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INTRODUCTION

The purpose of the present paper is to determine the capture region for the equation of the following form:

$$\frac{d}{dt}(m(\varepsilon,t)\dot{x}) + k(\varepsilon,t)\rho'(x) = \varepsilon f(\varepsilon,t,x,\dot{x})\dot{x}\ , \qquad /*/$$

where ε is a small parameter, which may be considered not negative whereas the functions $m(\varepsilon,t)$ and $k(\varepsilon,t)$ if $\varepsilon = 0$ are independent of t. Let x_e be the equilibrium position for the equation /*/. Then the capture region for a given moment of time t_0 and for a given equilibrium position x_e is understood here as the sets of the values of the initial conditions (x_0 , $\dot{x}_0$), such that the solutions of the equation /*/ determined by them will steadily oscillate with respect to x_e . The problems of finding the capture regions arise in the calculations of the charged particle accelerators of different types from where the term "capture region" is taken. The thing is that the particles accelerated in any resonance accelerator undergo the so-called "phase oscillations" described by a certain non-linear equation, and the capture region for this equation determines the number of particles captured into the acceleration regime. The equation for phase oscillations itself may be interpreted as an equation of type /*/ by introducting the small parameter in a natural way. Thus, the analysis of the equation /*/ may be used for still more exact determination of the capture region in the accelerator x/. From a mathematical point of view a certain analog of the separatrisses is introduced in this paper which makes it possible to separate the motions of different types and the method for the practical determination of these separatrisses is given.

The method applied by the physicists for the solution of this task /see, e.g., [1], [2], [3]/ for case $f(\varepsilon,t,x,\dot{x}) \equiv 0$ consisted in the following: since ε is small then the change of the functions $m(\varepsilon,t)$ and $k(\varepsilon,t)$ for the time of one period of phase oscillations which is of the order εT (T is the period of phase oscillations) is also small and the time dependence was neglected. In such a case the energy integral may be found, with the help of which the capture region may be also easily found. However, since $\rho'(x)$ is a non-linear function, T depends both upon x_0 and $\dot{x}_0$ and infinitely increases when x_0 and $\dot{x}_0$ approach the boundary of the capture region. Thus εT becomes at any fixed $\varepsilon \neq 0$ whatever great. Therefore, the main premise for the speculations of such a kind is not correct.

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x/ One can acquaint oneself with such a calculation in the note written together with Yu. S. Sajasov. This note will be published in JTP.

One may add also that the capture regions for the equation/*/if $E = 0$ and if $E \neq 0$ are essentially different: in the first case it is, generally speaking, limited, in the second it is not.

Yu. S. Sajasov determined the capture region by means of the numerical integration of equation/*/. He has obtained an appreciable increase of the capture region and set a problem of determining the capture region using the analtic method.

The equation /*/ may be interpreted from a mechanical standpoint as an equation describing the motion of the material point of a variable mass under the action of the non-linear elestic force weakly time dependent and the small force of friction. Therefore, the class of physical problems described by the equation is very wide. To make this method applicable it is only necessary that in the fulfillment of certain natural limitations the parameter E characterizing the closeness of the equation /*/ to the conserved would be small.

In conclusion I take the opportunity to express my gratitude to Yu. S. Sajasov who is the chief of this work, to S.V. Fomin and L.A. Chudov for their critical remarks. I am grateful also to V.V. Nemytsky and to the members of his seminar for some useful remarks.

CHAPTER I.

Let us make some remarks concerning the equation /*/ . First of all everywhere further we shall consider that the functions $m(\varepsilon,t)$, $K(\varepsilon,t)$, $\rho'(x)$ and $f(\varepsilon,t,x,\dot{x})$ are continuous and possess the continuous particular derivatives over all the variables in the necessary region of change E , x , $\dot{x}$, t up to the necessary order. In particular, we shall suppose that the theorem of the existence and uniqueness of the solution determined by the initial conditions: $x_\varepsilon(t_o)=x_o$ and $\dot{x}_\varepsilon(t_o)=\dot{x}_o$ is correct. The functions $m(\varepsilon,t)$ and $K(\varepsilon,t)$ will be assumed positive.

It is clear from the kind of the equation /*/ that the nulls of the function $\rho'(x)$ and only they are the equilibrium position. Further we shall suppose that all the nulls of the function $\rho'(x)$ are simple, i.e., if $\rho'(x_e)=0$ then $\rho''(x_e)\neq 0$. It follows from here immediately that all the equilibrium positions are isolated. On depending upon the sign of $\rho''(x_e)$ we shall speak that x_e is the equilibrium position of the saddle type if $\rho''(x_e)<0$, and x_e is the equilibrium position of the focus type if $\rho''(x_e)>0$. For the first case we introduce the notation x_s , for the second x_f . The equilibrium positions of the saddle type take turns in one with the equilibrium positions of the focus type. Further it would be very convenient to interpret any solution of the equation as a curve of the parameter t on the plane / x, $\dot{x}$ /. Let us make some remarks about this interpretation. Let $x_\varepsilon(t)$ be a certain solution of the equation /*/. We shall state that x' is a return point of the solution $x_\varepsilon(t)$, if there be found such a moment of time t' that $x_\varepsilon(t')=x'$ for all t sufficiently close to t' , the difference $x_\varepsilon(t)-x'$ conserves the constant sign. One may obtain from here that $\dot{x}_\varepsilon(t')=0$. Thus, the vanishing of the first derivative is the necessary condition of the return point. One can easily see that it is also sufficient. Really, if $\dot{x}_\varepsilon(t')=0$ then $\ddot{x}_\varepsilon(t')\neq 0$ otherwise $x_\varepsilon(t')$ would be an equilibrium position and we have the solution entering into the finite moment of time into the equilibrium position that is impossible according to the theorem about the uniqueness of the solution. Thus, the sufficient exctremum condition turns out to be fulfilled. One can draw a certain conclusion about the behaviour of the solutions on the phase plane. Namely: if the solution $x_\varepsilon(t)$ in the finite moment of time t' met the straight line $\dot{x}=0$ then it inevitably passes from the half-plane $\dot{x}>0$ to the half-plane $\dot{x}<0$ and vice versa. The latter circumstance depends upon the sign $\rho'(x_\varepsilon(t'))$. Namely, if $\rho'(x_\varepsilon(t'))>0$ then the solution passes from the upper half-plane to the lower one, if $\rho'(x_\varepsilon(t'))<0$ it is vice versa.

It follows immediately between two subsequent return points of the arbitrary solution $x_\varepsilon(t)$ from here that there is an odd number of changes of the function sign $\rho'(x)$ i.e., the odd number of equilibrium positions. The equilibrium positions of the focus type will be always by one more than the positions of the saddle type. The solution $x_\varepsilon(t)$ of the equation /*/ is called ω -oscillating if it has infinitely many return points if $t > 0$. It is easily seen that the moments of time on the axis when the return of motion occurs have no limiting points. Indeed, let us suppose contrary: let t_K be the moment of time in which the K -return of motion of a certain solution $x_\varepsilon(t)$ occurs and $t_K \to t'$ at $K \to \infty$. Then there is at least one equilibrium position between $x_\varepsilon(t_K)$ and $x_\varepsilon(t_{K+1})$ and with K sufficiently great -only one. Let it be x_e. Then it follows from the continuity of the solution that $x_\varepsilon(t') = x_e$ that contradicts the theorem about the uniqueness of the solution. We shall speak that the solution $x_\varepsilon(t)$ oscillates steadily if any its three return points following one another $x_\varepsilon(t_1)$, $x_\varepsilon(t_2)$ and $x_\varepsilon(t_3)$ $(t_1 < t_2 < t_3)$ satisfy the following condition:

the number of the equilibrium positions of the equation which are found between $x_\varepsilon(t_2)$ and $x_\varepsilon(t_3)$ does not exceed that of the equilibrium positions between $x_\varepsilon(t_1)$ and $x_\varepsilon(t_2)$. We shall speak about the ω-oscillating solution of the equation /*/ that it oscillates with respect to the equilibrium position x_e, if one of any its two successive return points is situated to the right from x_e and the other to the left. We shall speak that the solution $x_\varepsilon(t)$ tends to the point x, if $\sup_{\tau \geq t} |x_\varepsilon(\tau) - x| \to 0$ with $t \to \infty$. It easily follows from the above mentioned statements that ω- oscillating solutions may tend only to the equilibrium position of the focus type. Thus, in the equilibrium position of the saddle type the solution may enter only monotonously, i.e. so that from a certain moment of time $|\dot{x}_\varepsilon(t)| > 0$. In this paper we shall always suppose that the solution may tend monotonously only to the equilibrium position of the saddle type. It means first of all that the solution cannot monotonously tend to the point which is not an equilibrium position, secondly, it cannot tend to the equilibrium position of the focus type. The second limitation implies that we exclude the cases of great friction. It is clear that this limitation is not quite natural, however, its neglect considerably changes the results. We intend to devote a separate paper to the consideration of this case.

The first limitation will hold since the cases when it is not valid seem to be of no interest. Now we shall formulate the sufficient conditions for performing the first limitation. These conditions are obtained from the comparison with the analogous conditions for the equation $x''(t) - K(t)x(t) = 0$, where $K(t) \geq 0$. The divergence of the integral $\int_{t_0}^{\infty} t_K(t)dt$ is the necessary and sufficient condition that there exist no solutions tending to the finite limit different from zero(see, [4]). One may reduce the equation $\frac{d}{dt}[m(t)\dot{x}] - K(t)x = 0$ ($(m(t) > 0)$ to the form $\frac{d^2x}{ds^2} - m(t(s))K(t(s))x = 0$ by the substitution $s = \int_{t_0}^{t} \frac{d\tau}{m(\tau)}$ and if the integral $\int_{t_0}^{\infty} \frac{dt}{m(t)}$ is divergent then the divergence of the integral $\int_0^\infty s m(t(s))K(t(s))ds$ is the necessary and sufficient condition for performing the first limitation. This is equal to the divergence of the integral

$$\int_{t_0}^{\infty} K(t)\left(\int_{t_0}^{t} \frac{d\tau}{m(\tau)}\right)dt$$

In case when the integral $\int_{t_0}^{\infty} \frac{d\tau}{m(\tau)}$ is convergent as is shown in /6/ the divergence of the integral

$$\int_{t_0}^{\infty} \frac{1}{m(t)}\left(\int_{t_0}^{t} K(\tau)d\tau\right)dt$$

is the necessary and sufficient condition for performing the first limitation. Summing up all what was said we can state that in order the equation $\frac{d}{dt}[m(t)\dot{x}] - \kappa(t)x = 0$ would satisfy the first limitation it is necessary and sufficient that both integrals $\int_{t_0}^{\infty} \kappa(t)\left(\int_{t_0} \frac{dt}{m(t)}\right)dt$ and $\int_{t_0}^{\infty} \frac{1}{m(t)}\left(\int_{t_0}^{t} \kappa(\tau)d\tau\right)dt$ would be divergent simultaneously.

