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THE NONANALICITY OF THE NONRELATIVISTIC SCATTERING AMPLITUDE AND THE POTENTIAL

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As is known the scattering amplitude for given value of ℓ in the nonrelativistic case is not an analytic function in the upper half-plane (of the variable k *). There exists a quite simple proof which demonstrates an interesting connection between the dispersion relations and the inverse problem of the scattering theory.

Let us assume reverse that the amplitude of s-scattering** $S(k)-1=\exp 2i\delta(k)-1$ is analytic in the upper half-plane and decreases not too fast at $|k|\to\infty$ in this half-plane, (not faster than a polimial of k). Then its harmonic expansion has only negative frequencies:

$$s(z)=(2\pi)^{-1}\int_{-\infty}^{+\infty}(S(k)-1)\exp ikz\,dk=0 \qquad z>0 \tag{1}$$

This can be easily seen if we close the circuit by a large semicircle in the upper half-plane of the complex variable k.

On the other hand, it is known from the theory of the inverse problem (Gelfand and Levitan [1/] Marchenko[2/] that the function $s(z)$ if $z>0$ determines the relation between $S(k)$ and the exact solution of the Schrödinger equation /and, therefore, reconstructs the potential/. The fundamental equation in the Marchenko form is as follows***

$$A(x,y)-s(x+y)-\int_x^{\infty}A(x,t)s(t+y)dt=0 \tag{2}$$

* To be short we assume that there are no bound states.

** The proof is analogous for ℓ different from zero.

*** Marchenko makes use of the function f(x + y) = − s(x + y).

where $A(x,y)$ is the coefficient in the expansion of the exact solution $\psi(x,k)$ (x is the distance from the origin) into the series of nonorthogonal functions $\varphi(x,k)=(2/\pi)^{1/2}\sin[ky+\delta(k)]$

$$\psi(x,k)=\varphi(x,k)+\int_x^\infty A(x,y)\varphi(y,k)dy. \qquad (3)$$

If condition (1) is valid, $A(x,y)$ vanishes and the exact wave function $\psi(x,t)$ coincides everywhere with the asymptotic one $\varphi(x,t)$. It means that such an amplitude discribes the scattering without the potential(contact interaction). Thus,(1) cannot be made valid for the scattering on the potential.

It can be shown for the potential limited in the space $V(x)=0(x>a)$ that $\mathfrak{J}(z)=0$ when $z>2a$. It follows from the fact that $\mathfrak{J}(z)$ is the measure of the nonorthogonality of the functions $\varphi(x,k)$:

$$\int\varphi(x,k)\varphi(y,k)dk=\delta(x-y)-\mathfrak{J}(x+y) \qquad (4)$$

and that these functions are orthogonal when $x,y>a$.
Then it follows that one may write the dispersion relations for the function $(S(k)-1)\exp 2ak$ the result obtained by Van Kampen (3).

New results are obtained if one passes from the scattering amplitude given for one value of ℓ to the scattering amplitude $M(E,\tau)$ which is considered as the function of the energy E and of the transferred momentum τ. The analytic properties of such an amplitude were studied by Khuri[4], who showed that the following relations are valid*

$$2\int\delta_+(E'-E)M(E',\tau)dE' \equiv$$
$$\equiv \operatorname{Re} M(E,\tau)-\frac{1}{\pi}P\int_0^\infty \operatorname{Im} M(E',\tau)(E'-E)^{-1}dE'=\frac{1}{4\pi}\int e^{i\vec{\tau}\vec{y}}V(y)d\vec{y} \qquad (5)$$

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* The Khuri's relation is derived under conditions which are equivalent to the assumption that for $|k|\to\infty$ the scattering amplitude is given by the first Born approximation.

This relation is correct for $\tau > 2\alpha$ where α is the maximum positive number at which has the integral exists

$$\int_0^\infty \exp(\alpha y)|V(y)|y\,dy. \qquad (6)$$

Thus, the dispersion relation determines the $\vec{\tau}$ -Fourier component of the potential. It means that the knowledge of the scattering amplitude from the experiment at all angles and energies (i.e. for all $\underset{\sim}{\ell}$) makes it possible to reconstruct the potential directly with the accuracy up to the Fourier components with $\tau > 2\alpha$).

On the other hand, such a reconstruction is possible from the knowledge of the scattering amplitude integrated over all angles (s-scattering). In this case, however, one faces the problem of solving integral equation (2). The knowledge of the amplitude at all ℓ replaces in this sense the solution of the integral equation. That means that the equation (2) gives effectively the relation between scattering amplitudes for various values of ℓ .

There remains only to note that, generally speaking, both equation (2) and the dispersion relations cannot be immediately made use of since the Scrödinger equation itself exists only for small energies E . Therefore, it is still necessary to answer the question what information we obtain when measuring the cross section of the scattering (elastic and) or inelastic only in the finite energy interval $0 \leq E \leq E_{max}$.

We shall be concerned with these problems in a more detailed paper.

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