205

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P - 205

THE SIMULATING OF MULTIPLE PRODUCTION PROCESS

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P - 2o5

THE SIMULATING OF MULTIPLE PRODUCTION PROCESS

A B S T R A C T

An idea of constructing the "table of random stars" capable of reproducing the theoretical concepts about the multiple production of elementary particles in the form suitable for comparing with experiments is discussed. The possibility of making such a table in the energy range up to IO BeV is shown.

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A majority of papers on the multiple production theory is devoted to the solution of two problems: to the calculation of the statistical weights of different reactions and to the obtaining of momentum distributions. Apart from this there remains a problem about the angular distributions and correlations of various kinds between the directions and velocities of the secondary particles. Meanwhile, its solution would make it possible to judge more definitely about the applicability of the theory to the explanation of the multiple production process. Indeed, the correlations between the directions of the particle motion must depend upon the character of the interaction between particles at the moment of their production ; in the first Fermi model of the statistically independent secondary particles the correlations must depend only upon the limitations due to the conservation laws; in the concept of the resonance interaction of a nucleon and meson accepted now the magnitude of the correlations must be greater. Comparison of the quantitative expressions for the correlations with experiment would make, perhaps, the character of the resonance interaction more exact or would outline the limits of the isobar theory.

The qualitative evaluation of the correlations "forwardbackward" was made by Fermi/I/ taking into account the law of angular momentum conservation. However, the complexity of the necessary calculations is the obstacle for the quantitative conclusions. If they could be performed under simple assumptions then the complication of the form of the interaction matrix /the natural way of developing the theory/ would make the calculation more difficult. Under these conditions the simulating of the multiple production process would be an evident way out. However, since our idea about the multiple production mechanism is too vague to find a large scale process with similar features it is necessary to make use of the numerical simulating. Such a simulating must be capable of reproducing the whole class of the assumptions about the form of the interaction.

An idea of such simulating was suggested by M.I. Podgoretsky and M.Danysz in the form of a "table of random stars" according to an isobar model. This idea consists in the tabulating of the random numbers which obey the same regularities as the momenta of the secondary particles in the isobar theory. The table of random stars must contain a number of lines ; each of them is the description of one event of particle production; i.e. it contains the magnitudes

and directions of the momenta of secondary particles. The statistical treatment of such a table will yield the same data about the multiple production process – the statistical weights, momentum, angular and charge distribution etc., as well as the treatment of the real stars in the photoemulsion or in the chamber. The comparison of such data will answer the question about the agreement of the theory with experiment.

The table of randon stars includes in principle all the data which can be obtained from the real stars. This is the advantage over other calculation methods. Any distributions or correlations which can be obtained from the stars in the photoemulsion may be repeated in the corresponding table. The shortcoming of the method is a limited accuracy, a great deal of calculations necessary for making up and analyzing the table and the impossibility of obtaining the analytic expressions for the results.

It is shown in this paper how to realize the above idea by means of the computer. When there are 5-6 secondary particles and under some simplifying assumptions one may draw a table of 100-200 lines using the abovementioned method by means of the hand-operated calculation.

The method under consideration takes into account the energy and momentum conservation in the reactions. It can be generalized over a rather wide class of ideas about the particle interaction in their multiple production. This enables to give a positive answer to the question about the possibility of the multiple production simulating.

Paragraph I concerns drawing up the table of the reactions in which the number of secondary particles is equal to three, the Fermi model being oonsidered correct. This simple example illustrates the main features of the simulating method under consideration.

In Paragraph 2 the method is generalized for n secondary particles and for the arbitrary form of the square F of the matrix interaction element; some details which make the calculations simpler are given in brief. At the same time some kinematic characteristics of the system of n particles are derived.

In Paragraph 3 the procedure for drawing up the table of random stars suitable for the comparison with experiment is outlined.

§ I. TABLE OF THREE PARTICLE REACTIONS

Consider the reaction of two particle interaction with the total energy E and zero momentum. As a result of this reaction three particles with masses m_i, momenta $\vec{p}_i$ and total energies e_i / $i = 1,2,3$/ are produced. According to Fermi theory the probability of this reaction is proportional to

$$W(E) = \int d^3\vec{p}_1\, d^3\vec{p}_2\, d^3\vec{p}_3\, \delta\Big(\sum_1^3 e_i - E\Big)\delta\Big(\sum_1^3 \vec{p}_i\Big) \qquad /I.1/$$

The factor $\delta\Big(\sum_1^3 e_i - E\Big)\delta\Big(\sum_1^3 \vec{p}_i\Big)$ is the momentum distribution density in the momentum nine-dimensional space, the problem of simulating consists in realizing such distribution. But the narrow stripes along the intersection of the surfaces $\sum_1^3 e_i - E = 0$ and $\sum_1^3 \vec{p}_i = 0$ in the nine-dimen-

sional rectangle have too small volume for the points with the coordinates $\vec{p}_i$ chosen at random to hit these stripes sufficiently often. But the integral obtained from /I.I/ by simple calculations

$$W(E)\int_{\mathcal{D}_3} dp_1\, dp_2 \cdot 8\pi^2 p_1 p_2 (E - e_1 - e_2) \qquad /I2/$$

makes it possible to realize the necessary momentum distribution. The integration region $\mathcal{D}_3$ in it /Fig. I/ is bounded by the curves $\cos\vartheta = \pm 1$ / ϑ is the angle between $\vec{p}_1$ and $\vec{p}_2$ /. Their equations are

$$\varepsilon^+ = 0 \ , \ \varepsilon^- = 0 , \qquad /I.3/$$

where

$$\varepsilon^\pm = e_1 + e_2 + \sqrt{(p_1 \pm p_2)^2 + m_3^2} - E. \qquad /I.4/$$

It follows from /I.2/ that the density distribution of the probability to have the momentum p_1 in dp_1 and at the same time p_2 in dp_2 /i.e. the density of points $M(p_1, p_2)$ in D_3 /is proportional to the function

$$W(p_1, p_2) = p_1 p_2 (E - e_1 - e_2). \qquad /I.5/$$

The points M , according to /I.5/, are distributed in D_3 not uniformly. We shall have to be concerned with the nonuniform distributions below. Let us consider the ways of their realization with the help of the uniform ones.