Comparing the equation /*/ with the equation $\frac{d}{dt}(m(t)\dot{x}) - \kappa(t)x = 0$ we may obtain the sufficient conditions that the equations /*/ would satisfy the first limitation. On the basis of the mean-value theorem one may write

$$\frac{d}{dt}\left(m(\varepsilon,t)\dot{x}\right) + \Big\{\kappa(\varepsilon,t)p''(x_s + \theta(t)(x(t) - x_s)) -$$
$$\varepsilon f'_x\left(\varepsilon,t,x_s + \theta(t)(x(t)-x_s),\ \theta(t)\dot{x}(t)\right)\theta(t)\dot{x}(t)\Big\}(x(t)-x_s) =$$
$$= \varepsilon\Big\{f'_{\dot{x}}(\varepsilon,t,x_s + \theta(t)(x(t)-x_s),\ \theta(t)\dot{x}(t))\theta(t)\dot{x}(t) +$$
$$+ f(\varepsilon,t,x_s + \theta(t)(x(t)-x_s),\ \theta(t)\dot{x}(t)\Big\}\dot{x}(t)$$

where $0 < \theta(t) < 1$. Let for any $c > 0$ and for the sufficiently small fixed $h_o > 0$ there exist the functions $\bar{f}_c(\varepsilon,t)$, $\tilde{f}_c(\varepsilon,t)$ and $\kappa_c(\varepsilon,t) < 0$ such that

$$\kappa_c(\varepsilon,t) \geqslant \kappa(\varepsilon,t)p''(x) - \varepsilon f'_x(\varepsilon,t,x,\dot{x})\dot{x},$$

$$\tilde{f}_c(\varepsilon,t) \geqslant f'_{\dot{x}}(\varepsilon,t,x,\dot{x})\dot{x} + f(\varepsilon,t,\dot{x},\dot{x}) \geqslant \bar{f}_c(\varepsilon,t)$$

at any $|x - x_s| \leq h_o$ and $|\dot{x}| \leqslant c$

whereas both integrals

$$\int_{t_o}^{\infty} \frac{1}{m(\varepsilon,t)\bar{h}_c(\varepsilon,t)}\left(\int_{t_o}^{t} \kappa_c(\varepsilon,\tau)\bar{h}_c(\varepsilon,\tau)d\tau\right)dt$$

and

$$\int_{t_o}^{\infty} \kappa_c(\varepsilon,t)\bar{h}_c(\varepsilon,t)\left(\int_{t_o}^{t} \frac{d\tau}{m(\varepsilon,\tau)\bar{h}_c(\varepsilon,\tau)}\right) dt$$

is divergent if

$$\bar{h}_c(\varepsilon,t) = \exp\left(-\varepsilon\int_{t_o}^{t} \bar{f}_c(\varepsilon,\tau)\,dt\right)$$

and

$$\tilde{h}_c(\varepsilon,t) = \exp\left(-\varepsilon\int_{t_o}^{t} \tilde{f}_c(\varepsilon,\tau)\,dt\right).$$

These conditions are sufficient for the equation /*/ to satisfy the first limitation/cf./5/ /. Further everywhere we shall consider them to be performed and call them the conditions / α /.

Let us denote by E_s the sets of the equilibrium positions of the saddle type of the equation /*/. For every $x_s \in E_s$ let us take all the solutions $x_\varepsilon(t)$ of the equation satisfying the condition $x_\varepsilon(t) \to x_s$ with $t \to \infty$ (the existence of such solutions will be proved further). It follows from the fact that the solution may enter only monotonously into the equilibrium position of the saddle type that for such solutions $\dot{x}_\varepsilon(t) \to 0$ with $t \to \infty$. Let us draw the curve $x = x_\varepsilon(t)$, $\dot{x} = \dot{x}_\varepsilon(t)$ on the plane $(x, \dot{x})$ for every such solution. We obtain a certain family of curves which are, generally speaking, crossing. For a certain fixed moment of time t_o on each curve of our family we take the point $(x = x_\varepsilon(t_o), \dot{x} = \dot{x}_\varepsilon(t_o))$.

Thus we obtain a certain set $\Gamma(t_o)$. If time does not explicitly enter into Eq./*/then $\Gamma(t_o)$ consists of the whole trajectories and is independent of t_o. Otherwise it is wrong. The role of $\Gamma(t_o)$ will be clear from the following Theorem.

Theorem I. Let $\Delta = \tilde{E}^2 \setminus \Gamma(t_o)$, where $\tilde{E}^2$ is the plane $(x, \dot{x})$, from which the points of type $(x_e, 0)$ are taken away. Then there exists the unique representation $\Delta = \bigcup_\alpha \Delta_\alpha$ where each Δ_α is a linearly connected set open in Δ and $\Delta_\alpha \cap \Delta_{\alpha'} = 0$, with $\alpha \neq \alpha'$. Suppose that there exists the point $(x_o, \dot{x}_o) \in \Delta_{\alpha_o}$ such that the solution of the equation /*/ beginning there at the moment of time t_o steadily oscillates with respect to Δ_{α_o} /not with standing to what type of equilibrium positions x_e belongs). Then any other solution of the equation beginning at the moment of time t_o from an arbitrary point belonging to Δ_{α_o} will steadily oscillate with respect to x_e.

To prove this Theorem the following Lemmas are necessary.

Lemma I. Let $\Sigma \subset \Delta$ be a connected set not crossing with the straight line $\dot{x} = 0$ and let x' be the first return point with $t > t_o$ of a certain solution $x_\varepsilon(t)$ of the equation /*/ beginning at the moment of time t_o from the point $(x_o, \dot{x}_o) \in \Sigma$. Then the first with $t > t_o$ return point of the arbitrary solution $x_\varepsilon(t)$ of the equation /*/ beginning at the moment of time t_o from an arbitrary point belonging to Σ is situated between x'_e and x''_e, where x'_e is the first equilibrium position to the left from x', and x''_e is the first from the right.

Proof:

Consider the case when Σ lies in the half-plane $\dot{x} > 0$. The case when Σ lies in the half-plane $\dot{x} < 0$ is considered analogously.

It follows from the definition of the set Σ that any solution $x_\varepsilon(t)$ beginning at the moment of time t_o from the point $(x_o, \dot{x}_o) \in \Sigma$ either has the return point at $t > t_o$ or goes over into the plus infinity. Really, if $\dot{x}_\varepsilon(t)$ vanishes at $t' > t_o$ then $x_\varepsilon(t')$ is a return point, if the $\dot{x}_\varepsilon(t_o)$ does not vanish then $x_\varepsilon(t')$ monotonously goes over into plus infinity. $x_\varepsilon(t)$ cannot tend to some finite value under the assumptions about the behaviour of the solutions of the equation /*/ and in virtue of the choice of the set Σ. Let us denote by Σ' the set of points $\{\sigma \in \Sigma\}$ such that in the solutions emerging from them at the moment of time t_o the first at $t > t_o$ return point lies to the left from x'_e, by Σ'' the set of points such that in the solutions determined by them the first at $t > t_o$ return point lies between x'_e and x''_e, by Σ''' such that either in the solutions determined by them the first at $t > t_o$ return point lies to the right from x''_e or the solution goes over into plus infinity. The latter set may be determined as a set of such $\{\sigma \in \Sigma\}$ that every of the solutions determined by them passes at a finite moment of time t_σ a certain point $x_\sigma > x''_e$ with the velocity different from zero. It follows from the theorem of the continuous dependence of the solution upon the initial conditions for a finite interval of the values t that Σ', Σ'' and Σ''' are open sets.

On the other hand, $\Sigma'' \cap (\Sigma' \cup \Sigma''') = 0$ and $\Sigma'' \cup (\Sigma' \cup \Sigma''') = \Sigma$. According to the definition of the connected set it is possible only when either Σ'' or $(\Sigma' \cup \Sigma''')$ are emply sets. Since $\Sigma'' \neq 0$ then $(\Sigma' \cup \Sigma'') = 0$, thus $\Sigma'' = \Sigma$. Thereby the Lemma is proved.

Lemma 2. Let the connected set K lie on the straight line $\dot{x} = 0$ and involve no equilibrium position and let $t_o(K)$ is a continuous function for K . Let X' be the first at $t > t_o(K_o)$ return point of a certain solution $X_\varepsilon(t)$ beginning at the moment of time $t_o(K_o)$ from the point $K_o \in K$. Then the first with $t > t_o(K)$ return point of the arbitrary solution $\tilde{X}_\varepsilon(t)$ beginning at the moment of time $t_o(K)$ from the point $K \in K$ is situated between x'_e and x''_e , where x'_e is the first to the left from X' equilibrium position, whereas x''_e is the first one to the right.

The proof of this Lemma is analogous to that of the preceding one and therefore it is omitted.

The proof of the Theorem.

Let $(x_1, \dot{x}_1)$ be an arbitrary point belonging to Δ_{α_o}. Denote by $f(\xi) = \{x = f_1(\xi), \dot{x} = f_2(\xi)\}$ mapping the continuous of the segment $[0,1]$ into the set Δ_{α_o} such that $f(0) = (x_o, \dot{x}_o)$ and $f(1) = (x_1, \dot{x}_1)$. Denote by G the set $\{\xi \in [0,1]\}$ such that the solution of the equation /*/ beginning at the moment of time t_o in the point $f(\xi)$ will steadily oscillate with respect to x_e . We shall show that G is open in the segment $[0,1]$. Let ξ' be an arbitrary point beloning to G . Let further $f(\xi') = (x', \dot{x}')$. If $\dot{x}' \neq 0$ then take $\delta > 0$ so small that the image of the interval $(\xi' - \delta, \xi' + \delta)$ in the mapping would not cross with the straight line $\dot{x} = 0$. It can be done in virtue of the continuity of $f(\xi)$. If $\dot{x}' = 0$ then denoting by t_1 the moment of time at which there occurs the first at $t > t_o$ return of motion of the solution $x_\varepsilon(t)$ beginning in the point $f(\xi)$ at the moment of time t_o let us take $t'_o < t_1$, but greater than t_o . Then $\dot{x}_\varepsilon(t'_o) \neq 0$. According to the Theorem about the continuous dependence of the solution upon the initial conditions for the finite interval of values t one may choose $\delta' > 0$ so small that the sign of the first derivative in the point t'_o of any solution beginning at the moment of time t_o from an arbitrary point belonging to $f((\xi' - \delta', \xi' + \delta'))$ would coincide with the sign $\dot{x}_\varepsilon(t'_o)$. Then the case $\dot{x}' = 0$ will be reduced to the case $\dot{x}' \neq 0$ if substitute t_o for t'_o . Applying Lemma I we obtain that the first at $t > t_o$ return point of any solution beginning at the moment of time t_o from an arbitrary point of the set $\Sigma = f((\xi' - \delta, \xi' + \delta))$ lies between the same two equilibrium positions as well as the first at $t > t_o$ return point of the solution beginning in the point $f(\xi')$. Let $t_1(\xi)$ be equal to the moment of time at which there occurs the first at $t > t_o$ return of motion of the solution $x_\varepsilon(t, \xi)$ beginning in the point $f(\xi)$ where $\xi \in (\xi' - \delta, \xi' + \delta)$. It follows from the Theorem about the continuous dependence of the solution upon the initial conditions that $t_1(\xi)$ is continuous.

Thus, $t_1(\xi)$ and $K_1 = \{x_\varepsilon(t_1(\xi), \xi), 0\}$ satisfy the conditions of Lemma 2. Applying Lemma 2 we may determine the continuous function $t_2(\xi)$ by putting it equal to the moment of time at which there occurs the first at $t > t_1(\xi)$ return of motion of the solution $x_\varepsilon(t, \xi)$, and the set $K_2 = \{x_\varepsilon(t_2(\xi), \xi), 0\}$, which, evidently, satisfy the conditions of Lemma2 . Now it

is clear how to determine $f_{n+1}(\xi)$ and K_{n+1} if the function $f_n(\xi)$ and the sets K_n are available. Thus, we get the mutually-digit consistence between the return points of the solution $x_\varepsilon(t,\xi')$ and any other solution $x_\varepsilon(t,\xi)$, where $\xi\in(\xi'-\delta,\ \xi'+\delta)$, such that the K-return point of the arbitrary solution $x_\varepsilon(t,\xi)$ lies between the same two equilibrium positions as well as the K return point of the solution $x_\varepsilon(t,\xi')$. Since $x_\varepsilon(t,\xi')$ oscillates steadily with respect to x_e under the assumption, then $x_\varepsilon(t,\xi)$ for $\xi\in(\xi'-\delta,\ \xi'+\delta)$ will oscillate steadily, i.e., G is open in $[0,I]$. As is known([8] , page I30), any open set on the straight line is a combinaticr of not more than a numerable set of the noncrossing intervals by pairs. Thus, G is a combination of not more than a numerable set of the non-crossing intervals by pairs and of not more than two half-intervals which have no common points either with the intervals or with each other. There are no segments among the components G since in this case $G=[0,1]$ and everything is proved. To finish the proof we shall show that boundary points belonging to any interval or half-interval entering into G also belong to G . Let g_{i_0} be an arbitrary interval or half-interval entering into G and ξ_0 is one of its boundary points, for definiteness we shall consider it the left one. If it is necessary one may choose $\delta>0$ so small by substituting t_0 for $t_0'>t_0$ that firstly $(\xi_0,\xi_0+\delta)<g_{i_0}$, thatndly that the connected set $\Sigma=\{x_\varepsilon(t_0',\xi),\dot{x}_\varepsilon(t_0',\xi)\}$ would not cross with the straight line $\dot{x}=0$, i.e., would lie either in the half-plane $\dot{x}>0$ or in the half-plane $\dot{x}<0$. Let us take $\xi'=\xi_0+\frac{\delta}{2}$. Then the solution $x_\varepsilon(t,\xi')$ will steadily oscillate with respect to x_e . Having repeated almost word for word the first part of the proof we can easily obtain that $x_\varepsilon(t,\xi_0)$ oscillates steadily with respect to x_e
Now it is not difficult to show that $G=[0,1]$. Suppose contrary : let $G\neq[0,1]$. Since G is open and $(\xi=0)\in G$ then there exists the maximum half-interval g_{i_1} containing the point $(\xi=0)$. $g_{i_1}=[0,\bar{\xi})$. Since together with any half-interval $g_i\in G$ its boundary points belong to G then $[0,\bar{\xi}]\subset G$. Therefore, $\bar{\xi}<1$. But since G is open in $[0,I]$ then such $\delta>0$ will be found that $[0,\bar{\xi}+\delta)\subset G$. Therefore, g_{i_1} is not the maximum half-interval containing the point $(\xi=0)$. The obtained contradiction completes the proof of our Theorem.

