p}^2-(\mathcal{M}+\mu)^2][(p_0-t)^2-\vec{p}^2-\gamma^2\mu^2]>0 \quad if \quad p_0+|\vec{p}|\leq t\frac{\alpha}{\beta}.$$

Thus it is proved that the domain $G_t(\gamma)$ contains all points of the domain $G_t^0(\gamma)$.

The similar treatments (with the corresponding simplifications) lead to the same domain $G_t(\gamma)$ for the case $2t\leq \mathcal{M}+\mu+\gamma\mu$.

Repeating similar considerations for the function $\tilde{\mathcal{F}}_a$ we get an analitical function $\tilde{\mathcal{F}}_a(k)$ which is holomorphic in the domain $G_t(\gamma)$. But as it has just been proved the domain $G_t(\gamma)$ itself contains all points of $G_t^0(\gamma)$ together with their (complex) neighbourhoods. Hence the constructed functions $\tilde{\mathcal{F}}_j(k)$ are holomorphic in some complex neighbourhood of the domain $G_t^0(\gamma)$. But for real p from the domain $G_t^0(\gamma)$ these functions coincide as generalized functions and therefore due to their holomorphy property they coincide at each point. Hence the functions $\tilde{\mathcal{F}}_j(k), j=z,a$ define in fact a single analytical function $\tilde{\varphi}(k)$ which is holomorphic in the domain $G_t(\gamma)$ and coincides with the functions $\tilde{\mathcal{F}}_j$ at the real points k = p from the domain $G_t^0(\gamma)$. The theorem is proved completely.

Note a consequence useful for applications. It was stated in the course of proving the theorem. Under the conditions of the theorem the function $\tilde{\varphi}(k)$ is holomorphic in a complex neighbourhood of the domain $G_t^0(\gamma)$. The stated result was first obtained by N.N. Bogolubov (|4|, Mathematical appendix, theorem I); further this result was proved by other methods in papers|22,5,6|. In|6| theorems of this type were called the "edge of the wedge" theorems.

§ 3. Proof of the main Theorem

The proof of the main theorem is similar to that of theorems II and III in paper|5|. Instead of theorem I from|5| we use here the auxiliary theorem proved in Sec. 2. Preliminarily we shall briefly state those parts of the proof which to some extent imitate the corresponding considerations in paper|5|.

As the generalized functions $\mathcal{F}_j(x_1,...,x_4)$ are invariant under translations they can be treated as generalized functions of 3 four-vectors, eq.

$$y_1=x_1-x_3,\ y_2=x_4-x_2,\ y_3=x_1-x_2+x_3-x_4.$$

Therefore, we obtain functions

$$\varphi_{ij}(y_1,y_2,y_3)=\varphi_{ij}(x_1-x_3,x_4-x_2,x_1-x_2+x_3-x_4)=\mathcal{F}_{ij_1}(x_1,\dots,x_4) \tag{3.1}$$

$$\tilde{\varphi}_{ij}(q_1,q_2,q_3)=2^4\tilde{\mathcal{F}}_{ij_1}(q_1+q_3,-q_2-q_3,q_3-q_1,q_2-q_3) \tag{3.2}$$

where $j_1=r$ if $j=a$ and $j_1=a$ if $j=r$.

Taking

$$p_1=q_1+q_3,\ p_2=-q_2-q_3,\ p_3=q_3-q_1,\ p_4=q_2-q_3$$

we get

$$q_3=\frac{p_1+p_3}{2},\quad p_1^2=(q_1+q_3)^2,\ p_2^2=(q_2+q_3)^2,\ p_3^2=(q_3-q_1)^2,\ p_4^2=(q_2-q_3)^2, \tag{3.3}$$
$$(p_1+p_2)^2=(q_1-q_2)^2.$$

Due to conditions (1.4) and to relations (3.3) the functions $\tilde{\varphi}_{ij}(q_1,\dots,q_3)$ vanish on a domain where

$$q_3^2<\left(\frac{M+\mu}{2}\right)^2 \quad \text{or} \quad q_{30}<0, \tag{3.4}$$

i.e. outside the future light cone. Therefore and due to the Lorentz invariancy of the functions $\tilde{\varphi}_{ij}$ one can always choose a coordinate system in such a way that vector q_3 would be equal to

$$q_3=(t,\vec{o}). \tag{3.5}$$

In this coordinate system the generalized functions $\tilde{\varphi}_{ij}(q_1,q_2,q_3)$ will be written in the form $\tilde{f}_{ij}(q_1,q_2;t)$. Thus we obtain four generalized functions $f_{ij}(y_1,y_2;y_0)$ of 9 variables $(y_{10},\vec{y}_1,y_{20},\vec{y}_2,y_0)$ which due to (1.1), (1.3), (1.4) and (3.1)-(3.5) possess the following properties:

1. $$\left.\begin{aligned} f_{rr}&=0 \quad \text{if} \quad y_1\lesssim 0 \quad \text{or} \quad y_2\lesssim 0,\\ f_{ra}&=0 \quad \text{if} \quad y_1\lesssim 0 \quad \text{or} \quad y_2\gtrsim 0,\\ f_{ar}&=0 \quad \text{if} \quad y_1\gtrsim 0 \quad \text{or} \quad y_2\lesssim 0,\\ f_{aa}&=0 \quad \text{if} \quad y_1\gtrsim 0 \quad \text{or} \quad y_2\gtrsim 0.\end{aligned}\right\} \tag{3.6}$$

2. $$\tilde{f}_{rj}=\tilde{f}_{aj},\ j=r,a \quad \text{if} \quad q_1\in G_t^{\circ}(\gamma_1). \tag{3.7}$$
$$\tilde{f}_{ir}=\tilde{f}_{ia},\ i=r,a \quad \text{if} \quad q_2\in G_t^{\circ}(\gamma_2). \tag{3.8}$$

3. $$f_{ij}=0,\ i,j=r,a \quad \text{if} \quad t<\tfrac{1}{2}(M+\mu). \tag{3.9}$$

4. The functions $\tilde{f}_{ij}(q_1,q_2;t)$ are invariant under space rotations and reflections of vectors $\vec{q}_1$ and $\vec{q}_2$.

We may conclude due to (3.6) and (3.7) that for two pairs of functions $\tilde{f}_{tj}(q_1,q_2;t)$ and $\tilde{f}_{aj}(q_1,q_2;t)$ $(j=t,a)$ the conditions of the auxiliary theorem are fulfilled with respect to the variable q_1. Using this theorem we obtain two functions

$$\tilde{\varphi}_j(k_1,q_2;t)\ ,\quad j=t,a \tag{3.10}$$

which are holomorphic with respect to the variable k_1 in the domain $G_t(\delta_1)$ and are generalized with respect to the variables (q_2,t). By (3.6), (3.8) and (3.9) the functions (3.10) satisfy anew the conditions of the auxiliary theorem with respect to variable q_2. Applying again this theorem to the functions (3.10) we get the function $\tilde{\varphi}(k_1,k_2;t)$ which is holomorphic in the domain $G_t(\delta_1)\times G_t(\delta_2)$ with respect to variables (k_1, K_2) and is generalized with respect to the real variable t. Here we have used the Hartog's theorem (see e.g. |7|, ch. VII) which states: if a function $f(z_1,z_2,\dots,z_n)$ is holomorphic in each variable when others are fixed then it is holomorphic with respect to the variables $(z_1,z_2,\dots,z_n)$.

The obtained function $\tilde{\varphi}(k_1,k_2;t)$ for real $(k_1,k_2)=(q_1,q_2)$ from the domain $G_t^\circ(\delta_1)\times G_t^\circ(\delta_2)$ coincides with the functions $\tilde{f}_{ij}(q_1,q_2;t)$.

Now we make use the invariance conditions under space rotations and reflections. Due to these conditions the function $\tilde{\varphi}(k_1,k_2;t)$ depends on (k_1,k_2) only by means of five variables

$$k_{10},\ k_{20},\ \vec{k}_1^{\,2},\ \vec{k}_2^{\,2},\ \vec{k}_1\vec{k}_2\,.$$

One may introduce an equivalent system of variables $z=(z_1,z_2,\dots,z_5)$ instead of them taking

$$z_1=(k_{10}+t)^2-\vec{k}_1^{\,2},\quad z_2=(k_{20}+t)^2-\vec{k}_2^{\,2},\quad z_3=(k_{10}-t)^2-\vec{k}_1^{\,2},$$
$$z_4=(k_{20}-t)^2-\vec{k}_2^{\,2},\quad z_5=(k_{10}-k_{20})^2-(\vec{k}_1-\vec{k}_2)^2. \tag{3.11}$$

