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M.E. Mayer

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EXTENDED INVARIANCE PROPERTIES OF QUANTUM FIELDS

I. LAGRANGIAN FORMALISM

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*) On leave of absence from the Faculty for Mathematics and Physics of the C.I. Parhon University, Bucharest and the Institute for Atomic Physics of the Academy of the Rumanian People's Republic, Bucarest. This work was started at these Institutions in 1956.

ABSTRACT

This is the first paper from a series in which the consequences of invariance of Quantum Field Theory under an f-parameter Lie group are investigated. Part I deals with the consequences of invariance of a general Lagrangian under such a group. It is shown that the invariance leads to the conservations of f "current density operators", the corresponding charge operators being a representation of the Lie group on the state vector space. Invariance under extended transformations, i.e. transformations in which the parameters are functions of the space-time point leads to the necessity of introducing f noncommuting vectors fields. The general structure of the Lagrangian corresponding to these fields is established and their possible physical counterparts are discussed. Part II will contain a similar investigation in the framework of an S-matrix theory, and Part III will be devoted to a detailed classification of possible transformation groups and the corresponding classification of particle fields and their interactions.

I. Introduction

During the last years considerable progress has been made in the study of elementary particle interactions and the selection rules governing them, and there have been several, more or less successful attempts at theoretical interpretations or classifications of the corresponding interactions, the best - known of which is the Gell-Mann - Nishijima interpretation in terms of the Strangeness quantum number, and the various modifications of this scheme.

Although it has been clear that any conservation law in quantum field theory must be the consequence of some invariance property of the fields under investigation, there have been relatively few attempts to study in quite a general form the consequences of the invariance of a theory (we will specify below what exactly is meant by this term) and to investigate the possibility of obtaining all possible fields and their interactions on the basis of the fundamental postulates of quantum field theory and the structure of the admissible transformation groups, which, as will be shown below, are Lie groups which turn out to possess the important (from the mathematical point of view) property of being compact and which are well studied in the mathematical literature.

Such an investigation has been started by the author in 1956 and some of the results obtained have been quoted in an unpublished dissertation(1). Essentially the same results have been published at about the same time by Utiyama[2] and are essentially a generalization of Schwinger's analysis of con - stant phase transformations and their extension to gauge trans- formations, which leads to the introduction of the electromagnetic field [3],[4]. These results, together with some general properti- es of the groups which leave invariant the Lagrangian form of quantum field theory, form the contents of Secs. 2 and 3 of this paper. It is shown that the "internal degrees of freedom" (as contrasted to the space-time properties) of the fields are such that the "internal space" is a finite dimensional Euclidian spa- ce so that the corresponding transformations are orthogonal li- near transformations. The invariance of the (quantum) Lagrangian with respect to such transformations leads to the conservation of f charges, where f is the number of parameters characterizing the given group, and the commutation relations of the correspon- ding operators coincide with those of the infinitesimal opera - tors of the group, so that the charge operators are a representa- tion of these infinitesimal operators in the state-vector space of the system.

In Sec.3 the notion of "extended invariance" is introduced. This means that invariance of the Lagrangian is required not on- ly for transformations in which the parameters are constant in space-time, but also under point-dependent transformations. Phy- sically such a requirement can be motivated by imposing the con- dition that if a theory is invariant under a group of internal symmetry transformations, this invariance must not depend upon the point of location of the observer, so that two observers in distinct space-time points should be able to carry out inde - pendently arbitrary transformations from the group under consideration, without destroying the invariance of the observable quantities. In order to ensure the invariance of the Lagrangian formalism under such extended transformations it becomes necessary to introduce supplementary operator fields (in the same manner as the Maxwell field can be introduced to reestablish the invariance of a charged field Lagrangian, when extending constant phase transformations to gauge transformations). The covariant vector fields introduced in this manner are subjected under our group to generalized gauge transformations of the second kind (3.7). The possible general form of the "free Lagrangian" corresponding to these vector fields is also investigated and it turns out, that it leads to complicated nonlinear field equations, which have not been investigated in the general case.

A similar analysis, but in the framework of an S-matrix theory is the subject of Part II of this series and Part III will contain a detailed analysis of the structure of admissible transformation groups with the final scope of obtaining a classification of "elementary" particles and the structure of the corresponding Lagrangian (this was in part carried out by Schwinger[(3)]) or S-matrices. This investigation is only a preliminary step to the construction of a Quantum field theory in which physically meaningless quantities, such as interacting field operators etc. no longer occur and the basic quantities of the theory are closely connected with group representations and invariant transition amplitudes (in this connection attention must be called to attempts in this direction by Segal[11], which are however in a much too abstract form as yet).

