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A method of using indefinite metric in the quantum field theory has been suggested in the previous paper by N.N. Bogoliubov and authors.[1] The purpose of this note is to explain the meaning of the suggested approach by considering an instructive analogy in the framework of the classical field theory.

Let us consider two classical fields, for instance, the complex field $\psi(x)$ and the real $\chi(x)$ with the interaction Lagrangian:

$$\mathcal{L}_{int} = g \int \psi^*(x)\,\psi(x)\,\chi(x)\,dx \qquad (1)$$

We assume the field $\psi(x)$ to be a real physical field, whereas the field $\chi(x)$ a fictitious one (in the sense of[1]) and exhibit $\chi(x)$ in the form

$$\chi(x) = \sum_n c_n \varphi_n(x) \qquad * \qquad (2)$$

Evidently the analogy to the "fields with indefinite metric" in the classical theory is the field with negative energy or, equivalently, with the opposite sign in the free* Lagrangian. Accordingly we write now the full Lagrangian in the form

$$\mathcal{L} = \int \left(\partial_\mu \psi^*(x)\cdot\partial^\mu \psi(x) - M^2 \psi^*(x)\psi(x)\right)dx +$$

$$+ \frac{1}{2}\sum_n \epsilon_n \int dx \left(\partial_\mu \varphi_n(x)\cdot\partial^\mu \varphi_n(x) - m_n^2 \varphi_n^2(x)\right) +$$

$$+ g \sum_n c_n\, \psi^*(x)\,\psi(x)\,\varphi(x)\,dx \qquad (3)$$

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* One may assume, of course, that the field $\chi(x)$ involves the "physical" component $\varphi(x)$, too. Then we would approach the model of the field theory with the usual Yukawa-like interaction.

(we denote by M the mass of the field $\psi(x)$ and by m_n the masses of the fictitious fields $\varphi_n(x)$, where $\epsilon_n = \pm 1$; in the quantum theory the indefinite metric would correspond to the fields with $\epsilon_n = -1$.

Applying variational principle to (3), we obtain

$$(\Box - M^2)\psi(x) = -g\sum_n c_n \varphi_n(x)\psi(x) = -J(x) \qquad (4.1)$$

$$(\Box - m_n^2)\varphi_n(x) = -g\, c_n \epsilon_n \psi^*(x)\psi(x) = -j_n(x) \,. \qquad (4.2)$$

In the manner of Yang-Feldman formalism we rewrite (4.2) in the integral form:

$$\varphi_n(x) = \varphi_n^{in}(x) - \int \mathcal{D}^{ret}_{m_n}(x-x')\, j_n(x')\, dx' \qquad (5.1)$$

or

$$\varphi_n(x) = \varphi_n^{out}(x) - \int \mathcal{D}^{adv}_{m_n}(x-x')\, j_n(x')\, dx' \,, \qquad (5.2)$$

where, as usual, the incoming (outgoing) fields $\varphi_n^{in}(x)$ $(\varphi_n^{out}(x))$ obey the free field equations and coincide with $\varphi_n(x)$ at $t \to -\infty$ $(t \to +\infty)$. Now, introducing the symmetrical Green functions $\bar{\mathcal{D}}_n(x)$ and the Pauli-Jordan functions $\mathcal{D}_n(x)$ with the masses m_n by the usual relations

$$\mathcal{D}^{ret}_{m_n}(x) = \bar{\mathcal{D}}_n(x) + \frac{1}{2}\mathcal{D}_n(x)\,; \quad \mathcal{D}^{adv}_{m_n}(x) = \bar{\mathcal{D}}_n(x) - \frac{1}{2}\mathcal{D}_n(x) \qquad (6)$$

and adding and subtracting (5.1,2), we obtain

$$v_n(x) \equiv \frac{\varphi_n^{out}(x) - \varphi_n^{in}(x)}{2} = \frac{1}{2}\int \mathcal{D}_n(x-x')\, j_n(x')\, dx' \qquad (7.1)$$

and

$$u_n(x) = \frac{\varphi_n^{out}(x) + \varphi_n^{in}(x)}{2} = \int \bar{\mathcal{D}}_n(x-x')\, j_n(x')\, dx' - \varphi_n(x) \,. \qquad (7.2)$$

According to the program outlined in [1] the fictitious fields if carry energy, momentum and other dynamic characteristics then the interchange of these quantities between physical and non-physical fields during all the time of collision must be strongly forbidden. In other words we would like to require both the asymptotic values at $t=+\infty$ and at $t=-\infty$ of such dynamic characteristics of the fictitious fields to coincide. But all such characteristics (we assume, of course, that at $t\to\pm\infty$ the interaction is switched on and off with the help of the adiabatic hypothesis) at $t=\pm\infty$ are expressed as the sums (integrals) of the terms such as

$\varphi_n^{out}(x)\varphi_n^{out}(x')$ and, respectively, $\varphi_n^{in}(x)\varphi_n^{in}(x')$.