To obtain the random distributed numbers α in / a, b / with the density $f(\alpha)$/Fig. 2/ there are two possibilities: A/ rejection technique and B/ a "direct" method[5].

A. Two numbers are picked simultaneously : α uniformly in / a, b / and β uniformly in / $0, M'$ /, where the constant $M' \geq f(\alpha)$ for all α on (a, b). Those pairs α, β for which $f(\alpha) < \beta$, are rejected, then the remaining values of α will be distributed with the density $f(\alpha)$. Indeed, at great number of samples the number of values of α close to $\alpha = \alpha_1$ tends to be equal to the number of $\beta < f(\alpha_1)$, i.e. to the quantity $f(\alpha_1)/M'$ since β are distributed over the ordinate $\alpha = \alpha_1$ uniformly.

This method is also suitable for many-dimensional distributions, however, at sharp maxima of the function $f(\alpha)$ the number of the discarded pairs α, β may be found too great; if $f(\alpha)$ tends to infinity then this method is not suitable at all/ or gives only approximate results/.

B. β is selected uniformly in /0,I/ and the equation $\int_0^\alpha f(\alpha)\, d\alpha = \beta$ is solved each time. Then the distribution density of the solutions of this equation is proportional to $f(\alpha)$ since $dn/d\alpha = (dn/d\beta) \cdot f(\alpha) \sim f(\alpha)$

Through this method is applicable only to the one- dimensional distributions, it is suitable for infinite densities and is not connected with the necessity of rejecting the chosen values of β .

The two-dimensional distribution /I.5/ is convenient to realize using the rejection technique. For this it is necessary to know the maximum of $W(p_1, p_2)$ for D_3. One can easily find that the values $p_{1,2} = \bar{p}_{1,2}$, at which this maximum is achieved are the roots of

the system

$$p_1^2/e_1 = p_2^2/e_2 = E - e_1 - e_2 \qquad /I.6/$$

if they are found inside D_3. Otherwise $\bar{p}_{1,2}$ must lie on the boundary D_3, on the curve $\varepsilon^- = 0$, and satisfy the equation system

$$\frac{p_2[E - e_1 - e_2 - (p_1^2/e_1)]}{p_1[E - e_1 - e_2 - (p_2^2/e_2)]} = \frac{p_2 - (E - e_2)(p_1/e_1)}{p_1 - (E - e_1)(p_2/e_2)}, \quad \varepsilon^- = 0. \qquad /I.7\text{-}I.8/$$

At high energies E one has to solve /I.6/, at low energies -/I.7-I.8/.

A pair of values p_I, p_2 determines the momenta of all three particles by their magnitudes and directions. Therefore, the sequence of obtaining one line of three -particle star table corresponding to the Fermi model is the following :

I/ The random numbers p_I, p_2 in the intervals $(0, \tilde{p}_{1\,max})$, $(0, \tilde{p}_{2\,max})$ are chosen uniformly. Here

$$\tilde{p}_{1\,max} = \left\{\left[E^2 - (m_1 + m_2 + m_3)^2\right]\left[E^2 - (m_1 - m_2 - m_3)^2\right]\right\}^{1/2} \Big/ 2E, \qquad /I.9/$$

and an analogous form holds for $\tilde{p}_{2\,max}$.

2/ it would be verified whether the point (p_I, p_2) lies in D_3. One may do this in the simplest way by checking the fulfillment of the inequalities

$$\varepsilon^- < 0, \quad \varepsilon^+ > 0 \qquad /I.10/$$

If they are satisfied then one oan always find such an angle ϑ between $\vec{p}_1$ and $\vec{p}_2$ that

$$e_1 + e_2 + \sqrt{(\vec{p}_1 + \vec{p}_2)^2 + m_3^2} = E. \qquad /I.11/$$

If /I.II/ is not satisfied then the pair (p_I, p_2) is rejected, instead of it a new one is selected.

3/ in $(0, W_{max} = W(\bar{p}_1, \bar{p}_2))$ the random number β is pioked and the inequality

$$\beta \leq W(p_1, p_2) \qquad /I.12/$$

is verified.

If it is not fulfilled then the pair / p_I, p_2 / is discarded, if it is fultilled then :

4/ from /I.II/

$$\cos\vartheta = (p_3^2 - p_1^2 - p_2^2)/2p_1p_2 \qquad /I.13/$$

is calculated.

The momenta of three particles may now be considered equal to

$$\vec{p}_1 = \{p_1\cos\psi, -p_1\sin\psi, 0\}, \quad \vec{p}_2 = \{p_2\cos(\vartheta - \psi), p_2\sin(\vartheta - \psi), 0\} \qquad /I.14/$$

$$\vec{p}_3 = -\vec{p}_1 - \vec{p}_2,$$

where ψ is an random angle uniformly distributed in $/0, 2\pi/$. It is introduced in order the directions of any momentum to be isotropic in the plane $(\vec{p}_1, \vec{p}_2, \vec{p}_3)$.

