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APPLICATION OF DISPERSION RELATIONS TO $\pi - N$ SCATTERING AT LOW ENERGIES

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Abstract

The consequences of dispersion relations for scattering at low energies are investigated without passing to the Low type equation. The relations between the scattering lengths in different states have been obtained from the values of the derivatives of the dispersion relations when $k^2 = 0$.

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1. When applying the dispersion relations to the low energy region it became customary to pass to the Chew-Low equations [1] and then to investigate the consequences of these equations.

In the present paper the consequences of the dispersion relations for πN scattering at low energies are considered without using the Low equations.

Let us consider the dispersion relations for πN -scattering [2] written in the form

$$D_\pm(K)-\frac{1}{2}\left(1+\frac{\omega}{\mu}\right)D_\pm(0)-\frac{1}{2}\left(1-\frac{\omega}{\mu}\right)D_\mp(0)=K^2 J_\pm(\omega)\pm\frac{2f^2}{\mu^2}\cdot\frac{K^2}{\omega\mp\frac{M^2}{2M}}, \quad (1)$$

where the dispersion integral

$$J_\pm(\omega)=\frac{1}{4\pi^2}P\int_\mu^\infty\frac{d\omega}{K'}\left[\frac{\sigma_\pm(\omega')}{\omega'-\omega}+\frac{\sigma_\mp(\omega')}{\omega'+\omega}\right] \quad (2)$$

ω is the total energy of mesons in the lab. system, other notations are clear. Let us assume the meson energy in (1) to be small $\omega-\mu \ll \mu$. With $\eta^2=\frac{K^2}{\mu^2}\to 0$ both sides of (1) vanish. Therefore, in order to obtain the consequences of the dispersion relations for $\eta^2\to 0$ let us take the values of the derivatives over k^2 of (1) and then put $\eta^2\to 0$. When calculating the derivatives we make use of the representation of the energy dependence of the phase shifts in the form which the effective range theory |3|

$$\eta_B^{2\ell+1} \operatorname{ctg} \delta_\ell = \frac{1}{a_{2\ell+1}} + P_\ell \eta_B^2 + Q_\ell \eta_B^4 \tag{3}$$

where $a_{2\ell+1}$ is the scattering length in ℓ -state, $K_B = \eta_B \mu$ is the meson momentum in the c.m.s. and

$$\frac{K}{K_B} = \frac{\eta}{\eta_B} = \left(1 + \frac{2\omega}{M} + \frac{\mu^2}{M^2}\right)^{1/2} \tag{4}$$

The derivatives of the second and of the third term in the left-hand side of (1) will be as follows

$$\frac{1}{2}\left(\frac{\omega}{\mu}\right)^{(n)}\left[D_\mp(0) - D_\pm(0)\right] = \pm\frac{1}{2}\left(\frac{\omega}{\mu}\right)^{(n)}\cdot\left[D_{-}(0) - D_{+}(0)\right].$$

Since

$$\left(\frac{\omega}{\mu}\right)' = \frac{1}{2}\left(\frac{\mu}{\omega}\right),\ \left(\frac{\omega}{\mu}\right)'' = -\frac{1}{4}\left(\frac{\mu}{\omega}\right)^3,\ \left(\frac{\omega}{\mu}\right)''' = \frac{3}{8}\left(\frac{\mu}{\omega}\right)^5, \ldots$$

/ $'$ everywhere means the differentiation over η^2 /

then when $\eta^2 \to 0$ we have

$$\pm\frac{1}{4}\left[D_-(0) - D_+(0)\right] = \pm\frac{\lambda_c}{6}\left(1 + \frac{\mu}{M}\right)(a_1 - a_3);\quad \left(\lambda_c = \frac{\hbar}{\mu e}\right) \tag{5}$$

for the first derivative, and

$$\mp\frac{1}{8}\left[D_-(0) - D_+(0)\right] = \mp\frac{\lambda_c}{12}\left(1 + \frac{\mu}{M}\right)(a_1 - a_3) \tag{6}$$

for the second one. a_3 and a_1 denote, as usual, the scattering lengths in the S-states with the isotopic spin T = 3/2 and T = 1/2, respectively.