C H A P T E R 2.

After all what was said we proceed to the finding of the way for calculating $\Gamma(t_0)$. For this it is necessary to find the way for finding the solutions of the equation /*/ satisfying the condition $x_\varepsilon(t)\to x_s$ with $t\to\infty$:
We shall try to find such solutions expanding them in a series by the powers of the small parameter E :

$$x_\varepsilon(t)=\bar{x}_0(t)+\varepsilon\bar{x}_1(t)+\frac{\varepsilon^2}{2!}\bar{x}_2(t)+\cdots \qquad /2.I/$$

The equality /2.I/ is, generally speaking, divergent. Besides, the calculation of $\bar{x}_K(t)$ is rather clumsy, and we may, in fact, calculate only a certain small number of the terms of a series /2.I/. Assuming

$$x_\varepsilon^{(n)}(t)=\sum_{K=0}^{n}\bar{x}_K(t)\frac{\varepsilon^K}{K!} \qquad /2.2/$$

we must evaluate

$$R_n(t,\varepsilon) = x_\varepsilon(t) - x_\varepsilon^{(n)}(t).$$

As is well-known from the analysis

$$R_n(t,\varepsilon) = \left.\frac{\partial^{n+1} x_{\bar\varepsilon}(t)}{\partial \bar\varepsilon^{n+1}}\right|_{\bar\varepsilon=\theta\varepsilon} \frac{\varepsilon^{n+1}}{(n+1)!}, \quad where \quad 0<\theta<1$$

The validity of the expression /2.2/ for the calculations will follow from $|R_n(t,\theta)| \leq N_n(t)\varepsilon^{n+1}$... it will be shown further where $N_n(t)$ is limited when $t \geqslant t_0$ and $\lim_{t\to\infty} N_n(t) = 0$. Therefore, at the sufficiently small ε the term $|R_n(t,\varepsilon)|$ may be made whatever small and, hence, expression /2. 2/ whatever slightly is different from the exact solution. In order to prove this fact we shall prove that there exist the derivatives over ε up to the necessary order in the solutions satisfying the condition $x_\varepsilon(t) \to x_s$ with $t \to \infty$. We intend to evaluate their behaviour for great t. For this it is necessary to be able to distinguish the solutions of the equation/*/tending at $t \to \infty$ to the same equilibrium position of the saddle type, i.e. to find the conditions of uniqueness. The following Theorem will present these conditions.

Theorem 2. Let the expression $k(\varepsilon,t)p''(x) - \varepsilon f_x'(\varepsilon,t,x,\dot x)\dot x < 0$ with $t \geqslant t_0$ for $-\infty < \dot x < +\infty$, $0 \leq \varepsilon \leq \varepsilon_0$ and $(x - x_s) \leqslant h_0$. It can be easily seen that h_0 must be so small that the segment $[x_s - h_0, x_s + h_0]$ would involve not a single equilibrium position different from x_s. Then the solution of the equation/*/satisfying the conditions $x_\varepsilon(t_0) = x_0$ where $|x_0 - x_s| < h_0$, $\operatorname{sign} \dot x_\varepsilon(t) = \operatorname{sign}(x_s - x_0)$ with $t \geqslant t_0$ and $x_\varepsilon(t) \to x_s$ with $t \to \infty$ is unique.

Proof:

Suppose contrary: let $\bar x_\varepsilon(t)$ and $\tilde x_\varepsilon(t)$ be the solutions of the equation /*/ satisfying the conditions of the Theorem. Then $\xi(t) = \bar x_\varepsilon(t) - \tilde x_\varepsilon(t)$ vanishes if $t = t_0$ and tends to zero if $t \to \infty$. It follows from here that $\xi(t)$ in a certain point $t_1 > t_0$ has either the positive maximum or the negative minimum, i.e., $\dot\xi(t_1) = \dot{\bar x}_\varepsilon(t_1) - \dot{\tilde x}_\varepsilon(t_1) = 0$. Substituting first $\bar x_\varepsilon(t)$ then $\tilde x_\varepsilon(t)$ into the equation /*/ and subtracting the second expression from the first one we get at $t = t_1$:

$$m(\varepsilon,t_1)\ddot\xi(t_1) + \Big\{k(\varepsilon,t_1)p''(\tilde x_\varepsilon(t_1) + \theta\xi(t_1)) -$$
$$- \varepsilon f_x'(\varepsilon,t_1,\ \tilde x_\varepsilon(t_1) + \theta\dot\xi(t_1),\ \dot{\tilde x}_c(t_1))\dot{\tilde x}_\varepsilon(t_1)\Big\}\xi(t_1) = 0,$$

where $\theta < \theta < 1$. Because sign $\dot{\bar x}_\varepsilon(t) = \operatorname{sign}\dot{\tilde x}_\varepsilon(t) = \operatorname{sign}(x_s - x_0)$ with $t \geqslant t_0$ then $|x_s - \tilde x_\varepsilon(t_1) - \theta\xi(t_1)| < h_0$ and, hence, $k(\varepsilon,t_1)p''(\tilde x_\varepsilon(t_1) + \theta\xi(t_1)) -$

$$- \varepsilon f_x'(\varepsilon,t_1,\ \tilde x_\varepsilon(t_1) + \theta\xi(t_1),\ \dot{\tilde x}_\varepsilon(t_1))\dot{\tilde x}_\varepsilon(t_1) < 0$$

Thus, we have

$$\ddot\xi(t_1) - \omega^2(t_1)\xi(t_1) = 0, \qquad /2.3/$$

where $\omega^2(t_1) > 0$ and $\xi(t_1) \neq 0$. From here we immediately pass to the contradiction since if $\xi(t_1) > 0$, then $\ddot\xi(t_1) \leq 0$ if $\xi(t_1) < 0$ then $\ddot\xi(t_1) \geqslant 0$. Both these possibilities contradict equation /2.3/. The obtained contradiction proves our assertion. Everywhere further we shall consider that Theorem 2 is correct for any t_0 . Now we shall be concerned with the problem about the existence of the solutions $x_\varepsilon(t)$ satisfying the conditions $x_\varepsilon(t_0) = x_0$, $x_\varepsilon(t) \to x_s$

if $t \to \infty$ and $\operatorname{sign} \dot{x}_\varepsilon(t) = \operatorname{Sign}(x_5 - x_0)$ for $t \geqslant t_0$. We shall call these conditions /A/. The problem about the existence of the solutions satisfying the conditions /A/ has been considered in /5/. However, we approach it from a different standpoint and the result which we shall obtain will be somewhat different from the results of /5/. Taking the point x_0 $(|x_0 - x_5| < h_0)$ we denote by L_α the arc of the parabola $\dot{x} = \alpha \operatorname{Sign}(x_5 - x_0)\sqrt{|x - x_0|}$ connecting the points $(x = x_0,\ \dot{x} = 0)$ and $(x = x_5,\ \dot{x} = \alpha \operatorname{sign}(x_5 - x_0)\sqrt{|x_5 - x_0|})$ by $\alpha > 0$.

Theorem 3. In the set L_α there exists the point from which the solution satisfying the conditions /A/ emerges when $t = t_0$.

Proof:

Denote by L'_α the set of the points of the arc L_α such that

The solutions beginning in them at $t = t_0$ would cross the straight line $x = x_5$ making not a single return of motion before this at $t \geqslant t_0$. Denote by L''_α the sets of points of the arc L_α such that the solutions beginning in them at $t = t_0$, not reaching the straight line $x = x_5$ at $t \geqslant t_0$ have the return point. It follows from the theorem of continuous dependence of the solution upon the initial conditions that L'_α and L''_α are open sets, neither of them is empty /the point $(x = x_0,\ x' = 0)$ belongs to L''_α, whereas the point $(x = x_5, \dot{x} = \alpha \operatorname{Sign}(x_5 - x_0)\sqrt{|x_5 - x_0|})$ belongs to L'_α and their intersection is equal to zero. Therefore, if one assumes that $L'_\alpha \cup L''_\alpha = L_\alpha$, then we immediately obtain the contradiction with the connectedness of the segment. Therefore, there exists at least one point, which does not belong to $L'_\alpha \cup L''_\alpha$. The solution emerging from it at $t = t_0$ must monotously tend to a certain point confined between x_0 and x_5. Since there are no equilibrium positions between x_0 and x_5 and we assumed that there were no solutions entering monotonously into the point different from the equilibrium position then our solution enters into x_5. Thereby the Theorem is proved.

Further we shall be in need of the following Lemma.

Lemma 3. Let $\bar{x}_\varepsilon(t)$ and $\tilde{x}_\varepsilon(t)$ be two different solutions of the equation /*/ satisfying the conditions (A) and let at $t \geqslant t_0$ $|\bar{x}_\varepsilon(t) - x_5| \leqslant h_0$ and $|\tilde{x}_\varepsilon(t) - x_5| \leqslant h_0$ where h_0 so small that at $t \geqslant t_0$ the conditions of Theorem 2 are fulfilled. Then at any $t \geqslant t_0$ the expression

$$\{\bar{x}_5(t) - \tilde{x}_\varepsilon(t)\}\{\dot{\bar{x}}_\varepsilon(t) - \dot{\tilde{x}}_\varepsilon(t)\} < 0 \qquad /2.4/$$

Proof :

It follows from Theorem 2 that at any $t \geqslant t_0$ the expression $\bar{x}_\varepsilon(t) - \tilde{x}_\varepsilon(t) \neq 0$ if at a certain $t_1 \geqslant t_0$ $\bar{x}_\varepsilon(t_1) = \tilde{x}_\varepsilon(t_1)$ then according to Theorem 2 and $\dot{\bar{x}}_\varepsilon(t_1) = \dot{\tilde{x}}_\varepsilon(t_1)$ that contradicts our assumption that $\bar{x}_\varepsilon(t)$ and $\tilde{x}_\varepsilon(t)$ are different solutions since otherwise it would contradict the Theorem about the uniqueness of the solution. Suppose now contrary : let at $t = t'$ equation /2.4/ be not negative. Without restricting the community we shall consider that $\bar{x}_\varepsilon(t') - \tilde{x}_\varepsilon(t') > 0$. Then $\dot{\bar{x}}_\varepsilon(t') - \dot{\tilde{x}}_\varepsilon(t') \geqslant 0$. Let us consider the case $\dot{\bar{x}}_\varepsilon(t') - \dot{\tilde{x}}_\varepsilon(t') = 0$. Then with $t > t'$ but sufficiently close to t' $\dot{\bar{x}}_\varepsilon(t) - \dot{\tilde{x}}_\varepsilon(t) > 0$ since if $\dot{\bar{x}}_\varepsilon(t) - \dot{\tilde{x}}_\varepsilon(t) \leqslant 0$ then $\ddot{\bar{x}}_\varepsilon(t) - \ddot{\tilde{x}}_\varepsilon(t) \leqslant 0$ that contradicts equation /2.3//see, the proof of Theorem 2/.

