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by

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A b s t r a c t: The diagram with four external lines in a two-dimensional nonlinear fermion theory, analogous to Thirring's model, is calculated to the second order of perturbation theory. It is shown that the corresponding S-matrix element contains no ultraviolet divergences and that hence the coupling constant is not subject to renormalization. An improved expression for the matrix element for the scattering of two particles, is obtained by means of the renormalization group method and turns out to be very similar to the corresponding exact result of Thirring.

April, 1958.

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1. Introduction

Following Thirring[1] we consider a nonlinear spinor theory, in two-dimensional space-time, with the following interaction Lagrangian

$$L(x) = g : \bar{\psi}(x)\sigma^n\psi(x)\bar{\psi}(x)\sigma^n\psi(x) : \qquad (1)$$

Here $\sigma^0 = I$ and $\sigma^1, \sigma^2, \sigma^3$ are the usual 2x2 Pauli matrices, with the summation convention

$$\sigma^n \times \sigma^n = I \times I - \sigma^1 \times \sigma^1 - \sigma^2 \times \sigma^2 - \sigma^3 \times \sigma^3 \qquad (2)$$

x denotes a point in two-dimensional space-time with coordinates $x_0 = t, x_1 = z$ and $\psi, \bar{\psi}$ are two-component spinors, satisfying the two-dimensional Dirac equation

$$(\hat{p} - m)\psi(p) = \bar{\psi}(p)(\hat{p} - m) = 0 \quad ; \ \bar{\psi} = -\overset{*}{\psi}\sigma^2,$$

where

$$\hat{p} = -\sigma^2 p_0 - i\sigma^1 p_1 . \qquad (3)$$

This choice of matrices corresponds to the notations in[1].

The Lagrangian (1) is the only possible expression which is symmetrical with respect to the interchange of two ψ's or two $\bar{\psi}$'s (cf. the Appendix). As proved in the Appendix, Eq. (1) can also be written in the form

$$L(x) = 4g : \bar{\psi}(x)\psi(x)\bar{\psi}(x)\psi(x) : , \qquad (1')$$

which coincides, up to a numerical factor, with the limiting form of Thirring's Lagrangian [1], when his two fields ψ_1 and ψ_2 coincide.

We will consider the S-matrix element corresponding to the scattering of two particles, supposing m = 0. The expression of this element, following from (1) has the form

$$S = \frac{ig}{4\pi^2}\int \bar{\psi}_\alpha(p')\psi_\beta(q)\bar{\psi}_\gamma(q')\psi_\delta(p)\,\delta(p'+q'-p-q)\times$$

$$\times\,\Gamma_{\alpha\beta,\gamma\delta}(p',q',p,q)\,dp'\,dp\,dq'\,dq\,, \qquad (4)$$

where $dp = dp_0\,dp_1$, the function Γ possesses the following antisymmetry properties

$$\Gamma_{\alpha\beta,\gamma\delta}(p',q',p,q) = -\Gamma_{\gamma\beta,\alpha\delta}(q',p',p,q) = -\Gamma_{\alpha\delta,\gamma\beta}(p',q',q,p), \qquad (5)$$

and in the lowest approximation has the form (cf.formula (A.4) in the Appendix)

$$\Gamma^{(1)}_{\alpha\beta,\gamma\delta}(p',q',p,q) = \sigma^n_{\alpha\beta}\times\sigma^n_{\gamma\delta} = -\sigma^n_{\alpha\delta}\times\sigma^n_{\gamma\beta}\,. \qquad (6)$$

To the first order in g one can obtain an improved formula for the function Γ without carrying out an analysis of the properties of the one-particle Green's function. Therefore, we will compute in the following section the second order perturbation expression for the function Γ and in Sec. 3 we will obtain an improved formula by means of the renormalization group method.

