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A B S T R A C T

The dispersion relation for the elastic spin-flip forward scattering of pions by deuterons is derived and compared with the experimental data by means of Chew's impulse approximation. Unlike the former treated case of the elastic forward scattering without spin-flip the contribution from the non-observable region in the dispersion relation is now large and allows a determination of the pion-nucleon coupling constant. Its value is in accordance with that following from dispersion relations for pion scattering on free nucleons. The investigations show the practical possibility for treating bound states in the framework of the theory of dispersion relations, for which, however, it is necessary to perform explicity the analytic continuation because of the divergences appearing in the matrix elements of the non-observable region.

I. Introduction

In two preceding papers[1),2)] we have derived the dispersion relation for the elastic no-spin-flip forward scattering of pions by unpolarized deuterons and compared this with the experimental data by means of Chew's impulse approximation. It was shown, that the dispersion relation is consistent with the results of the impulse approximation, but a detailed test of the theory was not possible, because the contribution from the non-observable part turned out only as small. Therefore it was suggested in Ib to investigate the case of elastic forward scattering with spin-flip, since for that the contribution from the non-observable region in the dispersion relation must be large and comparable, which followed from the corresponding results of the impulse approximation.

In the present work we derive the dispersion relation for the suggested case of elastic spin-flip forward scattering of pions by deuterons under the same assumptions as in Ia. Like there it is necessary to perform explicity the analytic continuation because of the divergences appearing in the matrix elements of the current with respect to the deuteron system in the non-observable region, which is done again by means of Bogoliubov's method[3)]. As in Ia the whole non-observable region yields a continuous contribution (without pole contributions) that is calculated in a sufficient approximation. As in Ia the dispersion relation is again independent from the pion charge.

The comparison of the dispersion relation with the results of the impulse approximation requires as in Ib, that the contribution

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1) F. Kaschluhn, Z.Naturforsch. (in print), hereafter reffered to as Ia.

2) F. Kaschluhn, Nuclear Physics (in print), hereafter reffered to as Ib.

3) N.N. Bogoliubov, B.V.Medvedev, M.K.Polivanov, Problems of the Theory of Dispersion Relations, Gostekhisdat Moscow (in print); shortened translation into German in Fortschr.d.Phys., Akademie-Verlag Berlin (in print).

from the non-observable part must be equal to that in the dispersion relation for the sum of the spin-flip amplitudes for forward scattering of positive and negative pions by free nucleons, which, how - ever, is (as known) a pure pole contribution, so that the required accordance is quite a non-trivial matter. Now the contribution from the non-observable part in the dispersion relation for the elastic spin-flip forward scattering of pions by deuterons resulting from the analytic continuation shows indeed the required behaviour over the whole range of energy. In other words the dispersion relation for spin-flip scattering of pions by deuterons allows a determination of the pion-nucleon coupling constant using Chew's impulse approximation. Its value is $g^2/\hbar c = 18$ as found by Davidon and Goldberger[4] from the dispersion relation for spin-flip scattering of pions by free nucleons, at which the error is of course a little larger in our case.

The present investigations show, that it is practically possible also to handle bound states in the framework of the theory of dispersion relations, for which, however, it is necessary to carry out explicitly the analytic continuation because of the divergences appearing in the matrix elements of the current in the non-observable region.

2. The Spin-Flip Amplitude

As suggested in section 4 of Ib we define the spin-flip amplitude for the elastic scattering of pions by deuterons by

$$\tilde{T} = \frac{1}{2i|\vec{q}\,'\times\vec{q}|}\{\langle\chi_{1,1}|T|\chi_{1,1}\rangle - \langle\chi_{1,-1}|T|\chi_{1,-1}\rangle\} \qquad \text{(I)}$$

T is the usual transition amplitude for elastic pion-deuteron scattering, $\vec{q}\,'$ and $\vec{q}$ final and initial momentum of the pion and $\chi_{1,1}$

4) W.C. Davidon, M.L. Goldberger, Phys. Rev. 104, 119(1956).

