JOINT INSTITUTE FOR NUCLEAR RESEARCH

Laboratory of Theoretical Physics

V.K. Melnikov, Yu.S. Sajasov*)

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THE THEORY OF PARTICLE CAPTURE INTO SYNCHROTRON ACCELERATION REGIME WITH ACCOUNT OF NONCONSERVATION OF MOTION EQUATIONS

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V.K. Melnikov, Yu.S. Sajasov*)

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*) Institute of Chemical Physics of the USSR Academy of Sciences.

As is known the phase motion of particles in all the resonance accelerators which are based on the phase-self stability principle is described by the one-type equation which may be generally written as follows*)

$$\frac{d}{dt}\left[m(t)\dot{\psi}\right]+f(t)\,\mathcal{U}'(\psi)=0, \qquad /I/$$

where $\mathcal{U}'(\psi)=\frac{d\mathcal{U}}{d\psi}=-\frac{1}{\sin\varphi_s}\left[\cos(\varphi_s+\psi)-\cos\varphi_s\right]$, φ_s – is the synchrone phase, ψ – is the difference between the phases of the considered particle and the synchrone one.

One of the principal problems of the theory of accelerators is to determine the capture region, i.e. the region of the initial values of ψ^0, $\dot{\psi}^0$, from which with $t=t_0$ the solutions oscillating with respect to the position of stable equilibrium $\psi=0$ emerge.

The knowledge of the capture region makes it possible to evaluate the part of the injected particles incident into the synchrone acceleration regime and, therefore, leaving the accelerator with an energy close to the calculated one. Since in equation /I/ the functions $m(t)$ and f(t) vary in the concrete cases rather slowly then in a certain approximation they may be considered constant what enables to integrate equation /I/ (the conservation approximation) quite easily. Then the boundary of the capture region is determined by the formula:

$$\dot{\psi}^0=\pm\sqrt{2\mathcal{U}_m-2\mathcal{U}(\psi^0)}, \quad \mathcal{U}(\psi^0)=\int_0^{\psi^0}\mathcal{U}'(\xi)\,d\xi, \qquad /2/$$

where $\mathcal{U}_m=\mathcal{U}(-2\varphi_s)$ is the maximum of $U(\psi)$ (it is assumed that $m(t_0)=f(t_0)=I$ that does not violate the generality). At $\dot{\psi}^0=0$ the width of the capture region is evidently $\Delta\psi_c=2\varphi_s+\psi_m^0$, where ψ_m^0 and $-2\varphi_s$ are the roots of the equation $\mathcal{U}(\psi)=\mathcal{U}_m$ /the root $-2\varphi_s$ is two fold/. Particularly, if $\varphi_s \ll 1$ / in fact it is quite sufficient to assume that $\varphi_s<30°$/, then $\psi_m^0\approx\varphi_s$ and $\Delta\psi_c\approx 3\varphi_s$.

The motion of particles for which the initial values ψ^0, $\dot{\psi}^0$ lie inside the egg-shaped region limited by the curves /2/ will be stable.

Point out that equation /I/ corresponds to the simple mechanical model of the particle moving in the potential well limited from the left by the barrier U_m /Fig. I/. From this standpoint the motion of particles the energy of which $\mathcal{E}=\frac{1}{2}(\dot{\psi}^0)^2+\mathcal{U}(\psi^0)$ does not exceed $\mathcal{U}_m$, and $-2\varphi_s<\psi^0<\psi_m^0$ is stable. On the phase plane the trajectories of such particles are closed. The trajectories the initial values of which lie outside the region determined by /2/ are not stable and go over to the infinity /Fig.2/ except the trajectories which correspond to the initial values $\psi^0, \dot{\psi}^0=+\sqrt{2\mathcal{U}_m-2\mathcal{U}(\psi^0)}$ lying to the left from the barrier of the potential well. Indeed, the particles, the initial coordinates of which lie to the left from the barrier

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*) Note that in equation /I/ we neglect the terms of type of external force F(t), and do not take into account the effect of the volume charge.

U_m, and the initial velocity satisfies the relation $\dot{\psi}^{o}=+\sqrt{2U_m-2U(\psi^o)}$ move along the trajectories asymptotically tending to the point of the unstable equilibrium $-2\varphi_s$.