Therefore, assuming for t' somewhat later moment of time we reduce the case $\dot{\bar{x}}_\varepsilon(t')-\dot{\tilde{x}}_\varepsilon(t')=0$ to the case $\dot{\bar{x}}_\varepsilon(t')-\dot{\tilde{x}}_\varepsilon(t')>0$. We shall show that it is also impossible. Really, $\xi(t)=\bar{x}_\varepsilon(t)-\tilde{x}_\varepsilon(t)$ satisfy the following conditions: $\xi(t')>0$, $\dot{\xi}(t')>0$ and $\xi(t)\to 0$ at $t\to\infty$. Therefore, $\xi(t)$ possess the positive maximum what is impossible. We could have already become convinced in this when proving Theorem 2. Thereby the Lemma is proved.

Corollary I. Since any two points situating on the same arc L_α satisfy the condition $(x_1-x_2)(\dot{x}_1-\dot{x}_2)>0$ at any $\alpha>0$, then the set $L_\alpha\setminus(L'_\alpha\cup L''_\alpha)$ consists of the only point at any t_o and at any $\alpha>0$.

The following theorem will be useful further.

Theorem 4. Let equation /*/ if x_o and t_o are chosen possess the unique solution $x_\varepsilon(t)$ satisfying the conditions /A/. Then this solution is continuously dependent upon ε, i.e., $x_\varepsilon(t)\to x_{\varepsilon_o}(t)$ and $\dot{x}_\varepsilon(t)\to\dot{x}_{\varepsilon_o}(t)$ at $\varepsilon\to\varepsilon_o$ uniformly by t on every limited set of the values t.

The following lemma is required to prove this theorem.

Lemma 4. Let equation /*/ at the chosen x_o and t_o possess the unique solution $x_\varepsilon(t)$ satisfying the conditions /A/. Then any solution $\tilde{\tilde{x}}_\varepsilon(t)$ of the equation /*/ emerging at $t=t_o$ from the point $(x_o,\dot{x}_o)$ if $\dfrac{\dot{x}_o}{\dot{x}_\varepsilon(t_o)}>1$ would cross the straight line $x=x_s$, without making at $t\geqslant t_o$ a single return, whereas at $1>\dfrac{\dot{x}_o}{\dot{x}_\varepsilon(t_o)}>0$, not reaching at $t\geqslant t_o$ the straight line $x=x_s$ it will have a return point.

Proof:

Let us give the proof for case $\dfrac{\dot{x}_o}{\dot{x}_\varepsilon(t_o)}>1$.

The case $1>\dfrac{\dot{x}_o}{\dot{x}_\varepsilon(t_o)}>0$ is proved analogously. Suppose contrary: let $\dot{x}_o\left(\dfrac{\dot{x}_o}{\dot{x}_\varepsilon(t_o)}>1\right)$ be found and such that the corresponding solution $\tilde{\tilde{x}}_\varepsilon(t)$ not reaching at $t\geqslant t_o$ the straight line $x=x_s$ will have the return point. Let L_α, as earlier be an arc of the parabula $\dot{x}=\alpha\,\mathrm{Sign}(x_s-x_o)\sqrt{|x-x_o|}$ which connects the points $(x=x_s,\ \dot{x}=\alpha\,\mathrm{Sign}(x_s-x_o)\sqrt{|x_s-x_o|}$ and $(x=x_o,\ \dot{x}=0)$. Let t'_α be the least at $t>t_o$ moment of time, when the solution $x_\varepsilon(t)$ would cross L_α. It can be easily seen that taking $\alpha>0$ sufficiently great one may get that $\dfrac{\dot{x}_o}{\dot{x}_\varepsilon(t)}$ at $t\in[t_o,t'_\alpha]$ would be > I, whereas the solution $\tilde{x}_\varepsilon(t)$ satisfying the conditions:

$$\tilde{x}_\varepsilon(t'_\alpha)=x_o-\left(\frac{\dot{x}_o}{\alpha}\right)^2\mathrm{Sign}(x_s-x_o),\quad \dot{\tilde{x}}_\varepsilon(t'_\alpha)=\dot{x}_o$$

not reaching at $t\geqslant t'_\alpha$ the straight line $x=x_s$ will have a return point. Let L'_α be a part of the arc L_α, confined between the points $(x=x_s,\ \dot{x}=\alpha\,\mathrm{Sign}(x_s-x_o)\sqrt{|x_s-x_o|}$ and $(x=x_o-\left(\frac{\dot{x}_o}{\alpha}\right)^2\mathrm{Sign}(x_s-x_o),\ \dot{x}=\dot{x}_o$. It can be easily seen that the point $(x_\varepsilon(t'_\alpha),\dot{x}_\varepsilon(t'_\alpha))$ does not belong to L'_α. Applying the considerations of Theorem 3 to L'_α we may easily prove the existence of the solution $x_\varepsilon(t)$ satisfying the conditions /A/ and different from $x_\varepsilon(t)$ that contradicts Corollary I. The obtained contradiction proves the Lemma.

Proof of the Theorem :

For proof it is suffucient to prove that $\dot{x}_\varepsilon(t_o)\to\dot{x}_{\varepsilon_o}(t_o)$ with $\varepsilon\to\varepsilon_o$. We shall try to prove it. Suppose the contrary: let $|\dot{x}_{\varepsilon_k}(t_o)-\dot{x}_{\varepsilon_o}(t_o)|$ remains more than a certain $\Delta>0$

for a certain sequence $\varepsilon_K \to \varepsilon_o$ with $K \to \infty$ and let $\frac{\dot{x}_{\varepsilon_K}(t_o)}{\dot{x}_{\varepsilon_o}(t_o)} > 1+\bar{\Delta}\,(\bar{\Delta}>0)$ /the case when $0<\frac{\dot{x}_{\varepsilon_K}(t_o)}{\dot{x}_{\varepsilon_o}(t_o)}<1-\bar{\Delta}$ is proved in quite analogously./ Let $\dot{x}_o$ be an arbitrary point, such that $1+\bar{\Delta} > \frac{\dot{x}_o}{\dot{x}_{\varepsilon_o}(t_o)} > 1$. Denote by $\bar{x}_{\varepsilon_K}(t)$ the solution of the equation with $\varepsilon = \varepsilon_K$ satisfying the conditions $\bar{x}_{\varepsilon_K}(t_o)=x_o,\ \dot{\bar{x}}_{\varepsilon_K}(t_o)=\dot{x}_o$. Then the solution $\bar{x}_{\varepsilon_o}(t)$ would cross the straight line $x=x_5$, without making at $t \geqslant t_o$ a single return of motion, since $\frac{\dot{\bar{x}}_{\varepsilon_o}(t_o)}{\dot{x}_{\varepsilon_o}(t_o)} = \frac{\dot{x}_o}{\dot{x}_{\varepsilon_o}(t_o)} > 1$. Therefore according to the conventional Theorem about the continuous dependence of the solution upon the initial conditions all $\bar{x}_{\varepsilon_K}(t)$ with K greater than a certain k_0 would cross the straight line $x=x_5$ without making a single return of motion. But on the basis of Lemma 4 all $\bar{x}_{\varepsilon_K}(t)$ not reaching at $t > t_o'$ the straight line $x=x_5$ have the return point since $0<\frac{\dot{\bar{x}}_{\varepsilon_K}(t_o)}{\dot{x}_{\varepsilon_K}(t_o)}<1$. The obtained contradiction proves the Theorem.

Let L_α as earlier, denotes the arc of the parabola connecting the points $(x=x_o,\ \dot{x}=0)$ and $(x=x_5,\ \dot{x}=\alpha\, \mathrm{sign}(x_5-x_o)\sqrt{|x_5-x_o|})$ $(\alpha>0,\ |x_o-x_5|<h_o)$.

Let us take the straight line $x=x_o$. Then for any t_o determine t_o' as a moment of time at which the solution $x_\varepsilon(t)$ emerging at $t=t_o$ from a certain point lying on L_α and satisfying the conditions /A/, would cross the straight line $x=x_o$. It is evident that $t_o'<t_o$. Since according to Corollary I for any t_o there exists only one solution emerging with $t=t_o$ from a certain point of the arc L_α and satisfying the conditions /A/, then the function $t_o'=f_\alpha(t_o)$ is one valued.

Denote the point on the arc L_α by $(x(t_o),\dot{x}(t_o))$ from which the solution satisfying the conditions /A/ emerges with $t=t_o$. Then it is easy to prove by means of the considerations analogous to those given for the prove of Theorem 4 that the point $(x(t_o),\dot{x}(t_o))$ continuously depends upon t_o . From here making use of the Theorem about the continuous dependence of the solution upon the initial conditions we obtained that the solution $x_\varepsilon(t)$ of the equation /*/ emerging with $t=t_o+\Delta t_o$ from the point $(x_\varepsilon(t_o+\Delta t_o),\ \dot{x}_\varepsilon(t_o+\Delta t_o))$ would cross the straight line $x=x_o$ at the moment of time continuously dependent upon Δt_o , i.e., we obtain that $f_\alpha(t_o)$ is a continuous function. Since $f_\alpha(t_o)<t_o$ then $f_\alpha(t_o)\to-\infty$ with $t\to-\infty$. Show that $f_\alpha(t_o)\to+\infty$ with $t_o\to+\infty$. Thereby it will be shown that the set of the values of the function $f_\alpha(t_o)$ is all the numerical straight line i.e., it will be proved that equation/*/at any t_o' and x_o sufficiently close to x_5 possesses the solution satisfying the following conditions : $x_\varepsilon(t_o)=x_o,\ x_\varepsilon(t)\to x_5$ with $t\to\infty$ and $\mathrm{sign}\,\dot{x}_\varepsilon(t)=\mathrm{sign}(x_5-x_o)$ at any $t \geqslant t_o$. It follows from Theorem 2 that this solution is unique.

To prove this fact suppose contrary: let $f_\alpha(t_o)$ be limited from above and let $\tilde{t}_o'$ be an exact upper edge. Show that $f_\alpha(t_o)$ fails to reach its exact upper edge. Suppose contrary: let $f_\alpha(t_o)$ reach its upper edge $\tilde{t}_o'$ in a certain point $\tilde{t}_o$. This means that at any $t_o'>\tilde{t}_o'$ and $\dot{x}_o$ the solution $\hat{x}_\varepsilon(t)$ satisfying the oonditions $\hat{x}_\varepsilon(t_o')=x_o$ and $\dot{\hat{x}}_\varepsilon(t_o')=\dot{x}_o$ does not satisfy the conditions /A/. Show that it is not so.

Let $x_\varepsilon(t)$ be a solution of the equation /*/ satisfying the conditions /A/ and the condition $x_\varepsilon(\tilde{t}_o')=x_o$. Denote by $\bar{x}_\varepsilon(t)$ the solution of the equation /*/ satisfying the conditions $\bar{x}_\varepsilon(\tilde{t}_o')=x_o$, $\dot{\bar{x}}_\varepsilon(\tilde{t}_o')=\frac{3}{2}\dot{x}_\varepsilon(\tilde{t}_o')$; and the solution satisfying the conditions $\bar{\bar{x}}_\varepsilon(\tilde{t}_o')=x_o$,

$\bar{\bar{x}}_\varepsilon(\tilde{t}_o') = x_o$, $\dot{\bar{\bar{x}}}_\varepsilon(t_o') = \frac{1}{2}\dot{x}_\varepsilon(\tilde{t}_o')$ by $\bar{\bar{x}}_\varepsilon(t)$.

