JOINT INSTITUTE FOR NUCLEAR RESEARCH

M. LOKAJICEK *

P - 207

THE ALGEBRA OF ELEMENTARY PARTICLES

Dubna, 1958.

* Prague.

JOINT INSTITUTE FOR NUCLEAR RESEARCH

M. LOKAJICEK *

P - 207

THE ALGEBRA OF ELEMENTARY PARTICLES

Dubna, 1958.

* Prague.

Introduction

This paper represents a mathematical supplement to the paper /I/. It deals with the properties of the matrix-algebra used in the paper /I/ for a unified description of the isobaric structure of the system of elementary particles and also gives a generalisation of this algebra for the case of a foar-dimensional space.

In Section A, starting with the fundamental relations (A1-4) which define the algebra in isospace, a deduction is given of other relations, especially of (A7,8,I4,I6,34,35). On the basis of these auxiliary relations, some of which have been mentioned in /I/ without general proof, all irreducible representations of the algebra can easily be enumerated.

In Section B a similar algebra is defined in a fourdimensional space by a straightforward generalisation of the fundamental relations of Section A. Again further relations can be deduced from the fundamental ones by a suitable generalisation of the procedure used in Section A. These relations, from which the irreducible representations admitted by this "fourdimansional" algebra can be seen directly, are collected at the end of Section B.

A physical application of the fourdimensional algebra (to a unified description of the space-time properties of elementary particles) will be given elsewhere.

A. The algebra in isospace.

Let us have seven operators ω_j, λ_j $(j=1,2,3)$ and U; satisfying the relations

$$\sum_p (\omega_j \omega_k \omega_l - \delta_{jk}\,\omega_l) = 0, \tag{AI}$$

$$[\lambda_j, \lambda_k] = i\,\varepsilon_{jkl}\,\lambda_l, \tag{A2}$$

$$[\omega_j, \lambda_k] = i\,\varepsilon_{jkl}\,\omega_l, \tag{A3}$$

$$\lambda_j \omega_k + \lambda_k \omega_j = \delta_{jk}\, U \tag{A4}$$

where $\sum_p$ denotes the sum over all six permutations of the indices j, k, l and ε_{jkl} is the antisymmetrical unit tensor $(\varepsilon_{123} = +1)$.

First of all let us notice that the product $\varepsilon_{jkl}\,\varepsilon_{j'k'l'}$ can be expressed as follows

$$\begin{aligned}\varepsilon_{jkl}\,\varepsilon_{j'k'l'} = {} & \delta_{jj'}(\delta_{kk'}\delta_{ll'} - \delta_{kl'}\delta_{lk'}) + \\ & + \delta_{jk'}(\delta_{kl'}\delta_{lj'} - \delta_{kj'}\delta_{ll'}) + \\ & + \delta_{jl'}(\delta_{kj'}\delta_{lk'} - \delta_{kk'}\delta_{lj'}).\end{aligned} \tag{A5}$$

Then, putting $j'=j$ (and summing over j) we obtain

$$\varepsilon_{jkl}\,\varepsilon_{jk'l'} = \delta_{kk'}\delta_{ll'} - \delta_{kl'}\delta_{lk'} \tag{A5'}$$

and putting further $k'=k$ (and summing over k) we obtain

$$\varepsilon_{jkl}\,\varepsilon_{jkl'} = 2\delta_{ll'} \;. \tag{A5"}$$

Multiplying (A2) by $\varepsilon_{jkl'}$ and using (A5") we get

$$\varepsilon_{jkl}\,\lambda_j\,\lambda_k = i\,\lambda_l \tag{A2a}$$

(which is only an alternative, equivalent way of writting (A2)).
From (A3) we have

$$\omega_j\,\lambda_k = \lambda_k\,\omega_j + i\,\varepsilon_{jkl}\,\omega_l$$
$$\omega_k\,\lambda_j = \lambda_j\,\omega_k - i\,\varepsilon_{jkl}\,\omega_l$$

so that also

$$\omega_j\lambda_k + \omega_k\,\lambda_j = \lambda_j\,\omega_k + \lambda_k\,\omega_j = \delta_{jk}\,U \;. \tag{A4a}$$

Putting $k=j$ (and summing over j) we obtain from (A4a) immediately

$$U = \frac{2}{3}\,\lambda_j\,\omega_j = \frac{2}{3}\,\omega_j\,\lambda_j \;. \tag{A6}$$