The table each line of which is computed by /I.14/ will be the table of plane stars. When making the analysis of the experiment this will require to bring all three- particle

stars to one plane and only after this to compare the statistics of the experiment with that of the Table. To make the Table of three- particle stars which do not lie in the same plane it is, evidently, necessary to rotate the end of the vector of the unit normal to the plane / $\vec{p_1}, \vec{p_2}, \vec{p_3}$ / randomly about a certain axis.

The uniform distribution is most simply achieved if the cosine η of the angle between the normal and the axis OZ is chosen uniformly in (- I,I) as well as the angle φ between the axis OX and the projection of the normal to the plane XOY uniformly in $(0, 2\pi)$.

Then the number of hits of the normal to the element of the surface / η, φ / will appear proportional to the area of this element ($d\eta\, d\varphi$).

The angles φ and $\cos^{-1}\eta$ together with the angle ψ form three Eiler angles describing the transformation of the system $OXYZ$ in $OX'Y'Z'$ /2/. In Table of solid stars the components of the momenta must therefore be calculated by the formula $\vec{p}' = A\vec{p}$, where A is a well-known transformation matrix

$$A = \begin{pmatrix} \cos\varphi\cos\psi - \eta\sin\varphi\sin\psi & \sin\varphi\cos\psi + \eta\cos\varphi\sin\psi & \sqrt{1-\eta^2}\sin\psi \\ -\cos\varphi\sin\psi - \eta\sin\varphi\cos\psi & -\sin\varphi\sin\psi + \eta\cos\varphi\cos\psi & \sqrt{1-\eta^2}\cos\varphi \\ \sqrt{1-\eta^2}\sin\varphi & -\sqrt{1-\eta^2}\cos\varphi & \eta \end{pmatrix} \qquad /I.I5/$$

while the momenta may be put equal to

$$\vec{p_1} = \{p_1, 0, 0\}, \quad \vec{p_2} = \{p_2\cos\vartheta, p_2\sin\vartheta, 0\} \qquad /I.I6/$$

Another means for transforming the plane stars into those uniformly distributed in space is stated in § 2. The estimate of the efficiency of the supposed methods for obtaining the stars is given there too.

2. TABLE OF REACTIONS WITH n SECONDARY PARTICLES

General case. Consider the case of the appearance of n secondary particles in the interaction which is characterized by the square of the matrix element $\mathcal{F}(\vec{p_1}, \ldots, \vec{p_n})$ tending to infinity nowhere. F may also depend upon the energy E and the parameters of the primary particles. But it will be not essential further. The momentum part of the expression for the statistical weight has the form

$$W(E,\vec{P}) = \int d^3\vec{p_1} \ldots d^3\vec{p_n}\, \mathcal{F}(\vec{p_1}, \ldots, p_n)\, \delta(\sum_1^n e_i - E)\, \delta(\sum_1^n \vec{p_i} - P), \qquad /2.I/$$

where $\vec{P}$ is the total momentum of the system. The integration region over the first / k -I/ momenta $\vec{p_1}, \ldots, \vec{p_{k-1}}$ will be denoted by $\mathcal{D}k$, at the same time no restrictions except the conservation laws are imposed upon the rest momenta $\vec{p_k}, \ldots, \vec{p_n}$. Let the integration region over $\vec{p_k}$ at fixed $\vec{p_1}, \ldots, \vec{p_{k-1}}$ and arbitrary $\vec{p_{k+1}}, \ldots, \vec{p_n}$ /with account of the conserv-

ation laws/ be $d\kappa$. Introduce special notations for energy and momentum which are left for the particles $\kappa, \ldots, n$ at the fixed momenta of the particles $1, \ldots, k-1$

$$E_\kappa = E - \sum_1^{\kappa-1} e_i, \qquad /2.2/$$

$$P_\kappa = \sum_1^{\kappa-1} \vec{p}_i - \vec{P}, \qquad /2.3/$$

as well as for characteristic /"limiting"/ energy and momentum of the particle k in the centre of mass system of particles k, . . . , n at fixed $\vec{p}_1, \ldots, \vec{p}_{\kappa-1}$:

$$E_\kappa^* = \left(M_\kappa^2 + m_\kappa^2 - \mathcal{M}_\kappa^2\right)/2M_\kappa \qquad (\kappa = 1, \ldots, n-1), \qquad /2.4/$$

$$E_n^* = (M_{n-1}^2 - m_{n-1}^2 + m_n^2)/2M_{n-1}, \qquad /2.4'/$$

$$p_\kappa^* = \sqrt{E_\kappa^{*2} - m_\kappa^2} = \sqrt{(M_\kappa^2 - m_\kappa^2 - \mathcal{M}_\kappa^2)^2 - (2M_\kappa \mathcal{M}_\kappa)^2}/2M_\kappa, \qquad /2.5/$$

where M_κ is the effective mass of a system of particles $\kappa, \ldots n$, whereas $\mathcal{M}_\kappa$ is the mass of the compound particle composed from $k+1, \ldots, n$:

$$M_\kappa^2 = E_\kappa^2 - P_\kappa^2, \qquad /2.6/$$

$$\mathcal{M}_\kappa = \sum_{\kappa+1}^{n} m_i . \qquad /2.7/$$

We shall set the components of the momentum $\vec{p}_\kappa$ in the spherical coordinate system $\vec{p}_\kappa = \{p_\kappa, \vartheta_\kappa, \varphi_\kappa\}$, connected with the sum of the momenta $\vec{p}_I, \ldots, \vec{p}_{\kappa-1}$ already chosen. Let the polar axis of the system to be directed along $\vec{P}_\kappa$; ϑ_κ is the angle between the vectors $\vec{P}_\kappa$ and $\vec{p}_\kappa$, the azimuth φ_κ is counted off the vertical plane in which $\vec{P}_\kappa$ lies /Fig.3/. This coordinate system is convenient because the module of the sum of momenta $\vec{P}_\kappa$ and $\vec{p}_\kappa$ does not depend upon the angle φ_κ

$$p_{\kappa+1}^2 = P_\kappa^2 + p_\kappa^2 + 2P_\kappa p_\kappa \cos\vartheta_\kappa \qquad /2.8/$$