Then there exists $\Delta t > 0$ such that the solution $\tilde{x}_\varepsilon(t)$ satisfying the conditions $\tilde{x}_\varepsilon(\tilde{t}_o'+\Delta t) = x_o$ and $\dot{\tilde{x}}_\varepsilon(t_o'+\Delta t) = \frac{3}{2}\dot{x}_\varepsilon(t_o)$ would cross the straight line $x = x_s$ without making at $t \geqslant t_o' + \Delta t$ a single return of motion, whereas the solution $\tilde{\tilde{x}}_\varepsilon(t)$ satisfying the conditions $\tilde{\tilde{x}}_\varepsilon(\tilde{t}_o'+\Delta t) = x_o$ and $\dot{\tilde{\tilde{x}}}_\varepsilon(\tilde{t}_o') = \frac{1}{2}\dot{x}_\varepsilon(\tilde{t}_o')$ not reaching at $t \geqslant \tilde{t}_o' + \Delta t$ the straight line $x = x_s$ will have the return point. Then by the considerations like those given when proving Theorem 3 it is easy to prove that there exists the point $\hat{\hat{x}}$ confined between $\frac{1}{2}\dot{x}_\varepsilon(\tilde{t}_o')$ and $\frac{3}{2}\dot{x}_\varepsilon(\tilde{t}_o')$ such that the solution emerging from the point $(x_o, \hat{\hat{x}})$ at $t = \tilde{t}_o' + \Delta t$ satisfies the conditions/A/. The obtained contradiction shows that $f_\alpha(t_o)$ fails to reach its exact upper edge. Let now $t_K' < \tilde{t}_o'$ and tend to $\tilde{t}_o'$ with $K \to \infty$. Denote by T_K the set of the values t_o such that $f_\alpha(t_o) = t_K'$. Since $\tilde{t}_o'$ is the exact upper edge then no T_K is empty and since any $t_o \in T_K$ satisfies the condition $t_o > t_K'$ then there exists the final $t_K = \inf t_o \ (t_o \in T_K)$ .. Since $f_\alpha(t_o)$ is a continuous function then $f_\alpha(t_K) = t_K'$ and since $f_\alpha(t_o)$ fails to reach its upper edge then $t_K \to \infty$ with $K \to \infty$. Consequently, $t_K - t_K' > t_K - \tilde{t}_o' \to \infty$ with $K \to \infty$. It can be easily seen that t_K is the first at $t \geqslant t_K'$ moment of time, when the corresponding solution emerging at $t = t_K'$ from the point lying on the straight line $x = x_o$ will be incident on the arc L_α , i.e., at $t \in [t_K', t_K]$ $\dot{x}_\varepsilon^{(K)}(t) > \alpha\sqrt{|x - x_o|}$ if $x_o < x_s$, $\dot{x}_\varepsilon^{(K)} < -\alpha\sqrt{|x-x_o|}$ if $x_o > x_s$ i.e., $dt < \dfrac{dx}{\alpha\sqrt{|x_o - x|}}$ if $x_o < x_s$ and $dt < -\dfrac{dx}{\alpha\sqrt{|x_o - x|}}$, if $x_o > x_s$. By integrating we obtain that $\displaystyle\int_{x_o}^{x_\varepsilon^{(K)}(t_K)} \frac{d\xi}{\alpha\sqrt{|x_o - \xi|}} > t_K - t_K'$ if $x_o < x_s$ and $\displaystyle -\int_{x_o}^{x_\varepsilon^{(K)}(t_K)} \frac{d\xi}{\alpha\sqrt{|x_o - \xi|}} > t_K - t_K'$ if $x_o > x_s$. Since $x_\varepsilon^{(K)}(t_K)$ is confined in any of the two cases between x_o and x_s and the integrands are positive then extending the integration over the whole interval from x_o up to x_s we shall only increase the magnitudes of the integrals in the left-hand sides of the inequality, i.e., $\displaystyle t_K - t_K' < \int_{x_o}^{x_s} \frac{d\xi}{\alpha\sqrt{|x_o - \xi|}}$ with $x_o < x_s$ and $\displaystyle t_K - t_K' < \int_{x_s}^{x_o} \frac{d\xi}{\alpha\sqrt{|x_o - \xi|}}$ with $x_o > x_s$. So, one can see that in both cases $t_K - t_K' < \frac{2}{\alpha}\sqrt{|x_o - x_s|}$ i.e., at any K is limited from above.

The obtained contradiction proves our assertion. Thus, depending upon what side from x_s the point x_o is situated we have one family of the solutions of the equation /~~*~~/ dependent upon one parameter t_o . These two families contain all the solutions entering into the given equilibrium position of the saddle type.

Denote by $g^+_{t_o, x_s}(t_o')$ the continuous mapping of the straight line t_o' on the plane $(x, \dot{x})$ which can be determined in the following way:

we take the point $(x = x_\varepsilon(t_o), \dot{x} = \dot{x}_\varepsilon(t_o))$ on the solution $x_\varepsilon(t)$ of the equation satisfying the conditions: $x_\varepsilon(t_o') = x_o$, $x_o < x_s$, $|x_o - x_s| < h_o$ $x_\varepsilon(t) \to x_s$ at $t \to \infty$ and $\operatorname{sign} \dot{x}_\varepsilon(t) = \operatorname{sign}(x_s - x_o)$ at $t \geqslant t_o'$.

It will be a continuous and mutually-digit mapping on the plane $(x, \dot{x})$. $g^-_{t_o, x_s}(t_o')$ for $x_o > x_s$ is determined analogously. Thus, $\Gamma(t_o) = \bigcup_{x_s}(L'_{x_s} \cup L''_{x_s})$ where L'_{x_s} and L''_{x_s} is a topological image of the straight line.

CHAPTER III.

Further we shall be in need of some properties of the solutions of the following equation

$$\frac{d}{dt}\left(m(t)\dot{x}\right) - k(t)x = -k(t)\xi(t), \qquad /3.1/$$

where $m(t)>0$ and $k(t)>0$ are continuous functions. The corresponding homogeneous equation has the unique eqûilibrium position $x=0$. It will be the equilibrium position of the saddle type. The aim of this chapter is to prove the following Theorem.

Theorem 5.

If the homogeneous equation /3.1/ satisfies the conditions (α) , and $\xi(t)$ is limited and tending to zero with $t\to\infty$ then for any x_0 and t_0 there exists a solution $x(t)$ of the inhomogeneous equation /3.1/ satisfying the following conditions : $x(t_0)=x_0$ and $x(t)\to 0$ at $t\to\infty$.

Note immediately that if such a solution of the equation /3.1/ exists then it is unique. It follows from the uniqueness of the solution satisfying the conditions $x(t_0) = 0$ and $x(t)\to 0$ with $t\to\infty$ of the corresponding homogeneous equation. The proof of this Theorem for case when the integral $\int_{t_0}^{\infty}\frac{dt}{m(t)}$ is divergent is somewhat different from an analogous proof in case when the integral $\int_{t_0}^{\infty}\frac{dt}{m(t)}$ is convergent. Proving this Theorem for each of the possible cases we shall try, however, to reduce to the minimum all possible repeatitions. By the substitution $S=\int_{t_0}^{t}\frac{d\tau}{m(\tau)}$ equation /3.1/ will be reduced to the form

$$\frac{d^2x}{dS^2} - \tilde{k}(S)x = -\tilde{k}(S)\xi(t(S)) \qquad /3.2/$$

where $\tilde{k}(S) = k(t(S))m(t(S))$

Lemma 5. Let the integral $\int_{t_0}^{\infty}\frac{d\tau}{m(\tau)}$ be divergent, $\xi(S)$ is continuous together with their derivatives up to the second order for any $S\geqslant 0$ and tends to zero at $S\to\infty$ and let $\frac{d^2\xi(S)}{dS^2}$ has an identical sign with $\xi(S)$ at any $S\geqslant 0$. Then equation /3.2/ has the solution tending to zero at $S\to\infty$.

Proof:

It is easy to see that under these assumptions $\xi(S)$ does not reverse its sign since if at $S=S_0$ $\xi(S)=0$ then at $S > S_0$ $\xi(S)\equiv 0$. Since in this case the proof is trivial, we shall suppose that $\xi(S)$ at $S\geqslant 0$ does not vanish and for the sake of definiteness is positive. The curve $x=\xi(S)$ divides the half-plane $S\geqslant 0$ of the plane (S,x) into two parts, in one of which the expression $x-\xi(S)$ is positive and in the other it is negative. Let us integrate /3.2/ over S from $S_0\geqslant 0$ up to S .

We obtain

$$\frac{dx(S)}{dS} - \frac{dx(S_0)}{dS} = \int_{S_0}^{S}\tilde{k}(\tau)\left\{x(\tau)-\xi(\tau)\right\}d\tau \qquad /3.3/$$

It can be seen from formula /3.3/ that any solution $x(s)$ of the equation /3.2/ with the initial conditions $x(s_o)=x_o \geq \xi(s_o)$ and $x'(s)=x'_o \geqslant 0$ is passing monotonously into $+\infty$. Analogously any solution $x(s)$ with the initial conditions $x(s_o)=x_o \leq \xi(s_o)$ and $x'(s_o)=x'_o \leq \xi'(s_o)$ such that $|\xi(s_o)-x_o|+|\xi'(s_o)-x'_o|>0$ is going monotonously into $-\infty$.

Let us take $x_o > \xi(0) > 0$. If the solution $x(s)$ of the equation /3.2/ with the initial conditions $x(0)=x_o$, $x'(0)<0$ meets the curve $x=\xi(s)$ then $x(s)$ is going monotonously into $-\infty$. It is sufficient to show that at the moment of their meeting s_1 $x'(s_1)<\xi'(s_1)$. Evidently $x'(s_1) \leq \xi'(s_1)$

It is sufficient, therefore, to prove that $x'(s_1) \neq \xi'(s_1)$. Suppose contrary : let $x'(s_1)=\xi'(s_1)$. Then $x(s)=x(s_1)+x'(s_1)(s-s_1)+o((s-s_1)^2)$, since $x''(s_1)=0$ and $\xi(s)=\xi(s_1)+\xi'(s_1)(s-s_1)+\xi''_{(s_1)}(s-s_1)\frac{(s-s_1)^2}{2!}+o((s-s_1)^2)$ i.e., $\xi(s)-x(s)=\xi''(s_1)\frac{(s-s_1)^2}{2!}+o((s-s_1)^2)$ With s sufficiently close to s_I the obtained expression >0 since $\xi''(s_1)>0$. On the other hand with $s<s_1$ $\xi(s)-x(s)<0$. The obtained contradiction proves our assertion.

Let $x_o > \xi(0) > 0$ and $x'_o < \xi'(0) < 0$. Denote by $x_\tau(s)\,(\tau\in[0,1])$ the solution of the equation /3.2/ with the initial conditions $x_\tau(0)=x_o+\tau(\xi(0)-x_o)$ and $x'_\tau(0)=\tau x'_o$. Denote by T_1 sets of values $\{\tau\in[0,1]\}$ such that $x_\tau(s)$ meets with the curve $x=\xi(s)$ and, consequently, goes over into $-\infty$. Denote by T_2 the sets of values $\{\tau\in[0,1]\}$ such that the solution $x_\tau(s)$ not reaching the curve $x=\xi(s)$ has the return point and goes over into $+\infty$. It follows from the Theorem about the continuous dependence of the solution upon the initial conditions that T_1 and T_2 are open sets in $[0,I]$. Neither of them is empty since according to the above-mentioned statements $\tau=1$ belongs to T_1, whereas $\tau=0$ belongs to T_2. It is evident that $T_1 \wedge T_2 = 0$. It follows from the connectedness of the segment that $[0,1]\setminus(T_1\cup T_2)\neq 0$. Let $\tau_o\in[0,1]\setminus(T_1\cup T_2)$. Then it is clear that $x_{\tau_o}(s)>\xi(s)>0$ and $x'_{\tau_o}(s)<0$ at $s\geqslant 0$, i.e., $x_{\tau_o}(s)$ is tending to a certain not negative limit. Multiplying /3.2/ by s and integrating over s from zero up to s, we obtain

$$s x'_{\tau_o}(s)+x_{\tau_o}(0)-x_{\tau_o}(s)=\int_0^s \tau\,\tilde{k}(\tau)\{x_{\tau_o}(\tau)-\xi(\tau)\}\,d\tau. \qquad /3.4/$$

The left-hand side of the equality /3.4/ at any $s \geqslant 0$ does not exceed $x_{\tau_o}(0)$, since $s x'_\tau(s)-x_{\tau_o}(s)<0$ and consequently, the right-hand side of the equality does not exceed $x_{\tau_o}(0)$. Since the integrand is not negative it means that the integral $\int_0^\infty s\tilde{k}(s)\{x_{\tau_o}(s)-\xi(s)\}ds$ is convergent. Taking into account the conditions (α) we can easily obtain that it is possible only when $x_{\tau_o}(s)-\xi(s)\longrightarrow 0$ with $s\longrightarrow 0$ since $x_{\tau_o}(s)$ is a monotonously decreasing function. Therefore, $x_{\tau_o}(s)\longrightarrow 0$ with $s\longrightarrow\infty$. Thus our Lemma is proved.