2. Lie Groups, Infinitesimal Transformations, Structural Constants and Current Densities

We take as our starting point Schwinger's general form of the Lagrangian density (3), (4)

$$L(\chi) = \tfrac{1}{2}\left[\chi A^{\mu}\partial_{\mu}\chi - \partial_{\mu}\chi A^{\mu}\chi\right] - H(\chi), \tag{2.1}$$

where A^{μ} are a set of antihermitian matrices and χ sets of hermitian field operators. The matrix indices of A^{μ} refer only to the space-time transformation properties of the operators χ.

In order to be able to describe various physical properties of the particles associated with the fields χ, the latter must possess some internal degrees of freedom (i.e. they must consist of sets of components suitably labelled) which can be subjected to various transformation groups. The following natural requirements are to be imposed on such "internal transformations:
i) L must be invariant under the transformations of the group G, or at most invariant up to the addition of a diverence; ii) all physically significant relations, such as commutation relations,

expectation values of S-matrix elements etc, should also be left invariant. From these requirements it follows immediately that the internal space, i.e. the system of components of χ , must have some sort of scalar product defined in it, which shall as - sure the positive definite character of e.g. the anticommutator of fermion operators or the energy density of bosons (cf.also (5)).

Thus the internal space must be a Riemannian manifold, but if we do not want to get entangled with nonlinear field equations it is reasonable to suppose it a finite-dimensional Euclidian space. The corresponding transformations of the internal space are consequently linear orthogonal transformations = rotations.

Consider the *f*-parameter group of linear transformations of the *n*-dimensional internal space

$$'\chi_r = \sum_{s=1}^{n} M_{rs}(\lambda_1, \dots, \lambda_f)\, \chi_s \tag{2.2}$$

The infinitesimal operators of this group can be written in the form

$$M_{rs}(\delta\lambda_1 \dots \delta\lambda_f) = \delta_{rs} - i \sum_{a=1}^{f} \delta\lambda_a T^a_{rs} \, , \tag{2.3}$$

or in obvious matrix notation

$$'\chi = (I - i\,\delta\lambda_a T^a)\,\chi \, . \tag{2.4}$$

In order to preserve the Hermitian character of the field operators it is necessary that the matrices T^a be imaginary.

The A^μ commute with the T^a, as they refer to different degrees of freedom, so that the invariance requirement on the Lagrangian leads to the Hermitian property of the infinitesimal matrices

$$T^{aT} = -T^a , \quad T^{a+} = T_a , \tag{2.5}$$

Let the structural constants of the group, c^{ij}_k be defined by the relations

$$[T^i, T^j] = T^i T^j - T^j T^i = \sum_{k=1}^{f} c^{ij}_k T^k , \tag{2.6}$$

with the well known properties

$$c^{ij}_k = -c^{ji}_k ,$$

$$\sum_m \left(c^{im}_k c^{jn}_m + c^{jm}_k c^{ni}_m + c^{nm}_k c^{ij}_m \right) = 0. \tag{2.7}$$

A variation of the Lagrangian with respect to λ^a_α can be put into the form[x)]

$$\delta_\lambda L = -i \sum_{a=1}^{f} (\chi A^\mu T^a \chi)\, \partial_\mu \delta\lambda_a(x) , \tag{2.8}$$

and defines the current densities

$$j^\mu_a = -i(\chi A^\mu T^a \chi) , \qquad a = 1, 2, \ldots, f, \tag{2.9}$$

[x)] Here $\delta\lambda_a(x)$ are arbitrary functions of the coordinates, vanishing on the two limiting spacelike surfaces which intervene in the action principle; cf. for example (4) §21.

which satisfy the conservation laws

$$\int d^4x\, \partial_\mu j^\mu(x) = 0 ;$$

$$Q^a = \int_\sigma d\sigma^\mu\, j^a_\mu(x) = \text{const.} \tag{2.10}$$

Repeating the usual argument ((4) § 22) it can be seen that

$$[\chi, Q^a] = T^a \chi , \tag{2.11}$$

and (2.6) implies

$$[Q^a, Q^b] = \sum_c c^{ab}_c Q^c , \tag{2.12}$$

that is the various charges do not commute, in general.