In papers devoted to the theory of various resonance accelerators untill now the authors restricted themselves only by the determination of the capture region in the conservation approximation according to /2//see,e.g. /I/ /.*)

One may understand to what consequences the increase of m(t) and f(t) /this usually occurs/ leads by means of the already used model of a particle in the potential well. As concerns the particles moving in the conservation approximation /on the plane ψ, $\dot{\psi}$ / along the closed curves inside the egg-shaped region defined by /2/ they, evidently, turn out into spirals for which at $t\to\infty$, $\psi\to 0$ and $\dot{\psi}\to 0$. The law of the variation of the amplitude of these oscillations may be found using the method of adiabatic invariant /see/I//. The boundary of the region of the initial values ψ^o, $\dot{\psi}^o$ to which such stable trajectories correspond as a result of the increase of $m(t)$ and $f(t)$ must also considerably charge. Indeed, consider the particle which at the initial moment is in the point ψ^o situated some what to the right from ψ_m^o /see. Fig.I/ $(\dot{\psi}^o<0)$. Evidently, it will move to the barrier U_m which for the time of its approaching somewhat rises, and, therefore, the particle may appear to be captured in the potential well if its energy $\varepsilon=\frac{1}{2}(\dot{\psi}^o)^2+U(\psi^o)$ is close enough to U_m. These qualitative considerations are confirmed by the results of the numerical calculations/2/, from which it follows that by the increase of m(t) and f(t) the capture region somewhat extends.

In this paper the calculation of the capture region is given with account of the nonconservation of equation /I/. This calculation is based upon the assumption that the change of m(t) and f(t) is slow. This is understood in the sense that the time τ of the considerable change of m(t) and f(t) is great if compared with the period $2\pi/\Omega$ of the linear phase oscillations. This means that equation /I/ may be written in the form:

$$\frac{d}{dt}\left[m(\varepsilon t)\dot{\psi}\right]+f(\varepsilon t)U'(\psi)=0, \qquad /Ia/$$

where

$$\varepsilon=\frac{2\pi}{\Omega\tau}$$

Equation /Ia/ belongs to the equations with "slow time", to which many papers by N.N. Bogolubov, Yu.A. Mitropolsky 6/, and V.M. Volosov7/ are devoted. The methods for investigating these equations developed in /6,7/ are, however, based upon the assumption that their solutions are oscillating and the problem about the boundary of the region of the initial values to which such stable solutions correspond was not in general set up by these authors.

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*) Exceptions are the papers (2)-(5), containing some numerical calculations concerning the role of variability of coefficients m(t), f(t) in eq.(I).

For the equations of the considered type /Ia/ with the increasing $m(\varepsilon t)$ and $f(\varepsilon t)$ one may find the boundary of the capture region with the help of the following physical considerations. It is evident that any particle coming from the right to the vertex of the barrier with the velocity different from zero must be unstable. /see, Fig.I/.

On the other hand, the particles which move from the right to this vertex and fail to reach it are stable. Therefore, it is natural to define the separatrix limiting the region of stable ψ^o , $\dot\psi^o$ as a set of such ψ^o , $\dot\psi^o$ that the trajectories outgoing from them at $t=t_0$ asymptotically approach the unstable equilibrium position $-2\varphi_s$. It is clear that there exists an infinite number of trajectories /we shall call them boundary trajectories which satisfy the condition $\psi\to-2\varphi_s$, whereas $\dot\psi\to 0$ at $t\to\infty$. The problem of further investigation is to find the functional dependence $\dot\psi^o=\dot\psi^o(\psi^o)$ for them. Since equation /Ia/ involves a small parameter ε then one may try to find boundary trajectories by means of the formal expansion of the solutions of equation /Ia/ in powers of E. Since the considered region of the change of independent variable is infinite, the Poincarè theorem about the analiticity of the solutions of equations of type /Ia/ with respect to E may be incorrect. /see/8/ /.

V.K. Melnikov has shown,[9] however, that the formal expansion $\psi=\psi_0+\varepsilon\psi_1+\dots$ satisfying the auxiliary conditions $\psi_0\to-2\varphi_s$, $\dot\psi_0\to 0$, $\psi_i\to 0$, $\dot\psi_i\to 0\,(i=1,2\dots)$ at $t\to\infty$ and $\psi_i=0\,(i=1,2\dots)$ at $\psi_0=\bar\psi_0$ / $\bar\psi_0$ - is an arbitrary point lying sufficiently close to $-2\varphi_s$ to the right from $-2\varphi_s$ for the trajectories approaching $\psi=-2\varphi_s$ to the right and from $-2\varphi_s$ for the trajectories approaching $\psi=-2\varphi_s$ from the left /is asymptotic with respect to E and really determines the separatrix in the sense that the initial values of the boundary trajectories divide the plane $\psi^o,\dot\psi^o$ into the regions from which the solutions of one type emerge.*) Evidently, the behaviour of boundary trajectories is essentially different for $\dot\psi^o<0$ /the initial velocity is directed from the right to the left /Fig.I/ or $\dot\psi^o>0$ / the initial velocity is directed from the left to the right /Fig.I/ since in the case $\dot\psi^o>0$ there may exist a turning point /where $\dot\psi=0$ / absent in the case $\dot\psi^o<0$.