The non-observed region gives for the first derivative

$$\pm 2f^2\lambda_c\left\{\frac{1}{\frac{\omega}{\mu}\mp\frac{\mu}{2M}}-\frac{\eta^2}{\left(\frac{\omega}{\mu}\mp\frac{\mu}{2M}\right)^2}\cdot\frac{1}{2}\left(\frac{\mu}{\omega}\right)\right\}_o^{\mp}=\pm\frac{2f^2\lambda_c}{\left(1\mp\frac{\mu}{M}\right)} \qquad (7)$$

and

$$\mp\frac{2f^2\lambda_c}{\left(1\mp\frac{\mu}{2M}\right)^2} \qquad (8)$$

for the second one. The sign o everywhere implies that the expression with $\eta^2=0$ is meant. If we restrict ourselves by the values of only first two derivatives with $\eta^2=0$ it is sufficient to present for example, $D_+(K)$, in the form

$$2K_\beta^2 D_+(K)=K\left\{\sin 2\alpha_3+\sin 2\alpha_{31}+2(\sin 2\alpha_{33}+\sin 2\delta_{33})+3\sin 2\delta_{35}\right\} \qquad (9)$$

where δ_{33} and δ_{35} denote the phase shifts in the state $d_{3/2}$ and $d_{5/2}$ with T=3/2, whereas for the S and P-states phase shifts the conventional notations are used. Making use of |3| let us present (9) in the form

$$K_\beta D_+(K)=\lambda_c K\left\{\frac{A_3}{A_3^2+x}+\frac{xA_{31}}{A_{31}^2+x^3}+2\left[\frac{xA_{33}}{A_{33}^2+x^3}+\frac{x^2B_{33}}{B_{33}^2+x^5}\right]+\frac{3x^2B_{35}}{B_{35}^2+x^5}\right\} \qquad (10)$$

where A_3, A_{31}, A_{33}, B_{33}, B_{33} and B_{35} denote the righthand sides of (3) for the states $S_{1/2}$, $P_{1/2}$, $P_{3/2}$, $d_{3/2}$ and $d_{5/2}$ with $T=3/2$ whereas $x=\eta_\beta^2$

For the values of the first derivative of (10) when x = 0 we obtain

$$D'_{+0}(K)=\frac{\lambda_c}{\left(1+\frac{\mu}{M}\right)}\left\{2a_{33}+a_{31}+\frac{\mu}{2M}a_3-P_3a_3^2-a_3^3\right\} \qquad (11)$$

Collecting (6), (7) and (11) from the value of the first derivative of (1) when $\eta^2 = 0$ we have the expression

$$\frac{\lambda_c}{(1+\frac{\mu}{M})}\left\{\overset{\circ}{2a_{33}} + a_{31} + \frac{\mu}{2M}a_3 - P_3 a_3^2 - a_3^3\right\} + \frac{\lambda_c}{6}\left(1+\frac{\mu}{M}\right)(a_1 - a_3) =$$

$$= \left\{K^2 J_+\right\}_0' + \frac{2f^2\lambda_c}{1-\frac{\mu}{2M}} \qquad (12)$$

which establishes the relation between the meson scattering lengths in different states resulting from the dispersion relations. Further we denote $K^2 J_\pm(\omega)$ by $F_\pm(\omega)$. For the scattering of negative mesons on hydrogen, expressing $D_-(K)$ in terms of the scattering amplitudes in the states with the definite values of the isotopic spin

$$3D_-(K) = 2D_1(K) + D_3(K)$$

we obtain analogously

$$\frac{\lambda_c}{3(1+\frac{\mu}{M})}\left\{\frac{\mu}{2M}(2a_1 + a_3) - (2a_1^3 + a_3^3) - (2P_1 a_1^2 + P_3 a_3^2) + 4a_{13} + \right.$$

$$\left. + 2(a_{33} + a_{11}) + a_{31} - \frac{1}{2}\left(1+\frac{\mu}{M}\right)^2 (a_1 - a_3)\right\} = F_{-0}' - \frac{2f^2\lambda_c}{(1+\frac{\mu}{2M})} \qquad (13)$$