On the other hand, the transformation (2.2) can be expressed as a unitary transformation of the field operators

$$'\chi = M\chi = U^+ \chi U , \quad U^+ U = U U^+ = I \tag{2.13}$$

with U acting on the state vectors (e.g. in the occupation number representation U can be expressed in terms of the creation and annihilation operators). For infinitesimal transformations (2.3) U can be expressed by means of Hermitian operators Q^a in the form

$$U = I + i \sum_a Q^a \delta\lambda_a \tag{2.14}$$

It is easy to verify that the operators Q^a satisfy the commutation relations (2.11) and (2.12) and are thus identical with the operators (2.10). Thus for finite transformations we obtain

$$U = \exp\left[i \sum_a Q^a \lambda_a \right] = \exp\left[i \sum_a \int d\sigma^\mu j_\mu^a \lambda_a \right]. \tag{2.15}$$

Now let us consider briefly the structure of the possible transformation groups. As shown in Chapter XI of (6) the Lie algebra of an Euclidian space is compact and decomposes into the direct sum of its center (the commutative subgroup) and a set of simple compact noncommutative algebras which are ideals in the original algebra. The vector space of the Lie algebra is unitary with the Hermitian scalar product

$$(a,b) = -\sum_{i,j,\alpha,\beta} C^i_{\alpha j} C^j_{\beta i} a^\alpha b^\beta \equiv g_{\alpha\beta} a^\alpha b^\beta. \tag{2.16}$$

Thus it is sufficient to consider separately commutative and purely noncommutative algebras.

The commutative groups, which can be considered as rotations around an axis in the internal space, or alternatively as constant phase transformations, have been thoroughly studied in (3) and (4) and will not be considered here. We only note that the "charges" associated with various phase or gauge transformations commute and can be used for classification of the states of a system (this problem will be treated in detail in Part III of this series of papers). Examples of noncommutative groups have been investigated by different authors (7) - (10), and some special examples will be investigated below (part III).

3. Extended Group Invariance and Associated Vector Fields

As a generalization of the extension of the group of constant phase transformations that leads to the introduction of the electromagnetic field with its own gauge transformation of the second kind (cf.(4) sec. 22 or Schwinger's original work (3)) we will now consider "extended transformation groups" in which the parameters λ_α are arbitrary functions of the space-time co-ordinates. This extension is based upon the independence of the internal degrees of freedom on the space-time transformation properties and means that we may apply to each space-time point its own internal transformation.

Consider the extended transformations

$$'\chi(x) = M(\lambda_1(x), \ldots, \lambda_\ell(x))\, \chi(x), \qquad (3.1)$$

or their infinitesimal form

$$'\chi(x) = \left(I - i\sum_a \delta\lambda_a(x) T^a\right)\chi(x) = \chi(x)\left(I + i\sum_a \delta\lambda_a(x) T^a\right), \qquad (3.2)$$

where we have made use of the antisymmetry (2.5) of the matrices T^a . The variation of the Lagrangian has the form

$$\delta_\lambda L = -i\sum_a \chi A_\mu T^a \chi\, \partial^\mu \delta\lambda_a(x) = \sum_a j_\mu^a(x)\, \partial^\mu \lambda_a(x) \qquad (3.3)$$

which is no longer zero, as $\lambda(x)$ does not vanish on the boundary surfaces.

However we can reestablish the invariance of the Lagrangian by introducing supplementary fields U^μ_a and a supplementary term in the Lagrangian, L_1, such that

$$\delta(L+L_1)=0 \quad , \quad L_1 = -\sum_a j^a_\mu U^\mu_a . \tag{3.4}$$

It is clearly insufficient to subject the field U^μ_a to the gauge transformations of the second kind known from electrodynamics

$$U^\mu_a \rightarrow {}'U^\mu_a - \partial^\mu \lambda_a(x), \tag{3.5}$$

as in this case

$$\begin{aligned} \delta L_1(\chi, U) &= -\sum_a j^a_\mu \partial^\mu \lambda_a(x) + i\sum_{a,j} \chi A_\mu [T^a, T^j] U^\mu_a \lambda_j(x) \\ &= -\sum_a j^a_\mu \partial^\mu \lambda_a(x) - \sum j^k_\mu(x)\, c^{ij}_k\, U^\mu_i(x) \lambda_j(x). \end{aligned} \tag{3.6}$$