Using (A6,2) and (A3) we can write

$$\lambda_j\,U = \frac{2}{3}\lambda_j\,\lambda_k\,\omega_k = \frac{2}{3}\lambda_k\,\lambda_j\,\omega_k + \frac{2i}{3}\,\varepsilon_{jkl}\,\lambda_l\,\omega_k =$$
$$= U\lambda_j + \frac{2i}{3}\,\varepsilon_{jkl}\left(\lambda_k\,\omega_l + \lambda_l\,\omega_k\right) = U\lambda_j \;,$$

i.e.

$$[\lambda_j\,, U] = 0 \;. \tag{A7}$$

Using (A6) and (A4a) we have

$$\omega_j\,U = \frac{2}{3}\,\omega_j\,\lambda_k\,\omega_k = \frac{2}{3}\,U\omega_j - \frac{2}{3}\,\omega_k\,\lambda_j\,\omega_k$$

and similarly

$$U\omega_j = \frac{2}{3}\,\omega_k\,\lambda_k\,\omega_j = \frac{2}{3}\,\omega_j\,U - \frac{2}{3}\,\omega_k\,\lambda_j\,\omega_k \;.$$

Subtracting these two equations we obtain

$$[\omega_j , U] = 0 \tag{A8}$$

and then also

$$\omega_k \lambda_j \omega_k = -\frac{1}{2}\omega_j U = -\frac{1}{2} U \omega_j \tag{A9}$$

Multiplying (A2) by λ_j we obtain easily

$$[\lambda_j^2 , \lambda_k] = i\varepsilon_{jkl}(\lambda_j \lambda_l + \lambda_l \lambda_j) = 0 . \tag{AI0}$$

Similarly, multiplying (A3) by ω_j we obtain

$$[\omega_j^2 , \lambda_k] = i\varepsilon_{jkl}(\omega_j \omega_l + \omega_l \omega_j) = 0 . \tag{AII}$$

Now multiply (A4) by $\lambda_j \lambda_k$:

$$\lambda_j \lambda_k \lambda_j \omega_k + \lambda_j \lambda_k^2 \omega_j = \lambda_j^2 U .$$

Using (A2,2a,I0,6) and (A7) we get

$$\lambda_j^2 U = U \lambda_j^2 = \frac{3}{4} U . \tag{AI2}$$

From (AI) we have

$$\omega_j^2 \omega_k + \omega_j \omega_k \omega_j + \omega_k \omega_j^2 = 5 \omega_k . \tag{AI'}$$

Multiplying (AI') by λ_k and using (A4a,6) and (A8) gives

$$\omega_j^2 U = U \omega_j^2 = 3U . \tag{AI3}$$

Now multiply (AI) by λ_l from the right. Then using (A6,4a,II,I') and (AI3) we find after some rearrangements the relation

$$(\omega_j \omega_k + \omega_k \omega_j - 2\delta_{jk}) U = 0 . \tag{AI4}$$

Multiplying (AI4) from the left by λ_k and using (A4,6,I3,8) we obtain

$$\lambda_j U = \frac{1}{2}\omega_j U^2 . \tag{AI5}$$

Multiplying finally (AI5) by λ_j and using (A6) and (AI2) we find that

$$U^3 = U . \tag{AI6}$$

Now let us return to the equation (A9). Using (A3) we can rewrite (A9) in the form

$$\omega_k^2 \lambda_j + i\varepsilon_{jkl}\omega_k \omega_l = -\frac{1}{2} U \omega_j . \tag{A9a}$$

Multiplying (A9a) by ω_j from the right and using (A6,I3) we obtain

$$\varepsilon_{jkl}\, \omega_j \omega_k \omega_l = 6 i U \quad . \tag{AI7}$$

Multiplying this equation by $\varepsilon_{j'k'l'}$ and using (A5) yields the relation

$$\Sigma_c\, \omega_j \omega_k \omega_l - \Sigma_c\, \omega_j \omega_l \omega_k = 6 i\, \varepsilon_{jkl}\, U \tag{AI7a}$$

where Σ_c means the sum over the three cyclic permutations of the factors.

The equation (AI) can be written in the form

$$\Sigma_c\, \omega_j \omega_k \omega_l + \Sigma_c\, \omega_j \omega_l \omega_k = 2(\delta_{jk}\omega_l + \delta_{jl}\omega_k + \delta_{kl}\omega_j) . \tag{AIa}$$

Adding (AI7) and (AIa) yields the relation

$$\Sigma_c \left(\omega_j \omega_k \omega_l - \delta_{jk}\, \omega_l \right) = 3 i\, \varepsilon_{jkl}\, U \quad . \tag{AI8}$$

Now multiply (AI8) from the right by $\lambda_l \lambda_k$. Using (A2a,3,4a,6,7) and (AI5) we obtain

$$i\, \varepsilon_{kll'}\, \omega_l \omega_j \omega_{l'} \lambda_k + i\, \varepsilon_{jll'}\, \omega_k \omega_l \omega_{l'} \lambda_k = 2 \omega_j U^2 \quad . \tag{AI9}$$