For recounting the components of the momentum into the rectangular coordinate system one may use the formulas following from Fig. 3.

$$\vec{P}_\kappa = \{X_\kappa, Y_\kappa, Z_\kappa\}$$
$$R_\kappa = (X_\kappa^2 + Y_\kappa^2)^{1/2}, \qquad /2.9/$$

$$x_\kappa = p_\kappa\left[\frac{X_\kappa}{P_\kappa}\cos\vartheta_\kappa + \left(\frac{Z_\kappa}{P_\kappa}\frac{X_\kappa}{R_\kappa}\cos\varphi_\kappa - \frac{Y_\kappa}{R_\kappa}\sin\varphi_\kappa\right)\sin\vartheta_\kappa\right], \qquad /2.10/$$

$$y_\kappa = p_\kappa\left[\frac{Y_\kappa}{P_\kappa}\cos\vartheta_\kappa + \left(\frac{Z_\kappa}{P_\kappa}\frac{Y_\kappa}{R_\kappa}\cos\varphi_\kappa + \frac{X_\kappa}{R_\kappa}\sin\varphi_\kappa\right)\sin\vartheta_\kappa\right], \qquad /2.11/$$

$$z_\kappa = p_\kappa\left[\frac{Z_\kappa}{P_\kappa}\cos\vartheta_\kappa - \frac{R_\kappa}{P_\kappa}\cos\varphi_\kappa\sin\vartheta_\kappa\right]. \qquad /2.12/$$

Now let us transform /2.I/. After the integration over $\vec{p}_n$ and over ϑ_{n-1} δ-functions in the integrand vanish:

$$W(E,\vec{p}) = \int d^3\vec{p}_1 \dots d^3\vec{p}_{n-2}\, d p_{n-1}\, d\varphi_{n-1}\, p_{n-1}\frac{E_n}{P_{n-1}}\mathcal{F}(\vec{p}_1,\dots,\vec{p}_{n-1},-\vec{p}_{n-1}-\vec{p}_{n-1}), \qquad /2.I3/$$

where the vector $\vec{p}_{n-1}$ is of the form

$$\vec{p}_{n-1} = \left\{ p_{n-1}\ \cos^{-1}\frac{E_n^2-m_n^2-p_{n-1}^2-p_{n-1}^2}{2\,p_{n-1}\,p_{n-1}},\ \varphi_{n-1} \right\}, \qquad /2.I4/$$

whereas the integration region $\mathcal{D}_n$ is limited by the surfaces $\cos\vartheta_{n-1} = \pm 1$, i.e.,

$$\sqrt{(p_{n-1} \pm p_{n-1})^2 + m_n^2} = E_{n-1} - e_{n-1}. \qquad /2.I5/$$

In principle expression /2.I3/ makes it possible to solve the problem of the realization of the distribution required by the model with the interaction F . In /2.I3/ there are no dependent integration variables, therefore the sets of values of $\vec{p}_1,\dots,\vec{p}_{n-2},p_{n-1},\varphi_{n-1}$ must satisfy certain inequalities instead of equalities/ as distribution /2.I/ would have required/, what increases the efficiency of sampling . However, the amount of the calculations necessary for creating the distribution over $\vec{p}_1,\dots,\vec{p}_{n-2},p_{n-1},\varphi_{n-1}$ of the density $p_{n-1}E_n\mathcal{F}/P_{n-1}$ is still rather great. All further transformations must be aimed at reducing this amount of calculations. An evident way for this is to perform in /2.I3/ the maximum possible number of integrations. Due to this the dimension of the integration region falls increasing thereby the ratio of the volume of the region to the volume of the rectangle circumscribed. However, this aim cannot be achieved if the form of the function F is arbitrary. Having in view to suggest the method applicable for any F we shall choose another way of reducing the calculations, and namely, we present $d^3\vec{p}_k$ as $p_k^2\,dp_k\,d\cos\vartheta_k\,d\varphi_k$ using the spherical ooordinate system described above. Then the distribution density over $p_1,\cos\vartheta_1,\varphi_1;p_2,\dots;\dots;\dots,\varphi_{n-2};p_{n-1},\varphi_{n-1}$ is of the form:

$$\Phi(p_1,\dots,\varphi_{n-1}) = p_{n-1}\frac{E_n}{P_{n-1}}\mathcal{F}\cdot\prod_{k=1}^{n-2}p_k^2, \qquad /2.I6/$$

where the arguments F are the same as in /2.3/. It is essential that the regions $\mathcal{D}_2$, ... , $\mathcal{D}_k$, ... , $\mathcal{D}_n$ of the allowed values of $\vec{p}_1;\dots;\vec{p}_1,\dots,\vec{p}_{k-1};\dots;\vec{p}_1,\dots,\vec{p}_{n-1}$ are independent of $\varphi_1,\dots,\varphi_{n-1}$. Let us prove this.

Let the momenta $\vec{p}_1,\dots,\vec{p}_{k-1}$ $(k\le n-2)$ be fixed. Find the region α_k of the change of the momentum $\vec{p}_k$. For the remaining particles $k, k+1,\dots, n$ the equations

$$\sum_k^n e_i = E_k \qquad /2.I7/$$

$$\sum_k^n \vec{p}_i = -\vec{P}_k \qquad /2.I7'/$$

are correct.