For the half-sum of (11) and (12) representing $\pi^0 N$ scattering one gets

$$\frac{1}{3}\,\frac{\lambda_c}{(1+\frac{\mu}{M})}\left\{\frac{\mu}{2M}(a_1 + 2a_3) - (a_1^3 + 2a_3^3) - (P_1 a_1^2 + P_3 a_3^2) + 2(a_{13} + a_{31}) + \right.$$

$$\left. + a_{11} + 4a_{33}\right\} = \left\{\frac{F_+' + F_-'}{2}\right\}_0 + \frac{\mu}{2M}\,\frac{2f^2\lambda_c}{\left[1-\left(\frac{\mu}{2M}\right)^2\right]} \qquad (14)$$

The substitution of the experimental data about the phase shifts [4]

$\alpha_3 = -(0,105 \pm 0,010)\eta_b \quad P_{33} = 0,6 \quad d_{33} = 0,0035$

$\alpha_1 = (0,165 \pm 0,012)\eta_b \quad Q_{33} = -0,8 \quad d_{35} = -0,0035$

$a_{33} = 0,235$

with $2f^2 = 0,16$ (and $2f^2 = 0,19 \pm 0,01$[31]) and other coefficients equal to zero gives for $D'_{\pm 0}$ and $F'_{\pm 0}$ the values

$$D'_{+0} = 0,40\lambda_c \quad , \quad D'_{-0} = 0,14\lambda_c$$

and

$$F'_{+0} = 0,28(0,25)\lambda_c \quad , \quad F'_{-0} = 0,24(0,27)\lambda_c$$

The contribution of the nonobserved region -0.173(0.206) and + 0.149(0.177) being considerable. For the half-sum

$$\left\{\frac{F'_+ + F'_-}{2}\right\}_0 = 0,26\lambda_c$$

the contribution of the f^2 term is only $-0.01\ \lambda_c$. The calculation of $F'_{\pm 0}$ by the data about the total cross sections is discussed in the next section.

For the second derivative of (10) we have

$$D''_{+0}(k) = \lambda_c\left(1 + \frac{\mu}{M}\right)\left[2a_3^3(Q_3 + P_3)(a_3 + 2P_3) - 2a_3^2(Q_3 + a_3P_3^2) - \right.$$

$$\left. - 3(a_{31}^2P_{31} + 2a_{33}^2P_{33}) + 2(2d_{33} + 3d_{35})\right] + \frac{\mu}{M}\cdot\frac{\lambda_c}{\left(1+\frac{\mu}{M}\right)^3}\left[2a_{33} + \right. \tag{15}$$

$$\left. + a_{31} - P_3a_3^2 - a_3^3\right] - \frac{\mu}{4M}\cdot\frac{\lambda_c}{\left(1+\frac{\mu}{M}\right)}\left[1 + \frac{\mu}{M}\left(1+\frac{\mu}{M}\right)^2\right]a_3$$

where the scattering lengths in the states $d_{3/2}$ and $d_{5/2}$ are denoted by d_{33} and d_{35}. With the help of (10,),(6) and (8) from the magnitude of the second derivative of (1) when $\eta = 0$ we obtain the second group of the relations

$$D''_{+0}(K) - \frac{\lambda_c}{12}\left(1+\frac{\mu}{M}\right)(a_1 - a_3) = F''_{+0} - \frac{2f^2\lambda_c}{\left(1-\frac{\mu}{2M}\right)^2} \qquad (16)$$

for $\pi^+ - p$ -scattering

$$D''_{-0}(K) + \frac{\lambda_c}{12}\left(1+\frac{\mu}{M}\right)(a_1 - a_3) = F''_{-0} + \frac{2f^2\lambda_c}{\left(1+\frac{\mu}{2M}\right)^2} \qquad (17)$$

for the negative meson scattering and

$$\left\{\frac{D''_+ + D''_-}{2}\right\}_0 = \left\{\frac{F''_+ + F''_-}{2}\right\}_0 - \frac{\mu}{M}\cdot\frac{2f^2\lambda_c}{\left[1-\left(\frac{\mu}{2M}\right)^2\right]} \qquad (18)$$

where D''_- is constructed by analogy with (15).

