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# RELATIVISTIC THEORY OF REACTIONS INVOLVING POLARIZED PARTICLES

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# A b s t r a c t

It is shown that the relativistic formulae for angular distribution, for polarization vectors and tensors in the reaction of $a + b \to c + d$ type in its rest system coincide mainly with the nonrelativistic ones if the spin of the particle is determined as the intrinsic angular momentum of the particle with respect to its centre of inertia. The squane of this intrinsic angular momentum is the Lorentz invariant.* Spins of the particles are arbitrary and they have non-vanishing rest-masses.

The main difference from the nonrelativistic case is that the description of the spin state is not identical in different Lorentz systems. Therefore, it is necessary to introduce the corrections into the nonrelativistic formal theory of the cascades of the reactions (for instance, for the experiments on double scattering). The relativistic changes in the angular correlations in the cascades of the type

$$\pi + p \to Y + K, \quad Y \to N + \pi$$

are pointed out.


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* Shirokov Yu.M. informed us that he had elaborated a similar relativistic theory of polarization and correlation effects, starting from the same description of the spin state, which he obtained from the theory of the irreducible representations of the inhomogeneous Lorentz group.

## Introduction

There are formal theories for the reactions of type $a + b \to c + d$. There theories express the angular distribution and the polarization state of the reaction products in terms of the polarization state of the beam and the target and in terms of the unknown parameters which are the S-matrix elements of the process $a + b \to c + d$. The simplest example is the well-known formula for the function $f(\theta)$ appearing in the expression

$$\psi(\vec{r}) = e^{i\vec{k}\vec{r}} + f(\theta)\, e^{ikr}/r \tag{I}$$

for the wave function of the stationary scattering process of particles with zero spins. The unknown parameters in this case are called the scattering phase-shifts.

These theories are based upon the use of the laws of |conservation| (mainly law of the angular momentum conservation). Coester and Jauch |1| were the first who obtained the formulae for angular distribution and polarization in the case of the arbitrary spins of a,b,c,d starting from the expression of the laws of conservation in the form of diagonality of the S-matrix elements with respect to conserving variables. Simon and Welton have obtained the same formulae but in some different manner (see, for instance, |2| ).

These formulae are non-relativistic ones but only because the spin state of particles is described in the Pauli approximation (so that it is the same in all Lorentz systems). The theory of scattering of spinless particles is actually relativistic. In order to obtain the angular distribution in any Lorentz system it is necessary to transform $\sigma(\Theta)=|f(\Theta)|^2$ from the system of inertia into the abovementioned system using the known formulae. For the relativistic generalization, therefore, it is necessary to determine the relativistic operator of the spin. The operator of the spin used further satisfies all the claims which may be required of the notion of the spin as the intrinsic angular momentum of the particle.

To obtain the relativistic formulae we use the Coester and Jauch method in the form presented in |3| . Let us point out that in this method it is necessary to be only able to describe the state of the free particle with the spin. One will not need the relativistic equations (for free particles), similar to a Dirac one (which plays a considerable fole in the Stapp's relativistic theory |4| for the scattering of particles with spin 1/2).

## § I. CONSERVING PHYSICAL VARIABLES IN RELATIVISTIC THEORY

The conservation laws express the fact that physical processes in the isolated physical system must be independent of the way of its describing, particularly, of the choice of the frame of reference. It is assumed, of course, that spacetime is homogeneous and isotropic $^{\not 0}$ (one may consider, however, that this assumption is involved into the notion of the isolated system).

In quantum mechanics this fact is displayed in the requirement that the S-matrix of the physical process must commute with ten operators of infinitesimal translations of the origins of space and time coordinates $P_\mu$ and the rotations in space-time $M_{\mu\nu}$. The commutation of the operator with the S-matrix indicates that the latter is diagonal in the eigenvalues of this operator[2/] and that, therefore, the corresponding physical variable is conserved, i.e., the internal processes do not change it.

Four conservation laws are of the clear physical meaning of the conservation of the total momentum - energy $P_\mu = \{P_x, P_y, P_z, iP_0\}$. Three operators $M_{k4}$ $(K = 1,2,3)$ out of six other ones $M_{\mu\nu}$ are of no immediate physical meaning and we introduce six other operators instead of $M_{\mu\nu}$ which will have the meaning of the coordinates of the centre of inertia of the physical system and its total angular momentum (a.m. in short) relative to this of centre of inertia.