Lemma 5'. Let the integral $\int_{t_o}^\infty \frac{d\tau}{m(\tau)}$ be convergent, $\xi(s)$ is continuous together with its derivatives up to the second order for any $s\in[0,s_o]$, where $s_o=\int_{t_o}^\infty \frac{d\tau}{m(\tau)}$ monotonously tends to zero with $s\longrightarrow s_o$ and let $\xi''(s)$ have the sign opposite to $\xi(s)$ at any $s \in [0, s_o]$. Then the equation /3.2/ has the solution tending to zero at $s \to s_o$.

Proof:

Similarly as in proof of Lemma 5 one may restrict oneself by the case when $\xi(s) > 0$ for any $s \in [0, s_o)$. It can be easily seen from the formula analogous to /3.3/ that any solution $x(s)$ of the equation /3.2/ with the initial conditions $x(s_o') = x_o \geqslant \xi(s_o')$ and $x'(s_o') = x_o' > \xi'(s_o')$ goes over monotonously into $+\infty$. Analogously any solution $x(s)$ with the initial conditions $x(s_o') = 0$, $x'(s_o') < 0$ goes over monotonously into $-\infty$. Denote $x_\tau(s)$ the solution of the equation /3.2/ with the initial conditions $x_\tau(0) = \xi(0)(1-\tau)$ and $x_\tau'(0) = -\tau + (1-\tau)x_o'$, $(x_o' > \xi'(0))$ by $\tau \in [0, 1]$. Denote by T_1 the set of values $\{\tau \in [0,1]\}$ such that the solution $x_\tau(s)$ at $s = s_o' \in [0, s_o]$ with the curve $x = \xi(s)$. Denote by T_2 the sets of values $\{\tau \in [0,1]\}$ such that the solutions $x_\tau(s)$ cross with the straight line $x = 0$. As well as in the proof of Lemma 5 it can be easily seen that if the solution $x_\tau(s)$ crosses the curve $x = \xi(s)$ then further it monotonously goes over into $+\infty$. Therefore, T_1 and T_2 are sets open in $[0, 1]$ and $T_1 \wedge T_2 = 0$.

The choice of a set of the initial values has been made so that neither of the sets would be empty. It follows from the connectedness of the segment that $[0,1] \setminus (T_1 \cup T_2) \neq 0$. Let $\tau_o \in [0,1] \setminus (T_1 \cup T_2)$. Then $x_{\tau_o}(s)$ is confined between $x = \xi(s)$ and $x = 0$, i.e., $x_{\tau_o}(s) \to 0$ with $s \to s_o$. Thereby Lemma is proved.

Lemma 6. Let $\bar{\xi}(s)$ be determined and confined at $s \geqslant 0$ and at $s \to \infty$ tends to zero when there exists $\xi(s) > 0$ such that $\bar{\xi}(s) > |\xi(s)|$ $\lim_{s\to\infty} \bar{\xi}(s) = 0$, $\bar{\xi}'(s) < 0$, $\bar{\xi}''(s)$ is piecewise in the gap points we have the limit on the right and on the left and $\bar{\xi}''(s) > 0$ for any $s \geqslant 0$.

Proof:

Let $|\xi(s)| < M$ at $s \geqslant 0$. Take $\xi_o = 4M$ and put $\varepsilon_k = \varepsilon_o \cdot 2^{-k}$ for $k > 0$. For every ε_k determine s_k such that $|\xi(s)| < \frac{\varepsilon_k}{4}$ for $s \geqslant s_k$ $(s_o = 0)$. Taking if necessary $s_k' > s_k$ $(s_o' = 0)$ we may obtain that $\Delta_k = \dfrac{\varepsilon_{k+1} - \varepsilon_k}{s_{k+1} - s_k}$ would satisfy the condition $\Delta_k < \Delta_{k+1}$. Supposing $\tilde{\Delta}_k = \frac{1}{2}\Delta_k$ we see that $\Delta_k < \Delta_{k+1}$ for any $k \geqslant 0$. Introduce two piecewise functions $\sigma_1(s) = \Delta_k$ for $s \in [s_k, s_{k+1}]$ and $\sigma_2(s) = \tilde{\Delta}_k$ for $s \in [s_k, s_{k+1}]$. Let $\sigma_3(s) = \dfrac{\varepsilon_{k+1}(s_{k+1} - s) + \varepsilon_k (s - s_k)}{s_{k+1} - s_k}$ and $\sigma_4(s) = \frac{1}{2}\sigma_3(s)$ for $s \in [s_k, s_{k+1}]$. It can be easily seen that $\sigma_3(s) = -\int_s^\infty \sigma_1(\tau)\,d\tau$ and $\sigma_4(s) = -\int_s^\infty \sigma_2(\tau)\,d\tau$.
It is evident that $\sigma_3(s) > \sigma_4(s) > |\xi(s)|$. Finally introduce $\sigma_5(s) = \dfrac{\Delta_k (s_{k+1} - s_k) + \tilde{\Delta}_k (s - s_k)}{s_{k+1} - s_k}$ for $s \in [s_k, s_{k+1}]$ and put $\bar{\xi}(s) = -\int_s^\infty \sigma_5(\tau)\,d\tau$. It can be easily seen that $0 > \sigma_2(s) \geqslant \sigma_5(s) \geqslant \sigma_1(s)$ i.e., $0 < -\sigma_2(s) \leq -\sigma_5(s) \leq -\sigma_1(s)$. Integrating the last inequality from s up to ∞ , we obtain that $-\int_s^\infty \sigma_2(\tau)d\tau \leq -\int_s^\infty \sigma_5(\tau)d\tau \leq -\int_s^\infty \sigma_1(\tau)d\tau$, i.e., $\sigma_4(s) \leq \bar{\xi}(s)$. Comparing the last inequality with $\sigma_4(s) > |\xi(s)|$ we obtain that $\bar{\xi}(s) > |\xi(s)|$. It is evident that $\bar{\xi}'(s) < 0$ and for $s \in (s_k, s_{k+1})$ $\bar{\xi}''(s) = \dfrac{\tilde{\Delta}_k - \Delta_k}{s_{k+1} - s_k} = \dfrac{\varepsilon_{k+1}}{(s_{k+1} - s_k)^2} > 0$. It follows from the fact that $\sigma_4(s)$ and $\sigma_3(s)$ are tending to zero at $s \to \infty$ that $\bar{\xi}(s) \to 0$ also at $s \to \infty$. The proof of the Lemma is completed by this.

Lemma 6^{I}. Let $\xi(S)$ be determined and limited for $S\in[0,S_0)$ and tend to zero at $S\to S_0$. Then there exists $\bar\xi(S)>0$ monotonously tending to zero at $S\to S_0$ such that $\bar\xi(S)>|\xi(S)|$, whereas $\bar\xi''(S)$ is piecewise, in the gap points it has the limit on the right and on the left and $\bar\xi''(S)<0$ for any $S\in[0,S_0)$.

Proof:

We take the half straight line $L: S=S_0,\ \xi>0$ on the plane (S,ξ) and the sequence of points on it $p_K=\{S_0,\xi_0\cdot 2^{-K}\}$ where $\xi_0=\sup_{S\in[0,S_0]}|\xi(S)|$ and $K\geq 0$. We draw a straight line $\xi=\xi_K+\gamma_K(S_0-S)$ through each point p_K so that the graph of the function $|\xi(S)|$ would be situated lower than our straight line. For this it is necessary to take γ_K sufficiently great. We subject the choice of γ_K to the following conditions: $(\gamma_{K+1}-\gamma_K)>\frac{1}{2}(\gamma_K-\gamma_{K-1})>\frac{3}{16}\frac{\xi_0}{S_0}$ and $\gamma_{K+1}>\frac{4}{3}\gamma_K$. Then, it can be easily varified $\bar S_{K+1}=S_0+\frac{3}{4}\frac{\xi_{K+1}-\xi_K}{\gamma_{K+1}-\gamma_K}$ and $\bar\xi_{K+1}=\frac{\xi_K\gamma_{K+1}-\gamma_K\xi_{K+1}}{\gamma_{K+1}-\gamma_K}$ at $K+1>0$, whereas $\bar S_0=0$ and $\bar\xi_0=\xi_0+\frac{4}{3}\gamma_0 S_0$ satisfy the following conditions: $\bar S_{K+1}>\bar S_K$, $\bar\xi_{K+1}<\bar\xi_K$ and $\gamma_K(\bar S_{K+1}-S_K)<(\bar\xi_K-\bar\xi_{K+1})<\gamma_{K+1}(\bar S_{K+1}-\bar S_K)$

The last conditions as can be easily seen are necessary and sufficient for the existence of the continuous positive function $\alpha_K(S)$ such that $\int_{\bar S_K}^{\bar S_{K+1}}\alpha_K(S)dS=\gamma_{K+1}-\gamma_K$ and $\int_{\bar S_K}^{\bar S_{K+1}}\int_{\bar S_K}^{\omega}\alpha_K(S)\,dS\,d\omega=(\bar\xi_K-\bar\xi_{K+1})-\gamma_K(\bar S_{K+1}-S_K)$.

Denote by $\gamma(S)$ the function equal to $\gamma_K+\int_{\bar S_K}^{S}\alpha_K(\tau)d\tau$ for $S\in[\bar S_K,\bar S_{K+1}]$ and let $\bar\xi(S)=\bar\xi_0-\int_0^S\gamma(\tau)d\tau$. Then it can be easily varified the obtained function $\bar\xi(S)$ satisfies all the conditions of Lemma.

Just note that Lemma 5 /Lemma 5^{I}/ remains valid if $\xi(S)$ satisfying the conditions of Lemma 5 /Lemma 5^{I}/ is replaced for $\bar\xi(S)$, constructed in Lemma 6 /Lemma 6^{I}/.

Making use of the last remark it will be quite easy to finish the proof of Theorem 5. Denote by $x(t)$ the solution of the homogeneous equation /3.I/ satisfying the following conditions $x(t_0)=1$, $\dot x(t)<0$ and $\lim_{t\to\infty}x(t)=0$. Then as is known the general solution of equation /3.I/ is of the form

$$\bar x(t)=C_1x(t)+C_2x(t)\int_{t_0}^{t}\frac{d\tau}{m(\tau)x^2(\tau)}-x(t)\int_{t_0}^{t}\left\{\frac{1}{m(\tau)x^2(\tau)}\int_{t_0}^{\tau}K(\omega)\cdot x(\omega)\xi(\omega)d\omega\right\}d\tau \qquad /3.5/$$

where $x(t)\int_{t_0}^{t}\frac{d\tau}{m(\tau)x^2(\tau)}$ is the second solution of the homogeneous equation /3.I/ linearly -independent of $x(t)$. It follows from the fact that the homogeneous equation /3.I/ satisfies the conditions (α) that $x(t)\int_{t_0}^{t}\frac{d\tau}{m(\tau)x^2(\tau)}\longrightarrow\infty$ at $t\to\infty$.
Therefore, expression /3.5/ may tend to zero only at $C_2=\int_{t_0}^{\infty}K(\omega)x(\omega)\xi(\omega)d\omega$. Under the assumption $\xi(S)$ is limited and tends to zero at $t\to\infty$. Then according to Lemmas 6 and 6^{I} there always exists $\bar\xi(S)>|\xi(S)|$ such that if in the equation /3.I/ $\xi(S)$ is substituted for $\bar\xi(S)$ then we obtain the equation which according to Lemmas 5 and 5^{I} has the solution tending to zero at $t\longrightarrow\infty$. This means that the expression

$$x(t)\int_{t_0}^{t}\left\{\frac{1}{m(\tau)x^2(\tau)}\int_{\tau}^{\infty}K(\omega)\bar\xi(\omega)\,x(\omega)\,d\omega\right\}d\tau$$

is tending to zero at $t \longrightarrow \infty$. Since $x(t)>0$ at $t \geqslant t_0$ then

$$x(t)\int_{t_0}^{t}\left\{\frac{1}{m(\tau)x^2(\tau)}\int_{\tau}^{\infty}\kappa(\omega)\xi(\omega)\,x(\omega)\,d\omega\right\}d\tau \geqslant$$

$$\geqslant x(t)\int_{t_0}^{t}\left\{\frac{1}{m(\tau)x^2(\tau)}\int_{\tau}^{\infty}\kappa(\omega)|\xi(\omega)|\,x(\omega)\,d\omega\right\}d\tau \geqslant$$

$$\geqslant x(t)\int_{t_0}^{t}\left\{\frac{1}{m(\tau)x^2(\tau)}\left|\int_{\tau}^{\infty}\kappa(\omega)\xi(\omega)x(\omega)\,d\omega\right|\right\}d\tau \geqslant$$

$$\geqslant\left|x(t)\int_{t_0}^{t}\left\{\frac{1}{m(\tau)x^2(\tau)}\int_{\tau}^{\infty}\kappa(\omega)\xi(\omega)x(\omega)\,d\omega\right\}d\tau\right|,$$

therefore, the expression

$$x(t)\int_{t_0}^{t}\left\{\frac{1}{m(\tau)x^2(\tau)}\int_{\tau}^{\infty}\kappa(\omega)\xi(\omega)x(\omega)\,d\omega\right\}d\tau$$

is tending to zero at $t \longrightarrow \infty$. Thus, the expression

$$x_0 x(t)+x(t)\int_{t_0}^{t}\left\{\frac{1}{m(\tau)x^2(\tau)}\int_{\tau}^{\infty}\kappa(\omega)\xi(\omega)x(\omega)\,d\omega\right\}d\tau$$

is the solution we have sought for. Thereby Theorem 5 is proved.

