JOINT INSTITUTE FOR NUCLEAR RESEARCH

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Abstract:

Using the Freese-Matthews-Salam equations for chronological products of field operators, equations for Green functions of many electrons and photons are written. It is shown that for finding any single Green function an infinite recursive system of these equations is to be solved. This system is reduced, after adding terms containing the external electric current and external electromagnetic potential to the Lagrangian, to one equation containing functional derivatives of higher orders. It is shown that all relations and equations become much simpler when the definition of the Green function is appropriately changed.

1. Introduction

The Green function G^{rs} of a system of many electrons and photons can be expressed in terms of the field operators in the following manner:

$$G^{rs}(x_1 \dots x_r, y_1 \dots y_r, \underset{\mu_1}{z_1} \dots \underset{\mu_s}{z_s}) =$$

$$= \frac{1}{\langle S \rangle_0} \langle T \{ \psi(x_1) \cdots \psi(x_r) \bar{\psi}(y_1) \cdots \bar{\psi}(y_r) A_{\mu_1}(z_1) \cdots A_{\mu_s}(z_s) S \} \rangle_0 \quad (1)$$

$$= \frac{1}{\langle S \rangle_0} \langle S T \{ \Psi(x_1) \cdots \Psi(x_r) \bar{\Psi}(y_1) \cdots \bar{\Psi}(y_r) \mathbf{A}_{\mu_1}(z_1) \cdots \mathbf{A}_{\mu_s}(z_s) \} \rangle_0 \quad (1')$$

where $\psi, \bar{\psi}, A_\mu$ and $\Psi, \bar{\Psi}, \mathbf{A}_\mu$ are the field operators in the interaction picture and in the Heisenberg one respectively. T is the symbol of the Wick's chronological product, the bracket $\langle \cdots \rangle_0$ denotes the vacuum expectation value and for the S -matrix eht following formula holds

$$S = T\left\{ e^{i\int L(x)dx} \right\}$$

where L is the in-
teraction Lagrangian of the system*.

As it is known, the Green function describes, for given r and s, the process which takes place among the particles as the result of the interaction. $2r$ has the meaning of the total number of electrons and positrons before and after the process and s means the same for the photons.

The Green functions satisfy equations which follow from the field equations for the operators ψ, $\bar{\psi}$ and A_μ.
If one takes the interaction Lagrangian in the form

$$L(x) = ie\, j_\mu(x) A_\mu(x)$$

where

$$j_\mu(x) = tr\left[\bar{\psi}(x), \psi(x)\right]$$

one obtains the following equations for the Green functions of the nearest order

$$D_x G^{10}(x; y; -) = -e\gamma_\mu(x) G^{11}(x; y; \underset{\mu}{x}) + \delta(x-y) \qquad (2)$$

$$K_x G^{01}(-; -; \underset{\mu}{x}) = e\, tr\, \gamma_\mu(x) G^{10}(\underline{x}, \underline{x}, -) \qquad (2')^{**}$$

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* If not necessary, the vector indices $\mu_1, \dots \mu_s$ are omitted. The spinor indices are always omitted.

** The symbol $\underline{x}$ is defined in Section 2.

where

$$D_x = i(\gamma_\mu \frac{\partial}{\partial x_\mu} + m), \ \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu},$$

$$\delta(x-y) = \delta(x_0 - y_0)\delta(\vec{x}-\vec{y}), \ K_x = -i(\Delta_x - \partial_0^2).$$

Thus, we have two equations for determining three unknown functions G^{10}, G^{01} and G^{11}. As the third equation, we could take either

$$K_z G^{11}(x; y; \underset{\mu}{z}) = -e\, tr \gamma_\mu(z) G^{20}(\underline{z}, x, \underline{z}, y; -)$$

or

$$D_x G^{11}(x; y; z) = -e\gamma_\mu(x) G^{12}(x; y; \underset{\mu}{x}, z) + \delta(x-y) G^{01}(-; -; z).$$

Both these equations, however, contain a higher Green function besides G^{11}. To determine it, another equation must be added, but this contains, again, another Green function and so on. In this way, an infinite recursive set of equations is to be solved for determining G^{10}, G^{01}. As it was shown by Schwinger |1|, this trouble can be avoided by constructing a certain general relation among Green functions of different orders with the help of which the G^{11} can be expressed in terms of the G^{10} and G^{01}. This can be done, for instance, by introducing external sources into the interaction Lagrangian. In this way, the S-matrix as well as all Green functions become functionals of the function which describes the distribution of the sources and, moreover, the higher-order Green function is expressible in terms of functional derivatives of the nearer-order Green functions. For instance, if we add the term $ie\mathcal{J}_\mu(x)A_\mu(x)$ (where $\mathcal{J}_\mu(x)$ describes the distribution of external sources of photons) to the Lagrangian, we obtain, using (1), the following general relation

$$\frac{\delta G^{rs}(x_1\dots x_r; y_1\dots y_r; z_1\dots z_s)}{\delta \mathcal{J}_\lambda(v)} = eG^{01}(-;-;\underset{\lambda}{v})G^{rs}(x_1\dots x_r; y_1\dots y_r; z_1\dots z_s)$$
$$-eG^{rs+1}(x_1\dots x_r; y_1\dots y_r; \underset{\lambda}{v}\, z_1\dots z_s) \quad (\mathcal{J})$$