2. Second Order Perturbation Calculations

The second-order term in the S-matrix expansion

$$S_2 = \frac{i^2}{2}\int T(L(x)L(y))\,dx\,dy =$$

$$= -\frac{g^2}{2}\int T\{\bar{\psi}(x)\sigma^n\psi(x)\bar{\psi}(x)\sigma^n\psi(x)\bar{\psi}(y)\sigma^m\psi(y)\bar{\psi}(y)\sigma^m\psi(y)\}\,dx\,dy, \quad (7)$$

contains two contributions of the form (4) in which the following combinations of contractions

$$\overbrace{\psi(x)\bar{\psi}(y)}, \quad \overbrace{\psi(y)\bar{\psi}(x)}$$

and

$$\overbrace{\psi(x)\bar{\psi}(y)}, \quad \overbrace{\psi(x)\bar{\psi}(y)},$$

occur, respectively. The corresponding Γ 's are of the form

$$\Gamma^a_{\alpha\beta,\gamma\delta}(p';q';p,q) = \frac{2ig}{\pi^2}\int d\kappa\,\frac{Sp\{\sigma^n(\hat{\kappa}+\hat{P})\sigma^m(\hat{\kappa}-\hat{P})\}}{(\kappa+P)^2(\kappa-P)^2}\,\sigma^n_{\alpha\beta}\,\sigma^m_{\gamma\delta} =$$

$$= \frac{2ig}{\pi^2}\int d\kappa\,\frac{2(\kappa^2-P^2)\sigma^n\times\sigma^n + 4\,(\hat{\kappa}\times\hat{\kappa}-\hat{P}\times\hat{P})}{(\kappa+P)^2(\kappa-P)^2}\Bigg|_{\alpha\beta,\gamma\delta} \quad (8)$$

and

$$\Gamma^b_{\alpha\beta,\gamma\delta}(p';q';p,q) = \frac{ig}{2\pi^2}\int d\kappa\,\frac{[\sigma^n(\hat{\kappa}+\hat{Q})\sigma^m]_{\alpha\delta}[\sigma^n(\hat{\kappa}-\hat{Q})\sigma^m]_{\gamma\beta}}{(\kappa+Q)^2(\kappa-Q)^2} =$$

$$= \frac{2ig}{\pi^2}(\sigma^n_{\alpha\beta}\times\sigma^n_{\gamma\delta})\int d\kappa\,\frac{Q^2-\kappa^2}{(Q+\kappa)^2(Q-\kappa)^2}, \quad (9)$$

with
$$\mathcal{P}=\frac{p'-p}{2}\,,\quad Q=\frac{p+q}{2}\,,\quad \Gamma^{(2)}=\Gamma^{a}+\Gamma^{b},$$

and where use has been made of formula (A.6) for the rearrangement of Γ^{b} to the permuted indices.

By means of Eqs. (A.5) and (A.8) the sum of (8) and (9) can be brought to the form

$$\Gamma^{(2)}_{\alpha\beta,\gamma\delta}=\frac{2ig}{\pi^2}\sigma^{n}_{\alpha\beta}\times\sigma^{n}_{\gamma\delta}\int dK\left\{\frac{K^2-P^2}{(K+P)^2(K-P)^2}-\frac{K^2-Q^2}{(K+Q)^2(K-Q)^2}\right\}-$$
$$-\frac{4ig}{\pi^2}\left[\hat{P}_{\alpha\beta}\times\hat{P}_{\gamma\delta}+\hat{P}_{\alpha\delta}\times\hat{P}_{\gamma\beta}\right]\int\frac{dK}{(K+P)^2(K-P)^2}\,,$$

where the ultraviolet divergences cancel explicitly.

A calculation of the integrals yields

$$\Gamma'^{(2)}_{\alpha\beta,\gamma\delta}=-\frac{2g}{\pi}(\sigma^{n}_{\alpha\beta}\times\sigma^{n}_{\gamma\delta})\ln\frac{P^2}{Q^2}-\frac{g}{\pi}\frac{(\hat{P}_{\alpha\beta}\times\hat{P}_{\gamma\delta}+\hat{P}_{\alpha\delta}\times\hat{P}_{\gamma\beta})}{P^2}\cdot C\,, \qquad (10)$$

where
$$C=\int_0^1\frac{dx}{x(1-x)}\,,$$

is a constant which contains an infrared divergence. In the case that the particles with momenta p and p' are real, i.e.

$$\bar{\psi}(p')\hat{p}'=\hat{p}\psi(p)=0\,,$$

the second term in the r.h.s. of (10) can be reduced to the form

$$\frac{g}{2\pi}(\sigma^{n}_{\alpha\beta}\times\sigma^{n}_{\gamma\delta})C\,,$$

by means of Eq. (A.5) and may be neglected if Γ is suitably normalized. As in electrodynamics, the infrared divergence could be avoided by introducing formally a small mass for the particles. However this would not modify our further results.