Lemma 7. Let the homogeneous equation /3.I/ satisfy the conditions (α), whereas $x(t)$ is the solution of the homogeneous equation /3.I/ satisfying the conditions $x(t_0)=1,\ x'(t)<0$ at $t \geqslant t_0$ and $x(t)\longrightarrow 0$ at $t \longrightarrow \infty$. Then $m(t)\dot{x}(t)\longrightarrow 0$ at $t\longrightarrow\infty$.

Proof:

For case when the integral $\int_{t_0}^{\infty}\frac{d\tau}{m(\tau)}$ is divergent the proof is evident. Let us prove the validity of the Lemma for case when the integral $\int_{t_0}^{\infty}\frac{d\tau}{m(\tau)}$ is convergent. $m(t)x'(t)$ is the monotonously increasing function strictly less than zero for any $t \geqslant t_0$.
Let $\lim\limits_{t\to\infty} m(t)\dot{x}(t)=-c<0$. Then at any $t\geqslant t_0$ $m(t)\dot{x}(t)<-c$, i.e., $-\dot{x}(t)>\frac{c}{m(t)}$.
Integrating from t up to ∞, we obtain that $x(t)>c\int_{t}^{\infty}\frac{d\tau}{m(\tau)}$. Multiplying the last equality by $\kappa(t)$ and having integrated from t_0 up to ∞, we obtain that

$$\int_{t_0}^{\infty}\kappa(t)\left(\int_{t}^{\infty}\frac{d\tau}{m(\tau)}\right)dt<\frac{1}{c}\int_{t_0}^{\infty}\kappa(t)x(t)\,dt$$

The last integral as it has been noticed earlier is convergent. Therefore, the integral $\int_{t_0}^{\infty}\kappa(t)\left(\int_{t}^{\infty}\frac{d\tau}{m(\tau)}\right)dt$ is also convergent. It is known that from the convergence of the integral $\int_{t}^{\infty}\int_{\tau}^{\infty}\varphi(s)\,ds\,dv$ follows the convergence of the integral $\int_{0}^{\infty}s|\varphi(s)|\,ds$.
For $s=\int_{t}^{0}\kappa(\tau)\,d\tau$ and $\varphi(s)=(\kappa(s)\,m(s))^{-1}$ the first integral is equal to $\int_{t_0}^{\infty}\int_{t}^{\infty}\kappa(t)\,\frac{d\tau}{m(\tau)}\,dt$ and as proved is convergent. Therefore, the second integral is also convergent, i.e., the integral $\int_{t_0}^{\infty}\frac{1}{m(t)}\left(\int_{t_1}^{t}\kappa(\tau)\,d\tau\right)dt$ is convergent that contradicts the conditions $/\alpha/$. The obtained contradiction proves our Lemma.

Theorem 6. Let the equation

$$\frac{d}{dt}\left(m(t,\lambda)x'\right)-\kappa(t,\lambda)x=-\kappa(t,\lambda)\xi(t,\lambda) \tag{3.6}$$

satisfy the conditions of Theorem 5 for any $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta] (\delta > 0)$, the functions $m(t, \lambda)$, $\kappa(t, \lambda)$ and $\xi(t, \lambda)$ are uniformly tending to $m(t, \lambda_0), \kappa(t, \lambda_0)$ and $\xi(t, \lambda_0)$ with $\lambda \to \lambda_0$ on every segment $[t_0, t']$, whereas the function $\xi(t, \lambda)$ is limited by one constant for any $t \geqslant t_0$ and for any $\lambda \in [\lambda_0 - \delta, \lambda_0 + \delta]$. Denote by $x_\lambda(t)$ the solution of the equation /3.6/ satisfying the conditions $x_\lambda(t_0) = x_0$ and $x_\lambda(t) \to 0$ at $t \to \infty$. Then $x_\lambda(t)$ is continuosly dependent upon λ, i.e., $x_\lambda(t) \to x_{\lambda_0}(t)$ with $\lambda \to \lambda_0$ is uniformly on every final segment of the values t.

Proof:

According to Theorem 5

$$x_\lambda(t) = x_0 \bar{x}_\lambda(t) + \bar{x}_\lambda(t) \int_{t_0}^{t} \left\{ \frac{1}{m(t,\lambda)\bar{x}_\lambda^2(t,\lambda)} \int_{\tau}^{\infty} \kappa(\omega, \lambda) \xi(\omega, \lambda) \bar{x}_\lambda(\omega, \lambda) d\omega \right\} d\tau, \qquad /3.7/$$

where $\bar{x}_\lambda(t)$ is the solution of the homogeneous equation /3.6/ satisfying the oonditions $\bar{x}_\lambda(t_0) = 1$ and $\bar{x}_\lambda(t) \to 0$ at $t \to \infty$. From the fact that $\bar{x}_\lambda(t)$ and $\dot{\bar{x}}_\lambda(t)$ at $\lambda \to \lambda_0$ are tending to $\bar{x}_{\lambda_0}(t)$ and $\dot{\bar{x}}_{\lambda_0}(t)$ correspondingly uniformly on every segment $[t_0, t']$ one may write using Lemma 7

$$\int_t^\infty \kappa(\tau, \lambda) \bar{x}_\lambda(\tau) d\tau = -m(t, \lambda) \dot{\bar{x}}_\lambda(t) \to -m(t, \lambda_0) \dot{\bar{x}}_{\lambda_0}(t) = \int_t^\infty \kappa(\tau, \lambda_0) \bar{x}_{\lambda_0}(\tau) d\tau$$

$$with \qquad \lambda \to \lambda_0$$

is uniform on every finite segment of the values t / from the fact that the written expressions are the functions t, monotonously tending to zero follows that the oonvergence will be uniform on all the semiaxis $t \geqslant t_0$ /. Making use of the limitation of the function $\xi(t, \lambda)$ uniform over λ it can be easily shown that

$$\int_t^\infty \kappa(\tau, \lambda) \xi(\tau \lambda) \bar{x}_\lambda(\tau) d\tau \to \int_t^\infty \kappa(\tau, \lambda_0) \xi(\tau, \lambda_0) \bar{x}_{\lambda_0}(\tau) d\tau$$

at $\lambda \to \lambda_0$ is uniform at every finite segment of the values t. The latter oircumstance implies that Exp. /3.7/ with $\lambda \to \lambda_0$ is tending uniformly to an analogous expression for $\lambda = \lambda_0$ at every finite segment of the values t. Thus Theorem is proved.

CHAPTER 4.

The results of the previous chapter give the possibility to finish the outlined program. It is the aim of this chapter to prove the following Theorem.

Theorem 7. Let the functions $m(\varepsilon, t), \kappa(\varepsilon, t), p'(x)$ and $\varepsilon f(\varepsilon, t, x, \dot{x}) x'$ possess the con[illegible]ous derivatives over ε, x and $\dot{x}$ up to the order κ and satisfy the conditions (β) /their meaning will be explained in the proof of the Theorem/. Then every solution of the equation /[illegible]/ satisfying the conditions /A/ has continuous derivatives over ε up to

K-order tending to zero with $t \to \infty$ and satisfying the equations obtained from /*/ by the successive differentiation over ε.

Proof:

Let us prove the assertion for the first derivative. For the derivatives of higher orders the proof is analogous. Let $x_\varepsilon(t)$ be the solution of the equation /*/ satisfying the conditions $x_\varepsilon(t_0)=x_0$ $(|x_0 - x_s|<h_0)$, $x_\varepsilon(t)\to x_s$ with $t\to\infty$ and $sign\, \dot{x}_\varepsilon(t) = sign\,(x_s - x_0)$ with $t \geq t_0$,
whereas $x_{\varepsilon+\Delta\varepsilon}(t)$ be the solution of the equation /*/, with the changed value of the parameter satisfying the same conditions. Let us substract the corresponding equation for $x_\varepsilon(t)$ from the equation for $x_{\varepsilon+\Delta\varepsilon}(t)$. Applying Adamar Lemma /see, for example, /7/, page 8I/ we obtain that $\frac{\Delta x_\varepsilon(t)}{\Delta\varepsilon} = \frac{x_{\varepsilon+\Delta\varepsilon}(t) - x_\varepsilon(t)}{\Delta\varepsilon}$ with $\Delta\varepsilon \neq 0$ satisfies the equation

$$\frac{d}{dt}\left(m(\varepsilon+\Delta\varepsilon,t)\,\frac{\Delta\dot{x}_\varepsilon(t)}{\Delta\varepsilon}\right) - \frac{\Delta\dot{x}_\varepsilon(t)}{\Delta\varepsilon}\left\{f(\varepsilon+\Delta\varepsilon,t,x_{\varepsilon+\Delta\varepsilon},\dot{x}_{\varepsilon+\Delta\varepsilon}) + \right.$$

$$\left. + \dot{x}_\varepsilon\int_0^1 f'_{\dot{x}_\varepsilon}(\varepsilon+\Delta\varepsilon,t,x_{\varepsilon+\Delta\varepsilon},\dot{x}_\varepsilon+\xi\Delta\dot{x}_\varepsilon)\,d\xi\right\}(\varepsilon+\Delta\varepsilon) +$$

$$+ \frac{\Delta x_\varepsilon(t)}{\Delta\varepsilon}\left\{K(\varepsilon+\Delta\varepsilon,t)\int_0^1 P''(x_\varepsilon+\xi\Delta x_\varepsilon)\,d\xi - \right.$$

$$\left. - (\varepsilon+\Delta\varepsilon)\dot{x}_\varepsilon\int_0^1 f'_{x_\varepsilon}(\varepsilon+\Delta\varepsilon,t,x_\varepsilon+\xi\Delta x_\varepsilon,\dot{x}_\varepsilon)\,d\xi\right\} = \qquad /4.I/$$

$$= -\frac{d}{dt}\left(\dot{x}_\varepsilon\int_0^1 m'_\varepsilon(\varepsilon+\xi\Delta\varepsilon,t)\,d\xi\right) - P'(x_\varepsilon)\int_0^1 K'_\varepsilon(\varepsilon+\xi\Delta\varepsilon,t)\,d\xi +$$

$$+ f(\varepsilon,t,x_\varepsilon,\dot{x}_\varepsilon)\,\dot{x}_\varepsilon + (\varepsilon+\Delta\varepsilon)\dot{x}_\varepsilon\int_0^1 f'_\varepsilon(\varepsilon+\xi\Delta\varepsilon,x_\varepsilon,\dot{x}_\varepsilon)\,d\xi$$

with the boundary conditions:

$$\frac{\Delta x_\varepsilon(t_0)}{\Delta\varepsilon} = 0 \quad \text{and} \quad \frac{\Delta x_\varepsilon(t)}{\Delta\varepsilon} \longrightarrow 0 \quad \text{with} \quad t\to\infty$$