The properties of the centre of inertia follow from its very notion: motion of the system as a certain whole may be characterized first of all (in the very first approximation ) as the motion of the point the mass of which is equal to the rest-

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[2/] Write As-SA = 0 in the form of the matrix product and choose such a representation in which the A operator is diagonal (i.e., let us enumerate A elements by its proper values). Then

$$A_{ik}S_{kl} - S_{im}A_{ml} = (A_i - A_l)S_{il} = 0$$

i.e., $S_{il}$ must vanich if $i \neq l$.

mass (or energy) of the system and with the momentum which is equal to the total momentum $\vec{P}$ of the system. Therefore, the centre of inertia of the isolated system must move uniformly and rectilineary. Besides, in the quantum mechanics we should require the centre of inertia $\vec{R}$ to be the operator of the coordinate of a certain point particle, in particular, the known commutation relations between $R_x$, $R_y$, $R_z$, $P_x$, $P_y$, $P_z$ must be fulfilled.

Such an operator $\vec{R}$ may be obtained in the following way.

The following commutation relations which the operators $P_i$ and $M_{\mu\nu}$ must satisfy to are known (see |5| §3 and |6| ):

$$[P_i, P_j] = 0 \; ; \quad [P_i, E] = 0 \; ;$$

$$[M_i, P_j] = i\varepsilon_{ijk} P_k \quad (1.1); \quad [M_i, E] = 0 \; ;$$

$$[M_i, M_j] = i\varepsilon_{ijk} M_k \quad (1.2); \quad [N_i, P_j] = i\delta_{ij} E \; ;$$

$$[N_i, E] = iP_i \; ; \quad [M_i, N_j] = i\varepsilon_{ijk} N_k \quad (1.3) \; ;$$

$$[N_i, N_j] = -i\varepsilon_{ijk} M_k \quad (1.4) \qquad \text{(I)}$$

The notations: $[A, B] = AB - BA$ ; $h = c = 1$ ; $i, j, k$ assume the values 1,2,3.

$$\{M_1, M_2, M_3\} \equiv \{M_{23}, M_{31}, M_{12}\}, \quad N_j = M_{j4}, \quad E = (P^2 + m_0^2)^{1/2}.$$

$\varepsilon_{ijk}$ is the tensor antisymmetrical with respect to all indices, $\varepsilon_{123} = 1$. It is meant that we are concerned only with those state vectors $\Psi_0$, which describe the states with the definite rest mass $m_0$, i.e., for which $P_\mu P_\mu \Psi_0 = -m_0^2 \Psi_0$ or $(P_0 - E)\Psi_0 = 0$ (the time translation operator $P_0$ coincides with E). Note that since $[N_i, E] = iP_i$ the mean value of $N_i$ is linearly time dependent. Therefore, $N_i$ is "conserved" in the sense, that the internal processes affect this time dependence in no way.

We introduce three new operators $R_x$, $R_y$, $R_z$, for which

$$[R_i, R_j] = 0 \quad \text{and} \quad [R_i, P_j] = i\delta_{ij}$$

(and then $[R_i, E] = iP_i/E$)
Representing $\vec{M}$ in the form $M_k = \sum_{ij} \varepsilon_{kij} R_i P_j + J_k$ from (1.1) we obtain that $[J_i, P_j] = 0$. If one requires $R_x$, $R_y$, $R_z$ to be the components of a space vector, they must satisfy the following commutation relation: $[M_i, R_j] = i\varepsilon_{ijk} R_k$ Therefore $[J_i, R_j] = 0$ and $J_x, J_y, J_z$ are also the three dimensional vector. It follows from (1.2) that $[J_i, J_j] = i\varepsilon_{ijk} J_k$
By analogy, representing $\vec{N}$ in the form $N_i = 1/2(R_i E + ER_i) + K_i \equiv R_i E - iPi/2E + K_i$ we obtain that $|K_i, P_j| = 0$. Therefore, $|K_i, E| = 0$ and the mean value of $K_i$ is time independent.

Now we intend to express $\vec{M}$ and $\vec{N}$ in terms of the operators $\vec{R}$ and $\vec{J}$ introduced above. For $\vec{M}$ it has been already done. There remains only to express the space polar vector $\vec{K}$ in terms of $\vec{P}$ and $\vec{J}$. It can be shown that if $R_x$, $R_y$, $R_z$, are the first three components of any four-vector (i.e., for instance, $|N_i, R_j| = 0$ (if $i \neq j$) then $\vec{K}$ cannot be expressed only in terms of $\vec{J}$ and $\vec{P}$ so that all the commutations for $\vec{N}$ would be satisfied. It means that if $\vec{K}$ are composed only of $\vec{P}$ and $\vec{J}$ then $[\vec{R} \times \vec{P}]_k$ and $J_k$ are not the space components of the second rank fourtensor.

The simplest $\vec{K}$ (namely linear with respect to $J_x, J_y, J_z$) satisfying (1.3) and (1.4) is of the form: $\vec{K} = [\vec{P} \times \vec{J}] (E \pm m)^{-1}$ (compare |7| and |6| )3/.