Let us investigate first the boundary trajectories for $\dot\psi^o<0$. We shall assume $t_0=0$, whereas $m(0)=f(0)=1$ (this can be always achieved in the proper definition of t). All the calculations must be made by restricting by the quantities of the first order over E.

The equations which the functions $\psi_0, \psi_1, \dots \psi_i, \dots$ of the formal expansion satisfy have the form

$$\ddot\psi_0+\mathcal{U}'(\psi_0)=0 \qquad /3/$$

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*) Point out that this theorem was proved in /9/ also for the equations of more general form than /Ia/.

$$\ddot{\varphi}_1 + \mathcal{U}''(\varphi_0)\varphi_1 = -f'(0)t\,\mathcal{U}'(\varphi_0) - m'(0)\frac{d}{dt}(\dot{\varphi}_0 t) \qquad /4/$$

$$\ddot{\varphi}_i + \mathcal{U}''(\varphi_0)\varphi_i = \mathcal{F}_i(t, \varphi_0, \ldots \varphi_{i-1}) \qquad /5/$$

It follows from equation /3/

$$\frac{\dot{\varphi}_0^2}{2} + \mathcal{U}(\varphi_0) = C$$

Making use of the conditions $\varphi_0 \to -2\mathcal{Y}_s$ and $\dot{\varphi}_0 \to 0$ at $t \to \infty$ that gives $C = \mathcal{U}_m$, we find

$$\frac{\dot{\varphi}_0^2}{2} = \mathcal{U}_m - \mathcal{U}(\varphi_0) \quad , \; t = \int_{\varphi_0}^{\chi} \left(\sqrt{2\mathcal{U}_m - \mathcal{U}(\xi)}\right)^{-1} d\xi \qquad /6/$$

/ χ is the parameter defining one of the boundary trajectories, ξ is the integration variable/.

Equations /4/ and /5/ can be easily integrated if introduce φ_0 as an independent variable connected with formula (6). Indeed, as can be easily shown

$$\frac{d}{d\varphi_0}\left[2\,\frac{d\varphi_i}{d\varphi_0}\left(\mathcal{U}_m - \mathcal{U}(\varphi_0)\right) + \mathcal{U}'(\varphi_0)\varphi_i\right] = \ddot{\varphi}_i + \mathcal{U}''(\varphi_0)\varphi_i$$

and, therefore, equation /4/ is reduced to the equation of the first order

$$2\left(\mathcal{U}_m - \mathcal{U}(\varphi_0)\right)\frac{d\varphi_1}{d\varphi_0} + \mathcal{U}'(\varphi_0)\varphi_1 = G(\varphi_0)$$

where

$$G(\varphi_0) = -m'(0)\int_{-2\mathcal{Y}_s}^{\varphi_0} \dot{\varphi}_0(\xi)\,d\xi + (m'(0) - f'(0))\int_{-2\mathcal{Y}_s}^{\varphi_0} t(\xi)\,\mathcal{U}'(\xi)\,d\xi \qquad /7/$$

From /7/ using the condition $\varphi_1 = 0$ at $\varphi_0 = \bar{\varphi}_0$ we find

$$\varphi_1(\varphi_0) = \sqrt{2\mathcal{U}_m - 2\mathcal{U}(\varphi_0)}\cdot\int_{\bar{\varphi}_0}^{\varphi_0} \frac{G(\xi)\,d\xi}{(2\mathcal{U}_m - 2\mathcal{U}(\xi))^{3/2}} \qquad /8/$$

An analogous formula holds for φ_i $(i=2,3\ldots)$. Thus, the equation of the boundary trajectories at $\dot{\varphi}^o < 0$ is determined with the accuracy up to the terms of the first order over E by means of the formulas

$$\varphi = \varphi_0(t,\chi) + \varepsilon\varphi_1(t,\chi) + \ldots$$

$$\dot{\varphi} = -\sqrt{2\mathcal{U}_m - 2\mathcal{U}(\varphi_0)} + \varepsilon\left[-\frac{G(\varphi_0)}{\sqrt{2\mathcal{U}_m - 2\mathcal{U}(\varphi_0)}} + \mathcal{U}'(\varphi_0)\int_{\bar{\varphi}_0}^{\varphi_0}\frac{G(\xi)\,d\xi}{(2\mathcal{U}_m - 2\mathcal{U}(\xi))^{3/2}}\right] \qquad /9/$$