When using the data about the phase shifts given earlier we have

$$\lambda_c^{-1} D''_{+0} = \left(1+\frac{\mu}{M}\right)\left[2a_3^5 - 6a_{33}^2 P_{33} + 2(2d_{33} - 3d_{35})\right] + \\ + \frac{\mu/M}{\left(1+\frac{\mu}{M}\right)^3}(2a_{33} - a_3^3) - \frac{\mu/4M}{\left(1+\frac{\mu}{M}\right)}\left[1+\frac{\mu}{M}\left(1+\frac{\mu}{M}\right)^2\right]a_3 \qquad (19)$$

$$3\lambda_c^{-1} D''_{-0} = \left(1+\frac{\mu}{M}\right)\left[2(a_1^5 + a_3^5) - 6a_{33}^2 P_{33} + 4(2d_{13} + d_{33}) + \right. \\ \left. + 6(2d_{15} + d_{35})\right] + \frac{\mu}{M}\cdot\frac{2a_{33} - 2a_1^3 - a_3^3}{\left(1+\frac{\mu}{M}\right)^3} - \frac{\mu}{4M}\,\frac{\left[1+\frac{\mu}{M}\left(1+\frac{\mu}{M}\right)^2\right](2a_1 + a_3)}{\left(1+\frac{\mu}{M}\right)} \qquad (20)$$

The substitution of the experimental data about the phase shifts gives

$$D''_{+0} = -0{,}180\lambda_c\ ,\quad D''_{-0} = -0{,}052\,\lambda_c\ ,\quad \left\{\frac{D''_+ + D''_-}{2}\right\}_0 = -0{,}116\,\lambda_c$$

$$F''_{+0} = -0{,}019\,\lambda_c\,(2f^2=0{,}16),\quad F''_{+0} = +0{,}016\,\lambda_c\ (2f^2=0{,}19)$$

$$F''_{-0} = -0{,}162\,\lambda_c\,(2f^2=0{,}16),\quad F''_{-0} = -0{,}187\,\lambda_c\,(2f^2=0{,}19)$$

$$\left\{\frac{F''_+ + F''_-}{2}\right\}_0 = -0{,}090\,\lambda_c(2f^2=0{,}16),\quad \left\{\frac{F''_+ + F''_-}{2}\right\}_0 = -0{,}085\,\lambda_c\,(2f^2=0{,}19)$$

the contribution of the nonobserved region is found to be very essential for the first two values and is about 35% for the half-sum. The value of the second derivative of F_+ is mainly obtained as a difference of the resonance contribution and that of the nonobserved region (the contribution of which is 0.188 λ_c and 0.222 λ_c). In view of the decrease of the role of the resonance transition for the scattering of negative mesons the contribution of the nonobserved region equal to −0.138 λ_c and −0.161 λ_c is a principal one.

2. In the previous section the values of the derivatives of F_+ have been obtained from the experimental data on the phase shifts. The same magnitudes may be calculated directly from (2) and the experimental data on the total cross sections. In such a manner one may check the consistence of the experimentally determined phase shifts with the dispersion relations. On the other hand, one may hope, that the discrepancy which Puppi and Stanghellini pointed out, if any will

be expressed in the most explicit form.