The first term of this equation can further be performed using (A3,4a,8,5',7,I7,I6,2a,I5) into

$$(\omega_j \omega_k^2 - 3\, \omega_k \omega_j \omega_k - 5 \omega_j U^2) \quad .$$

Similarly, using (A3,5',6,I6,I5,7), the second term is easily performed into

$$(\omega_k^2 \omega_j - \omega_k \omega_j \omega_k - 3\, \omega_j U^2) \quad .$$

Hence the equation (AI9) takes the form

$$\omega_k^2 \omega_j - 4 \omega_k \omega_j \omega_k + \omega_j \omega_k^2 = 10 \omega_j U^2 \tag{AI9a}$$

This can be combined with (AI') giving

$$\omega_k \omega_j \omega_k = \omega_j - 2 \omega_j U^2 \quad . \tag{A20}$$

Now let u, t, v be three indices for which

$$u \neq t, \quad t \neq v, \quad v \neq u$$

is valid (and over which no summations can be performed).

The equation (A20) can then be written as follows:

$$\omega_u \omega_t \omega_u + \omega_t^3 + \omega_v \omega_t \omega_v = \omega_t - 2 \omega_t U^2 \quad . \tag{A20a}$$

Putting $j=k=\ell=t$ in (AI8) we obtain

$$\omega_t^3 = \omega_t \;, \tag{AI8'}$$

whereas with $j=k=u$, $\ell=t$ we have

$$\omega_u^2\,\omega_t + \omega_u\,\omega_t\,\omega_u + \omega_t\,\omega_u^2 = \omega_t \tag{AI8"}$$

and finally with $j=v$, $k=u$, $\ell=t$

$$\Sigma_c\left(\omega_v\,\omega_u\,\omega_t - i\,\varepsilon_{vut}\,U\right) = 0\,. \tag{AI8'''}$$

From (A2) we have

$$(\lambda_u\lambda_v - \lambda_v\lambda_u)\,U = i\,\varepsilon_{uvt}\,\lambda_t\,U\;.$$

Using (AI5,8,I6) and (AI4) this equation can be performed into

$$\omega_u\,\omega_v\,U = i\,\varepsilon_{uvt}\,\omega_t\,U^2\,. \tag{A2I}$$

Multiplying (A2I) by ω_t from the right and using (A8) and (AI4) (with $j=k=t$) yields

$$\omega_u\,\omega_v\,\omega_t\,U = i\,\varepsilon_{uvt}\,U^2\,. \tag{A2Ia}$$

Now let us return to the equation (A20a). Combining it with (AI8') gives

$$\omega_u\,\omega_t\,\omega_u + \omega_t\,U^2 = -\left(\omega_v\,\omega_t\,\omega_v + \omega_t\,U^2\right)$$

or

$$a_{ut} = -\,a_{vt}\;, \tag{A22}$$

if we introduce the operator

$$a_{ut} \equiv \omega_u\,\omega_t\,\omega_u + \omega_t\,U^2\,. \tag{A23}$$

If we insert for $\omega_u\,\omega_t\,\omega_u$ from (A23) into (AI8"), this relation can be put in the form

$$\omega_u^2\,\omega_t + \omega_t\,\omega_u^2 = -a_{ut} + \omega_t + \omega_t\,U^2\,. \tag{A24}$$

Using (A23,8) and (A24) we can now write

$$\begin{aligned} a_{ut}\,\omega_u &= \omega_u\,\omega_t\,\omega_u^2 + \omega_t\,\omega_u\,U^2 = \\ &= \left(-\omega_u^3 + \omega_u\right)\omega_t - \omega_u\,a_{ut} + \left(\omega_u\,\omega_t + \omega_t\,\omega_u\right)U^2 \end{aligned}$$

and with regard to (AI8') and (AI4)

$$a_{ut}\,\omega_u = -\omega_u\,a_{ut} \qquad \text{(A25)}$$

Changing u into v gives

$$a_{vt}\,\omega_v = -\omega_v\,a_{vt}$$

and using (A22) yields

$$a_{ut}\,\omega_v = -\omega_v\,a_{ut} \qquad \text{(A26)}$$

Now using (A23,8,I8''',I8',I4,2Ia) and (AI6) we can write

$$\omega_v\,a_{ut}\,\omega_u = \omega_v\,\omega_u\,\omega_t\,\omega_u^2 + \omega_v\,\omega_t\,\omega_u\,U^2 =$$

$$= -\omega_t\,\omega_v\,\omega_u^3 - \omega_u\,\omega_t\,\omega_v\,\omega_u^2 + 3i\,\varepsilon_{vut}\,\omega_u^2\,U + \omega_v\,\omega_t\,\omega_u\,U^2 =$$

$$= -\omega_t\,\omega_v\,\omega_u - \omega_u\,\omega_t\,\omega_v\,\omega_u^2 + 3i\,\varepsilon_{vut}\,U + i\,\varepsilon_{vtu}\,U\;.$$