It is known[3] that the limiting values of the momentum of one of the particles at their multiple production are obtained when all other secondary particles move as a whole, i.e., as a particle of mass equal to their total mass. For particle k this mass is equal to μ_k /see /27/ /. The conservation laws for two particles with masses m_k and μ_k, with the total energy E_k and the momentum $-\vec{P}_k$ are written as follows

$$\sqrt{p_k^2+m_k^2} + \sqrt{(-\vec{P}_k-\vec{p}_k)^2+\mu_k^2} = E_k.$$

This is an equation of a surface limiting the region of possible positions of the vector end $\vec{p}_k$. The position and form of the region d_k may be most explicitly obtained from the graphic representation of the conservation laws [4]. The sphere $p_k^* = Const.$ in the center of mass system is deformed when passing into the laboratory system into the ellipsoid of revolution extended along the direction of mutual motion of two frames of references. In the considered case the center of the ellipsoid is the point $0(-E_k^*\vec{P}_k/M_k)$, the major axis has the length $2p_k^*E_k/M_k$. It is directed along the vector $\vec{P}_k$ and is an axis of revolution, the semi-minor axis is equal to p_k^* /Fig.4/. Here E_k^* and p_k^* are defined by /2.4/ -/2.5/.

Now the region $\mathcal{D}_k$ may be found. Since the energy E_k^* of the particle k in the center of mass system of particles k, k+I, ... , n is not less than the mass of this particle m_k then /see /2.4/ /

$$M_k \geq \mu_{k-1} \quad (k \leq n-1). \qquad /2.18/$$

Condition /2.18/ is not only a necessary restriction upon $\vec{p}_1, \ldots, \vec{p}_{k-1}$, but a sufficient one: when it is fulfilled there may be always found such $\vec{p}_k, \ldots, \vec{p}_n$ that the conservation laws will be satisfied. It is possible for instance to direct the particles k+I, ..., n all together (in the frame of reference where $\vec{P}_k = 0$) in one direction whereas the particle k in the opposite one.

But M_k depends only upon the modulus /2.6/ which in its turn is independent of φ_k /2.8/. Therefore the form of the region $\mathcal{D}_k$ /2.18/ is independent of φ_k $(k \leq n-1)$. For $\mathcal{D}_n$ it follows from /2.15/.

The independence of the outlines of the regions $\mathcal{D}_k$ of φ_k for all k from 1 up to n makes it possible to reduce the dimension of each D_k as much as one and a half times. So, the dimension D_n will be 2(n-I) instead of 3(n-I). Thereby the ratio of the volume of the region to the volume of circumscribed rectangle of the same dimension considerably increases.

Write down in more detail the limits restricting the dimensions of D_k and of the rectangle circumscribed. It is clear from Fig.4 that at fixed $\vec{p}_1, \ldots, \vec{p}_{\kappa-1}$ the limits of the change of p_k are given by the formulas

$$p_{max}^{(\kappa)} = (E_\kappa^* p_\kappa + p_\kappa^* E_\kappa)/M_\kappa \qquad \text{at} \quad \kappa = 1, \ldots, n-1,$$

$$p_{min}^{(\kappa)} = \begin{cases} (E_\kappa^* p_\kappa - p_\kappa^* E_\kappa)/M_\kappa & \text{at } E_\kappa/M_\kappa \geq E_\kappa^*/m_\kappa \text{ and } \kappa < n-1 \\ 0 & \text{at } E_\kappa/M_\kappa \leq E_\kappa^*/m_\kappa \text{ and } \kappa < n-1 \end{cases} \qquad /2.19/$$

$$p_{min}^{(n-1)} = \left| E_{n-1}^* p_{n-1} - p_{n-1}^* E_{n-1} \right| / M_{n-1}$$

/The restrictions upon p_{n-1} differ from those upon p_κ / $\kappa < n-1$/ because the system of particles n-I and n is kinematically definite/.

The limits of the change of $\cos\vartheta_\kappa$ at the fixed $\vec{p}_1, \ldots, \vec{p}_{\kappa-1}, p_\kappa$ have the following form

$$(M_\kappa E_\kappa^* - E_\kappa e_\kappa)/p_\kappa p_\kappa \geq \cos\vartheta_\kappa \geq -1 \quad \text{at} \quad E_\kappa/M_\kappa \geq E_\kappa^*/m_\kappa \quad \text{or}$$

$$\text{at} \quad E_\kappa/M_\kappa \leq E_\kappa^*/m_\kappa \text{ and } p_\kappa \geq (p_\kappa^* E_\kappa - E_\kappa^* p_\kappa)/M_\kappa, \qquad /2.20/$$

$$1 \geq \cos\vartheta_\kappa \geq -1 \quad \text{at} \quad E_\kappa/M_\kappa \leq E_\kappa^*/m_\kappa \text{ and } p_\kappa \leq (p_\kappa^* E_\kappa - E_\kappa^* p_\kappa)/M_\kappa.$$

The boundaries of the circumscribed rectangle are obtained by the same considerations but under the assumption that no momentum is fixed. For the momentum of the particle k (k=I,..,n) a following chain of formulae is obtained

$$M^2 = E^2 - p^2, \qquad /2.21/$$

$$e_\kappa^* = [M^2 + m_\kappa^2 - (\sum_{i \neq \kappa}^{n} m_i)^2]/2M, \qquad /2.22/$$

$$p_\kappa^* = \sqrt{e_\kappa^{*2} - m_\kappa^2}$$

$$\tilde{p}_{\kappa\,max} = (E p_\kappa^* + p e_\kappa^*)/M \qquad /2.22'/$$

$$\tilde{p}_{\kappa\,min} = \begin{cases} (p e_\kappa^* - E p_\kappa^*)/M & \text{at } E/M \geq e_\kappa^*/m_\kappa \\ 0 & \text{at } E/M \leq e_\kappa^*/m_\kappa \end{cases} \qquad /2.23/$$

The limits of the change of $\cos\vartheta_\kappa$ are $\pm$ I /except $\cos\vartheta_1$/.