Therefore

$$\tilde{\varphi}(k_1,k_2;t)=\varphi(z_1,z_2,\dots,z_5;t)\,. \tag{3.12}$$

From (3.11) we get

$$k_{10}=\frac{z_1-z_3}{4t},\quad k_{20}=\frac{z_2-z_4}{4t},\quad (\vec{k}_1-\vec{k}_2)^2=-z_5+\left(\frac{z_1-z_2-z_3+z_4}{4t}\right)^2,$$
$$\vec{k}_1^{\,2}=t^2-\frac{z_1+z_3}{2}+\left(\frac{z_1-z_3}{4t}\right)^2,\quad \vec{k}_2^{\,2}=t^2-\frac{z_2+z_4}{2}+\left(\frac{z_2-z_4}{4t}\right)^2. \tag{3.13}$$

Thus in order to prove the main theorem it is sufficient owing to (1.6) to show that for arbitrary numbers t, τ and Δ^2 from intervals $t \geq \frac{\mu + \mu}{2}$, $\tau \leq 0$ and from ellipse (1.7) respectively there exists at least one complex point (k_1, k_2) satisfying the equations

$$\vec{k}_j^2 = t^2 - \frac{\mu^2 + \tau + \tau_j^\circ}{2} + \left(\frac{\mu^2 - \tau - \tau_j^\circ}{4t}\right)^2 = \varphi^2(t, \tau + \tau_j^\circ),$$
$$k_{j0} = \frac{\mu^2 - \tau - \tau_j^\circ}{4t}, \quad (\vec{k}_1 - \vec{k}_2)^2 = 4\Delta^2 + \left(\frac{\tau_1^\circ - \tau_2^\circ}{4t}\right)^2, \quad j = 1, 2 \tag{3.14}$$

and belonging to the domain $G_t(\gamma_1) \times G_t(\gamma_2)$.

It follows from (3.14) that the real and imaginary parts of the complex vectors $\vec{k}_j = \vec{p}_j + i\vec{q}_j$ are orthogonal, $\vec{p}_j \vec{q}_j = 0$, $j = 1, 2$.
Hence the equations (3.14) may be written in the form

$$p_{j0} = \frac{\mu^2 - \tau - \tau_j^\circ}{4t}, \quad q_{j0} = 0, \quad \vec{p}_j^2 - \vec{q}_j^2 = \varphi^2(t, \tau + \tau_j^\circ), \quad j = 1, 2; \tag{3.15}$$

$$\vec{p}_1^2 + \vec{p}_2^2 - 2|\vec{p}_1||\vec{p}_2|\mu_1 - \vec{q}_1^2 - \vec{q}_2^2 + 2|\vec{q}_1||\vec{q}_2|\mu_2 + 2i|\vec{q}_1||\vec{p}_2|\nu_1 +$$
$$+ 2i|\vec{p}_1||\vec{q}_2|\nu_2 = 4\Delta^2 + \left(\frac{\tau_1^\circ - \tau_2^\circ}{4t}\right)^2 \tag{3.16}$$

where the numbers μ_j and ν_j, $j = 1, 2$ are cosines of the angles between the corresponding vectors,

$$\mu_1\nu_2 + \nu_1\mu_2 + \alpha_1\alpha_2 = 0, \quad \mu_j^2 + \nu_j^2 = 1 - \alpha_j^2, \quad |\mu_j| \leq 1, \quad |\nu_j| \leq 1, \quad |\alpha_j| \leq 1, \quad j = 1, 2. \tag{3.17}$$

Excluding the values $|\vec{q}_j|$ from (3.16) we get owing to (3.15) the relation

$$4\Delta^2 = \varphi^2(t, \tau + \tau_1^\circ) + \varphi^2(t, \tau + \tau_2^\circ) - \left(\frac{\tau_1^\circ - \tau_2^\circ}{4t}\right)^2 - 2|\vec{p}_1||\vec{p}_2|\mu_1 + 2\sqrt{\vec{p}_1^2 - \varphi^2(t, \tau + \tau_1^\circ)} \times$$
$$\times \sqrt{\vec{p}_2^2 - \varphi^2(t, \tau + \tau_2^\circ)}\,\mu_2 + 2i|\vec{p}_1|\sqrt{\vec{p}_2^2 - \varphi^2(t, \tau + \tau_2^\circ)}\,\nu_2 + 2i|\vec{p}_2|\sqrt{\vec{p}_1^2 - \varphi^2(t, \tau + \tau_1^\circ)}\,\nu_1. \tag{3.18}$$