Since the problem has the solution then:

1) $\vec{R}$ is "conserved" (in the same sence, as $\vec{N}$) since it may be expressed in terms of the conserving operators $M_{\mu\nu}$(see Appendix ). $\vec{R}$ may be called the operator of the centre of inertial. It coincides with the definition (e) of the centre of inertia in Pryce's papers |8|.

2) $\vec{J}$ is also conserved and, what it is especially important for us, $J^2 = J_x^2 + J_y^2 + J_z^2$ is the Lorentz invariant, as $[\vec{N}, J^2] = 0$. Emphasize that it is correct for any $\vec{K} = \vec{K}(\vec{P}, \vec{J})$.

## § 2. USE OF LAWS OF CONSERVATION.

## RELATIVISTIC DETERMINATION OF THE SPIN OF A PARTICLE

Four laws of the total momentum- energy conservation and three laws of the centre of inertia conservation may be expressed

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3/ We were able to show, that no other $\vec{K}$ exist for $J^2 = \frac{1}{2}(\frac{1}{2}+1)$ and $J^2 = 2$. Starting from other considerations, L.G. Zastavenko seems to have proved that $\vec{K}$ is unique (whatever eigenvalue $J^2$ has). We are grateful to him for the discussion of this question.

very simply. As usual the consideration is being carried on in the Lorentz system $K_s$, where the (conserving) total momentum is equal to zero (the so called c.m. system). The origin of the coordinate frame of axes may be taken in the point of the centre of inertia (to be more exact, in the point of the mean value of the operator $\vec{R}$ 4/) of the particles a and b (or c and d). Then $\vec{J}$ is the total a.m. Since the commutations between $J_x, J_y, J_z$ are the same as for the any a.m. (e.g. as for the Pauli spin matrices) then the eigenvalues of $\hat{J}^2$ and $\hat{J}_z$ are equal to

$h^2 J(J+1)$ and $M = J, J-1, \dots -J$

respectively.

The law of conservation of $\vec{J}$ is expressed as the diagonality of the S-matrix in the eigenvalues of $\hat{J}^2$ and $\hat{J}_z$

$$(\dots J'M'|S|\dots JM) = (\dots|S^{JM}|\dots)\cdot\delta_{J'J}\cdot\delta_{M'M} \qquad (2)$$

and, besides, as the independence of $(\dots|S^{JM}|\dots)$ of M, following from $[J_x, S] = 0$

If one needs to find, for instance, the angular distribution of c and d it is necessary to know the S-matrix

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4/ The law of $\vec{R}$ conservation means something more than the conservation of mean value. The requirement $|R, S| = 0$ implies that it the system is in the state with the definite $\vec{R}$ (note that in the interaction picture the wave function of the external behaviour of the physical system does not change in time) the interaction ~~picture the wave function~~ processes in the system do not take it out of this state. This property is not used explicitly but such $\vec{R}$ is necessary to define the conserving $\vec{J}$ — intrinsic a.m. of the system (and spin of a particle, see further).

elements in the representation of the particles momenta. In order to express these elements through the elements (2) we must first of all write out the remaining variables of the complete set (denoted by dots in (2)), commuting with $J^2$ and $J_z$ and with each other.

The initial and final states of the process a + b $\rightarrow$ c + d are the states of the two free not interacting particles. S-matrix elements are in fact the transition amplitudes between such states. Therefore for the total set of variables enumerating the S-matrix elements we must take the quantum-mechanical variables describing the free particles a and b or c and d. The total a.m. $\vec{J}$ (in the system $K_s$) is presented in the form $\vec{J} = \vec{j}_1 + \vec{j}_2$ where $\vec{j}$ is one-particle total a.m. in $K_s$.

The procedure of obtaining the conserving angular momentum with respect to the centre-of-inertia which was set forth in § 1 may be applied for the system of any physical nature (e.g., for physical fields). It is only sufficient to know the specific representation of the operators $P_\mu$ and $M_{\mu\nu}$. It is reasonable, therefore, to apply this procedure to the "elementary" particle, the physical nature of which is unknown at all (according to the notion of the "elementarity"). Apart from the coordinate of the centre of inertia $\vec{r}$ and the momentum $\vec{p}$ we obtain then in the only way one more internal conserving characteristic of the microparticle $\vec{S}$ - the a.m. of the particle relative to its centre of inertia $\vec{r}$. Defining the spin of the particle as $\vec{S}$ we only attribute more exact meaning to the notion of the spin as an intrinsic momentum of the particle.