and $\chi_{1,-1}$ the spin functions for the triplett states[5]. We suppose as in section 4 of Ib, that the deuteron is polarized (contrary to Ia) perpendicular to the scattering plane in the direction of $\vec{q}' \times \vec{q}$. As usually[3] we choose that frame, in which the sum of the initial and final momentum of the deuteron vanishes

$$\vec{p}\,' + \vec{p} = 0. \tag{2}$$

For the dispersion relation derived in the next section we need the antihermitic part of the spin-flip amplitude (I) in the nonobservable region. The antihermitic part of the transition matrix T is generally given by (compare[3] and Ia (13) for notation)

$$A_{\varkappa\omega}(E,\vec{q}) =$$
$$= \mathrm{tr}\Big\{ \sum_n \langle -\vec{p},s'|j_\varsigma(0)|\vec{p}+\vec{q},n\rangle\langle\vec{p}+\vec{q},n|j_\varsigma(0)|\vec{p},s\rangle\, \delta\big(E-\sqrt{M_n^2+(\vec{p}+\vec{q})^2}+\sqrt{M_D^2+\vec{p}^{\,2}}\big) \tag{3}$$
$$- \sum_n \langle -\vec{p},s'|j_\varsigma(0)|-\vec{p}-\vec{q},n\rangle\langle -\vec{p}-\vec{q},n|j_\varsigma(0)|\vec{p},s\rangle\, \delta\big(E+\sqrt{M_n^2+(\vec{p}+\vec{q})^2}-\sqrt{M_D^2+\vec{p}^{\,2}}\big)\Big\}$$

where we have already used the fact, that according to Ia the scattering amplitude is diagonal in the isotopic spin indices of the pion. Since we shall restrict us in the following section for practical reasons to forward scattering, we may take over directly the results from Ia, that in the non-observable region only the two-nucleon-terms contribute. The needed general matrix element of the current is again given by Ia (20).

The calculation of the antihermitic part of the spin-flip amplitude (I) using (3) yields in the non-observable region after carrying out the summation over spin and isotopic spin states of the two-nucleon system and specializing to forward scattering (compare the correspo-nding considerations in Ia leading to Ia (27))

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5) We remark that (I) corresponds to the second term in the general expression Ib (5) exhibiting the spin structure of the transition matrix.

$$\tilde{A}(E,\vec{q}) = \mathrm{Im}\,\tilde{T} = \frac{\pi}{2}\sum_{z}\left\{\left|1-\frac{E_z}{M_D}\right|\left[B_-(\vec{q},z)+B_+(\vec{q},z)\right]\left[\delta(E+E_z)+\delta(E-E_z)\right]\right. \tag{4}$$

with

$$B_{\mp}(\vec{q},z) = \frac{g^2}{M^2}\left|\int d^3\vec{x}_r\,\psi'^{*}_{z,\binom{t}{s}}\,\frac{\sin}{\cos}\left(\frac{\vec{q}\vec{x}_r}{2}\right)\psi_D\right|^2 \tag{5}$$

($0< E< m$). E_z is defined by Ia (30) and the investigation of the possible contributions from the individual δ -functions in the non-observable region can be performed in exactly the same way as in section 3 of Ia. We notice, that contrary to the no-spin-flip scattering amplitude Ia(27) the two δ -functions in (4) have the same sign, which will be very important in the following. Further the wave functions $\psi'_{z,t}$, $\psi'_{z,s}$ of the continuous spectrum of Ia(3) may again be replaced exactly by plane waves due to Ia (43) and the calculation of the matrix elements in (5) using as in Ia the Hulthén function Ia (44) for ψ_D leads to the expressions Ia (45), (46). The results from section 3 of Ia (especially those remarked at the end) concerning the divergences in the matrix elements of the current in the non-observable region are valid here without any change and the analytic continuation will be performed in the following section in the same way as in Ia section 4 for the case of no-spin-flip scattering.

3. The Dispersion Relation

The dispersion relation for the elastic spin-flip forward scattering of pions by deuterons, which will be written down for the present again for $\tau < 0$, has unlike Ia (47) the form

$$\tilde{D}(E,\tau) = \frac{2}{\pi}E\cdot P\left[\int_{|E_{min}(\tau)|}^{\infty} dE'\,\frac{A(E',\tau)}{E'^2-E^2} + \int_{|E_{max}(\tau)|}^{|E_{min}(\tau)|} dE'\,\frac{A(E',\tau)}{E'^2-E^2}\right] \tag{6}$$

$\tilde{D}(E,\tau)$ is the hermitic part of the spin-flip amplitude (I). The subtraction needed for convergence with respect to large E' in the case of no-spin-flip scattering is unnecessary here (compare also[4]).