Taking into account the antisymmetry of Γ , expressed by Eq. (5), and the property (6), we obtain finally

$$\Gamma^{(2)}_{\alpha\beta,\gamma\delta}(p',q',p,q) = \tag{11}$$
$$= -\frac{g}{\pi}\left(\sigma^n_{\alpha\beta}\times\sigma^n_{\gamma\delta}\right)\ln\frac{(p'-p)^2(p'-q)^2}{(p+q)^4}.$$

3. The Improved Expression for Γ

Comparing formulas (6) and (11) and taking into account the above remark concerning the term with the infrared divergence, we may write

$$\Gamma_{\alpha\beta,\gamma\delta}(p',q';p,q) = (\sigma^n_{\alpha\beta}\times\sigma^n_{\gamma\delta})\Gamma(p',q';p,q)$$

$$\Gamma(p',q';p,q) = 1 - \frac{g}{\pi}\ln\frac{(p'-p)^2(p'-q)^2}{(p+q)^4} + gc + \dots, \quad (12)$$

where c is an arbitrary finite constant, corresponding to the arbitrariness in the definition of the T-product in (7).

Let us now improve the approximating properties of the expression (12) by means of the renormalization group method [2].

In the theory under consideration, the admissible finite multiplicative counterterms to the Lagrangian are of the form

$$\delta L = (z_1-1)\bar\psi\hat p\psi + (z_2-1)g\,\bar\psi\sigma^n\psi\bar\psi\sigma^n\psi .$$

The introduction of such counterterms is equivalent to the following finite renormalization of the Green's function d and the 4-vertex function Γ

$$d_1 \to d_2 = z_1 d_1\ , \quad \Gamma_1 \to \Gamma_2 = z_2^{-1}\Gamma_1\ , \qquad (13)$$

(here d is the scalar part of the Green's function: $G(p) = d(p^2)/\hat{p}$). The transformations (13) have the same form as the renormalization transformations in the four-dimensional nonlinear meson theory, which is obtained from the two-charge meson theory [2] by switching off the meson nucleon coupling.

The functional equations are of the form [3]

$$d(x,g) = d(t,g)\,d\left(\frac{x}{t}, g\varphi(t,g)\right), \tag{14}$$

$$\Gamma(x_1,\dots x_6,g) = \Gamma(t,\dots t,g)\,\Gamma\left(\frac{x_1}{t},\dots \frac{x_6}{t}, g\varphi(t,g)\right), \tag{15}$$

$$\varphi(t,g) = d^2(t,g)\,\Gamma(t,\dots t,g). \tag{16}$$

Here $x_1,\dots,x_6$ are the dimensionless, scalar, independent arguments of the function Γ , which can for example be chosen in the following manner

$$x_1 = p^2/\lambda^2, \quad x_2 = p'^2/\lambda^2, \quad x_3 = q^2/\lambda^2, \quad x_4 = q'^2/\lambda^2,$$

$$x_5 = (p'-p)^2/\lambda^2, \quad x_6 = (p'-q)^2/\lambda^2, \tag{17}$$

where the normalization momentum λ is such that

$$d(1,g) = 1, \quad \Gamma(1,\dots 1,g) = 1. \tag{18}$$

The function φ introduced in (16) plays the role of an "invariant charge". Eqs. (14), (15), (16) lead to the following functional and differential equations for φ

$$\varphi(x,g) = \varphi(t,g)\varphi\left(\frac{x}{t}, g\varphi(t,g)\right), \tag{19}$$

$$\frac{\partial\varphi(x,g)}{\partial x} = \frac{\varphi(x,g)}{x}\Phi(g\varphi(x,g)), \tag{20}$$