This can be further improved by the help of (AI8") (with v instead of t) :

$$\omega_v\,a_{ut}\,\omega_n = -\omega_t\,\omega_v\,\omega_n - \omega_u\,\omega_t\,\omega_v + \omega_u\,\omega_t\,\omega_u\left(\omega_u\,\omega_v + \omega_v\,\omega_u\right) + 2i\,\varepsilon_{vut}\,U\;.$$

Using again (AI8'',23) and (AI4) we obtain

$$\omega_v\,a_{ut}\,\omega_u = \omega_v\,\omega_u\,\omega_t - i\,\varepsilon_{vut}\,U + a_{ut}\left(\omega_u\,\omega_v + \omega_v\,\omega_u\right)$$

and with regard to (A26) finally

$$\omega_v\,\omega_u\,\omega_t - i\,\varepsilon_{vut}\,U = -a_{ut}\left(\omega_u\,\omega_v + 2\omega_v\,\omega_u\right)\;. \qquad \text{(A27)}$$

The expression $\omega_v\,a_{ut}\,\omega_u$ can be performed also using (A22) instead of (A23). Thus we obtain

$$\omega_v\,a_{ut}\,\omega_u = \omega_v\left(-\omega_v\,\omega_t\,\omega_v - \omega_t\,U^2\right)\omega_u =$$

$$= -\omega_v^2\left(-\omega_v\,\omega_u\,\omega_t - \omega_u\,\omega_t\,\omega_v + 3i\,\varepsilon_{tvu}\,U\right) - \omega_v\,\omega_t\,\omega_u\,U^2 =$$

$$= \omega_v\,\omega_u\,\omega_t + \omega_v^2\,\omega_u\,\omega_t\,\omega_v - 3i\,\varepsilon_{vut}\,U - i\,\varepsilon_{vtu}\,U =$$

$$= \omega_v\,\omega_u\,\omega_t + \left(-\omega_u\,\omega_v^2 - \omega_v\,\omega_u\,\omega_v + \omega_u\right)\omega_t\,\omega_v - 2i\,\varepsilon_{vut}\,U =$$

$$= -\omega_t \omega_v \omega_u - (\omega_u \omega_v + \omega_v \omega_u)\omega_v \omega_t \omega_v + i\,\varepsilon_{vut}\, U =$$

$$= -\omega_t \omega_v \omega_u + a_{ut}(\omega_u \omega_v + \omega_v \omega_u) + i\,\varepsilon_{vut}\, U$$

or, with regard to (A26),

$$\omega_t \omega_v \omega_u - i\,\varepsilon_{tvu}\, U = a_{ut}(\omega_u \omega_v + 2\,\omega_v \omega_u) \ . \tag{A28}$$

Adding (A27) and (A28) and using (A18‴) we obtain finally

$$\omega_u \omega_t \omega_v = i\,\varepsilon_{utv}\, U \ . \tag{A29}$$

Multiplying (A29) by U^2 and using (A16) gives

$$\omega_u \omega_t \omega_v U^2 = i\,\varepsilon_{utv}\, U$$

and subtracting this equation from (A29) yields

$$\omega_u \omega_t \omega_v (1 - U^2) = 0 \ . \tag{A30}$$

Using (A23,8,29,21) and (A16) we can now write

$$\begin{aligned} a_{ut}\,\omega_v &= \omega_u \omega_t \omega_n \omega_v + \omega_t \omega_v U^2 = \\ &= i\,\varepsilon_{tuv}\,\omega_u U + i\,\varepsilon_{tvu}\,\omega_u U = 0 \ . \end{aligned} \tag{A31}$$

From (A31,22,18',8) and (A14) we have further

$$\begin{aligned} 0 = a_{ut}\,\omega_v^2 &= -(\omega_v \omega_t \omega_v + \omega_t U^2)\omega_v^2 = \\ &= -\omega_v \omega_t \omega_v - \omega_t U^2 = -a_{vt} \end{aligned}$$

i.e.