In the case $r=1$, $s=0$ this relation can be used for eliminating G^{11} from (2):

$$\left[D_x + e\gamma_\mu(x)G^{01}(-;-;\underset{\mu}{x}) - \gamma_\mu(x)\frac{\delta}{\delta\mathcal{J}_\mu(x)}\right]G^{10}(x;y;-) = \delta(x-y). \tag{3}$$

This is the well-known Schwinger equation. Due to the above mentioned change of the Lagrangian, eq. (2') is to be replaced by

$$K_x G^{01}(-;-;\underset{\mu}{x}) = e\,\mathrm{tr}\,\gamma_\mu(x)G^{10}(\underline{x};\underline{x};-) - e\mathcal{J}_\mu(x). \tag{3'}$$

There are, of course, other possibilities of introducing external sources into the Lagrangian. For instance, the functions $\eta(x)$ and $\bar{\eta}(x)$, which describe the distribution of sources of the electrons and positrons, are frequently used. With the help of all the three ones, namely $\mathcal{J}_\mu$, η and $\bar{\eta}$, it is possible to replace the equations for the Green functions by functional differential equations for the S-matrix (see [2]).

However, the η and $\bar{\eta}$ have the formal meaning only without any deeper physical interpretation as it is in the case of $\mathcal{J}_\mu$, which describes the external electric current. Also, their mathematical character is not clear. In fact, these c-numbers must anticommute with one another and also with $\psi(x)$ and $\bar{\psi}(x)$, in order to obtain the right mathematical sense for the field equations. It is, therefore, very desirable to manage without η and $\bar{\eta}$ in the theory. So has been done by Anderson |3| , Polivanov |4| and Berestetski and Galanin |5|, which have deduced the Schwinger equations (3), (3') using the $\mathcal{Y}_\mu$ only.

Till now, all our considerations have been related to the case of one particle. Now we shall show the situation in the case of many particles.

For one-particle Green functions we have the equations (2) and (2'), which contain the following functions

eq. (2) . . . . G^{10}, G^{11}

eq. (2') . . . . G^{01}, G^{10}

Using ($\mathcal{J}$) we have obtained the following scheme ~~eq. (3)~~

eq. (3) . . . . G^{10}, G^{01}

Eq. (3') . . . . G^{01}, G^{10}

i.e. a system from which two unknown functions G^{10} and G^{01} can be determined without resorting to the above mentioned recursive system of equations. On the other hand, in the case of many particles, it will be shown that equations corresponding to (2) and (2') contain the functions

$$G^{r+1s},\ G^{r+1s+1},\ G^{rs}$$

and G^{rs+1}, G^{r+1s}, G^{rs}, G^{rs-1}

respectively (see eqs. (6) and (6')). Using ($\mathcal{J}$) we shall obtain the following scheme

$$G^{r+1s},\ G^{01},\ G^{rs}$$

and G^{rs+1}, G^{r+1s}, G^{rs}, G^{rs-1}

(see eqs (7) and (6')). We see that in the general case this method is not effective because we obtain a system of two equations from which four quantities are to be determined. Thus, for avoiding the infinite recursive system, we must eliminate more than one term from the equations.

In the present paper two possible ways are shown how to perform the above-mentioned elimination without using the η and $\bar{\eta}$. The first one (Section 3) is the simplest possibility, which apparently has only the formal meaning. The second one (Section 4) leads to a high-order functional differential equation, which contains only one Green function. It is rather complicated but may be simplified by redefining the photon part of the Green function.

2. Equations for general Green functions

In this section it will be shown how it is possible to deduce the equations for Green functions of many particles by generalizing the method of Berestetski and Galanin |5| and using the Fresse-Mathhews-Salam equations |6| for T -products of the field operators.

If choosing the Lagrangian in the form

$$L'(x) = ie(j_\mu(x) + \mathcal{J}_\mu(x))A_\mu(x)$$

the Freese-Mathhews-Salam equations have the following form:

$$D_{x_1} T\{\Psi(x_1)\ldots\Psi(x_r)\bar{\Psi}(y_1)\ldots\bar{\Psi}(y_r)A_{\mu_1}(z_1)\ldots A_{\mu_s}(z_s)\} =$$

$$= -e\gamma_\mu(x_1)\left\{\Psi(x_1)\ldots\Psi(x_r)\bar{\Psi}(y_1)\ldots\bar{\Psi}(y_r)A_\mu(x_1)A_{\mu_1}(z_1)\ldots A_{\mu_s}(z_s)\right\}+$$

$$+\sum_{\rho=1}^{r}(-1)^{r+\rho}\delta(x_1-y_\rho)T\left\{\Psi(x_2)\ldots\Psi(x_r)\bar{\Psi}(y_1)\ldots\bar{\Psi}(y_{\rho-1})\bar{\Psi}(y_{\rho+1})\ldots\bar{\Psi}(y_r)A_{\mu_1}(z_1)\ldots A_{\mu_s}(z_s)\right\}; \tag{4}$$

$$K_{z_1}T\left\{\Psi(x_1)\ldots\Psi(x_r)\bar{\Psi}(y_1)\ldots\bar{\Psi}(y_r)A_{\mu_1}(z_1)\ldots A_{\mu_s}(z_s)\right\}=$$

$$=(-1)^r e\,tr\,\gamma_{\mu_1}(z_1)T\left\{\underline{\Psi(z_1)}\Psi(x_1)\ldots\bar{\Psi}(x_r)\underline{\bar{\Psi}(z_1)}\bar{\Psi}(y_1)\ldots\bar{\Psi}(y_r)A_{\mu_1}(z_1)\ldots A_{\mu_s}(z_s)\right\}- \tag{4'}$$