So, $\vec{j} = [\vec{r} \times \vec{p}] + \vec{S}$ and in the system $K_s$, where $\vec{p}_1 = -\vec{p}_2 = \vec{p}$ we obtain

$$\vec{J} = [\vec{r}_1 \times \vec{p}_1] + \vec{S}_1 + [\vec{r}_2 \times \vec{p}_2] + \vec{S}_2 = [(\vec{r}_1 - \vec{r}_2) \times \vec{p}] + \vec{S}_1 + \vec{S}_2 = \vec{\ell} + \vec{S}_1 + \vec{S}_2 \qquad (3)$$

Now we may introduce in $K_s$ the operator of the total spin $\vec{S} = \vec{S}_1 + \vec{S}_2$ formally by a complete analogy with the non-relativistic consideration. (The square of this spin is, no longer, however, a Lorentz invariant). The eigenfunctions of its square and projection $S_z$ may be expanded by the products $\Psi_{i_1 m_1} \cdot \Psi_{i_2 m_2}$ of the eigenfunctions of the squares and of the $z$-projections of the operators $\vec{S}_1$ and $\vec{S}_2$ respectively (the eigenvalues of $\hat{S}_1^2$ are denoted by $h^2 i_1(i_1+1)$). Since the commutation relations for $\vec{S}, \vec{S}_1, \vec{S}_2$ are of the usual form $[S_x, S_y] = iS_z$ etc., the expansion coefficients will be the well-known Clebsch-Gordan coefficients $(i_1 i_2 m_1 m_2 | i_1 i_2 S m)$ which are simultaneously the transformation functions from the representation in the variables $i_1, i_2, m_1, m_2$ into $i_1, i_2, S, m$ - representation (and vice versa). The coefficient $(\ell S \mu m | \ell S J M)$ has an analogous meaning.

As variables marked by the dots in (2) we may take $S$, $\ell$ and the momentum module in $K_s$ (or the total energy of the system which in $K_s$ coincides with the relativistic invariant-the rest mass of the system).

## § 3. FORMULAE FOR CROSS SECTION, VECTOR AND TENSORS OF POLARIZATION. RELATIVISTIC ROTATION OF SPIN

Now we may express the S-matrix elements in the representation of the momenta of particles and of their spin projections in terms of the elements $(p_c, s', \ell', J', M' | S | p_a, s, \ell, J, M)$

The transformation function from the representation in the variables $p, s, \ell, J, M$ into the representation in the momenta and the spin projections is the product of three transformation functions written out in the following formula (compare [9]).

$$
\begin{aligned}
&(\vec{p}_c, m_c, m_d | S | \vec{p}_a, m_a, m_b) = \\
&= (\vartheta_c \varphi_c p_c | \ell' \mu' p_c)(i_c i_d m_c m_d | i_c i_d s' m')(\ell' s' \mu' m' | \ell' s' J M) \times \\
&\quad \times (s' \ell' | S^{J, E(p_a)} | s \ell) \times \qquad\qquad (4) \\
&\times (\ell s J M | \ell s \mu m)(i_a i_b s m | i_a i_b m_a m_b)(\ell \mu p_a | \vartheta_a \varphi_a p_a).
\end{aligned}
$$

Expression (2) and the law of total energy conservation are used. $\vec{p}_c$ and $\vec{p}_a$ are the momenta of the particles c and d and a and b correspondingly in $K_s$; $\vartheta_c, \varphi_c, p_c$; $\vartheta_a, \varphi_a, p_a$ are their spherical angles and modules. It is implied that $p_c$ is the function of $p_a$: $\sqrt{p_a^2 + \varkappa_a^2} + \sqrt{p_a^2 + \varkappa_b^2} = \sqrt{p_c^2 + \varkappa_c^2} + \sqrt{p_c^2 + \varkappa_d^2}$. The sum over $\ell', \mu', s', m', J, M, \ell, \mu, s, m$ is implied.

$$
(\vartheta \varphi p | \ell \mu p_0) = \frac{2\pi h \sqrt{2R}}{\sqrt{V}} \cdot \frac{i^{-\ell}}{p} \cdot Y_{\ell\mu}(\vartheta, \varphi) \cdot (p | p_0)
$$

where $Y_{\ell\mu}(\vartheta, \varphi)$ is the spherical function.
See the other notations in [3] (particularly in Appendix II).

Making use of the formula $\rho' = S\rho S^{+}$ we may obtain now the density matrix $\rho'$ of the reaction products in the representation of their momenta and spin projections ( $\rho$ -is the density matrix of the beam-target in the same representation). The problems of normalization and the obtaining of the cross section in the system $K_s$ are solved in the same way as in the nonrelativistic case (see |3|). The statistical polarization tensors may be introduced instead of the density matrices just in a similar manner. All the formulae will be of the same form as the nonrelativistic ones |3|. The difference is that the spin projections $m_a, m_b, m_c, m_d$ (or $\tau, \nu$ ) as well as the total spin are refered to the $K_s$ system. The same spin state is of another form in the other Lorentz frame $\tilde{K}$ (for instance, in the Laboratory one).