As in Ia (50), (5I) we can write the contribution of the second term in (6), that yields the non-observable part after the analytic continuation $\tau \to m^2$, in the form (we suppress in the following the P-symbol)

$$-\frac{g^2}{M^2}E\int_0^P \left|1-\frac{E_z(\tau)}{M_D}\right| \left\{ \left| \int d^3\vec{x}_r \frac{e^{i\vec{p}_r\vec{x}_r}}{(2\pi)^{3/2}} \sin\frac{\vec{q}'(\tau)\vec{x}_r}{2} \Psi_D \right|^2 \right. \tag{7}$$

$$\left. + \left| \int d^3\vec{x}_r \frac{e^{i\vec{p}_r\vec{x}_r}}{(2\pi)^{3/2}} \cos\frac{\vec{q}'(\tau)\vec{x}_r}{2} \Psi_D \right|^2 \right\} \frac{d^3\vec{p}_r}{E^2-E_z^2(\tau)}$$

with

$$\vec{q}'^2(\tau) = E_z^2(\tau) - \tau > 0 \tag{8}$$

where P is given by Ia(36) and $E_z(\tau)$ by Ia(40).

The analytic continuation $\tau \to m^2$ can be performed in exactly the same way as in Ia (52)-(56) and (7) becomes like Ia(57)

$$-\frac{g^2}{M^2}E\frac{4\alpha\beta(\alpha+\beta)}{\pi(\alpha-\beta)^2}\frac{P^{-1}}{m}\int_0^1 dx\, Z(x) \tag{9}$$

with

$$Z(x) = (1-\kappa)^2\left(1-\frac{m}{M_D}\frac{\kappa-x^2}{1-\kappa}\right)\left[\frac{1}{f_1^2+g^2}+\frac{1}{f_2^2+g^2}- \right. \tag{I0}$$

$$\left. -\frac{1}{(f_2-f_1)g}\left(\mathrm{arctg}\frac{2f_1g}{f_1^2-g^2}-\mathrm{arctg}\frac{2f_2g}{f_2^2-g^2}\right)\right]\frac{x^2}{\left(\frac{E(1-\kappa)}{m}\right)^2-(\kappa-x^2)^2}$$

The functions f_I, f_2 and g as well as the constants P and κ are given in Ia (58), (59), α and β by Ia (44).

Then the dispersion relation for elastic spin-flip scattering appears in the final form

$$\tilde{D}(E) = \frac{2}{\pi} E \int_m^\infty dE' \frac{\tilde{A}(E')}{E'^2 - E^2} - \frac{g^2}{M^2} \frac{4\alpha\beta(\alpha+\beta)}{\pi(\alpha-\beta)^2} \frac{P^{-1}}{m} \int_0^1 dx\, Z(x) \qquad (II)$$

where Z(x) is defined by (I0).

4. Comparison with the Impulse Approximation

For comparison of the dispersion relation (II) with the results of the impulse approximation we calculate at first integral over Z(x) appearing in (II) which can be done relatively easy in a sufficient approximation. We neglect the small quantity κ (= 0,023) in (I0) and replace the second bracket involving the ratio m/M_D by I. The error resulting from these is smaller than 5% especially - as we shall see immediately - Z(x) is large only for small x. We consider the function

$$Z(x) = \left[\frac{1}{f_1^2+g^2} + \frac{1}{f_2^2+g^2} - \frac{1}{(f_2-f_1)g}\left(\operatorname{arctg}\frac{2f_1g}{f_1^2-g^2} - \operatorname{arctg}\frac{2f_2g}{f_2^2-g^2}\right)\right]\frac{x^2}{(E/m)^2-x^4} \qquad (I2)$$

The function

$$z(x) = Z(x)\left[(E/m)^2 - x^4\right] \qquad (I3)$$

independent from E is represented in Fig. I.

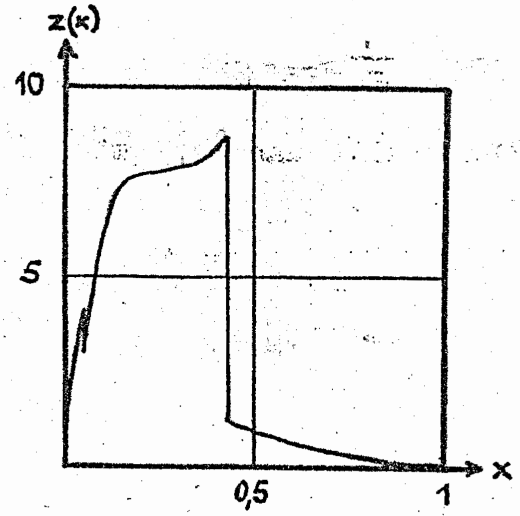

Fig. I

The two discontinuities at $x = 0,05$ and $x = 0,43$ come from the function arc tg $[2f_1 q/(f_1^2 - q^2)]$ (compare Ia, especially footnote I5) continued analytically according to Ia (56).