The first term cancels with (3.3) and the second cancels only if we subject the vector fields U^μ_a to the generalized gauge transformations of the second kind

$${}'U^\mu_a(x) = U^\mu_a(x) - \partial^\mu \lambda_a(x) + \sum_{i,j} c^{ij}_a U^\mu_i(x) \lambda_j(x)^{x)} \tag{3.7}$$

In order to have a complete escription of the U^μ_a - field we must investigate the possible form of the corresponding "free

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x) Essentially the same result was obtained by Utiyama (2) by means of a less elementary proof. For the special case of the 3-dimensional iso-rotation group the same result has been obtained by Yang and Mills (7).

Lagrangian", imposing the natural conditions that it be of the general form (2.1) or reducible to that form by means of introducing suitable supplementary variables (since we are looking for first order equations), and that the Lagrangian be invariant under Lorentz transformations and extended internal transformations (here invariant means as usual, invariance up to possible appearance of expressions in the form of a four-dimensional divergence).

As well known from electrodynamics, or the theory of the Proca field, the supplementary variables needed to obtain first order equations for the vector field are antisymmetrical tensors of the second rank, so that besides U^{μ}_{α} we introduce the f antisymmetrical tensor fields $Z^{\mu\nu}_{\alpha}$, for which we suppose the following transformation law under the extended transformation (3.7)

$$'Z^{\mu\nu}_{a} = Z^{\mu\nu}_{a} + i\sum_{k,l} c^{kl}_{a} Z^{\mu\nu}_{k} \lambda_{l} \tag{3.8}$$

(there are no terms involving the derivatives of λ_{i}, because of the antisymmetry). This means that the internal space components of the vector U^{μ} and the tensor $Z^{\mu\nu}$ transform cogrediently under our group (leaving aside the derivative of λ_{α}, which for our present purpose can be considered as a "constant").

Using the well-known facts from the theory of Lie groups, that the structural constants are "tensors" under the transformations of the group, and that, for fixed index α they form a matrix representation of the generators of the Lie group (one can easily verify that the identity (2.7) is just the representation of (2.6) in an f-dimensional Euclidean space with the scalar product defined by (2.16). We are able to write the equations (3.2), (3.7) and (3.8) in the form

$$
\begin{aligned}
'\chi_a &= \sum_b \left(\delta_a^b - i \sum_c c_a^{cb} \lambda_c \right) \chi_b , \\
'U_a^\mu &= \sum_b \left(\delta_a^b - i \sum_c c_a^{cb} \lambda_c \right) U_b^\mu - \partial^\mu \lambda_a , \qquad (3.9) \\
'Z_a^{\mu\nu} &= \sum_b \left(\delta_a^b - i \sum_c c_a^{cb} \lambda_c \right) Z_b^{\mu\nu} ,
\end{aligned}
$$

and the "metric tensor"

$$
g^{al} = - \sum_{i,j} c_i^{aj} c_j^{li} = \sum_{i,j} c_i^{aj} c_j^{il} , \qquad (3.10)
$$

can be used for lowering or raising internal indices and for forming invariants.

Taking into account all the mentioned facts, we see that the only interesting bilinear Lorentz invariants, linear in the first derivatives of U_a^μ and $Z_a^{\mu\nu}$ are

$$
\begin{aligned}
I_1 &= \sum_a \left(\partial_\mu U_\nu^a - \partial_\nu U_\mu^a \right) Z_a^{\mu\nu} = \sum_{a,b} g^{ab} \left(\partial_\mu U_a^\nu - \partial_\nu U_a^\mu \right) Z_b^{\mu\nu} , \\
I_2 &= \tfrac{1}{2} \sum Z_{\mu\nu}^a Z_a^{\mu\nu} , \quad I_3 = \sum U_a^\mu \partial^\nu Z_{\mu\nu}^a , \qquad (3.11) \\
I_4 &= \tfrac{1}{2} \sum_{a,b,c} \left(U_a^\mu U_b^\nu - U_a^\nu U_b^\mu \right) Z_{\mu\nu}^c c_c^{al}
\end{aligned}
$$