The connection between the initial values of these trajectories corresponding to t=0 is obtained at $\varphi_0 = \chi$, i.e., determined by the parametric formula

$$\psi_0 = x + \varepsilon \psi_1(0,x) + \ldots$$

$$\dot\psi^o = -\sqrt{2U_m - 2U(x)} + \varepsilon \left[\frac{Q(x)}{\sqrt{2U_m - 2U(x)}} + U'(x) \int_{\psi_0}^{x} \frac{Q(\xi)}{(2U_m - 2U(\xi))^{3/2}} \right] + \ldots \qquad /10/$$

where

$$Q(x) = \frac{m'(0) + f'(0)}{2} \int_{-2\varphi_s}^{x} \sqrt{2U_m - 2U(\xi)}\, d\xi$$

Let us raise $\dot\psi^o$ determined by /10/ to the second power restricting by the terms of the first order over E . This gives

$$(\dot\psi^o)^2 = 2(U_m - U(x)) + 2\varepsilon\left[Q(x) - U'(x)\psi_1(0,x)\right] \qquad /11/$$

Now pass to the direct dependence between $(\dot\psi^o)^2$ and ψ^o with the accuracy up to the terms of the first order over E . According to /10/ in this approximation $x = \psi^o - \varepsilon\psi_1(0,\psi^o)$ Substituting this equation into /11/ we find the formula for the separatrix at $\dot\psi^o < 0$:

$$(\dot\psi^o)^2 = 2(U_m - U(\psi^o)) + \bar\varepsilon \int_{-2\varphi_s}^{\psi^o} \sqrt{2U_m - 2U(\xi)}\, d\xi, \quad \text{where } \bar\varepsilon = (m'(0) + f'(0))\varepsilon \qquad /12/$$

In particular, the width of the capture region at $\dot\psi^o = 0$ is equal to $\Delta\psi = \psi_m^1 + 2\varphi_s$ where ψ_m^1 and $-2\varphi_s$ are the roots of the equation

$$2(U_m - U(\psi)) + \bar\varepsilon \int_{-2\varphi_s}^{\psi} \sqrt{2U_m - 2U(\xi)}\, d\xi = 0$$

It can be easily seen that

$$\psi_m^1 = \psi_m^o + \frac{\bar\varepsilon}{2U'(\psi_m^o)} \int_{-2\varphi_s}^{\psi_m^o} \sqrt{2U_m - 2U(\xi)}\, d\xi = \psi_m^o + \delta\psi, \qquad /13/$$

where $\delta\psi$ is the increment of the capture region due to the nonconservation.

At sufficiently small φ_s one may assume that $U = \frac{\psi^2}{2} + \frac{\psi^3}{6\varphi_s}$. Introducing new quantities $\dot x^o = \frac{\dot\psi^o}{\varphi_s}$, $x^o = \frac{\psi^o}{\varphi_s}$ represent the equation of separatrix /12/ at $\dot x^o < 0$ in the form

$$(\dot x^o)^2 = 2\left(\frac{2}{3} - U(x^o)\right) + \bar\varepsilon \int_{-2}^{x^o} \sqrt{\frac{4}{3} - 2U(x)}\, dx \quad , \qquad /14/$$

where $\quad U(x) = \frac{x^2}{2} + \frac{x^3}{6}, \qquad 2(U_m - U) = \frac{1}{3}(1-x)(2+x)^2,$

$$\int_{-2}^{x^o} \sqrt{\frac{4}{3} - 2U(x)}\, dx = \frac{12}{5} - \frac{2}{5\sqrt3}(1-x^o)^{3/2}(4+x^o)$$

At $\dot x^o = 0$ the width of the capture region $\Delta x = \frac{\Delta\psi}{\varphi_s} \approx 3 + \delta x$,

where $\quad \delta x = \frac{\bar\varepsilon}{2U'(1)} \int_{-2}^{1} \sqrt{(1-x)(2+x)^2}\, dx = \frac{4}{5}\bar\varepsilon$

In Fig. 2 the graphs of the separatrix are given at $\bar E$=0,1 and $\bar E$=0,2 in the variables

x^o, $\dot{x}^o$ found by the formula /I4/.

Now let us investigate the behaviour of the boundary trajectories at $\dot{\psi}^o > 0$ /the initial velocity is directed from left to right/. In this case two possibilities should be indicated:

I/ The particles start to move on the left from the barrier $\mathcal{U}_m$ $(\psi^o < -2\varphi_s)$ and come to the vertex of the barrier with zero velocity. Evidently, geometrical place of the initial values ψ^o, $\dot{\psi}^o$ for such trajectories is obtained from the curve $\dot{\psi}^o = \sqrt{2\mathcal{U}_m - 2\mathcal{U}(\psi^o)}$ by some shift due to the nonconservation.

2/ the particles start to move to the left from the barrier and come to it with the velocity different from zero $\dot{\psi} > 0$ or start to move on the left from the barrier $\mathcal{U}_m$ with $\dot{\psi}^o > 0$ In this case the boundary trajectory must have at a certain moment of time t_I at $\psi > 0$ a turning point where $\dot{\psi} = 0$, and further it moves with $\dot{\psi} < 0$ along the trajectory determined by the formula found earlier /I2/.