From Fig. I we can see that Z(x) contributes to the integral only for such x where it is possible to neglect in a sufficient approximation x^4 against $(E/m)^2$ in (I2). Furthermore we contend us with the estimate

$$\int_0^1 z(x)dx = 3,5 \qquad \text{(I4)}$$

based on Fig.I, so that we get finally

$$\int_0^1 dx\, Z(x) = 3,5\, \frac{m^2}{E^2} \qquad \text{(I5)}$$

Then the contribution from the non-observable part in (II) becomes

$$-\frac{g^2}{M^2} E \frac{4\kappa\beta(\kappa+\beta)}{\pi(\kappa-\beta)^2}\, \frac{P^{-1}}{m} \int_0^1 dx\, Z(x) = -\,0,91\, \frac{g^2}{M^2}\frac{1}{E} \qquad \text{(I6)}$$

The comparison of the dispersion relation (II) with the impulse approximation requires as in Ib, that the contribution from the non-observable part in (II) must be equal to that in the dispersion relation for the sum of the forward spin-flip amplitudes for scattering of positive and negative pions by free nucleons, i.e. equal to

$$\frac{2}{\pi} E \frac{\pi g^2}{2M^2} \frac{1}{\left(\frac{m^2}{2M}\right)^2 - E^2} = - \frac{g^2}{M^2} \frac{1}{E} \tag{17}$$

where we have neglected as in Ib the small quantity $(m^2/2M)^2$ against E^2.

Indeed the quantities (16) and (17) are in accordance within the needed accuracy in the whole range of energy, i.e. the dispersion relation (II) is compatible with the results of the impulse approximation.

5. Conclusion.

In the last section it was shown that the dispersion relation (II) for elastic spin-flip forward scattering of pions by deuterons is compatible with the results of the impulse approximation in the whole range of energy. For the value of the pion-nucleon coupling constant determined in this way we can take over directly the result of Davidon and Goldberger [4)]

$$g^2/\hbar c = 18 \tag{18}$$

where the error in our case is, of course, a little larger than that following from determination by means of dispersion relations for scattering of pions by free nucleons.

We emphasize once more, that the comparison of the dispersion relation (II) with the impulse approximation is quite a non-trivial matter, since the contribution from the non-observable part in (II) is a continuous one, that comes in the main from the first δ-function in (4) (compare Ia), whereas the contribution from the non-observable part in the dispersion relation for the sum of the spin-flip amplitudes for forward scattering of positive and negative pions by free nucleons is a pure pole contribution, that corresponds to the second δ - function in (4). Nevertheless, we got accordance within the requisite accuracy, for the analytically continued functions in (IO) showed such a behaviour, that according to Fig.I essential contributions to the integral in (IO) came only from small x. We remember, that in the case of no-spin-flip scattering there was necessarily a difference in sign in the two contributions from the non-observable parts, which we had to compare in Ib, since in the relation Ia (27) corresponding to (4) the δ -functions have different signs. The contradiction was solved in such a manner, that the two before-mentioned contributions from the non-observable parts were only small and not testable within the accuracy of the impulse approximation (compare Ib , section 3).

We notice further on, that very likely the contribution from the non-observable part in the dispersion relation (II) is rather insensitive to the special form of the wave function for the deuteron ground state (for that we have chosen for simplicity the Hulthén function Ia (44)) or to the special form of the corresponding two-nucleon potential respectively. This we can expect from the analogy to the well-known fact, that in the low-energy two-nucleon problems only the short range and the great strength of the nuclear forces are important (The modifications arising from the existence of the D-state are certainly unimportant with regard to the accuracy of the whole method).

Concluding we can say that the present investigation show the practical possibility of including bound states in the theory of dispersion relations, for which, however, it is necessary to carry out explicity the analytic continuation because the divergences appearing in the matrix elements of the current in the non-observable region. So the only problem remains to give a theoretical account for the special "renormalisation prescription" assumed in Ia for the matrix elements of the current with respect to the deuteron system.

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