After obtaining F_+ the values of the derivatives may be calculated by the direct differentiation. However, the differentiation of the curve calculated by the experimental data gives rise to large errors, therefore one may present the derivative in another form. Taking an interest in the values of the derivatives when $\eta^2 \to 0$, let us divide the dispersion integral $J_+(\omega)$ into three parts

$$4\pi^2 J_+(\omega) = P\int_\mu^z \frac{d\omega'}{K'}\frac{\sigma_+(\omega')}{\omega'-\omega} + \int_z^\infty \frac{d\omega'}{K'}\frac{\sigma_+(\omega')}{\omega'-\omega} + \int_\mu^\infty \frac{d\omega'}{K'}\frac{\sigma_-(\omega')}{\omega'+\omega} \qquad (21)$$

so that the second integral do not contain the singularity $(z > \omega)$ any longer. Now one may choose such a z that the expression for $\sigma_+(\omega)$ in the restriction by S and P -waves using (3) may be presented as follows

$$\sigma_+(\omega) = 4\pi\left\{a_3^2 + \frac{(a_{31}^2 + 2a_{33}^2)}{(1+\frac{\mu}{M})^4}K^4\right\} \qquad (22)$$

Substituting (22) into the first integral into (21) we obtain for it

$$\pi L(\omega) = (4\pi)^{-1} P\int_\mu^z \frac{d\omega'}{K'}\frac{\sigma_+(\omega')}{\omega'-\omega} = a_3^2 J_\alpha(K) + \frac{(a_{31}^2 + 2a_{33}^2)}{(1+\frac{\mu}{M})^4} J_\beta(K)$$

where

$$J_\alpha(K) = -\frac{1}{K}\ln\left[1 + \frac{(\omega-\mu)(z+\mu) + K\sqrt{z^2-\mu^2}}{\mu(z-\omega)}\right] \qquad (23')$$

$$\mu^4 J_\beta(K) = \frac{(z^2-\mu^2)^{3/2}}{3} + \frac{(z^2-\mu^2)^{1/2}}{2}(\omega z + K^2) + K^4 J_\alpha(K) - \mu\frac{\mu^2 - K^2}{2}\ln\left[\frac{\sqrt{z^2-\mu^2}+z}{\mu}\right] \qquad (23'')$$

At small k^2

$$\mu^2 J_\alpha(p) = -\sqrt{z^2-\mu^2}\left\{\frac{1}{\frac{z}{\mu}-1} + \frac{k^2}{6}\cdot\frac{2-\frac{z}{\mu}}{(\frac{z}{\mu}-1)^2} + \frac{k^4}{40}\,\frac{3(\frac{z}{\mu})^2-9(\frac{z}{\mu})+8}{[(\frac{z}{\mu})-1]^3} + \dots\right\} \qquad (23^a)$$

So that (when $z = 1{,}43\mu$)

$$J_\alpha(0) = -\frac{\sqrt{z^2-\mu^2}}{\mu(z-\mu)} = -2{,}38\,\lambda_c$$

$$\mu^4 J_\beta(0) = \frac{(z^2-\mu^2)^{3/2}}{3} + \frac{\mu z(z^2-\mu^2)^{1/2}}{2} - \frac{\mu^3}{2}\ln\left[\frac{\sqrt{z^2-\mu^2}+z}{\mu}\right] = 0{,}639\mu^3$$

and

$$J_+(\mu) = \frac{a_3^2}{\pi}J_\alpha(0) + \frac{(a_{31}^2+2a_{33}^2)}{\pi(1+\frac{\mu}{M})^4}J_\beta(0) + \frac{1}{4\pi^2}\int_z^\infty \frac{d\omega'}{k'}\,\frac{\sigma_+(\omega')}{\omega'-\mu} + \frac{1}{4\pi^2}\int_\mu^\infty \frac{d\omega'}{k'}\,\frac{\sigma_-(\omega)}{\omega'+\mu} \qquad (24)$$

The expression for $J_-(\mu)$ is obtained from (24) by the substitution

$$\sigma_+ \rightleftharpoons \sigma_-,\; 3a_3^2 \to a_3^2 + 2a_1^2;\; 3a_{33}^2 \to a_{33}^2 + 2a_{13}^2;\; 3a_{31}^2 \to a_{31}^2 + 2a_{11}^2. \qquad (25)$$

From (21) and (23) the expressions for the dispersion integrals follow which are convenient in the calculations in the low energy region $\omega-\mu \ll \mu$. Before making use of them for the calculations of the magnitudes of the derivatives when $\eta^2 \to 0$ let us note the following.