Let us consider the first from the mentioned possibilities. The boundary trajectory is described by the formal expansion $\psi = \psi_0 + \varepsilon\psi_1 + \dots$, where ψ_i obey the same auxiliary conditions and is determined as in the case $\dot{\psi}^o < 0$. The difference is that the point lying to the left from $-2\varphi_s$ should be taken as $\bar{\varphi}_0$. Then $t = \int_{\varkappa}^{\psi_0} \frac{d\xi}{\sqrt{2\mathcal{U}_m - 2\mathcal{U}(\xi)}}$ since at $t > 0$ $\dot{\psi}_0 > 0$, and $\psi_0 \geqslant \varkappa$. The boundary trajectories are described by the formulas:

$$\psi = \psi_0(t, \varkappa) + \varepsilon\psi_1(t, \varkappa) + \dots$$

$$\dot{\psi} = \sqrt{2\mathcal{U}_m - 2\mathcal{U}(\psi_0)} + \varepsilon\left[\frac{Q(\psi_0)}{\sqrt{2\mathcal{U}_m - 2\mathcal{U}(\psi_0)}} - \mathcal{U}'(\psi_0)\int_{\bar{\psi}_0}^{\psi_0} \frac{Q(\xi)\,d\xi}{(2\mathcal{U}_m - 2\mathcal{U}(\xi))^{3/2}}\right] \qquad /I5/$$

whereas their initial values are obtained from /I5/ at $t = 0$, i.e., at $\psi_0 = \varkappa$.

From here, reasoning in the same manner as in the derivation of /I2/ we obtain:

$$(\dot{\psi}^o)^2 = 2(\mathcal{U}_m - 2\mathcal{U}(\psi^o)) + \bar{\varepsilon}\int_{\psi^o}^{-2\varphi_s} \sqrt{2\mathcal{U}_m - 2\mathcal{U}(\xi)}\,d\xi \qquad /I6/$$

In particular, at $\varphi_s \ll 1$, introducing the quantities $x^o = \frac{\psi^o}{\varphi_s}$, $\dot{x}^o = \frac{\dot{\psi}^o}{\varphi_s}$, we rewrite /I6/ in the form:

$$(\dot{x}^o)^2 = 2\left(\frac{2}{3} - \mathcal{U}(x)\right) + \bar{\varepsilon}\int_{x^o}^{-2} \sqrt{\frac{4}{3} - 2\mathcal{U}(x)}\,dx \qquad /I7/$$

where $\mathcal{U}(x)$ and $\int_{x^o}^{-2} \sqrt{\frac{4}{3} - 2\mathcal{U}(x)}\,dx$ are defined in /I4/.

Finally, let us investigate the last case. We shall look for the motion of a particle in the interval (t_1, ∞), i.e. after the turning point just as in case $\dot{\psi}^o < 0$. Using the expansions obtained earlier /9/ we can calculate $\psi(t_1)$ /i.e., the value of ψ in the turning point / which as can be easily checked with the accuracy up to the terms of the first order over E is defined by /I3/ inclusive. After this we shall look for the solution $\psi(t)$ on the interval $(0, t_1)$ in the form of a series $\psi = \psi_0(t; t_1) + \varepsilon\psi_1(t, t_1) + \dots$

where the functions ψ_0, ψ_1, ψ_i satisfy the equations /3/, /4/ and /5/ with the initial conditions $\psi_0(t_1, t_1) = \psi(t_1)$, $\dot\psi_0(t_1, t_1) = 0$ and $\psi_i(t_1, t_1) = \dot\psi_i(t_1, t_1) = 0$ at $i = 1, 2 \dots$. /For this expansion the Poincare theorem about the analytic dependence upon the parameter E /see, /8/ / is correct/. It can be easily verified that up to the turning point the solution with the accuracy up to the term of the first order over E is of the form

$$\psi = \psi_0 + \varepsilon\sqrt{2\mathcal{E}_m - 2\mathcal{U}(\psi_0)} \int_{\psi_m^1}^{\psi^0} \frac{Q(\xi)\,d\xi}{(2\mathcal{E}_m - 2\mathcal{U}(\xi))^{3/2}} + \dots$$

$$\dot\psi = \sqrt{2\mathcal{E}_m - 2\mathcal{U}(\psi_0)} + \varepsilon\left[\frac{Q(\psi_0)}{\sqrt{2\mathcal{E}_m - 2\mathcal{U}(\psi_0)}} - \mathcal{U}'(\psi_0)\int_{\psi_m^1}^{\psi_0} \frac{Q(\xi)\,d\xi}{(2\mathcal{E}_m - 2\mathcal{U}(\xi))^{3/2}}\right] + \dots, \qquad /18/$$

where $\mathcal{E}_m = \mathcal{U}(\psi_m^1)$. From /18/ reasoning in the same manner as in the derivation of /12/ we obtain

$$(\dot\psi^0)^2 = 2(\mathcal{E}_m - \mathcal{U}(\psi^0)) + \bar\varepsilon \int_{\psi^0}^{\psi_m^1} \sqrt{2\mathcal{E}_m - 2\mathcal{U}(\xi)}\, d\xi \qquad /19/$$