where

$$\Phi(g) = \frac{\partial}{\partial \xi}\varphi(\xi,g)\Big|_{\xi=1}. \tag{21}$$

As usual, the function Φ in the r.h.s. of Eq. (20) is to be replaced by its perturbation-theoretical approximation. We will consider the approximation linear in g. As the expansion of <u>d</u> starts with terms proportional to g^2, we may replace φ in (21) by the expression for Γ obtained in Sec. 2, which, by (17) and (18) can be written as

$$\Gamma^{(2)}(x_1,\ldots x_6;g) = 1 - \frac{g}{\pi}\ln\frac{4x_5x_6}{(x_1+x_2+x_3+x_4-x_5-x_6)^2}, \tag{22}$$

so that

$$\Phi^{(2)}(g) = 0.$$

From (20) it follows, that

$$\varphi(x,g) = 1. \tag{23}$$

The result (23) is remarkable, as it shows, that in the first-order approximation in g, the coupling constant is not renormalized. This corresponds to the absence of divergences in $\Gamma^{(2)}$.

Let us now improve the expression for the function Γ corresponding to the scattering of real particles. In this case $x_1 = x_2 = x_3 = x_4 = 0$. Introducing the notation

$$\Gamma(0,0,0,0,x,y,g) = \Gamma(x,y,g),$$

we obtain from (22)

$$\Gamma^{(2)}(x,y,g) = 1 - \frac{g}{\pi} f\left(\frac{x}{y}\right), \quad f(z) = \ln \frac{4z}{(1+z)^2}.$$

Taking into account (23), Eq. (15) yields the following Lie differential equation for $\Gamma(x,y,g)$

$$\frac{\partial}{\partial x} \ln \Gamma(x,y,g) = \frac{1}{x} \frac{\partial}{\partial \xi} \ln \Gamma^{(2)}\left(\xi, \frac{y}{x}, g\right)\Big|_{\xi=1} = -\frac{g}{\pi} \frac{\partial}{\partial x} f\left(\frac{x}{y}\right).$$

In the r.h.s., only terms proportional to g have been retained. An elementary integration of this equation, taking into account the initial condition f(1) = 0, yields

$$\Gamma(x,y,g) = \left[\frac{4xy}{(x+y)^2}\right]^{-g/\pi},$$

or

$$\Gamma(p',q',p,q) = \left[\frac{(p'-p)^2(p'-q)^2}{(p+q)^4}\right]^{-g/\pi}. \tag{24}$$

Note that the only approximation made in the derivation of this formula was the neglect of higher powers of g. Thus, in a certain sense, the result (24) may be considered as exact in the limit of small g, in contradistinction from the usual results obtained by means of the renormalization group method, which are valid only in asymptotic regions of the momentum variables. This feature is due to the vanishing mass of the particles of the ψ -field.*

Formula (24) closely resembles Thirring's result (Eq. |4.11| in[1]) in the limit of small $\lambda = g$. Supposing that the correspondence between our results and those of Thirring will remain valid also in higher orders in g, which is a quite plausible assumption, as the two models differ only in their symmetry properties, we may conjecture, that taking into account the higher order approximations, we will obtain the following result

$$\Gamma(p',q';p,q) \sim \left[\frac{(p'-p)^2(p'-q)^2}{(p+q)^4}\right]^{-\frac{1}{\pi}\operatorname{arctg} g} . \qquad (25)$$

This would mean, that the charge does not suffer any renormalization and that Eq. (23) remains valid to all orders.

These last remarks apply unconditionally to Thirring's model and show us, that the latter possesses the following remarkable preperty: in contradistinction from all hitherto considered field theories[4] this model does not lead to inconsistencies of the weak coupling approximation or to the appearance of "ghost difficulties".

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* The result (24) is also a counterexample to the recipe given in[6] for obtaining momentum asymptotics. Cf. a detailed discussion of this problem in[3].

In conclusion, the authors would like to express their gratitude to Prof. W.E. Thirring, for sending them a preprint of his work and for some very useful remarks, and also to Prof. N.N. Bogolyubov and B.V. Medvedev for illuminating discussions.