$$a_{vt} = -a_{ut} = 0$$

or

$$\omega_v \omega_t \omega_v = -\omega_t U^2 \ . \tag{A32}$$

Multiplying (A32) by U^2 and using (A16) yields

$$\omega_v \omega_t \omega_v U^2 = -\omega_t U^2$$

and subtracting this equation from (A32) gives

$$\omega_v \omega_t \omega_v (1 - U^2) = 0 \ . \tag{A33}$$

The equations (A30) and (A33) can be collected into

$$\omega_j \omega_k \omega_l (1-U^2)=0, \ (j \neq k, \ k \neq l) \ . \tag{A34}$$

The equations (A30) and (A33) together with the equations (A18') and (A18") (which can be multiplied by $(1-U^2)$) are also sufficient to guarantee the equations

$$(\omega_j \omega_k \omega_l + \omega_l \omega_k \omega_j - \delta_{jk}\omega_l - \delta_{lk}\omega_j)(1-U^2)=0 \ . \tag{A35}$$

Thus we have deduced from (A1 - 4) the following most important equations :

$$[\lambda_j, U]=0, \tag{A7}$$

$$[\omega_j, U]=0, \tag{A8}$$

$$U^3=U, \tag{A16}$$

$$(\omega_j\omega_k+\omega_k\omega_j-2\delta_{jk})U=0, \tag{A14}$$

$$(\omega_j\omega_k\omega_l+\omega_l\omega_k\omega_j-\delta_{jk}\omega_l-\delta_{lk}\omega_j)(1-U^2)=0, \tag{A35}$$

$$\omega_j\omega_k\omega_l(1-U^2)=0, \ (j \neq k, \ k \neq l) \ . \tag{A34}$$

From (A7) and (A8) it follows that in each irreducible representation of our algebra the matrix U is a multiple of the unit matrix. From (A16) we see that its eigenvalues are only 0 and ± 1. In representations in which $U=+1$ or -1 the ω's fulfil the relations

$$\omega_j\omega_k+\omega_k\omega_j = 2\delta_{jk} \ . \tag{A1}$$

In representations with $U=0$ the ω's fulfil the relations

$$\omega_j\omega_k\omega_l+\omega_l\omega_k\omega_j = \delta_{jk}\omega_l+\delta_{lk}\omega_j$$

and (A11)

$$\omega_j\omega_k\omega_l=0, \ (j \neq k, \ k \neq l) \ .$$

For the sake of the generalization of the algebra to four dimensions it is convenient to have the fundamental relations in terms of the dual antisymmetrical quantities $\dot{\lambda}_{jk}$ and $\dot{U}_{jkl}$ which are connected with λ_j and U by the formulas

$$\lambda_j=\tfrac{1}{2}\varepsilon_{jkl}\dot{\lambda}_{kl} \ , \quad U=\tfrac{1}{6}\varepsilon_{jkl}\dot{U}_{jkl} \tag{A36a}$$

or conversely

$$\dot{\lambda}_{jk}=\varepsilon_{jkl}\lambda_l \ , \quad \dot{U}_{jkl}=\varepsilon_{jkl}U \ . \tag{A36b}$$

The equation (AI) remains the same

$$\Sigma_p\left(\omega_j\,\omega_k\,\omega_l - \delta_{jk}\,\omega_l\right) = 0 \quad . \qquad (\mathrm{A}\dot{1}) \equiv (\mathrm{A}1)$$

Inserting for λ_j and U into (A2,3,4,6) and removing the ε-tensors by the help of the formulas (A 5 - 5") we obtain the relations

$$[\dot\lambda_{jk}, \dot\lambda_{lm}] = i\left(\delta_{jl}\,\dot\lambda_{km} + \delta_{km}\,\dot\lambda_{jl} - \delta_{jm}\,\dot\lambda_{kl} - \delta_{kl}\,\dot\lambda_{jm}\right), \qquad (\mathrm{A}\dot{2})$$

$$[\dot\lambda_{jk}, \omega_l] = i\left(\delta_{jl}\,\omega_k - \delta_{kl}\,\omega_j\right), \qquad (\mathrm{A}\dot{3})$$

$$\begin{aligned}(\delta_{jl}\,\dot\lambda_{mn} + \delta_{jm}\,\dot\lambda_{nl} + \delta_{jn}\,\dot\lambda_{lm})\,\omega_k &+ \\ + (\delta_{kl}\,\dot\lambda_{mn} + \delta_{km}\,\dot\lambda_{nl} + \delta_{kn}\,\dot\lambda_{lm})\,\omega_j &= \delta_{jk}\,\dot U_{lmn} ,\end{aligned} \qquad (\mathrm{A}\dot{4})$$

$$\tfrac{2}{3}\left(\dot\lambda_{jk}\,\omega_l + \dot\lambda_{kl}\,\omega_j + \dot\lambda_{lj}\,\omega_k\right) = \dot U_{jkl} \quad . \qquad (\mathrm{A}\dot{6})$$

The same equation (A$\dot{6}$) is obtained by contracting the equation (A$\dot{4}$) in the indices j, k . The equation (A$\dot{4}$) can of course be contracted also in other ways (e.g. by putting $j = l$ and similarly). The expressions for U_{jkl} resulting from all such contractions are equivalent.