Remind, that $\mathcal{E}_m = \mathcal{U}(\psi_m^1)$ depends upon E, i.e., in /19/ the terms of higher orders over E are taken into account in an indirect way. However, as follows from the derivation of this formula the error has the order over E not lower than the second. For small ψ_s restricting in the expansion $\mathcal{U}(\psi)$ by the terms not higher than the second order and substituting $\dot x^0 = \frac{\dot\psi^0}{\psi_s}$, $x^0 = \frac{\psi^0}{\psi_s}$ we may reduce expression /19/ to the form:

$$(\dot x^0)^2 = \frac{4}{3} - 2\,\mathcal{U}(x^0) + \frac{12}{5}\bar\varepsilon + \bar\varepsilon \int_{x^0}^{1 + \frac{4}{5}\bar\varepsilon} \sqrt{\frac{4}{3} + \frac{12}{5}\bar\varepsilon - 2\mathcal{U}(x)}\, dx \qquad /20/$$

The set of formulas /12/, /16/ and /19/ makes it possible to construct on the plane / ψ^0, $\dot\psi^0$ / the curve which gives the boundary of the capture region. This curve in the variables x^0, $\dot x^0$ is given in Fig. 2 for $\bar\varepsilon\,(m'(0) + f'(0)) = 0{,}1$ and for $\bar\varepsilon\,(m'(0) + f'(0)) = 0{,}2$ using the formulas /14/, /17/ and /20/.

The question may arise why in the last case we had to introduce additionally new expansion /18/ why at $\dot\psi^0 > 0$ the expansion of type /9/ could not be used over all the interval $0 \le t \le \infty$. Everything becomes clear if we are concerned with the solution of equation /4/ which is given by formula /8/. In the turning point the dominator of the integrand has zero of order not lower than 3/2 due to which the integral has a singularity in the turning point which does not enable to make use of expansion /9/ on the interval $(0, t_I)$.

The obtained results make it possible to draw some interesting conclusions which in our opinion may be of technical interest:

1. As is seen from Fig. 2 at $\bar{\varepsilon}>0$ the capture region looses its symmetrical form with respect to the straight line $\dot{\psi}^0=0$. Now the greatest width of the capture is obtained /at sufficiently small $\bar{\varepsilon}>0$ / not at $\dot{\psi}^0=0$, but according to the formula /19/ at

$$\dot{\psi}_{min}=\dot{\psi}^0(-2\varphi_s)=\sqrt{2(\mathcal{E}_m-\mathcal{U}_m)+\bar{\varepsilon}\int_{-2\varphi_s}^{\psi_m^1}\sqrt{2\mathcal{E}_m-2\mathcal{U}(\xi)}\,d\xi}$$

It is defined by the root of the equation

$$(\dot{\psi}_{min})^2=2(\mathcal{E}_m-\mathcal{U}(\psi))+\bar{\varepsilon}\int_{\psi}^{\psi_m^1}\sqrt{2\mathcal{E}_m-2\mathcal{U}(\xi)}\,d\xi$$

nearest to ψ_m^1 and by the root of the equation

$$(\dot{\psi}_{min})^2=2(\mathcal{U}_m-\mathcal{U}(\psi))+\bar{\varepsilon}\int_{\psi}^{-2\varphi_s}\sqrt{2\mathcal{U}_m-2\mathcal{U}(\xi)}\,d\xi$$

nearest to $-2\varphi_s$. We shall look for the root of the first equation in the form $\psi=\psi_m^1+\delta_1\psi$, the root of the second one -in the form $\psi=-2\varphi_s+\delta_2\psi$. Substituting the obtained expressions into our equations we may easily find that in the first term of the expansion over $\bar{E}$

$$\delta_1\psi=-\frac{\bar{\varepsilon}}{\mathcal{U}'(\psi_m^1)}\int_{-2\varphi_s}^{\psi_m^1}\sqrt{2\mathcal{E}_m-2\mathcal{U}(\xi)}\,d\xi \quad \text{and}$$

$$\delta_2\psi=-\sqrt{-\frac{2\bar{\varepsilon}}{\mathcal{U}''(-2\varphi_s)}\int_{-2\varphi_s}^{\psi_m^1}\sqrt{2\mathcal{E}_m-2\mathcal{U}(\xi)}\,d\xi}$$

It can be easily seen that with the accuracy up to the terms of the first order over E $\delta_1\psi=-2\delta\psi$ where $\delta\psi$ is determined by the formula /13/.