* * *

A p p e n d i x

In this Appendix we collect all the formulas from the algebra of 2 x 2 σ -matrices which were employed in the main text. The defining properties of the σ -matrices are

$$\left.\begin{array}{ll} \sigma^0 = I \ , \ \sigma^\alpha\sigma^\beta + \sigma^\beta\sigma^\alpha = 2\delta^{\alpha\beta} & \alpha,\beta = 1,2,3 \\ \sigma^1\sigma^2 = i\sigma^3, \ldots \qquad (\sigma^i)^2 = 1 & \end{array}\right\} \tag{A.1}$$

Repeating an argument of Pauli [5], it is easy to show, that

$$\sum_{i=0}^{3} \sigma^i_{\rho\sigma}\,\sigma^i_{\sigma'\rho'} = 2\,\delta_{\sigma'\sigma}\,\delta_{\rho'\rho} \,. \tag{A.2}$$

Consider now the direct product of two arbitrary 2 x 2 matrices A and B. Introducing (A.2) into

$$A_{\alpha\beta}B_{\gamma\delta} = A_{\alpha\alpha'}\delta_{\alpha'\beta}\,\delta_{\gamma\delta'}B_{\delta'\delta} \,,$$

we obtain

$$A_{\alpha\beta}B_{\gamma\delta} = \frac{1}{2}\sum_{i=0}^{3} \sigma^i_{\gamma\beta}\,(A\sigma^i B)_{\alpha\delta} \,. \tag{A.3}$$

Formula (A.3) allows us to prove easily the symmetry of the Lagrangian (1), since it implies (the sum over n is according to Eq. (2))

$$\sigma^n_{\alpha\beta}\,\sigma^n_{\gamma\delta} = -\,\sigma^n_{\alpha\delta}\,\sigma^n_{\gamma\beta} \,. \tag{A.4}$$

This equation, together with the anticommutativity of the operators ψ and $\bar\psi$ proves the symmetry of L with respect to an interchange of two ψ's or two $\bar\psi$'s.

Substituting into (A.3) the matrix vector $\hat a$ in place of A and B we obtain (cf. the definition (4) of $\hat a$)

$$\hat a_{\alpha\beta}\hat a_{\gamma\delta} = \frac{a^2}{2}\sigma^n_{\alpha\delta}\sigma^n_{\gamma\beta} + \hat a_{\alpha\delta}\hat a_{\gamma\beta} =$$

$$= -\frac{a^2}{2}\sigma^n_{\alpha\beta}\sigma^n_{\gamma\delta} + \hat a_{\alpha\delta}\hat a_{\gamma\beta}\,. \qquad (A.5)$$

In the same manner one can derive the following identities (the sums over n and m are according to the rule (2))

$$(\sigma^n\hat a\sigma^m)_{\alpha\beta}(\sigma^n\hat b\sigma^m)_{\gamma\delta} = 4(ab)\sigma^n_{\alpha\delta}\sigma^n_{\gamma\beta},$$

or, taking into account (A.4)

$$\frac{1}{4}(\sigma^n\hat a\sigma^m)\times(\sigma^n\hat b\sigma^m) = -(ab)\sigma^n\times\sigma^n, \qquad (A.6)$$

and

$$\frac{1}{4}(\sigma^n\hat a\sigma^m)_{\alpha\beta}(\sigma^m\hat a\sigma^n)_{\gamma\delta} = \frac{a^2}{2}\sigma^n_{\alpha\delta}\sigma^n_{\gamma\beta} + \hat a_{\alpha\delta}\hat a_{\gamma\beta},$$

which, together with (A.5) yields

$$(\sigma^n\hat a\sigma^m)\times(\sigma^m\hat a\sigma^n) = 4\hat a\times\hat a. \qquad (A.7)$$

We record here some formulas in which the σ-matrices stand between spinor operators. First of all, note, that Eqs. (A.2) and (A.4), together with the anticommutativity of the spinorm, imply (1'). Secondly, for any matrix vector $\hat a$ that does not depend on the momenta p, p', q, q' we have the identity (under the sign of the integral)

$$\bar\psi(p')\hat a\psi(q)\bar\psi(q')\hat a\psi(p) = -\frac{a^2}{4}\bar\psi(p')\sigma^n\psi(q)\bar\psi(q')\sigma^n\psi(p), \qquad (A.8)$$

which is a consequence of (A.5).

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