These forms are not invariant under the extended group, because of the derivatives of $\lambda_a(x)$ appearing in (3.9) (they are however invariant, for constant λ_a) but they can be combined so as to yield an invariant, the various terms with derivatives cancelling with one another, or by using antisymmetry and neglecting terms of the form of a divergence. By these argumrnts we are led almost uniquely to the free Lagrangian for the U-Z -field, of the form

$$L(U,Z) = -\tfrac{1}{2}\Big[\tfrac{1}{2}\{ Z^{\mu\nu}_{a}, (\partial_\mu U^a_\nu - \partial_\nu U^a_\mu)\} - \{ U^a_\nu, \partial_\mu Z^{\mu\nu}_{a}\} - $$

$$ - \tfrac{1}{2}\{ Z^{\mu\nu}_{a}, Z^{a}_{\mu\nu}\} + \tfrac{1}{4}\{ Z^{c}_{\mu\nu}, C^{ab}_{c} (U^\mu_a U^\nu_b - U^\nu_a U^\mu_b)\}\Big], \qquad (3.12)$$

where summation over repeated indices is understood. The first two terms in the Lagrangian can be evidently transformed into one another, up to a divergence. The form (3.12) reduces to the one used by Schwinger (5) for $C^{ab}_c = 0$.

The resulting field equations (taking into account the coupling with the χ -field through L_1 (3.4)) are nonlinear generalizations of the Maxwell equations of the form first proposed by Yang and Mills (4) x)

$$Z^{\mu\nu}_c = \partial^\mu U^\nu_c - \partial^\nu U^\mu_c + \tfrac{1}{2} C^{ab}_c (U^\mu_a U^\nu_b - U^\nu_a U^\mu_b), \qquad (3.13)$$

and

$$\partial^\mu Z^a_{\mu\nu} = \tfrac{1}{2} C^{ab}_c \{ U^\lambda_b Z^c_{\lambda\nu} - U^c_\lambda Z^{\lambda\nu}_b \} = -j^a_\nu . \qquad (3.14)$$

x) Utiyama (2) has proved that the free Lagrangian of the field can depend only on the combination of U^a_μ and its derivatives in the r.h.s. of Eq. (3.13). It is evident that our Lagrangian (3.12) differs only by a divergence from the square of this expression. The latter would however lead to second order equations.

The nonlinear character of these field equations leads to great difficulties in analysing their quantum properties, which will not be investigated here. We only want to call attention to the fact, that due to the vanishing of the rest mass of the U — field (a term $U^a_\mu U_a^\mu$ would lead to noninvariance of the U-Lagrangian under the extended group) and the existence of the gauge-transformation, not all the components of the U -field will be dynamical variables (the same is well known for the electromagnetic field, which, as yet, is the only U -field known to have a physical counterpart).

As already mentioned in the introduction, the only special cases of U-fields treated in the literature so far, excepting of course the electromagnetic field, are the Yang-Mills B-field, associated with extended isotopic invariance ($\mathcal{T}$) and the gravitational field or more exactly, the field of the Riemann affine connection, which as shown by Utiyama, plays the role of the U-field for "extended Lorentz" transformations" i.e. Lorentz transformations with point-dependent coefficients (2). Utiyama also called attention to the fact, that in order to obtain the invariant interaction term between the field χ and the new U -field, one has simply to replace the ordinary derivative $\partial_\mu \chi$ by the "covariant derivative"

$$\nabla_\mu \chi = \partial_\mu \chi - i T_a \chi U_\mu^a . \tag{3.15}$$

For the extended Lorentz group this coincides with the usual notion of covariant derivative, and in the case of electrodynamics it reduces to the well known recipe of introducing gauge invariant interaction terms.

As far as the physical existence of the U-Z-fields is concerned, it must be stressed that one cannot expect to find real particles corresponding to each of them, as not all of the considered invariance properties are strictly universal. For example isotopic rotation invariance is true only as long as electromagnetic effects are negligible, and so one may doubt the real existence of the Yang-Mills field. On the other hand the conservation laws of baryonic and leptonic charge and the associated invariance properties of the fields are possibly rigorous laws, and should therefore lead to observable U-fields, coupled respectively to the baryons and leptons and possibly to the electromagnetic field. These two fields may in some respects resemble the Z-field, which according to Schwinger (5) is responsible for the weak interactions. These questions will be investigated in Part III.

From the point of view of renormalizability, the U-fields lead in general to nonrenormalizable nonlinear theories, which are difficult to investigate with the usual methods. However, for commutative groups, the difficulties will not be greater than in quantum electrodynamics. This question will also be investigated separately in Part III.

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