$$-e\gamma_{\mu_1}(z_1)T\left\{\Psi(x_1)\ldots\Psi(x_r)\bar{\Psi}(y_1)\ldots\bar{\Psi}(y_r)A_{\mu_2}(z_2)\ldots A_{\mu_s}(z_s)\right\}+$$

$$+\sum_{\sigma=2}^{s}\delta(z_1-z_\sigma)\delta_{\mu_1\mu_\sigma}T\left\{\Psi(x_1)\ldots\Psi(x_r)\bar{\Psi}(y_1)\ldots\bar{\Psi}(y_r)A_{\mu_2}(z_2)\ldots A_{\mu_{\sigma-1}}(z_{\sigma-1})A_{\mu_{\sigma+1}}(z_{\sigma+1})\ldots A_{\mu_s}(z_s)\right\}.$$

The symbol $(-1)^r T\left\{\underline{\Psi(z_1)}\Psi(x_1)\ldots\Psi(x_r)\underline{\bar{\Psi}(z_1)}\bar{\Psi}(y_1)\ldots\bar{\Psi}(y_r)\right\}$ has the following meaning

$$T\left\{\Psi(x_1)\ldots\Psi(x_r)\left[\Psi(z_1),\bar{\Psi}(z_1)\right]\bar{\Psi}(y_1)\ldots\bar{\Psi}(y_r)\right\} \tag{5}$$

where

$$T\left\{\Psi(x)\bar{\Psi}(x)\right\}=\Psi(x)\bar{\Psi}(x)$$

and

$$T\left\{\bar{\Psi}(x)\Psi(x)\right\}=\bar{\Psi}(x)\Psi(x).$$

The advantage of this notation is that the operators $\underline{\psi(x)}$ and $\underline{\bar{\Psi}(x)}$ may be regarded as anticommuting inside the T-product, analogically as the $\psi(x)$'s and $\bar{\Psi}(y)$'s with independent arguments.

From (4) and (4'), with the help of (1'), one immediately obtains corresponding equations for the Green functions of many particles:

$$D_{x_1} G^{rs}(x_1 \dots x_r ; y_1 \dots y_r ; z_1 \dots z_s) =$$

$$= -e\gamma_\mu (x_1) G^{rs+1}(x_1 \dots x_r ; y_1 \dots y_r ; \underset{\mu}{x_1} , z_1 \dots z_s) + \tag{6}$$

$$+ \sum_{\rho=1}^{r} (-1)^{r+\rho} \delta(x_1 - y_\rho) G^{r-1s}(x_2 \dots x_r ; y_1 \dots y_{\rho-1} y_{\rho+1} \dots y_r ; z_1 \dots z_s) ;$$

$$K_{z_1} G^{rs}(x_1 \dots x_r ; y_1 \dots y_r ; \underset{\mu_1}{z_1} \dots z_s) =$$

$$= (-1)^r e \, tr \, \gamma_{\mu_1}(z_1) G^{r+1s-1}(\underline{z_1} x_1 \dots x_r ; \underline{z_1} y_1 \dots y_r ; z_2 \dots z_s) -$$

$$- e \mathcal{J}_{\mu_1}(z_1) G^{rs-1}(x_1 \dots x_r ; y_1 \dots y_r ; z_2 \dots z_s) + \tag{6'}$$

$$+ \sum_{\sigma=2}^{s} \delta(z_1 - z_\sigma) \delta_{\mu_1 \mu_\sigma} G^{rs-2}(x_1 \dots x_r ; y_1 \dots y_r ; z_2 \dots z_{\sigma-1} z_{\sigma+1} \dots z_s) ,$$

in which the arguments $\underline{z_1}$, $\underline{z_1}$ originate in the operators $\underline{\psi(z_1)}$ and $\underline{\bar{\Psi}(z_1)}$. (6) and (6') are the generalization of the (2) and (2') respectively. Using ($\mathcal{J}$) we can, analogically to the oneparticle case, eliminate the term containing G^{rs+1} from (6) and obtain the equation

$$\mathbb{X}_{x_1} G^{rs}(x_1 \ldots x_r; y_1 \ldots y_r; z_1 \ldots z_s) =$$

$$= \sum_{p=1}^{r} (-1)^{r+p} \delta(x_1 - y_p) G^{r-1s}(x_2 \ldots x_r; y_1 \ldots y_{p-1} y_{p+1} \ldots y_r; z_1 \ldots z_s) \tag{7}$$

where

$$\mathbb{X}_1 = D_{x_1} + e\gamma_\mu(x_1) G^{01}(-;-;\underset{\mu}{x_1}) - \gamma_\mu(x_1) \frac{\delta}{\delta \mathcal{J}_\mu(x_1)} .$$

This equation becomes, in the case $r=1, \; s=0$, the Schwinger equation (3).

3. Reduction of the recursive system

We shall show that, if taking the Lagrangian in the form (L'), now it is possible to reduce the infinite recursive system (7), (6') to a system of two equations so that from a single equation (7) and a single eq. (6') it is possible to determine two Green functions of the given order.