We use now the auxiliary theorem from Sec.2. First we shall find out the conditions that the points K satisfying equations (3.15),

$$K=\left[\frac{M^2-\tau-\tau^\circ}{4t},\ \vec{p}+i\vec{\lambda}\sqrt{\vec{p}^{\,2}-\varphi^2(t,\tau+\tau^\circ)}\right], \tag{3.19}$$

belong to the domain $G_t(\gamma)$. In (3.19) $\vec{\lambda}$ is the real orth orthogonal to the vector $\vec{p}$ and $\vec{p}$ is such an arbitrary real vector that*

$$\max\left[0,\ \varphi^2(t,\tau+\tau^\circ)\right]\le \vec{p}^{\,2}. \tag{3.20}$$

In the considered case $q^2\le 0$.

It follows from (1.9) that inequalities (2.4) for points (3.19) are always fulfilled if $\tau^\circ<\gamma^2\mu^2$. The conditions (2.5) and (2.6) become respectively

$$|\vec{p}|<\sqrt{\varphi^2(t,\tau+\tau^\circ)+\frac{(2M+\mu)(\gamma^2\mu^2-\tau-\tau^\circ)}{4t^2-(M+\mu-\gamma\mu)^2}\mu}\equiv u_1(t,\tau+\tau^\circ) \tag{3.21}$$

$$\text{if}\quad \frac{M^2-\tau-\tau^\circ}{4t}+|\vec{p}|\le t\,\frac{M+\mu+\gamma\mu}{M+\mu-\gamma\mu}\ ;$$

$$u_3(t,\tau+\tau^\circ)\equiv\frac{(2M+\mu)\mu}{4t}-\frac{1}{4t}\sqrt{(2t+M+\mu)^2-\tau-\tau^\circ}\sqrt{(2t-M-\mu)^2-\tau-\tau^\circ}<|\vec{p}|<$$

$$<\frac{(2M+\mu)\mu}{4t}+\frac{1}{4t}\sqrt{(2t+M+\mu)^2-\tau-\tau^\circ}\sqrt{(2t-M-\mu)^2-\tau-\tau^\circ}\equiv u_2(t,\tau+\tau^\circ) \tag{3.22}$$

$$\text{if}\quad \frac{M^2-\tau-\tau^\circ}{4t}+|\vec{p}|\ge t\,\frac{M+\mu+\gamma\mu}{M+\mu-\gamma\mu}\ .$$

The function $u_1(t,\tau)$ decreases monotonously with the increase of τ. Therefore the inequality

$$\min_{t\ge\frac{1}{2}(M+\mu)}\left[\left(t+\frac{M^2-\tau^\circ}{4t}\right)^2-M^2+\frac{(2M+\mu)(\gamma^2\mu^2-\tau^\circ)}{4t^2-(M+\mu-\gamma\mu)^2}\mu\right]>0 \tag{3.23}$$

provides the positivity of the expression under the root in (3.21). The function $u_3(t,\tau)$ is negative for all

$$t\ge\frac{1}{2}(M+\mu)\ ,\quad \tau\le\gamma\mu\left(M+\mu-\frac{4t^2}{M+\mu-\gamma\mu}\right).$$

---

* When $\tau>\mu^2$ the function $\varphi^2(t,\tau)$ may become negative.

Therefore there is in fact no restriction on $|\vec{p}|$ from below in (3.22).

Let us consider in the 3-dimensional space of variables $(t, \tau, |\vec{p}|)$ three surfaces

$$|\vec{p}|=u_1(t,\tau+\tau^\circ),\quad |\vec{p}|=u_2(t,\tau+\tau^\circ),\quad |\vec{p}|=t\,\frac{M+\mu+\gamma\mu}{M+\mu-\gamma\mu}-\frac{M^2\tau+\tau^\circ}{4t}.$$

By the construction (see the auxiliary theorem) these surfaces intersect along a common curve the projection of which on the plane $|\vec{p}|=0$ is given by equation

$$u_1(t,\tau+\tau^\circ)=u_2(t,\tau+\tau^\circ)$$

or other performing

$$\tau+\tau^\circ=\gamma\mu\left(M+\mu-\frac{4t^2}{M+\mu-\gamma\mu}\right).$$

Due to the above stated conditions (3.21)-(3.22) can be rewritten in the form:

$$|\vec{p}|<\psi(t,\gamma,\tau+\tau^\circ)$$

where the continuous function ψ is defined by Eqs. (1.10).