So, it is possible to outline the following order of sampling n- particle reactions characterized by the square of the matrix element F.

p_I and $\cos\vartheta_1$ are selected in $(\tilde{p}_{1\,min}, \tilde{p}_{1\,max})$ and (-I, +I), then p_2 and $\cos\vartheta_2$ are chosen in $(\tilde{p}_{2\,min}, \tilde{p}_{2\,max})$ and (-I, + I), and the fulfillment of (2.I8) for k = 3 is

verified etc... p_k and cos ϑ_k are chosen in ($\widetilde{p}_{k\,min}$, $\widetilde{p}_{k\,max}$) and (-I,+I), E_{k+I} and P_{k+I} are calculated by (2.2) and (2.8), and (2.I8) is verified; when (2.I8) is not fulfilled the sampling begins again from p_I, otherwise one passes in to (k + I) etc. up to k=n-I, when only p_{n-1} are picked in ($p_{min}^{(n-1)}$, $p_{max}^{(n-1)}$) (not between $\widetilde{p}_{n-1,min}$, $\widetilde{p}_{n-1,max}$) and the verification of /2.I8/ is not necessary. Only after it $\varphi_1, \ldots, \varphi_{n-1}$ are drawn in $(0, 2\pi)$, cos ϑ_{n-1} is calculated by /2.I4/ and $\Phi(p_1, \cdots, \varphi_{n-1})$ by /2.I6/.

The realization of the density distribution using the rejection technique has in this case some pecularities. It is impossible to apply this technique directly since $\Phi_{max} = \infty$ for $p_{n-1} = 0$ if F is finite everywhere. But just at $p_{n-1} \to 0$ the interval of the change p_{n-1} (2.I9) contracts into a point, i.e. p_{n-1} need not be drawn. Let the accuracy for setting the momenta in Table be ε, then it is possible to find $p_{min}^{(n-1)}$ so small that the interval of the change p_{n-1} will be less than ε • , and for $p_{n-1} \leq p_{min}^{(n-1)}$ it may be supposed that $p_{n-1} = p_{n-1}^{*}$. If $p_{n-1} > p_{min}^{(n-1)}$ then the distribution is realized by a rejection technique, and the maximum /2.I6/ is taken as a maximum of Φ with the change of p_{n-1} for $p_{min}^{(n-1)}$. If a preliminary evaluation of Φ_{max} is too overestimated or impossible due to a complex form of F , it may be improved in the course of calculation. Having calculated $\Phi = \Phi^{(1)}, \ldots, \Phi^{(\nu)}$ for a sufficiently large number ν of sets of the components $\vec{p}_1, \ldots, \vec{p}_{n-1}$ one may assume the greatest value of Φ on the obtained assembly of sets as Φ_{max}. After that having choosen ν random numbers $\alpha^{(1)}, \ldots, \alpha^{(\nu)}$ on ($0, \Phi_{max}$) and comparing $\Phi^{(1)}, \ldots, \Phi^{(\nu)}$ with them, it is necessary to reject those sets for which $\alpha > \Phi$.

After the distribution of the density Φ has been realized the tabulation is completed by the calculation of $\vec{p}_n = -\vec{p}_{n-1} - \vec{p}_{n-1}$ and by transforming of all the momenta into the system $OXYZ$ by /2.9/-/2.I2/.

<u>The Fermi Model</u> $F \equiv I$ enables to simplify a little the calculation method. In this case one may in /2.I3/ integrate over φ_{n-1} and over p_{n-1} (more exactly, over e_{n-1} within the limits

$$e_{\substack{min \\ max}} = \frac{E_{n-1}^{*} E_{n-1} \mp p_{n-1}^{*} p_{n-1}}{M_{n-1}} \Bigg) \qquad /2.24/$$

and to obtain the density distribution over $\vec{p}_1, \ldots, \vec{p}_{n-2}$:

$$\Phi'(\vec{p}_1, \ldots, \vec{p}_{n-2}) = \frac{4\pi}{3} p_{n-1}^{*} \frac{3 E_{n-1}^{*} E_n^{*} E_{n-1}^{2} - p_{n-1}^{*2} p_{n-1}^{2}}{M_{n-1}^{3}} \cdot \prod_{i=1}^{n-2} p_i^2 . \qquad /2.25\ /^{I/}$$

Here the sequence of samplings is the same as in the general case, but comes to an end on the

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I/ If n= 3 , /2.25/ coincides with /4/ from /6/.

sampling ρ_{n-2}, $\cos\vartheta_{n-2}$. On obtaining the points uniformly distributed in D_{n-I} the nonuniform distribution of the density φ' is being realized using the "rejection technique". φ'_{max} may be calculated beforehand, and, namely, it may be shown that the greatest value of φ' is realized when $\vec{P}_{n-I} = 0$, and the momenta $\vec{p}_1, \ldots, \vec{p}_{n-2}$ satisfy the equation system

$$\frac{p_1^2}{e_1} = \frac{p_2^2}{e_2} = \ldots = \frac{p_{n-2}^2}{e_{n-2}} = \frac{2 p_{n-1}^{*2} E_{n-1}^* E_n^* E_{n-1}}{(E_{n-1}^* E_n^*)^2 + p_{n-1}^{*2}(E_{n-1}^{*2} - E_{n-1}^* E_n^* + E_n^{*2})}. \qquad /2.26/$$

This system is easily solved using the iteration method.