To perform it we shall use the relation ($\mathcal{J}$), which connects Green functions with different s. So, we can eliminate the G^{rs} in (6') by expressing it through G^{rs-1}. Then, by iterating, we eliminate the G^{rs-1} too, by expressing it in terms of G^{rs-2}. After this transformation having been performed, the eq. (6') contains only two Green functions, namely the G^{rs-2} and G^{r+1s-2}, and has the following form (after changing appropriately the symbols):

$$\left\{-\frac{1}{e}K_{v_1}\frac{\delta}{\delta \mathcal{J}_{\lambda_1}(v_1)}+K_{v_1}G^{01}\left(-;-;\genfrac{}{}{0pt}{}{v_1}{\lambda_1}\right)+e\mathcal{J}_{\lambda_1}(v_1)\right\}\times$$

$$\times\left\{-\frac{1}{e}\frac{\delta}{\delta \mathcal{J}_{\lambda_2}(v_2)}+G^{01}\left(-;-;\genfrac{}{}{0pt}{}{v_2}{\lambda_2}\right)\right\}G^{rs}(x_1\cdots x_r;y_1\cdots y_r;z_1\cdots z_s)=$$

$$=(-1)^r e\,\mathrm{tr}\,\gamma_{\lambda_1}(v_1)\left\{-\frac{1}{e}\frac{\delta}{\delta \mathcal{J}_{\lambda_2}(v_2)}+G^{01}\left(-;-;\genfrac{}{}{0pt}{}{v_2}{\lambda_2}\right)\right\}G^{r+1s}(\underline{v_1}x_1\cdots x_r;\underline{v_1}y_1\cdots y_r;z_1\cdots z_s)+$$

$$+\sum_{\sigma=1}^{s}\delta(v_1-z_\sigma)\delta_{\lambda_1\mu_\sigma}G^{rs}\left(x_1\ldots x_r;y_1\ldots y_r;z_1\ldots z_{\sigma-1}z_{\sigma+1}\ldots z_s\genfrac{}{}{0pt}{}{v_2}{\lambda_2}\right)+$$

$$+\delta(v_1-v_2)\delta_{\lambda_1\lambda_2}G^{rs}(x_1\ldots x_r;y_1\ldots y_r;z_1\ldots z_s). \tag{7'}$$

The equations (7) and (7') form a system which contains only two unknown functions because the G^{01} is considered as known from the system (3), (3'). Thus, the system (7), (7') is the many-particle analogue of (3), (3').

4. Replacement of the recursive system by one equation

In this section a more effective method will be given. Two following purposes are intended:

1. Having eliminated the term with G^{11} from (2) Schwinger has, firstly, lowered the number of unknown functions but, simultaneously, he has removed the interaction term from (2). Since the presence of an interaction term causes great troubles in every field equation, the eliminating of it also from eq. (3') (or the general eq. (6')) is desirable. In this way, we should obtain, instead of (6'), a functional differential equation which together with (7) forms a system of functional differen-

tial equations not containing the interaction terms. Consequently, there is in principle possible to solve the field equations without using the perturbation theory and express the solution in closed form.

2. In eq. (7) the operator X_1 is defined which lowers the index r by one, leaving the s unchanged. Thus, if the operator $X_r X_{r-1} \dots X_2 X_1$ is applied to (7) one obtains, on the right hand side, the function G^{os} i.e. a Green function which does not contain the electron part. If an analogous operator V will be found lowering the s and leaving the r unchanged, it will be possible to exclude the photon part of the Green function too and obtain an equation which contains, on the right, no Green function at all and, on the left, the G^{rs} for one value of r , s only.

Ad. 1. For excluding the interaction term in (6') let us choose the Lagrangian in a symmetrical form

$$L''(x) = ie(j_\mu(x) + \mathcal{J}_\mu(x))(A_\mu(x) + a_\mu(x))$$

where $a_\mu(x)$ denotes the external classical electromagnetic potential. ($L''(x)$ apparently represents the most general form of electromagnetic interaction). Then, analogously to ($\mathcal{J}$), the following relation is valid:

$$\frac{\delta G^{rs}(x_1 \dots x_r; y_1 \dots y_r; z_1 \dots z_s)}{\delta a_\lambda(v)} = e\,tr\gamma_\lambda(v) G^{10}(\underline{v}; \underline{v}; -) G^{rs}(x_1 \dots x_r; y_1 \dots y_r; z_1 \dots z_s) \tag{a}$$

$$+ (-1)^r e\,tr\gamma_\lambda(v) G^{r+1s}(\underline{v}x_1 \dots x_r; \underline{v}y_1 \dots y_r; z_1 \dots z_s)$$

With the help of (a), eq. (6') turns into

$$K_{z_1} G^{rs}(x_1 \ldots x_r; y_1 \cdots y_r; z_1 \cdots z_s) =$$

$$= \left\{ \frac{\delta}{\delta a_{\mu_1}(z_1)} + e\,tr\,\gamma_{\mu_1}(z_1) G^{10}(\underline{z_1}; \underline{z_1}; -) - e\mathcal{J}_{\mu_1}(z_1) \right\} G^{rs-1}(x_1 \cdots x_r; y_1 \cdots y_r; z_1 \cdots z_s) +$$

$$+ \sum_{\sigma=2}^{s} \delta(z_1 - z_\sigma)\delta_{\mu_1\mu_\sigma} G^{rs-2}(x_1 \ldots x_r; y_1 \cdots y_r; z_1 \cdots z_{\sigma-1} z_{\sigma+1} \cdots z_s).$$

This equation contains no interaction more. There are three values of the photon index and one value of the electron one in it.