Thus, at $\bar{\varepsilon}>0$ the width of the capture increases in the first term of the expansion over E by $-\delta_2\psi$.

In the variables x^0, $\dot{x}^0$, i.e., at small synchrone phases in the units φ_s the increment of the capture region is equal to $\delta_2 x=-\frac{\delta_2\psi}{\varphi_s}=\sqrt{\frac{24}{5}\bar{\varepsilon}}$. At $\bar{E}$ = 0,2 $\delta_2 x$ = 1.0, i.e., the width of the capture increases in comparison with its magnitude in the conservation case by 30%.

2. If the shift of the initial velocity exceeds /in the units $\dot{\psi}$ / $\dot{\psi}_{min}$ /see above / then the particles captured into the synchrone acceleration regime are grouped in two blobs /the sections (x_1,x_2) and (x_3,x_4)/ which can exist separately for a long period of time since the particles from the section (x_1, x_2) which have to pass through the position of unstable equilibrium $-2\varphi_s$, having there a small velocity are moving on the phase plane considerably slower than the particles from the section /x_3, x_4 /. It is possible that this circumstance

turns out to be essential in the theory taking into account the effect of the volume charge.

These conclusions may be of great interest if applied to the proton linear accelerator with the drift tubes. The equation of phase oscillations in this case may be written as follows /see, for instance, /II/ /:

$$\frac{d}{dt}(\vartheta_s^2\dot{\psi}) = \frac{eE\,\omega\vartheta_s}{m}\left[\cos(\varphi_s+\psi)-\cos\varphi_s\right] \qquad /2I/$$

where $\psi=\frac{\omega}{\vartheta_s}q$, q is the mean distance between the considered and the synchrone particles at the given stage, $E=\frac{1}{L}\int_L E_z\cos\frac{2\pi}{L}z\,dz$ / E_z - is the strength of the accelerating field on the axis of the accelerating section with the length L, along which the integration is made/, $\omega=\frac{2\pi}{T}$, T is the period of high-frequency oscillations, e , m is the charge and mass of the particle ϑ_s - is the velocity of the synchrone particle, t - is the time in sec. Note that the increment of its velocity on one accelerating section is determined by the formula $\Delta\vartheta_s=\frac{e}{m}ET\cos\varphi_s$. For simplicity we assume this quantity to be constant, i.e., $\vartheta_s=\vartheta_{so}+\frac{\Delta\vartheta_s}{T}t=\vartheta_{so}(1+\frac{t}{\tau})$; ϑ_{so} is the initial velocity of the synchrone particle $\tau=\frac{\vartheta_{so}}{\Delta\vartheta_s}T$. Let us introduce a dimensionless variable $t'=\Omega t$ $\Omega^2=\frac{e\omega\sin\varphi_s E}{m\vartheta_{so}}$ /it is evident, that Ω coincides with the frequency of the linear phase oscillations at $\vartheta_s=\vartheta_{so}$ /. Then equation /2I/ may be written with the accuracy up to the magnitudes of the first order over E in the form:

$$\frac{d}{dt'}\left[(1+2\varepsilon t')\frac{d\psi}{dt'}\right]+(1+\varepsilon t')U'(\psi)=0\,, \qquad /22/$$

where

$$U'(\psi)=-\frac{1}{\sin\varphi_s}\left[\cos(\varphi_s+\psi)-\cos\varphi_s\right]\,,\quad \varepsilon=\frac{1}{\Omega\tau}=\frac{1}{\sqrt{2\pi\,tg\varphi_s}}\sqrt{\frac{\Delta\vartheta_s}{\vartheta_{so}}}$$

Assuming

$$\mathcal{E}_o=\frac{m\vartheta_{so}^2}{2}=4\,mev\,,\quad \varphi_s=20^\circ\,,\quad \frac{\Delta\vartheta_s}{\vartheta_{so}}=0{,}04\,, \qquad ^{*)}$$

we find that $\varepsilon=0{,}4$. According to /I4/ the increment of the width of the capture δx at $\psi^o=0$ is /in the units of φ_s/ for the considered case $\delta x=\frac{4}{5}\varepsilon=0{,}3$, i.e., 10%.