Further it is necessary to select the momentum p_{n-1}. Instead of this it is better to choose the energy e_{n-1} using the direct method /B/, i.e., to solve the equation

$$2e_{n-1}^3 - 3E_{n-1} e_{n-1}^2 = (2e_{min}^3 - 3E_{n-1} e_{min}^2)(1-\alpha) + (2e_{max}^3 - 3E_{n-1} e_{max}^2)\alpha, \qquad /2.27/$$

where α is uniformly distributed in /0,I/.

Only after this it is meaningful to pick $\vartheta_1, \ldots, \vartheta_{n-1}$.

The characteristic feature of the method stated in § I-2 is the presence of the unsuccessful samplings, i.e., the realization of the nonuniform distributions using the rejection technique. Evidently, when the number of unsuccessful samplings is great the method becomes practically useless for tabulating. One fails to estimate the efficiency of the method in the general form. However, the numerical experiments has shown that for the reactions on meson production by protons at E = 4.72 BeV the efficiency of the method is 50% /the production of one pion / and I0% /two pions/. When the number of mesons is great the efficiency is too small to apply the method described above in the hand calculations. Still it is possible to make up the tables when 3-4 mesons are produced by the hand calculation if φ' is integrated some times more. The use of electronic computer will enable to make up large tables of 5-6 -particle reactions applying the method described here.

3. TABLE OF RANDOM STARS

The table of n -particle reactions is difficult to compare with experimental data since neutral particles fail to be detected in the experiment. It must be incorporated into the table of random stars created in the collision of the two given particles at the given energy E.

To make up such a table it is necessary to consider all kinds of reactions having an appreciable weight. The magnitudes of the statistical weights define the average number of lines for each reaction.

In the Fermi model the statistical weights may be calculated and are already known for many reactions. But for the simulating of the interaction with an arbitrary square of the

matrix element the problem of the calculation of the statistical weight may appear to be as difficult as the tabulation of random stars. However, the above method for the making up of the table of n-particle reactions enables in principle to determine the phase volume W - the main part of the expression for the statistical weight. Indeed, it is clear that the phase volume W is equal to the product of the volume of the region $\mathcal{D}n$ by an expected value Φ /2.I6/ over this region. The volume of the region $\mathcal{D}n$ is equal to the fraction of the successful hits of the sets $\vec{p}_1, \ldots, \vec{p}_{n-1}$ to this region multiplied by the volume of the rectangle with the known dimensions circumscribed around $\mathcal{D}n$. The application of this method for calculating the phase volumes would require a considerable number of samplings due to great spread in the values of Φ, - the smaller the phase volume the greater the spread.

If the statistical weights of the most important reactions occuring at the given energy are known then the making up of the table of random stars is not difficult. Let us arrange all the reactions $\mathcal{N}_1, \mathcal{N}_2, \ldots, \mathcal{N}_j, \ldots$ in an arbitrary order and give its own interval $\delta\mathcal{N}_j$ to each of them on the segment (0,I). The length of the interval must be proportional to the statistical weight of the reaction $\mathcal{N}_j$. Then for drawing the next line of the table one throws randomly a point on the segment (0,I). The number of the interval $\delta\mathcal{N}_j$ to which it will be incident points out the reaction which will be represented in this line. Besides it is being cleared up by sampling which of the momenta in this line should be considered belonging to the charged particles. The momenta of other particles as unobservable need not be introduced into the line.

In the table made up in such a way a/ the reactions will be shuffled randomly b/ the number of lines reffering to each reaction will fluctuate with respect to that expected by the statistical weight. The law of these fluctuations will be the same as in the observation of the real stars. It should be noted in order to increase the accuracy of small tables the fluctuations may be forbidden.

Using the computer giving the stars with $n = 6$ by the above method a majority of reactions induced by protons of the kinetic energy I0 BeV may be taken into account. The stars with different number of rays will be present.

The accuracy of the results obtained from the table of random stars is mainly determined by the same factors as in case of the real stars. There are no errors in the determination of energy, angles of particle emergence; one may neglect oscillations of the frequency of some reactions with respect to their weight. However, the results will be influenced by the error in the calculation of the statistical weights of all reactions. When the errors in the weights are of the order of I0% the making up of tables of more than I000 lines is

hardly worth while. The surplus of accuracy which is taken is necessary to obtain the conditional distributions and for the comparison of the distributions at different energies. The error in the statistical weights at different E is levelled since they are more or less systematical. Therefore, the results of the comparison of the distributions at different E may have a greater accuracy.

An interesting possibility of increasing the accuracy appears in the tabulation of random stars in the laboratory system of reference. Then there is no neoessity any more to transform real stars into the centre of mass system – this problem is solved at high energies rather roughly. Thus, the analysis of the experimental data becomes easier and the accuracy of comparing the table of random stars with the experimental data increases. The formulae for the calculations in both systems are identical as concerns their complexity /in § 2 everywhere $\vec{P} \neq 0$ /. Practically, however, at $\vec{P} \neq 0$ the interval of the allowed values of the momenta reduces/narrow beams!/ and the interval of the allowed values of the momenta $\vec{p}_k$ extends. Both these factors reduce the efficiency of the samplings $\vec{p}_k$. Therefore, it is better to transform the momenta in the table of stars in the centre of mass system into the laboratory system.