Ad 2. For constructing the operator V it is sufficient now to express the G^{rs} in terms of G^{rs-1} using ($\mathcal{J}$). After performing it and after changing appropriately the symbols one obtains

$$V_{\lambda_1 1} G^{rs}(x_1 \ldots x_r; y_1 \cdots y_r; z_1 \ldots z_s) =$$

$$= \sum_{\sigma=1}^{s} \delta(v_1 - z_\sigma)\delta_{\lambda_1\mu_\sigma} G^{rs-1}(x_1 \ldots x_r; y_1 \cdots y_r; z_1 \ldots z_{\sigma-1} z_{\sigma+1} \cdots z_s) \quad (8')$$

where

$$V_{\lambda_1 1} = -\frac{1}{e} K_{v_1} \frac{\delta}{\delta \mathcal{J}_{\lambda_1}(v_1)} + K_{v_1} G^{01}(-; -; \underset{\lambda_1}{v_1}) -$$

$$- \frac{\delta}{\delta a_{\lambda_1}(v_1)} - e\,tr\,\gamma_{\lambda_1}(v_1) G^{10}(\underline{v_1}; \underline{v_1}; -) + e\mathcal{J}_{\lambda_1}(v_1)$$

is the operator looked for. If taking the Lagrangian L'' instead of L' , eq. (7) turns into

$$X_1 G^{rs}(x_1 \dots x_r; y_1 \dots y_r; z_1 \dots z_s) =$$

$$= \sum_{\rho=1}^{r} (-1)^{r+\rho} \delta(x_1 - y_\rho) G^{r-1s}(x_2 \dots x_r; y_1 \dots y_{\rho-1} y_{\rho+1} \dots y_r; z_1 \dots z_s) \tag{8}$$

where

$$X_1 = D_{x_1} + e\gamma_\mu(x_1)\left(G^{01}(-;-;\underset{\mu}{x_1}) + a_\mu(x_1)\right) - \gamma_\mu(x_1)\frac{\delta}{\delta \mathcal{J}_\mu(x_1)} .$$

Now, using (8) and (8') we obtain the following equation:

$$V_{\lambda_s s} \dots V_{\lambda_1 1} X_r \dots X_1 G^{rs}(x_1 \dots x_r; y_1 \dots y_r; z_1 \dots z_s) =$$

$$= \sum_{P_\rho} \operatorname{sg}\rho\, \delta(x_1 - y_{\rho_1}) \dots \delta(x_r - y_{\rho_r}) \sum_{P_\sigma} \delta(v_1 - z_{\sigma_1}) \delta_{\lambda_1 \mu_{\sigma_1}} \dots \delta(v_s - z_{\sigma_s}) \delta_{\lambda_s \mu_{\sigma_s}} . \tag{9}$$

Here $\sum_{P_\rho}$ and $\sum_{P_\sigma}$ denote the summations over all permutations $\begin{pmatrix} 1 \dots r \\ \rho_1 \dots \rho_r \end{pmatrix}$ and $\begin{pmatrix} 1 \dots s \\ \sigma_1 \dots \sigma_s \end{pmatrix}$ respectively, $\operatorname{sg}\rho$ being the sign of the former permutation.

In this way, we have obtained an equation for the Green function describing the general system of electrons and photons, which equation contains only one value of the indices r , s and, consequently, is sufficient for determining G^{rs}. It is, however, rather complicated because of complexity of the X's and V's. It will be shown now that an appreciable simplification can be achieved if the photon-part of G^{rs} is redefined according to the following relations:

$$G(1) = g(1)$$

$$G(12) = g(12) + g(1)g(2)$$

$$G(123) = g(123) + g(1)g(23) + g(2)g(13) +$$

$$+ g(3)g(12) + g(1)g(2)g(3). \qquad (10)$$

and so on. Here $G(12\ldots S)$ denotes the old Green function $G^{os}(-;-;z_1\ldots z_S)$ and the $g(1\ldots S)$ denotes the new one. These relations are demonstrated graphically for $S=3$ in Fig. 1.

For the g -functions, instead of (6'), the following equations hold:

$$K_{z_1} g\binom{1}{\mu_1} = e\,tr\,\gamma_{\mu_1}(z_1)\,g(\underline{1};\underline{1};-) - e\,\mathcal{J}_{\mu_1}(z_1) \qquad (11)$$

$$K_{z_1} g\binom{1\ \ 2}{\mu_1\ \mu_2} = e\,tr\,\gamma_{\mu_1}(z_1)\,g(\underline{1};\underline{1};{}^{2}_{\mu_2}) + \delta(z_1 - z_2)\,\delta_{\mu_1\mu_2} \qquad (11')$$

$$K_{z_1} g(12\ldots S) = e\,tr\,\gamma_{\mu_1}(z_1)\,g(\underline{1};\underline{1};2\ldots S) \qquad (11'')$$

the last one being valid for s = 3,4,5,... . The function $g(\underline{1},\underline{1};2\ldots S)$ is defined according to Fig. 2. The proof is given in Appendix I.