If the initial energy of particles exceeds the energy E_0 by the quantity, corresponding to $\left(\frac{d\psi}{dt'}\right)_{min}=\frac{1}{\Omega}\dot{\psi}_{min}$ i.e., by the quantity $\delta\mathcal{E}=\mathcal{E}_o\varphi_s\sqrt{\frac{tg\varphi_s}{2\pi}\frac{\Delta\vartheta_s}{\vartheta_{so}}}\left(\frac{dx}{dt'}\right)_{min}$ /at small synchrone

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*) These data concern the Alvarez accelerator /II/.

phases/ then the increment of the width of capture is $\sqrt{\frac{24}{5}\,\bar{\varepsilon}}$. In other words, in the considered case the energy increase of the injected particles from 4 MeV up to I00 KeV must lead to the increase of the capture approximately by I.4, i.e., almost by 50% if compared with the conservation approximation corresponding to $\dot{\psi}^o = 0$.

It is interesting to note that in paper /I2/ devoted to the description of the linear accelerator meant for the injection into the Berkeley Bevatron it is indicated that the greatest capture is obtained if the initial proton energy is not the designed one/450 KeV/, but exceeds it by I0 kv*). It is possible that this phenomenon is explained by the abovementioned facts, i.e., by the increase of the capture with the increase of the initial energy if compared with the designed one due to the nonconservation of the phase motion.

In conclusion we should like to emphasize that the abovementioned technical considerations take place not only for linear accelerators but also for the cycle accelerators of phasotron and synchrophasotron type. In particular, one may believe that the increase of the field amplitude with time in the accelerating sections of the synchrophasotron and a certain increase of the initial proton energy injected if compared with the designed one may lead to a considerable increase of the number of particles captured into the regime of the synchrone accelerator.

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*) According to our formulas the maximum width of the capture concerning the data of this accelerator is obtained at the injection energy exceeding the designed one by 30 KeV.

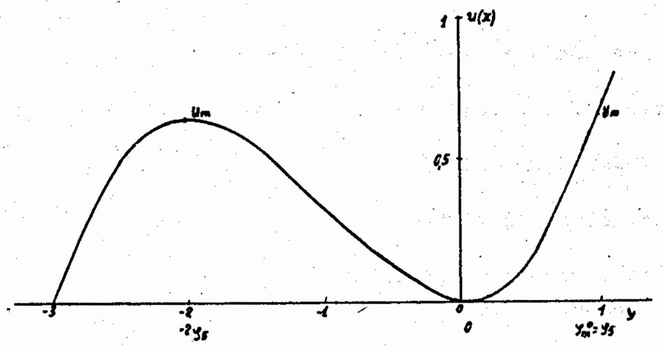

Fig.I

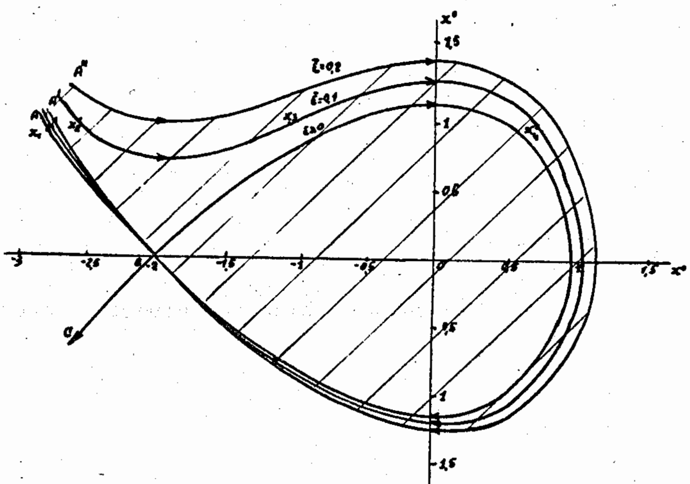

Fig. 2

R E F E R E N C E S

I. Twiss R.Q. and Frank N.H. , Rev.Sci. Instr. 20, I, (I949).

2. Yu.S. Sajasov, Report Institute of Chemical Physics Akad. Nauk USSR, /I950/.

3. W. Dallenbach, Ann. d. Phys. 3,89, /I947/.

4. T.R. Kaiser, Proc. Phys. Soc., 63 A, 52 /I950/.

5. G.I. Lubarsky, Report, Physical-Technical Institut Akad. Nauk USSR, /I950/.

6. N.N. Bogolubov and Yu.A. Mitropolsky, "Asymptotic methods in the Theory of Nonlinear Oscillations".. GTTI, I955.

7. V.V. Volosov, Mat. Sbornik 3(I955).

8. V.V. Golubev, Lectures on Analytical Theory of Differential Equations, GTTI, I950, page I53.

9. V.K. Melnikov, Mat. Sbornik /in print/.

IO. V.N. Smirnov, The course of high mathematics, Volume III, Part II, GTTI, I949.

II. L. Smith, R.L. Gluckstern, Rev. Sci. Instr. 26, 220, /I955/.

I2. B. Cork, Rev. Sci. Instr., 26, 2I0, /I955/.

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