C O N C L U S I O N

It is the purpose of this paper to clear up the possibility of making up the table of random stars. A way for it is given here. Evidently, in the first turn one will succeed in reproducing the Fermi model and that of isobar in the form of a table. One may hope that the reproduction of the reactions with 6-7 secondary particles for the modern computers is quite real. It means that it is possible to make up for instance, a table of stars created p-p̄ collisions with the IO BeV accelerator.

The way of making up such a table for these models is such that the distribution of the reaction products over the angles is isotropic. To explain the nonisotropy really observed one has to choose a definite suitable dependence of $\mathcal{F}$ upon momenta and angles.

The described method of making up the table of random stars is accepted in different kinds of the matrix element square F . The complexity of the function $\mathcal{F}$ is of no importance since a major period of time during the work of the computer has to be spent for rejecting the unsuccessful momentum componenta, but not for calculating the accepted momenta. Therefore there is a possibility of finding a satisfactory $\mathcal{F}$ from a phenomenologically standpoint form of interaction by testing different ideas about the interaction and comparing the obtained statistical weights, distributions, correlations with the experimental data Such a method might have been applied for studying the decay of unstable particles.

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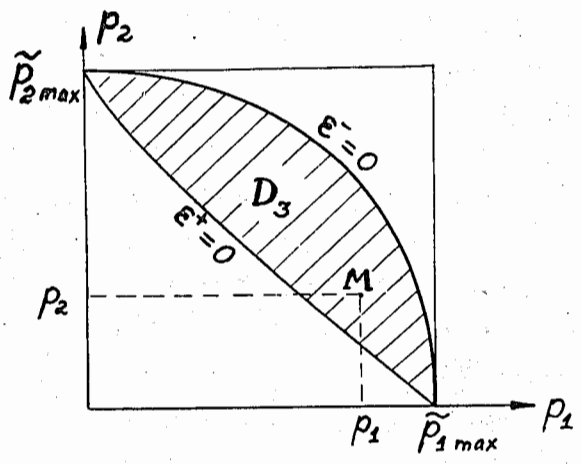
p_2
$\tilde{p}_{2\,max}$
$\varepsilon^- = 0$
D_3
$\varepsilon^+ = 0$
M
p_2
p_1
$\tilde{p}_{1\,max}$
p_1

Fig.I

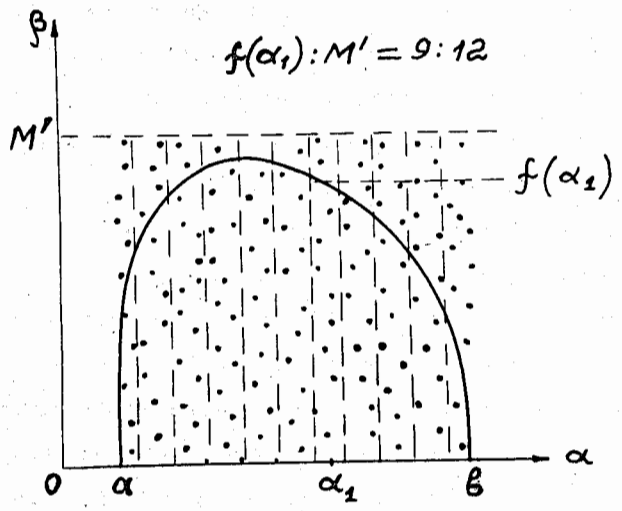
β
$f(\alpha_1) : M' = 9 : 12$
M'
$f(\alpha_1)$
0
α
α_1
β
α

Fig.2

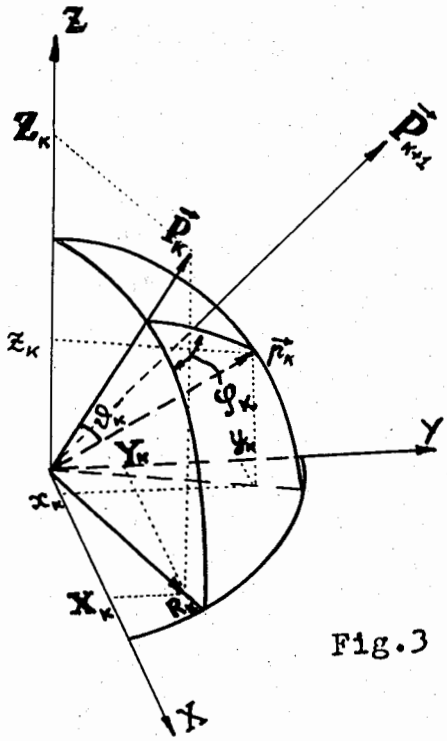

Fig.3

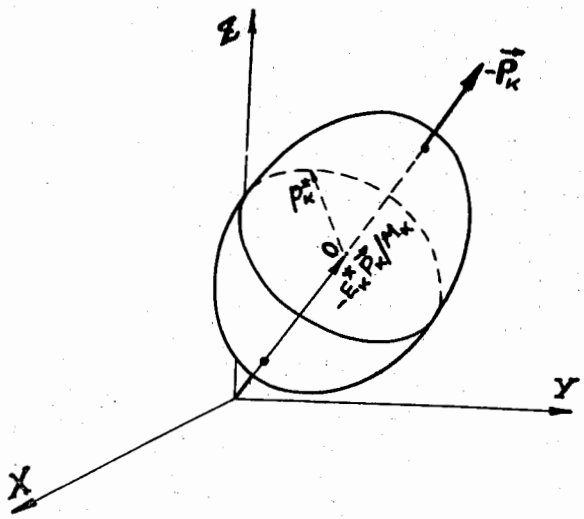

Fig.4

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