Instead of ($\mathcal{J}$) and (a) the following relations are valid:

$$\frac{\delta}{\delta \mathcal{J}_{\mu_1}(z_1)}\, g(2\cdots S) = -e\,g(12\cdots S) \qquad (\mathcal{J}')$$

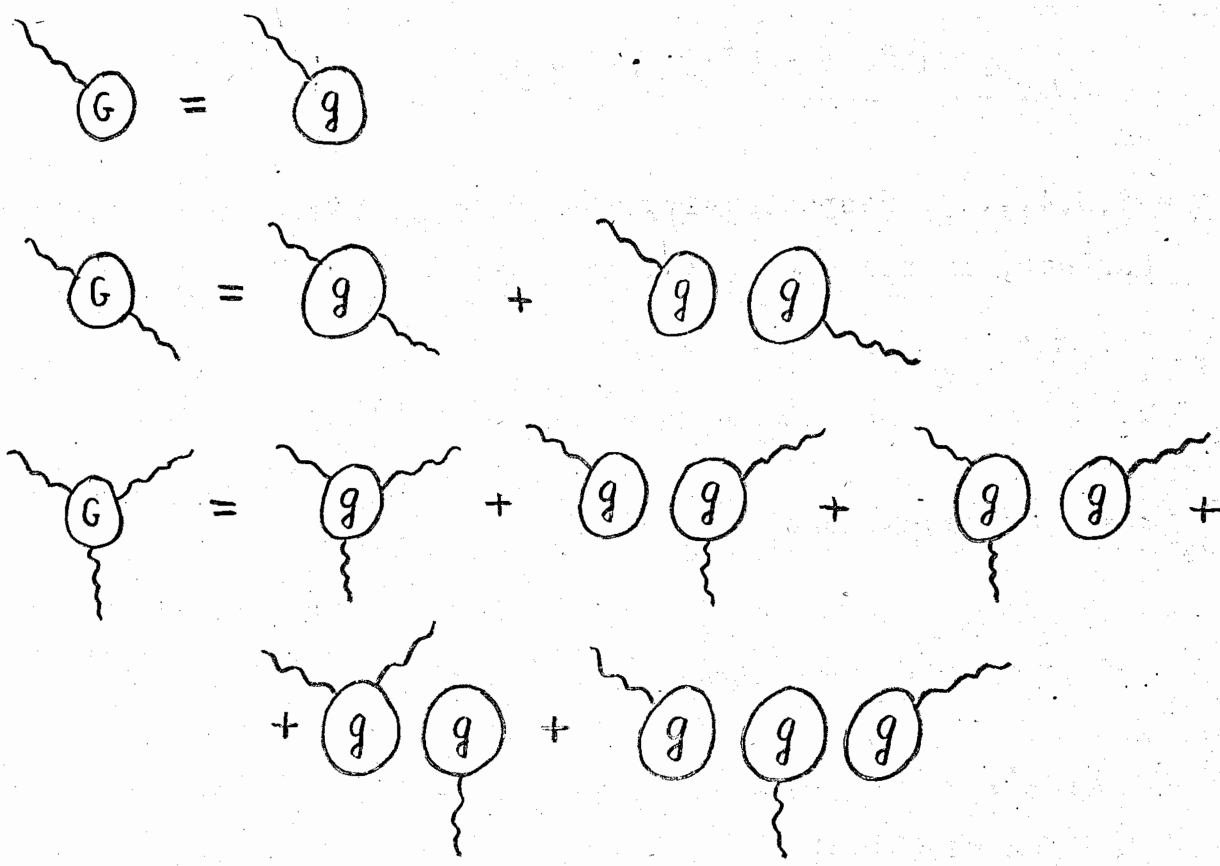

Fig. 1. The first three relations (10). The G -function includes all higher order graphs (radiation corrections) inclusive the cases in which several external lines are not connected one with another. On the other hand, the g -function includes only the connected graphs. The closed vacuum-vacuum diagrams are, of course, excluded in both cases by the factor $\frac{1}{\langle S\rangle_0}$ in (1) and (1').

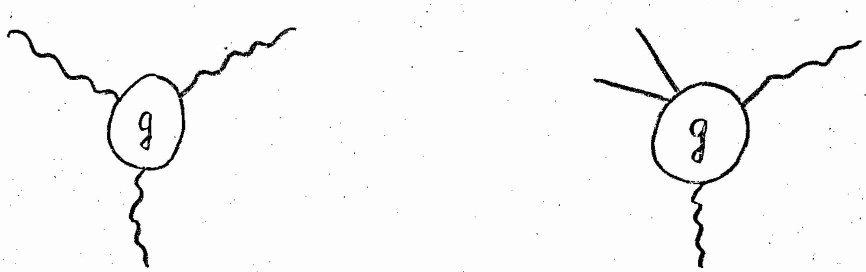

Fig. 2. The $g(1\ldots S)$ -function and the $g(\underline{1};\underline{1};2\ldots S)$ -function for the case s = 3.

and
$$\frac{\delta}{\delta a_{\mu_2}(z_1)} g(2\ldots s) = e\, tr\, \gamma_{\mu_1}(z_1)\, g(\underline{1},\underline{1},2\ldots s) \qquad (a')$$

(s = 2,3,4,...), respectively. The proof is given in Appendix II.