Let a certain spin state be defined in $K_s$. In order to know how it is described in $\tilde{K}$ it is necessary to find the transformation function from the representation in the eigenvalues of $S^2$ and $S_z$ in the system $K_s$ to the representation in the eigen values of $\tilde{S}^2$ and $\tilde{S}_z$ which are the square and the projection of the same operator but in the frame $\tilde{K}$. As $S^2$ is the Lorentz invariant the operator $\vec{\tilde{S}}$ is a vector rotated in comparison with $\vec{S}$. Therefore, the transformation function is the same as that obtained when solving the task of describing the given spin state in the rotated frame of space axes:

$$(\tilde{m}|m) = D^{i}_{\tilde{m},m}(\phi_2,\theta,\phi_1) = e^{-i\tilde{m}\phi_2}\, i^{m-\tilde{m}}\, P^{i}_{\tilde{m}m}(\cos\theta)\, e^{-im\phi_1} \qquad (5)$$

$P^{l}_{\tilde{m}m}(\cos\theta)$ is determined in |10| (formula (22) page 77. Note that the matrix $P^{1}_{mn}$ written out on page 78 in the explicit form does not correspond to (22) and is not correct). If the rotation is interpreted as the rotation of the vector with respect to the fixed coordinate frame it consists of 1) the rotation of the vector around the Z axis at the angle $\phi_1$. 2)the rotation around the y axis at the angle $\theta$; 3) the rotation around Z at $\Phi_2$. All the rotations are contraclockwise. It is shown in the Appendix how to find the axis of the rotation and the angle of the rotation $\Omega$ of the spin vector when transforming from $K_s$ into $\bar{K}$. To transform the spin state of the reaction product from $K_s$ into the lab. system we obtain such Euler angles of rotation $\{\phi_1, \theta, \phi_2\} = \{+\varphi, \Omega, -\varphi\}$

$$\sin\Omega = \frac{\beta v \sin\vartheta (1+\gamma+\gamma_\beta+\gamma')}{(1+\gamma)(1+\gamma_\beta)(1+\gamma')} \cdot \gamma \cdot \gamma_\beta \tag{6}$$

where $v = |\vec{p}|/\omega = \sqrt{\omega^2 - \varkappa^2}/\omega$; $\gamma = \omega/\varkappa$; $\gamma_\beta = (1-\beta^2)^{-1/2}$; $\gamma' = \omega'/\varkappa$.

$\omega'$ is the energy of the reaction product in the lab. system, $\vartheta$ and $\varphi$ are the spherical angles of its momentum $\vec{P}$ in $K_S$, defined with respect to the frame of axes with the axis $z \| \vec{\beta}$ /the axes x and y are chosen arbitrary/.

In Stapp's formula (48) in |4| for $\sin\Omega$ there is an error/ or a misprint/: the factors $\gamma \cdot \gamma_\beta$ are absent there ( $\gamma^{(a)} \cdot \gamma^{(b)}$ in his notations). If we repeat Stapp's calculations (in accordance with his arguments) we shall obtain namely formula (6). The rotation at the angle $\Omega$ must be made around the vector $[\vec{\beta} \times \vec{p}]$ contraclockwise, $\vec{\beta}$ is the velocity of the lab. system relative to the c.m.s. of the reaction (see Appendix).

The relativistic effect of the spin state rotation which was set forth above in no way display itself when transforming the angular distribution into the lab. system (since the angular distribution is the zero rank polarization tensor). It is only necessary to make the usual (kinematic) relativistic transformation of the angles from $K_s$ into the lab. system. The nonrelativistic theory of the angular distribution in the reactions with the nonpolarized beam and target remains also correct in the relativistic region (only the meaning of the quantities involving into the formulae changes or specified).

As for the vector and, moreover, the tensors of polarization they are not directly measured in the experiment.

In order to measure the polarization vector of the product c of the reaction a + b → c + d it is necessary to scatter c on the target e and to measure the asymmetry in the angular distribution of the scattered c. Then we obtain some information about the polarization vector $\vec{\mathcal{P}}'$ in the c.m.s. $K_s'$ of the reaction c + e → c + e . The polarization vector which we are looking for is obtained from $\vec{\mathcal{P}}'$ with the help of rotation. The angle of the rotation is found using formula /6/. Indeed, in the successive Lorentz transformations from Ks into the lab. system (with the help of the known velocity $\vec{\beta}$ ) and further into the system $K_s'$ (the velocity $\vec{\beta}'$ ) the rotation really occurs only in the first transformation, since the momentum $\vec{p}_c'$ of the particle c in the lab. system is parallel to $\vec{\beta}'$, so that $\sin\Omega_2 \sim \sin(\widehat{\vec{\beta}'\,\vec{p}_c'}) = 0$. This problem is examined in [4] in more detail; note that the consideration given there may be applied to any spin.