In the second term in (22)(the contribution of p waves) the transition from the c.m.s. to the lab. system is made approximately. More exactly, using (4) insead of $J_\beta(k)/(1+\mu/M)^4$ we get

$$L_\beta(k) = \int_\mu^z \frac{d\omega'}{\omega'-\omega}\cdot\frac{k'^3}{(1+\frac{2\omega'}{M}+\frac{\mu^2}{M^2})^2} = \frac{1}{(1+\frac{2\omega}{M}+\frac{\mu^2}{M^2})^2}\int_{\mu-\omega}^{z-\omega}\frac{dx\sqrt{R^3}}{x(1+\gamma x)^2}$$

where
$$\gamma = \frac{2\mu}{M} \cdot \frac{1}{\left(1+\frac{2\omega}{M}+\frac{\mu^2}{M^2}\right)^2} \quad , \quad R = k^2 + 2\omega x + x^2$$
or
$$\left(1+\frac{2\omega}{M}+\frac{\mathring{\mu}^2}{M^2}\right)^2 L_\beta(k) = J_\beta(k) - \gamma N_\beta = J_\beta(k) - \gamma\left\{\int_{\mu-\omega}^{z-\omega}\left[\frac{1}{1+\gamma_x}+\frac{1}{(1+\gamma_x)^2}\right]R^{3/2}dx\right\}. \quad (26)$$

Let us give the expression for $N_\beta(k)$ which is obtained in the expanision of the dominators of the integrands up to γ^2
$$N_\beta(k) = \left[\left(\frac{N^2}{2}-\frac{3}{4}\right)z\cdot N - \frac{3}{16}\ln(\omega+2z+N)\right] - 3\gamma\left[\frac{N}{5} - z\left(\frac{\omega N^3}{4}-\frac{3}{8}N\right) - \frac{3}{8}\ln(\omega+\right.$$
$$\left.+2z+N)\right] + 4\gamma^2\left[\left(\frac{z-1}{6}-\frac{7\omega}{30}\right)N^5 + \frac{(6\omega^2+1)}{6}\, z\left(\frac{N^3}{4}-\frac{3N}{8}\right) + \frac{6\omega^2+1}{16}\ln(\omega+2z+N)\right], \quad (27)$$

where $N=(z^2-1)^{1/2}$ $(\mu=1)$. The last term in (27) gives less than 6.5% of the value of N_β when k = 0. $\gamma N_\beta(0) = 0.085$ itself that leads to the replacement of $J_\beta(0) = 0.639$ for $L_\beta(0) = 0.554$ i.e., introduces the correction of about 15%. If one takes into account only the first term in (27) it gives $L_\beta(0) \cong 0.530$. from where it is clear that the last terms give < 5%. Since all the contribution from $J_\beta(0)$ is not great one may restrict oneself by (23) involving for the corrections the first term in (27).

Let us obtain now the expressions for the derivatives of $F_\pm$.

For the arbitrary momenta of

$$F'_+(\omega) = J_+(\omega) + \eta^2 J'_+(\omega) \quad (28)$$

It can be seen from (21), (23) and (27) that

$$\eta^2 J'_{+0}(\omega) = 0 \quad (29)$$

Since (28) is correct also for $\eta^2 J'_-$ one gets

$$F'_{\pm 0}(\omega) = J_{\pm}(\mu). \tag{30}$$

The last equation makes it possible to reduce the calculation of the derivative of F to the value of the dispersion integral at one point $\omega = \mu$. The value of the derivative is calculated from (30), (23) and (24). With $Z = 1{,}43\mu$ the contribution of the first two terms in (23) gives $- 0.026/\pi$ for the S-wave and $+ \frac{0{,}035}{\pi}$ for P-waves what is a very small fraction of the value $F'_{+0} = 0{,}28(0{,}25)\lambda_C$ obtained in the previous section.