Further, in place of (8'), equation

$$-\frac{1}{e} K_v \frac{\delta}{\delta \mathcal{J}_\lambda(v)} g\underset{\mu}{(1)} = \frac{\delta}{\delta a_\lambda(v)} g\underset{\mu}{(1)} + \delta(v - z_1)\delta_{\lambda\mu} \qquad (12)$$

for s = 1 and equations

$$-\frac{1}{e} K_v \frac{\delta}{\delta \mathcal{J}_\lambda(v)} g(12\ldots s) = \frac{\delta}{\delta a_\lambda(v)} g(12\ldots s) \qquad (12')$$

for s = 2,3,4,... are valid. Therefore, instead of (9), we obtain the following equation:

$$\mathcal{V}_\lambda X_r \ldots X_1 g^{r1} = \delta(v - z_1)\delta_{\lambda\mu} \sum_{P_p} \operatorname{sgp} \delta(x_1 - y_{p_1}) \ldots \delta(x_r - y_{p_r}) \qquad (13)$$

for s = 1 and

$$\mathcal{V}_\lambda X_r \ldots X_1 g^{rs} = 0 \qquad (13')$$

for s = 2,3,4 ... . The operator $\mathcal{V}_\lambda$ has the form

$$\mathcal{V}_\lambda = -\frac{1}{e} K_v \frac{\delta}{\delta \mathcal{J}_\lambda(v)} - \frac{\delta}{\delta a_\lambda(v)}$$

and g^{rs} denotes the Green function the electron part of which is defined by (1) or (1') and the photon part by (10).

5. Conclusion

If equations for one-particle Green functions are generalized to the case of many particles an infinite recursive set of equations arises, which must be solved for finding any single Green function. The system was reduced to the higher-order functional differential equation which contains only one Green function. The Lagrangian used contains two independent classical functions, which have a clear physical interpretation.

As to the redefinition of the Green function, it should be remarked that both G and g are used in literature. From the formal point of view the latter is more useful because all formulae are simpler for it. Physically, it could be considered as not so convenient as the former because it is not symmetrical with respect to electrons and photons. However, this question has not a great meaning because the G 's and the g 's are connected by comparatively simple relations (10).

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Appendix I

Let us prove the equations (11), (11') and (11"). Equations (11) and (11') may be proved immediately by expressing g through the G and inserting them into (6'). Eq. (11") for general $S \geqq 3$ will be verified by showing that the corresponding eq. (6') follows from it.

We shall write down different equations of the type (11") in order to obtain, by summarizing them and using the relation (10), the equation (6'). A product of k Green functions will be called the product of the k-th order.

First of all, the equation for one product of the first order is to be written down:

$$K_{z_1} g(12\ldots S) = e \operatorname{tr} \gamma_{\mu_1}(z_1) g(\underline{1}, \underline{1}, 2 \ldots S).$$

Then, equations for all possible products of the second order will be written:

$$K_{z_1} g(1) g(2\ldots s) = etr\gamma_{\mu_1}(z_1) g(\underline{1};\underline{1};-) g(2\ldots s) - e\mathcal{J}_{\mu_1}(z_1) g(2\ldots s)$$

$$K_{z_1} g(1\sigma) g(2\ldots\sigma^{-1}\ldots s) = etr\gamma_{\mu_1}(z_1) g(\underline{1};\underline{1};\sigma) g(2\ldots\sigma^{-1}\ldots s) + \delta(z_1 - z_\sigma)\delta_{\mu_1\mu_\sigma} g(2\ldots\sigma^{-1}\ldots s)$$

$$K_{z_1} g(1\sigma_1\sigma_2) g(2\ldots\sigma_1^{-1}\ldots\sigma_2^{-1}\ldots s) = etr\gamma_{\mu_1}(z_1) g(\underline{1};\underline{1};\sigma_1\sigma_2) g(2\ldots\sigma_1^{-1}\ldots\sigma_2^{-1}\ldots s)$$

. . . .

(and so on). The second of them represents by itself s - 1 equations for σ = 2,3,4, ..., s and the third one represents $\binom{s-1}{2}$ equations for different combinations of σ_1 , σ_2 . In a similar way all other "second order equations" will be written. Only the first of them contains the term with $\mathcal{J}_\mu(z_1)$ and only the s - 1 second equations contain the term with $\delta(z_1 - z_\sigma)$

Now, we shall write the equations for all products of the third order:

$$K_{z_1} g(1) g(2\ldots s_1) g(s_1'\ldots s) = etr\gamma_{\mu_1}(z_1) g(\underline{1};\underline{1};-) g(2\ldots s_1) g(s_1'\ldots s) -$$

$$- e\mathcal{J}_{\mu_1}(z_1) g(2\ldots s_1) g(s_1'\ldots s)$$

$$K_{z_1} g(1\sigma) g(2\ldots\sigma^{-1}\ldots s_1) g(s_1'\ldots s) = etr\gamma_{\mu_1}(z_1) g(\underline{1};\underline{1};\sigma) g(2\ldots\sigma^{-1}\ldots s_1) g(s_1'\ldots s) +$$

$$+ \delta(z_1 - z_\sigma) g(2\ldots\sigma^{-1}\ldots s_1) g(s_1'\ldots s)$$

and so on. The arguments are to be distributed among the factors in all possible ways (the equivalent possibilities being omitted). Similarly, equations for all higher-order products should be written. The last equation containing $\delta(z_1 - z_\sigma)$ is

$$K_{z_1} g(1\sigma)g(2)\ldots g(\sigma-1)g(\sigma+1)\ldots g(s) =$$

$$=(etr\gamma_{\mu_1}(z_1)g(\underline{1};\underline{1};\sigma)+\delta(z_1-z_\sigma))g(2)\ldots g(\sigma-1)g(\sigma+1)\ldots g(s)\,.$$