And in general, the relativistic rotation of the spin is essential, apparently, only when considering the cascades of the reactions. In the next paragraph we shall be concerned with the relativistic change of the angular correlations in the cascades of the $a + b \rightarrow c + d$, $c \rightarrow e + f$ type. ~~In co~~

In conclusion we note that in the transition from $K_s$ into the system $K_0$, where the particle is at rest, the description of spin state does not undergo any changes since in this case $\vec{\beta} \parallel \vec{p}$ and then $\Omega = 0$. Therefore one may consider that the quantities $m_a$, $m_b$ etc. describe the spin states of particles in their rest systems $K_0$. Such an interpretation is preferential than the former one: the spin state of the particles is described by the quantities the determination of which is independent of the system $K_s$, i.e., of what target the particle reacts with, what is its energy or what is the energetic balance of the reaction[5].

---

5/ In connection with this interpretation the following puzzle may occur. Since there exists only one system where the particle is at rest then in any reaction m signify the same: the projections of spins in their rest systems. Therefore, it looks like as if no transformations of spin state are not really necessary.

The point is that if $\vec{V}_{21}$ is the velocity of the system $K_2$ relative to $K_1$ and $\vec{V}_{32}$ the velocity of $K_3$ relative to $K_2$, then the velocity $\vec{V}_{31}$ (which is function of $\vec{V}_{21}$ and $\vec{V}_{32}$ of course) appeared not to be parallel to $\vec{V}_{13}$ if $[\vec{V}_{21} \times \vec{V}_{32}] \neq 0$ (see [11], §22). The transformation from $K_1$ into $K_3$ must have the form of the Lorentz transformation with the space rotation /ibid., formula(58)/. If the particle was at rest in $K_1$, then in $K_3$ it has the velocity $\vec{V}_{13}$ and it is possible to pass with the help of the usual Lorentz transformation with this velocity into the system $K_4$ where this particle is again at rest. The calculations show that the product of the transformations from $K_1$ into $K_3$ and further into $K_4$ has the form of a purely space rotation: $\vec{S}_{(4)} = D^{-1}\vec{S}_{(1)}$ if $D\vec{V}_{13} = -\vec{V}_{31}$ /of course, the space axes of the Lorentz systems $K_1, K_2, K_3, K_4$ are assumed to be parallel/.

## § 4. RELATIVISTIC ANGULAR CORRELATIONS IN THE CASCADES OF THE $a + b \rightarrow c + d$, $c \rightarrow e + f$ TYPE

Let us consider first the known cascade $\pi^- + p \rightarrow Y + k$, $Y \rightarrow N + \pi$. If the first reaction occurs near the threshold the correlation in the angle $\gamma$ between the direction of the incident $\pi^-$-mesons and that of the decay nucleon makes it possible to determine the spin $j$ of the hyperon $Y$. This correlation may be obtained if the explicit expression for the statistical tensors of the hyperon polarization $\rho(q,\nu)$ will be substituted into the expression

$$\mathcal{F}(\vartheta,\varphi) = \frac{w}{\sqrt{4\pi}} \sum_{q=0,2}^{2j-1} (2q+1)^{-1/2} Q(j,q) \sum_{\nu=-q}^{q} Y_{q\nu}(\vartheta,\varphi)\rho(q,\nu) \tag{7}$$

for the angular distribution of the hyperon decay products in its rest system $K_Y$. /see |12| and |3|/. In the centre of inertia system $K_s$ of the reaction $\pi^- + p \rightarrow Y + k$ near the threshold /the axis $Z$ is directed along the $\pi^-$-meson beam)

$$\rho_s(q,\nu) \sim Q(j,q)\,\delta_{\nu,0} \tag{8}$$

The nonrelativistic correlation in the angle $\gamma$ is obtained simply by substituting (8) into (7):

$$\mathcal{F}_{nr}(\vartheta,\varphi) \sim \sum_{q=0}^{2j-1} (2q+1)^{-1/2} Q^2(j,q)\, Y_{q,0}(\vartheta,0) \sim$$

$$\sim \sum_{q=0}^{2j-1} Q^2(j,q)\, P_q(\cos\gamma). \tag{9}$$

As a matter of fact, it is necessary to substitute into (7) not $\rho_S(q,\nu)$ but the statistical tensors of the hyperon refered to the system $K_Y$ :

$$\rho(q,\nu)=\sum_{\nu'} D^{q}_{\nu,\nu'}(\varphi_c,\Omega(\vartheta_c),-\varphi_c)\rho_S(q,\nu')=$$
$$=\sqrt{4\pi/2q+1}\;Y^{*}_{q,\nu}(\Omega,\varphi_c)Q(j,q). \tag{10}$$