We see that the equations (A$\dot{2}$,$\dot{3}$,$\dot{4}$,$\dot{6}$) appear more complicated than (A2,3,4,6). However the equations (A$\dot{1}$,$\dot{2}$,$\dot{3}$,$\dot{6}$,) can be used without any change also in fourdimensional space. Only the equation (A$\dot{4}$) will need some completion.

B. The algebra in space-time.

In four dimensions the elements ω_j are substituted by α_μ $(\mu = 1,2,3,4)$ and the elements $\dot\lambda_{jk}$, $\dot U_{jkl}$ by the antisymmetrical quantities $\dot\sigma_{\mu\nu}$, $\dot N_{\mu\nu\rho}$ respectively.
The relations (A$\dot{1}$,$\dot{2}$,$\dot{3}$,$\dot{6}$) are substituted by relations of exactly the same form :

$$\Sigma_p\left(\alpha_\mu\,\alpha_\nu\,\alpha_\rho - \delta_{\mu\nu}\,\alpha_\rho\right) = 0 , \qquad (\mathrm{B}\dot{1}) \equiv (\mathrm{B}1)$$

$$[\dot\sigma_{\mu\nu}, \dot\sigma_{\lambda\rho}] = i\left(\delta_{\mu\lambda}\,\dot\sigma_{\nu\rho} + \delta_{\nu\rho}\,\dot\sigma_{\mu\lambda} - \delta_{\mu\rho}\,\dot\sigma_{\nu\lambda} - \delta_{\nu\lambda}\,\dot\sigma_{\mu\rho}\right), \qquad (\mathrm{B}\dot{2})$$

$$[\dot\sigma_{\mu\nu}, \alpha_\lambda] = i\left(\delta_{\mu\lambda}\,\alpha_\nu - \delta_{\nu\lambda}\,\alpha_\mu\right) , \qquad (\mathrm{B}\dot{3})$$

$$\tfrac{2}{3}\left(\dot\sigma_{\mu\nu}\,\alpha_\lambda + \dot\sigma_{\nu\lambda}\,\alpha_\mu + \dot\sigma_{\lambda\mu}\,\alpha_\nu\right) = \dot N_{\mu\nu\lambda} \quad . \qquad (\mathrm{B}\dot{6})$$

The relations (A$\dot{1}$,$\dot{2}$,$\dot{3}$,$\dot{6}$) represent simply the space-part of (B$\dot{1}$,$\dot{2}$,$\dot{3}$,$\dot{6}$). On the other hand a correct generalization of (A$\dot{4}$) to four dimensions is as follows :

$$\left(\delta_{\mu\varrho}\dot{\sigma}_{\lambda\tau}+\delta_{\mu\lambda}\dot{\sigma}_{\tau\varrho}+\delta_{\mu\tau}\dot{\sigma}_{\varrho\lambda}\right)\alpha_\nu+$$
$$+\left(\delta_{\nu\varrho}\dot{\sigma}_{\lambda\tau}+\delta_{\nu\lambda}\dot{\sigma}_{\tau\varrho}+\delta_{\nu\tau}\dot{\sigma}_{\varrho\lambda}\right)\alpha_\mu=$$
$$=\delta_{\mu\nu}\dot{N}_{\varrho\lambda\tau}-\frac{1}{12}\left(\varepsilon_{\varrho\lambda\tau\mu}\varepsilon_{\alpha\beta\gamma\nu}+\varepsilon_{\varrho\lambda\tau\nu}\varepsilon_{\alpha\beta\gamma\mu}\right)\dot{N}_{\alpha\beta\gamma}\quad. \qquad (\mathrm{B}\dot{4})$$

Here $\varepsilon_{\mu\nu\varrho\tau}$ is antisymmetrical unit tensor $(\varepsilon_{1234}=+1)$.

We can see that the relations of the form (A$\dot{4}$) are again obtained from (B$\dot{4}$) if we restrict the values of the indices only to I,2,3, because in this case the additional terms on the right hand side of (B$\dot{4}$) vanish. These additional terms are however necessary in the "fourdimensional case" because otherwise (B$\dot{6}$) could not be obtained by contraction of (B$\dot{4}$).