The last equation containing $\mathcal{I}_\mu(z_1)$ is simultaneously the last one of all the system and has the form

$$K_{z_1} g(1)g(2)\ldots g(s)=(etr\gamma_{\mu_1}(z_1)g(\underline{1};\underline{1};-)-e\mathcal{I}_{\mu_1}(z_1))g(2)\ldots g(s).$$

Summing all these equations, one obtains of the left

$$K_{z_1} G(12\ldots s).$$

On the right, the first term will be

$$etr\gamma_{\mu_1}(z_1)G(\underline{1};\underline{1},2\ldots s).$$

The terms containing $\mathcal{I}_\mu(z_1)$ give the expression

$$-e\mathcal{I}_{\mu_1}(z_1)\{g(2\ldots s)+\sum g(2\ldots s_1)g(s_1'\ldots s)+$$

$$+\sum g(2\ldots s_1)g(s_1'\ldots s_2)g(s_2'\ldots s)+\sum\ldots+\ldots\}$$

where $\sum$ denotes the summation over all different possibilities in which the arguments 2,3,..., s can be distributed among the factors. Summing one obtains the expression

$$-e\mathcal{I}_\mu(z_1)G(2\ldots s),$$

which is identical with the second term on the right-hand side of (6'). In a similar way, the terms containing $\delta(z_1 - z_\sigma)$ give, for fixed σ,

$$\delta(z_1 - z_\sigma)\{g(2\ldots\sigma^{-1}\ldots s)+\sum g(2\ldots\sigma^{-1}\ldots s_1)g(s_1'\ldots s)+$$

$$+\sum g(2\ldots\sigma^{-1}\ldots s_1)g(s_1'\ldots s_2)g(s_2'\ldots s)+\ldots\}.$$

The expression in the bracket is, obviously, the function $G(2\ldots\sigma^{-1}\ldots s)$ so that summing over $\sigma = 2,3,\ldots,s$ one obtains

$$\sum_{\sigma=2}^{s} \delta(z_1 - z_\sigma)G(2\ldots\sigma^{-1}\ldots s)$$

i.e. an expression which is identical with the third term on the right-hand side of (6').

Appendix II.

The relation (y') can be proved analogically. Let us write down the (y')'s for products of different orders and for different distributions of arguments among the factors:

$$\frac{\delta}{\delta y_{\mu_1}(z_1)}g(2\cdots s) = -e\,g(12\cdots s)$$

$$\frac{\delta}{\delta y_{\mu_1}(z_1)}g(2\cdots s_1)g(s_1'\cdots s) = -e\{g(12\cdots s_1)g(s_1'\cdots s)+g(2\cdots s_1)g(1s_1'\cdots s)\}$$

$$\frac{\delta}{\delta y_{\mu_1}(z_1)}g(2\cdots s_1)g(s_1'\cdots s_2)g(s_2'\cdots s) = -e\{g(12\cdots s_1)g(s_1'\cdots s_2)g(s_2'\cdots s)+$$

$$+g(2\ldots s_1)g(1s_1'\ldots s_2)g(s_2'\ldots s)+g(2\ldots s_1)g(s_1'\ldots s_2)g(1s_2'\ldots s)\}$$

and so on. For each order, all inequivalent possibilities must be written. Summing the left-hand sides one obtains

$$\frac{\delta}{\delta \mathcal{Y}_{\mu_1}(z_1)} G(2\ldots s).$$

On the right, all terms have more by the argument 1 if compared with those on the left. All terms are different one from the other, Consequently, on the right there will be all terms which can be obtained by functional differentiation of all the products

$$g(2\ldots s_1)g(s_1'\ldots s_2)\ldots\ldots g(s_{k-1}'\ldots s_k)g(s_k'\ldots s)$$

of all orders and all different distributions of the arguments. The Green function $G(12\ldots s)$ contains all these terms but, moreover, it contains the terms

$$g(1)g(2\ldots s),\ g(1)g(2\ldots s_1)g(s_1'\ldots s),\ \ldots\ldots,$$

$$g(1)g(2\ldots s_1)g(s_1'\ldots s_2)g(s_2'\ldots s),\ \ldots\ldots, \tag{14}$$

and so on, which cannot be obtained by differentiating any term on the left. All other terms the Green function $G(1\ldots s)$ is made up of are obtainable by this differentiating. The sum of the terms (14) is equal to

$$g(1)G(2\ldots S) = G(1)G(2\ldots S)$$

so that the right-hand side can be expressed as

$$-e\left\{G(1\ldots S) - G(1)G(2\ldots S)\right\}.$$

By this, the relation (γ ') is demonstrated. The proof of (α ') can be performed in a similar way.

The equation (12) follows from (11') by using (γ ') and (α '). Analogously, (12') follows from (11") by using (γ ') and (α '). It should be remarked that if (γ ') and (α ') have been proved it is possible to deduce (12') from (11') without using the (11").

R E F E R E N C E S

|1| Schwinger J., Proc. Nat, Acad. Sci. USA, 37, 452 (1951).

|2| Fradkin E.S., DAN USSR 98, 47 (1954).

|3| Anderson J.L., Phys. Rev., 94, 703 (1954).

|4| Polivanov M.K., DAN USSR 100, 1061 (1955).

|5| Berestetski V.B., Galanin A.D., "Problemy sovremennoi fiziki" 1955, N 3, pag. 1.

|6| Freese E., Zs. f. Naturforschung, 8a, 776 (1953).
Matthews P.T., Salam A., Proc. Roy, Soc., A221, 128 (1954).

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