Therefore, to satisfy this requirement, it is sufficient to impose the condition

$$\varphi_n^{out}(x)\varphi_n^{out}(x') - \varphi_n^{in}(x)\varphi_n^{in}(x') = 0 \tag{8}$$

Substituting into (8) instead of $\varphi_n^{in}(x), \varphi_n^{out}(x)$ the fields $u(x)$ and $v(x)$ introduced by (7) we obtain the equivalent condition

$$u_n(x)v_n(x') + v_n(x)u_n(x') = 0. \tag{9}$$

So, in order to satisfy the requirement that the energy etc. would not be transferred to the non-physical fields it is sufficient to require

$$u_n(x) = \frac{\varphi_n^{out}(x) + \varphi_n^{in}(x)}{2} = 0 \tag{9a}$$

or $$V_n(x) = \frac{\varphi_n^{out}(x) - \varphi_n^{in}(x)}{2} = 0^{*}. \tag{9b}$$

From equations (7) we see that conditions (9a) and (9b) have quite a different character. Indeed, condition (9b) requires that the integral of the physical fields $\psi(x)$ in the right-hand side of (7.1) would vanish. Therefore it turns out to be a condition imposed also on the physical part of the system. It is clear that it can be fulfilled only if the physical part has certain specific properties.

On the contrary, (9a) does not impose any limitations on the physical part of the system. Since in the right-hand side of (7.2) besides the integral of the physical fields there stands also a non-physical field $\varphi_n(x)$ we can always satisfy (9a) by the choice of $\varphi_n(x)$ —equations (7.2) simply determine the non-physical fields $\varphi_n(x)$ in terms of integrals of the physical ones. So, we can always impose condition (9a) on the system without being afraid that some contradictions would arise**.

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* As a matter of fact it would not have been necessary to require the fulfillment of (9a) in all the points but only the vanishing of the definite kind integrals of the sums of the terms of such a kind. Since these linear combinations would be rather multiform then the detailed analysis of the problem about the existence of such a possibility would have been sufficiently complicated.

** Note, that prohibiting not only the energy etc. - interchange between the non-physical and physical states, but requiring the asymptotic vanishing of the non-physical states energy etc. , we should impose both the conditions (9a,b). From the preceding discussion it is clear that it does not lead to the contradiction, only in the case when the physical part has definite properties.

Making use of this observation we impose condition (9a) in order to exclude the non-physical fields $\varphi_n(x)$ at all and to deal further only with the physical field $\psi(x)$.

This procedure leads to the interaction Lagrangian

$$\mathcal{L}_{int} = g^2 \int dx dx' \, \psi^*(x)\psi(x) \, K(x-x') \, \psi^*(x')\psi(x') \tag{10}$$

and the equations of motion

$$(\Box - M^2)\psi(x) = -g^2 \int dx' \psi^*(x')\psi(x') \, K(x'-x)\psi(x) \tag{11}$$

with the nucleus

$$K(x-x') = \sum_n \epsilon_n c_n^2 \, \bar{\mathcal{D}}_n(x-x') \tag{12}$$

expressed in the form of a sum (or of an integral if a continuous set of the fictitious fields is introduced) of the symmetrical Green functions $\bar{\mathcal{D}}_n(x-x')$ with different masses m_n. Clearly with an appropriate choice of the coefficients c_n and sign factors ϵ_n we may make $K(x)$ either singular or regular to the extent desired. This possibility, of course, is provided only by the "indefinite" metric. In the case of continuous mass spectrum we obtain instead of (12)

$$K(x-x') = \int_0^\infty d(m^2)\rho(m^2) \, \bar{\mathcal{D}}_m(x-x') \tag{13}$$

where the spectral function $\rho(m^2)$ is not necessarily positive in virtue of the remards made above.

Thus, we see that excluding the non-physical fields $\varphi_n(x)$

from Lagrangian (1) initially local by means of condition (a) we obtain the theory of typically non-local form. (This result is quite natural- one may see a direct analogy between this result and the attempts of some authors[21] to exclude any idea about the photons formulating the electrodynamics as a pure action-in-distance theory supposing the requirement to use only the half sums of the retarded and advanced potentials. The obtained nucleus (12) or (13) is not an arbitrary function of $(x-x)^2$ since in virtue of the properties of the functions $\bar{\delta}$ it is in any case restricted by the requirement

$$K(x^2) = 0 \quad \text{for} \quad x^2 < 0.$$

Point out that the non-local character of equations (11) is essentially associated with the imposed non-local condition (9a). Indeed, since the functions $\bar{D}$ are the Green functions of the Klein-Gordon equation then it would have seemed that at any rate for the finite number of the fictitious fields differentiating (11) number of times needed one may have returned to the differential equation. However, the non-local boundary conditions would have turned to the imposed upon such an equation and the theory would remain non-local.

Note that when passing from the classical example considered here to the quantum case a new essential moment arises. As we have shown the non-local theory may be obtained imposing certain auxiliary conditions on the fields. In the quantum theory these conditions may be in principle imposed either on the field operators or on the state vectors, like Lorentz-condition in the electrodynamics.

All the usual investigations in the non-local field theories followed, in a certain sense, the first way. But imposing the auxiliary conditions the field operators we have always the risk of coming to a contradiction with the commutation relations. Here we see the reasons for failures of the non-local theories considered. It is not accidental that the difficulties of the Kristiensen-Møller-Bloch theory[3] are associated with the non-commutativity of the field operators.

General idea of the method suggested in[1] points out that the second way must be chosen which does not lead[1] to the similar difficulties. One hopes, therefore, that the method suggested in[1] will give the possibility of constricting the consistent theory with the non-local interaction. In this connection we should like to point out that recently some experimental indications have appeared which show the necessity of introducing the non-local interaction. So, Lee and Yang[4] found out recently that the experimental value of the Michel parameter in μ - meson decay may be easily explained by introducing the non-local nucleus into the four-fermion interaction, with indefinite metric.

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References

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4. T.D. Lee and C.N. Yang. Phys. Rev., 108, 1611 (1957).

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