Notice the following formulas for the tensor $\varepsilon_{\mu\nu\varrho\tau}$:

$$\begin{aligned}\varepsilon_{\lambda\mu\nu\varrho}\varepsilon_{\lambda\mu'\nu'\varrho'}=\;&\delta_{\mu\mu'}\left(\delta_{\nu\nu'}\delta_{\varrho\varrho'}-\delta_{\nu\varrho'}\delta_{\varrho\nu'}\right)+\\&+\delta_{\mu\nu'}\left(\delta_{\nu\varrho'}\delta_{\varrho\mu'}-\delta_{\nu\mu'}\delta_{\varrho\varrho'}\right)+\\&+\delta_{\mu\varrho'}\left(\delta_{\nu\mu'}\delta_{\varrho\nu'}-\delta_{\nu\nu'}\delta_{\varrho\mu'}\right),\end{aligned} \qquad (\mathrm{B}5)$$

$$\varepsilon_{\lambda\mu\nu\varrho}\varepsilon_{\lambda\mu\nu'\varrho'}=2\left(\delta_{\nu\nu'}\delta_{\varrho\varrho'}-\delta_{\nu\varrho'}\delta_{\varrho\nu'}\right), \qquad (\mathrm{B}5')$$

$$\varepsilon_{\lambda\mu\nu\varrho}\varepsilon_{\lambda\mu\nu\varrho'}=6\,\delta_{\varrho\varrho'}\quad. \qquad (\mathrm{B}5'')$$

As in the "threedimensional case" a relatively sipmler form of the relations (B$\dot{2}$,$\dot{3}$,$\dot{4}$,$\dot{6}$) can be obtained if we use instead of $\dot{\sigma}_{\mu\nu}$ and $\dot{N}_{\mu\nu\varrho}$ the dual quantities $\sigma_{\mu\nu}$ and N_μ which are defined by the relations

$$\dot{\sigma}_{\mu\nu}=\frac{1}{2}\varepsilon_{\mu\nu\lambda\varrho}\sigma_{\lambda\varrho}\quad,\quad \dot{N}_{\mu\nu\lambda}=\varepsilon_{\mu\nu\lambda\varrho}N_\varrho$$

or conversely

$$\sigma_{\lambda\varrho}=\frac{1}{2}\varepsilon_{\lambda\varrho\mu\nu}\dot{\sigma}_{\mu\nu}\quad,\quad N_\varrho=\frac{1}{6}\varepsilon_{\mu\nu\lambda\varrho}\dot{N}_{\mu\nu\lambda}\quad.$$

In fact, in terms of these quantities the equations (B$\dot{2}$,$\dot{3}$,$\dot{4}$,$\dot{6}$) take the simpler form

$$\left[\sigma_{\mu\nu},\sigma_{\lambda\varrho}\right]=\frac{i}{2}\varepsilon_{\mu\nu\alpha\beta}\varepsilon_{\lambda\varrho\alpha\gamma}\varepsilon_{\beta\gamma\sigma\tau}\sigma_{\sigma\tau}\;, \qquad (\mathrm{B}2)$$

$$\left[\sigma_{\mu\nu},\alpha_\lambda\right]=i\,\varepsilon_{\mu\nu\lambda\varrho}\alpha_\varrho\;, \qquad (\mathrm{B}3)$$

$$\sigma_{\mu\nu}\alpha_\lambda + \sigma_{\lambda\nu}\alpha_\mu = \delta_{\mu\lambda}N_\nu - \frac{1}{2}(\delta_{\mu\nu}N_\lambda + \delta_{\lambda\nu}N_\mu), \qquad (B4)$$

$$\frac{2}{3}\sigma_{\lambda\mu}\alpha_\lambda = N_\mu \,. \qquad (B6)$$

The equation (B6) can be obtained by contracting (B4) in any way. The equations (BI,2,3,4) are the appropriate fundamental relations defining the algebra in space-time. If we introduce the operator

$$N = \frac{1}{4} N_\mu^2 , \qquad (B7)$$

it is possible (by similar procedures as in the "threedimensional case") to deduce from (BI,2,3,4) the following relations

$$[\alpha_\mu, N] = 0, \qquad (B8)$$

$$[\sigma_{\mu\nu}, N] = 0, \qquad (B9)$$

$$[N_\mu, N] = 0, \qquad (BI0)$$

$$N^2 = N, \qquad (BII)$$

$$(\alpha_\mu\alpha_\nu + \alpha_\nu\alpha_\mu - 2\delta_{\mu\nu})N = 0, \qquad (BI2)$$

$$(\alpha_\mu\alpha_\nu\alpha_\lambda + \alpha_\lambda\alpha_\nu\alpha_\mu - \delta_{\mu\nu}\alpha_\lambda - \delta_{\lambda\nu}\alpha_\mu)(1-N) = 0, \qquad (BI3)$$

$$\alpha_\mu\alpha_\nu\alpha_\lambda(1-N) = 0, \quad (\mu \neq \nu, \ \nu \neq \lambda) \,. \qquad (BI4)$$