Here $\varphi_c$ and $\vartheta_c$ are the spherical angles of the hyperon direction of emission in the system $K_s'$. The angle $\Omega$ is determined by formula (6), since in the transformation from $K_s$ into the lab. system and further into $K_Y$ the rotation occurs only in the transitiön from $K_s$ into the lab. system. /Let us note that $\mathcal{F}(\vartheta,\varphi)$ is obtained by the transformation of the measured distributi on from the lab. system into $K_Y$, but not by the transformation from $K_s$ into the hyperon rest system/.

Substituting (10) into (7) we obtain

$$\mathcal{F}_\tau(\vartheta,\varphi)\sim\sum_{q=0,2}^{2j-1}(2q+1)^{-1}Q^2(j,q)\sum_{\nu=-q}^{q}Y_{q,\nu}(\vartheta,\varphi)Y^{*}(\Omega,\varphi_c)=$$
$$=1/4\pi\sum_{q=0}^{2j-1}Q^2(j,q)P_q(\cos\gamma_\tau) \tag{11}$$

where $\gamma_\tau$ is now the angle between the direction of the emission of the decay products and the direction $\{\Omega(\vartheta_c),\varphi_c\}$. Thus, the form of the correlation remains the old one if we change the determination of the angle $\gamma$ . $\Omega$ doës not exceed $1.5^0$ for the expreiment under discussion. If one constructs the distribution in $\gamma$ selecting only the cases with the fixed $\vartheta_c\cong 90^0$ and the fixed $\varphi_c$ , the difference between the nonrelativistic and relativistic correlations may be 3% for $j=3/2$

and 5% for $j = 5/2$ . Actually, all the cascade cases are used in the expyeriment for the construction of $\mathcal{F}(\gamma)$ . If (11) is integrated by $\varphi_c$ the difference of the obtained correlation $\mathcal{F}_\tau(\gamma, \Omega(\vartheta_c))$ from the nonrelativistic (9) will not exceed 0.1% for all values of $\gamma$ and $\vartheta_c$ ($j \leq 5/2$).

In the case of the $K^- + p \to Y + \pi$, $Y \to N + \pi$ cascade the angular correlation does not involve any unknown parameters and is dependent only upon the spin $j$ of hyperon if the energy of $K^-$-mesons does not exceed 20-30 MeV. But this energy must be sufficiently high ( $\gg$ 0,01 MeV) in order the K-mesoatom not to be produced (see in more detail /9/ ). In the c.m.s. of the reaction $K^- + p \to Y + \pi$ wìth Z axis parallel to the direction $\vec{n}_Y$ of the hyperon emission and the axis $y \parallel [\vec{n}_K \times \vec{n}_Y]$ where $\vec{n}_K$ is the direction of the incident K-meson beam

$$\rho_s(q,\tau) \sim Q(j,q) \cdot \delta_{\tau,0}$$

In the same set of the axes but in the rest system $K_Y$ of the hyperon

$$\rho(q,\tau) = \sum_{\tau'} D^q_{\tau,\tau'}(0,\Omega,0)\rho_s(q,\tau') =$$

$$= \sqrt{4\pi/2q+1}\; Y^*_{q,\tau}(\Omega,0)\, Q(j,q). \tag{12}$$

The difference of the relativistic correlation $\mathcal{F}_\tau(\vartheta,\varphi)$ from the nonrelativistic is in general the same: substituting /12/ into /7/ we obtain the correlation $\mathcal{F}_\tau(\Theta)$ in the angle $\Theta$ between the direction $\vec{n}$ of the decay product emission and the vector obtained by rotating $\vec{n}_Y$ at the angle $\Omega$ around the vector $[\vec{n}_k \times \vec{n}_Y]$ /i.e., in the plane of the reaction/. The nonrelativistic correlation had the same form but $\theta$ was the angle between $\vec{n}$ and $\vec{n}_Y$.

Although the correlation proposed by Adair |13| (see also |9| assumes the energies greater than those near the threshold it does not change in the relativistic consideration: the cases when the hyperons are emitted at small angles to the direction of the incident beam are used in Adair's method and then $\Omega \cong 0$.