Thus, owing to (3.20) and (3.24), we have obtained the following result: all points (3.19) for which enequality

$$\max\left[0,\varphi^2(t,\tau+\tau^\circ)\right]\le\vec{p}^{\,2}<\psi^2(t,\gamma,\tau+\tau^\circ) \tag{3.25}$$

is fulfilled belong to the domain $G_t(\gamma)$.

Now we shall derive the conditions which provide non-emptiness of the domain (3.25) for all $\tau\le 0$ and $t\ge\frac{1}{2}(M+\mu)$. It can be easily seen that the inequalities

$$\psi^2(t,\gamma,\tau+\tau^\circ)>0\;,\quad \psi^2(t,\gamma,\tau+\tau^\circ)-\varphi^2(t,\tau+\tau^\circ)>0 \tag{3.26}$$

are necessary and sufficient for it.

It follows from (1.9) and (1.10) that the function φ and ψ decrease with the increase of τ. It is evident for the first expression in (1.10). To the second one we have

$$\frac{\partial}{\partial\tau}(\psi^2-\varphi^2)=\frac{1}{2t}\left(t+\frac{M^2\tau}{4t}\right)+\frac{1}{8t^2}\left[(2M+\mu)\mu+\sqrt{(2t+M+\mu)^2-\tau}\sqrt{(2t-M-\mu)^2-\tau}\right]\times$$

$$\times\frac{\tau-4t^2-(M+\mu)^2}{\sqrt{(2t+M+\mu)^2-\tau}\sqrt{(2t-M-\mu)^2-\tau}}\qquad\text{if}\quad \tau\le\gamma\mu\left(M+\mu-\frac{4t^2}{M+\mu-\gamma\mu}\right).$$

This yields

$$\frac{\partial}{\partial\tau}(\psi^2-\varphi^2)<-\frac{(2M+\mu)\mu}{8t^2}<0\qquad\text{if}\quad \tau\le\gamma\mu\left(M+\mu-\frac{4t^2}{M+\mu-\gamma\mu}\right).$$

Thus conditions (3.26) can be rewritten as

$$\psi^2(t,\gamma,\tau^\circ)>0\,;\quad \psi^2(t,\gamma,\tau^\circ)-\varphi^2(t,\tau^\circ)>0.$$

However the latter inequalities are provided by the condition (3.23).

Thus the condition (3.23) provides the non-emptiness of the domain (3.25). Now make use of just obtained result to the points k_j satisfying condition (3.15). Due to (3.25) these points belong to the domain $G_t(\gamma_1)\times G_t(\gamma_2)$ only in that case when the inequalities

$$\max[0, \varphi^2(t,\tau+\tau_j^0)] \le \vec{p}_j^{\,2} < \psi^2(t,\gamma_j,\tau+\tau_j^0),\ j=1,2, \tag{3.27}$$

are fulfilled simultaneously. By the inequalities (1.5) the domain (3.27) is not empty.

Due to what has been said above only those values of Δ^2 (given $\tau\le 0$) belong to the domain of analiticity $\mathcal{D}_t$ of the function φ which may be represented in form (3.18) at least by one set of the numbers $\mu_j, \nu_j, |\vec{p}_j|, j=1,2$ satisfying the inequalities (3.17) and (3.27) respectively. In order to obtain the boundary of the corresponding domain it is evidently necessary to take in (3.18)

$$-\mu_1=\mu_2=\cos\delta,\ \nu_1=\nu_2=\sin\delta,\ |\vec{p}_j|=\psi(t,\gamma_j,\tau+\tau_j^0)$$

(then conditions (3.17) will be fulfilled by $\alpha_j=0$). Thus we get the ellipse (1.7).