From (B8,9,I0) it follows that in each irreducible representation of our algebra the matrix N is a multiple of the unit matrix. From (BII) we see that its eigenvalues are only I and 0. In the representation in which $N=1$ the α's fulfil the relations

$$\alpha_\mu\alpha_\nu + \alpha_\nu\alpha_\mu = 2\delta_{\mu\nu} \,.$$

In representations with N = 0 the α's fulfil the relations

$$\alpha_\mu\alpha_\nu\alpha_\lambda + \alpha_\lambda\alpha_\nu\alpha_\mu = \delta_{\mu\nu}\alpha_\lambda + \delta_{\lambda\nu}\alpha_\mu ,$$

and

$$\alpha_\mu\alpha_\nu\alpha_\lambda = 0 \quad (\mu \neq \nu, \ \nu \neq \lambda) \,.$$

From above relations the irreducible representations of our algebra can easily be determined.

In the course of the derivation of the relations (B,8-I4) one obtains also other useful relations, e.g.

$$[\sigma_{\mu\nu}^2, \sigma_{\lambda\varsigma}] = 0 , \qquad (BI5)$$

$$[\alpha_\lambda^2, \sigma_{\mu\nu}] = 0 , \qquad (BI6)$$

$$[\sigma_{\mu\nu}, N_\lambda] = i\,\varepsilon_{\mu\nu\lambda\varrho} N_\varrho \,, \tag{BI7}$$

$$(\alpha_\mu \alpha_\nu + \alpha_\nu \alpha_\mu - 2\,\delta_{\mu\nu}) N_\varrho = 0 \,, \tag{BI8}$$

$$N_\mu \alpha_\nu + \alpha_\mu N_\nu = 0 \,, \tag{BI9}$$

$$N_\mu \alpha_\nu + N_\nu \alpha_\mu = \frac{1}{2}\,\delta_{\mu\nu}\, N_\lambda \alpha_\lambda \,, \tag{B20}$$

$$(N_\mu N_\nu + N_\nu N_\mu - 2\delta_{\mu\nu}) N_\varrho = 0 \,, \tag{B2I}$$

$$(\sigma_{\mu\nu} + \frac{i}{4}\,\varepsilon_{\mu\nu\lambda\varrho}\, \alpha_\lambda \alpha_\varrho)\, N_\tau = 0 \,, \tag{B22}$$

$$\sum_c (\alpha_\mu \alpha_\nu \alpha_\lambda - \delta_{\mu\nu} \alpha_\lambda) = 3\,i\,\varepsilon_{\mu\nu\lambda\varrho} N_\varrho \,, \tag{B23}$$

$$\alpha_\nu \alpha_\mu \alpha_\nu = \alpha_\mu (1 - 3N) \,, \tag{B24}$$

$$N\alpha_\mu = \frac{2}{3}\,\sigma_{\mu\nu} N_\nu \;. \tag{B25}$$

From (B2I) we obtain (in addition to (BIO) and (BII)) also the formulas

$$N_\mu = N.N_\mu = N_\mu . N \,,$$

$$N_\mu^{\,3} = N_\mu \quad (\text{no summation})$$

and from (B23)

$$N_\varrho = \frac{1}{6i}\,\varepsilon_{\mu\nu\lambda\varrho}\, \alpha_\mu \alpha_\nu \alpha_\lambda \;.$$

Define

$$\alpha_5 = \frac{1}{24}\,\varepsilon_{\mu\nu\lambda\varrho}\, \alpha_\mu \alpha_\nu \alpha_\lambda \alpha_\varrho \quad \left(= \frac{i}{4} N_\varrho \alpha_\varrho = \frac{-i}{4} \alpha_\varrho N_\varrho \right) \;. \tag{B26}$$

Then we easily find that α_5 can be written also in the form

$$\alpha_5 = \alpha_1 \alpha_2 \alpha_3 \alpha_4 \tag{B26'}$$

and that

$$\alpha_5^2 = N$$

$$N_\mu = i\,\alpha_\mu \alpha_5 = -i\,\alpha_5 \alpha_\mu \;.$$

I am grateful to Professor V. Votruba for valuable discussions about related topics and some refinements of the given proofs.

Reference

/I/ V. Votruba and M. Lokajíček : An Algebraic System of Fundamental Particles. Preprint of the Joint Institute for Nuclear Research, April I958; to be published in Nuclear Physics.