Let $z_\varepsilon(t)$ be a solution of the equation

$$\frac{d}{dt}\left(m(\varepsilon,t)\dot{z}_\varepsilon\right) - \varepsilon\dot{z}_\varepsilon\left\{f(\varepsilon,t,x_\varepsilon,\dot{x}_\varepsilon) + f'_{\dot{x}_\varepsilon}(\varepsilon,t,x_\varepsilon,\dot{x}_\varepsilon)\dot{x}_\varepsilon\right\} + z_\varepsilon\left\{K(\varepsilon,t)P''(x_\varepsilon) - \varepsilon f'_{x_\varepsilon}(\varepsilon,t,x_\varepsilon,\dot{x}_\varepsilon)\dot{x}_\varepsilon\right\} =$$

$$= f(\varepsilon,t,x_\varepsilon,\dot{x}_\varepsilon)\dot{x}_\varepsilon + \varepsilon f'_\varepsilon(\varepsilon,t,x_\varepsilon,\dot{x}_\varepsilon)\dot{x}_\varepsilon - \frac{d}{dt}\left(m'_\varepsilon(\varepsilon,t)\dot{x}_\varepsilon\right) - K'_\varepsilon(\varepsilon,t)P'(x_\varepsilon) \qquad /4.2/$$

with the boundary conditions $z_\varepsilon(t_0)=0$ and $z_\varepsilon(t)\to 0$ with $t\to\infty$. In virtue of Theorem 4 the coefficients in equation /4.I/ with $\Delta\varepsilon\to 0$ are tending to the corresponding coefficients of equation /4.2/. The homogêneous equations /4.I/ and /4.2/ satisfy the conditions /α/ because the equations /*/ satisfy the conditions /α/. Therefore, for the applicability of Theorems 5 and 6 it is necessary that the expression

$$\left\{ f(\varepsilon,t,x_\varepsilon,\dot{x}_\varepsilon)\dot{x}_\varepsilon + (\varepsilon+\Delta\varepsilon)\dot{x}_\varepsilon \int_0^1 f'_\varepsilon(\varepsilon+\xi\Delta\varepsilon,t,x_\varepsilon,\dot{x}_\varepsilon)d\xi - \frac{d}{dt}\left(\dot{x}_\varepsilon \int_0^1 m'_\varepsilon(\varepsilon+\xi\Delta\varepsilon,t)d\xi\right) - p'(x_\varepsilon)\int_0^1 \kappa'_\varepsilon(\varepsilon+\xi\Delta\varepsilon,t)d\xi \right\} :$$

$$: \left\{ \kappa(\varepsilon+\Delta\varepsilon,t)\int_0^1 p''(x_\varepsilon+\xi\Delta x_\varepsilon)d\xi - (\varepsilon+\Delta\varepsilon)\dot{x}_\varepsilon \int_0^1 f'_{x_\varepsilon}(\varepsilon+\Delta\varepsilon,t,x_\varepsilon+\xi\Delta x_\varepsilon,\dot{x}_\varepsilon)d\xi \right\}$$

would be uniformly limited for the sufficiently small $\Delta\varepsilon$ and tending to zero with $t\to\infty$. This will be fulfilled if for any $c>0$ for $|x_\varepsilon - x_s|<h_0$, $|\dot{x}_\varepsilon|<C$ and for the sufficiently small $\Delta\varepsilon$ the expression $\Big\{|f(\varepsilon,t,x_\varepsilon,\dot{x}_\varepsilon)+(\varepsilon+\Delta\varepsilon)f'_\varepsilon(\varepsilon+\theta_1\Delta\varepsilon,t,x_\varepsilon,\dot{x}_\varepsilon)| + |\kappa'_\varepsilon(\varepsilon+\theta_1\Delta\varepsilon,t)|\Big\} : |\kappa_c(\varepsilon+\Delta\varepsilon,t)|$ $(0<\theta_1<1)$ /see chap.I/ will be limited with $t\geq t_0$ or increases slower than $\min\left\{\frac{1}{|\dot{x}_\varepsilon(t)|}, \frac{1}{|x_\varepsilon(t)-x_s|}\right\}$, whereas the expression $\frac{d}{dt}\left(\dot{x}_\varepsilon m'_\varepsilon(\varepsilon+\theta_2\Delta\varepsilon,t)\right):|\kappa_c(\varepsilon+\Delta\varepsilon,t)|$ $(0<\theta_2<1)$ is uniformly limited and $\to 0$ with $t\to\infty$. We call this condition $/\beta/$. We may finish the proof of the Theorem by the requirement of the validity of the condition (β) for the equation /*/ , since its proof for the derivatives of higher order is quite identical the validity of the condition analogous to the condition $/\beta/$ provided.

CHAPTER 5

Now let us be engaged with the actual calculation of $\bar{x}_\kappa(t)$. Differentiating the equation /*/ κ times over ε and assuming $\varepsilon=0$, we obtain

$$m_0\ddot{\bar{x}}_0 + \kappa_0 p'(\bar{x}_0) = 0 \qquad /5.0/$$

$$m_0\ddot{\bar{x}}_\kappa + \kappa_0 p''(\bar{x}_0)\bar{x}_\kappa = \mathcal{F}_\kappa(t,\bar{x}_0,\ldots\ddot{\bar{x}}_{\kappa-1}) \qquad /5.K/$$

Thus, having determined $\bar{x}_0(t)$ from /5.0/ we could using /5k/ have determined successively $\bar{x}_1(t)$, $\bar{x}_2(t)$ etc. However, the determination $\bar{x}_\kappa(t)$ considerably simplified if one in $(5,\kappa)$ takes $\bar{x}_0$ as an independent variable. Let us multiply /5.0/ by $\dot{\bar{x}}_0(t)$ and integrate from t'_0 up to t . We obtain

$$m_0\left(\frac{\dot{\bar{x}}_0^2(t)}{2} - \frac{\dot{\bar{x}}_0^2(t'_0)}{2}\right) = -\kappa_0\left\{p(\bar{x}_0(t)) - p(\bar{x}_0(t'_0))\right\} \qquad /5.I/$$

Passing in /5.I/ to the limit at $t\to\infty$ we obtain having in vien $\bar{x}_0(t)\to x_s$ and $\dot{\bar{x}}_0(t)\to 0$ with $t\to\infty$

$$-m_0\frac{\dot{\bar{x}}_0^2(t'_0)}{2} = -\kappa_0\left\{p(x_s) - p(\bar{x}_0(t'_0))\right\} \qquad /5.2/$$

Subtracting /5.2/ from /5.I/ we obtain that

$$m_0\frac{\dot{\bar{x}}_0^2(t)}{2} = \kappa_0\left\{p(x_s) - p(\bar{x}_0(t))\right\},$$ i.e.

$$t = \int_{x_0}^{\bar{x}_0} \frac{d\xi}{\sqrt{2\frac{\kappa_0}{m_0}(p(x_s)-p(\xi))}} + t'_0 ,$$

where t_0' is the moment of time, in which the solution $x_\varepsilon(t)$ satisfying the conditions / A / crosses the straight line $x = x_0$ for the last time.

We have

$$\dot{\bar{x}}_\kappa = \frac{d\bar{x}_\kappa}{d\bar{x}_0}\,\dot{\bar{x}}_0\,, \quad \ddot{\bar{x}}_\kappa = \frac{d^2\bar{x}_\kappa}{d\bar{x}_0^2}\,\dot{\bar{x}}_0^2 + \frac{d\bar{x}_\kappa}{d\bar{x}_0}\,\ddot{\bar{x}}_0$$

Substituting into /5.K/ we have

$$2\frac{d^2\bar{x}_\kappa}{d\bar{x}_0^2}\left(p(x_s)-p(\bar{x}_0)\right) - \frac{d\bar{x}_\kappa}{d\bar{x}_0}\,p'(\bar{x}_0) + \bar{x}_\kappa p''(\bar{x}_0) = \frac{1}{\kappa_0}\mathcal{F}_\kappa(\bar{x}_0, t_0') \qquad /5.3/$$

It is easy to verify that $\bar{x}_\kappa(\bar{x}_0) = \sqrt{2p(x_s)-2p(\bar{x}_0)}$ is the solution of the homogeneneons equation /5.3/ tending to zero with $\bar{x}_0 \to x_s$ /i.e. with $t \to \infty$ /. Then according to Theorem 5 the solution of the equation /5.3/ equal to zero at $\bar{x}_0 = x_0$ i.e., at $t = t_0'$ and tending to zero at $\bar{x}_0 \to x_s$, i.e., at $t \to \infty$ has the form

$$\bar{x}_\kappa(\bar{x}_0) = \sqrt{2p(x_s)-2p(\bar{x}_0)}\int_{x_0}^{\bar{x}_0} \frac{-\int_x^{x_s}\frac{1}{\kappa_0}\mathcal{F}_\kappa(\xi, t_0')\,d\xi}{\left[2p(x_s)-2p(x)\right]^{3/2}}\,dx \qquad /5.4/$$

In order to estimate the capture region for the moment of time t_0 , it is necessary to calculate $\bar{x}_\kappa(\bar{x}_0)$ and $\frac{d\bar{x}_\kappa}{d\bar{x}_0}\dot{\bar{x}}_0$ for the value $\bar{x}_0 = \bar{x}_0(t_0)$ for all t_0' . By t_0' one can determine $\bar{x}_0$ from the formula

$$\int_{x_0}^{\bar{x}_0} \frac{d\xi}{\sqrt{2\frac{\kappa_0}{m_0}\left(p(x_s)-p(\xi)\right)}} = t_0 - t_0' \qquad /5.5/$$

For $\bar{x}_0$ obtained in such a way we determine $\bar{x}_\kappa(x_0)$ and $\frac{d\bar{x}_\kappa}{dx_0}\dot{\bar{x}}_0$ from the formula /5.4/ substituting there the expression for t_0' by the formula /5.5/ The obtained corrections give good results for that part of the boundary of the capture region which goes into the point $(x_s, 0)$, i.e., for $t_0' < t_0$. For $t_0' > t_0$ we determine at first $\dot{x}_\varepsilon(t_0')$, then will look for our solution for $t_0 \le t \le t_0'$ taking as $\bar{x}_0$ the solution of the equation /5.0/ with the initial conditions: $\bar{x}_0(\bar{t}_0) = x_0$ and $\dot{\bar{x}}_0(t_0') = \dot{x}_\varepsilon(t_0')$ whereas for $\bar{x}_\kappa$ the solution of the equation /5.K/ with the initial conditions: $\bar{x}_\kappa(t_0') = \dot{\bar{x}}_\kappa(t_0') = 0 \quad (\kappa > 0)$. For this expansion the H.Poincare theorem is correct as well as usual theorems about the existence of the derivatives by the parameter.

Thus, in this case we may determine the point $\left(x_\varepsilon(t_0), \dot{x}_\varepsilon(t_0)\right)$ on the solution satisfying the conditions /A/.

C O N C L U S I O N

The reader could notice that we did not use the closeness of the equation (✱) to the conservation one anywhere except the last chapter.

Thus, the results of the first four chapters will hold also for the equation:

$$\frac{d}{dt}\left[\tilde{m}(\varepsilon,t)\dot{x}\right] + \tilde{\kappa}(\varepsilon,t)\,p'(x) = \tilde{f}(\varepsilon,t,x,\dot{x})\,\dot{x} \qquad (**)$$

with an assumption that at $\varepsilon = 0$ the functions $\tilde{m}(\varepsilon,t)$ and $\tilde{\kappa}(\varepsilon,t)$ are independent of t, and $\tilde{f}(\varepsilon,t,x,\dot{x})\dot{x}$ is identically equal to zero. Therefore, we could here first determine the capture region for the equation (*, *) at $\varepsilon = 0$, what is usually simpler, than at $\varepsilon \neq 0$,and then to construct the asymptotic expansions by the powers ε. However, the cases, when the expansion coefficients may be expressed in terms of integrals, as it holds for the equation (*) under these general assumptions seem to be extremely rare. The numerical integration of the linear equations of type (S.K) is very likely simpler than that of the equation (**).

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