Since the most general cawe of the cascade a + b → c + d, c → e + f when all the spins are arbitrary and the correlation is dependent upon the unknown parameters is of no practical interest we simply note without proving that the nonrelativistic form of the angular correlation may be conserved. For this we must find a particular set of axes for each case of the cascade (using measured angles of the direction of emission of the particle c). The emergence angles of the decay products of C are calculated in respect to this set of axes. The distribution over such recalculated angles has the old nonrelativistic form.

Of course, to make up for it, the prescription for the construction of the angular correlation is changed.

## Appendix

1. The expression of $\vec{S}$ in terms of $M_{\mu\nu}$ and $P_\mu$.

Let $\varkappa$ be the rest mass of the particle and $\omega = \sqrt{p^2+\varkappa^2}$.

$$\vec{M} = [\vec{r} \times \vec{p}] + \vec{S}$$

$$\vec{N} = \vec{r}\omega - i\,\vec{P}/2\omega + \frac{[\vec{p}\times\vec{S}]}{\omega+\varkappa} \qquad (A.1)$$

The four vector $\Gamma_\sigma = \frac{1}{2i}\,\varepsilon_{\mu\nu\sigma\lambda} M_{\mu\nu} P_\lambda$ ( $\varepsilon_{\mu\nu\sigma\lambda}$ is the completely antisymmetrical tensor of the fourth rank, $\varepsilon_{1234} = 1$ ) then has the form:

$$\vec{\Gamma} = \vec{S}\varkappa + \frac{(\vec{p}\,\vec{S})}{\omega+\varkappa}\,\vec{p}\,, \qquad \Gamma_4 = i(\vec{S}\cdot\vec{p}). \qquad (A.2)$$

Noting that $(\vec{\Gamma}\cdot\vec{p}) = \omega(\vec{S}\,\vec{p})$, we find from (A.2)

$$\vec{S} = \vec{\Gamma}/\varkappa - \frac{1}{\varkappa\omega(\omega+\varkappa)}(\vec{\Gamma}\cdot\vec{P})\cdot\vec{P}. \qquad (A.3)$$

All these operator equations ought to be understood in the momentum representation.

From the second relation in /A.1/ we obtain now

$$\vec{r}\omega = \vec{N} + i\,\vec{P}/2\omega - \frac{[\vec{P}\times\vec{\Gamma}]}{\varkappa(\omega+\varkappa)} \qquad (A.4)$$

2. The vector $\vec{\tilde{S}}$ in the new system $\tilde{K}$, which moves relative to $K_s$ with the velocity $\vec{\beta}$ (in the units of the velocity of light) may be found now in the following way.

Substituting into the right and into the left sides of the following expressions /see |11|, § 18, formula (25)/

$$\tilde{\vec{\Gamma}} = \vec{\Gamma} + \vec{\beta}\left\{(\vec{\Gamma}\cdot\vec{\beta})\frac{\gamma_\beta - 1}{\beta^2} - \gamma_\beta \; \Gamma_4/i\right\}$$

$$\tilde{\Gamma}_{4/i} = \gamma_\beta\left\{\Gamma_4/i - (\vec{\beta}\;\vec{\Gamma})\right\} \qquad (A.5)$$

the expressions (A.2) of $\vec{\Gamma}$ and $\tilde{\vec{\Gamma}}$ in terms of $\vec{S}$ , $P_\mu$ and $\tilde{\vec{S}}$ , $\tilde{P}_\mu$ respectively and substituting for $\tilde{P}_\mu$ their expressions through $P_\mu$ /which have the same form /A.5/) we obtain the expressions of $\tilde{\vec{S}}$ through $\vec{S}$ . First of all we ascertain that $\tilde{\vec{S}}$ if the linear combination of vector $\vec{S}$ , $\vec{\beta}$ and $\vec{P}$. It means that the vector $\tilde{\vec{S}}$ is obtained from $\vec{S}$ by means of the rotation around the axis perpendicular to $\vec{\beta}$ and $\vec{P}$. There remains only to find out the magnitude and the sign of the angle of the rotation around this axis. For this we choose a convenient set of the spatial axes /it is clear that the angle of the rotation must not depend upon the choice of the axes/: $z \parallel \vec{\beta}$ , $y \parallel [\vec{\beta} \times \vec{P}]$ . The rotation of a vector around the axis $y$ contraclockwise at an angle $\Omega$ must have the form

$$\tilde{S}_x = \cos\Omega\, S_x + \sin\Omega\, S_z$$

$$\tilde{S}_z = -\sin\Omega\, S_x + \cos\Omega\, S_z \, . \qquad (A.6)$$

Representing the expression $\tilde{\vec{S}}$ through $\vec{S}$(in the chosen set of the axes) in the form of (A.6) and finding the coefficnet which has $S_z$ in the expression for $\tilde{S}_x$(as one having the simplest form/ we obtain formula (6) in § 3 for $\sin\Omega$ .

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