Prove now that the interval (1.11) belongs to all ellipses (1.7). It is clear that all ellipses (1.7) contain real Δ^2 from the interval

$$\max_{\substack{t\ge\frac{1}{2}(\mathcal{M}+\mu)\\ \tau\le 0}} [\mathcal{A}(t,\tau)-\mathcal{B}(t,\tau)] < \Delta^2 < \min_{\substack{t\ge\frac{1}{2}(\mathcal{M}+\mu)\\ \tau\le 0}} [\mathcal{A}(t,\tau)+\mathcal{B}(t,\tau)]. \tag{3.28}$$

We shall show that the minimum of the right-hand side in (3.28) is realized only by $\tau=0$. To see this it is sufficient to establish that by any fixed t $(\ge\frac{1}{2}(\mathcal{M}+\mu))$ the functions $\mathcal{A}$ and $\mathcal{B}$ increase monotonously when τ (≤ 0) decreases. But this statement follows immediately from (1.8) and from the properties of the functions ψ,φ and $\psi^2-\varphi^2$ proved above.

The main theorem is proved completely.

We shall prove that $\Delta^2_{min}=0$ if the parameters γ_j and τ_j^0 satisfy relations (1.14). As

$$\lim_{t\to\infty}[\mathcal{A}(t,\tau)-\mathcal{B}(t,\tau)]=0$$

it is sufficient to establish the inequality

$$\mathcal{A}(t,\tau)-\mathcal{B}(t,\tau)\le 0$$

for all $t\ge\frac{1}{2}(\mathcal{M}+\mu)$ and $\tau\le 0$. Inequality (3.29) will be provided if we prove the following inequality

$$\begin{gathered}\varphi^2(t,\tau+\tau_2^0)[\psi^2(t,\gamma_1,\tau+\tau_1^0)-\varphi^2(t,\tau+\tau_1^0)]+\varphi^2(t,\tau+\tau_1^0)[\psi^2(t,\gamma_2,\tau+\tau_2^0)-\varphi^2(t,\tau+\tau_2^0)]+\\ +2\sqrt{\psi^2(t,\gamma_1,\tau+\tau_1^0)-\varphi^2(t,\tau+\tau_1^0)}\sqrt{\psi^2(t,\gamma_2,\tau+\tau_2^0)-\varphi^2(t,\tau+\tau_2^0)}\times\\ \times\left[\left(t+\frac{\mathcal{M}^2-\tau-\tau_1^0}{4t}\right)\left(t+\frac{\mathcal{M}^2-\tau-\tau_2^0}{4t}\right)-\mathcal{M}^2\right]-\mathcal{M}^2\left(\frac{\tau_1^0-\tau_2^0}{4t}\right)^2\ge 0.\end{gathered} \tag{3.30}$$

While proving the main theorem it has been established that the functions φ, ψ and $\psi^2-\varphi^2$ decrease monotonously in τ. Owing to $\tau_j^\circ \leq \mu^2$ the function φ^2 is non-negative. Hence the left-hand side of (3.30) decreases monotonously in τ. Therefore it is sufficient to establish inequality (3.30) for $\tau = 0$. But then the restriction from below for τ_j° in (1.14) indicates that the function ψ must be calculated by the first formula (1.10). It is easy to verify that the function $\psi^2(t,\gamma,\tau)$ increases nomotonously in γ for t and τ under consideration. Therefore it is sufficient to prove the inequality (3.30) for $\gamma_1=\gamma_2=2$. But then the left-hand side of (3.30) will be a symmetrical function of τ_1° and τ_2°. Taking into account the monotony property in τ_j° we conclude at last that it is sufficient to prove the inequality (3.30) for

$$\tau=0,\ \gamma_1=\gamma_2=2,\ \tau_1^\circ=\mu^2,\ -9\frac{\mathcal{M}+\mu}{\mathcal{M}-2\mu}\mu^2 \leq \tau_2^\circ \leq \mu^2.$$

By these values of the parameters the inequality (3.30) becomes

$$\left[\left(t+\frac{\mathcal{M}^2-\mu^2}{4t}\right)\sqrt{4\mu^2-\tau_2^\circ}+\left(t+\frac{\mathcal{M}^2-\tau_2^\circ}{4t}\right)\sqrt{3}\,\mu\right]^2 \geq \mathcal{M}^2\left(\sqrt{3}\,\mu+\sqrt{4\mu^2-\tau_2^\circ}\right)^2+\frac{\mathcal{M}^2(\mu^2-\tau_2^\circ)}{(2\mathcal{M}+\mu)\mu}\cdot\frac{4t^2-(\mathcal{M}-\mu)^2}{16t^2}$$

which is verified directly.

In conclusion the authors express their deep gratitude to N.N. Bogolubov for interesting discussions and remarks.

The Russian variant of this paper was received by Publishing Department on November 1, 1959.

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