For the interaction of negative mesons with protons we have analogously

$$J_-(\mu) = \frac{(a_3^2 + 2a_1^2)}{3\pi} J_\alpha(0) + \frac{a_{31}^2 + 2a_{11}^2 + 2a_{13}^2 + a_{33}^2}{3\pi\left(1+\frac{\mu}{M}\right)^4} L_\beta(0) +$$

$$+ \frac{1}{4\pi^2}\int_Z^\infty \frac{d\omega'}{k'}\cdot\frac{\sigma_-(\omega')}{\omega'-\mu} + \frac{1}{4\pi^2}\int_\mu^\infty \frac{d\omega'}{k'}\cdot\frac{\sigma_+(\omega')}{\omega'+\mu} \tag{31}$$

instead of (24).

For the contribution of the first two terms in (31) we have

$$\pi L_- = -0{,}051 + 0{,}012 = -0{,}039 .$$

Here the contributions of S and P-waves which are compatible turn out at $Z = 1{,}43\mu$ to be negligible in comparison with that of the integral components in (31) if one considers the results of the previous section. For the second derivative of F with the arbitrary momenta obtain

$$F''(\omega) = 2J'(\omega) + \eta^2 J''(\omega)$$

When $\eta^2 \to 0$ the second component vanishes so that

$$F_0''(\omega) = 2J'(\mu). \tag{32}$$

Dividing $J_+^{0\prime}(\omega)$ into the parts by analogy to (21), one may obtain from (23) for the contribution of S and p-waves ($M = 1$)

$$\pi L'_+(\mu) = -a_3^2 \frac{N(2-Z)}{6(Z-1)^2} + \frac{a_{31}^2 + 2a_{33}^2}{\pi(1+\frac{\mu}{M})^4}\left\{\frac{N}{2}\left(1+\frac{Z}{2}\right) + \frac{1}{2}\ln(N+Z)\right\} \tag{33}$$

and

$$J'_+(\mu) = L'(\mu) + \frac{1}{4\pi^2}\int_Z^\infty \frac{d\omega'}{k'}\,\frac{\sigma_+(\omega')}{(\omega'-\mu)^2} - \frac{1}{4\pi^2}\int_\mu^\infty \frac{d\omega'}{k'}\cdot\frac{\sigma_-(\omega')}{(\omega'+\mu)^2} \tag{34}$$

The contribution of $L'_+(\mu)$ is $= 0.002\,\lambda_c$. The expression for $J'_-(\mu)$ is obtained from (34) by the replacement of (25). The contribution of $L'_-(\mu)$ for $\pi^- p$ is $-0.002\,\lambda_c$.

3. The numerical values of the dispersion integrals are

$$J_+(\mu) = \frac{6{,}96}{4\pi^2} = 0{,}176$$

$$J_-(\mu) = \frac{4{,}99}{4\pi^2} = 0{,}126\,. \tag{35}$$

The comparison of (35) and (20) with the values obtained earlier, 0.28 (0.25) and 0.24 (0.27) makes possible the positive sign a_{33} to be reaffirmed once more. Moreover, it follows from (35) and (12) that a_{31} is a negative quantity, and

$$(2f^2=0{,}16)\quad a_{31} = -0{,}115 \qquad D'_{+0} = 0{,}258\,\lambda_c$$

$$(2f^2=0{,}19)\quad a_{31} = -0{,}080 \qquad D'_{+0} = 0{,}320\,\lambda_c\,. \tag{36}$$

It follows from (35) and (13) as a consequence of the dispersion relations that

$$(2f^2=0,16)\quad D'_{-0}=0,026\lambda_c\ ;\quad 2a_{13}+a_{11}=-0,139$$

$$(2f^2=0,19)\quad D'_{-0}=-0,00(4)\lambda_c\ ;\quad 2a_{13}+a_{11}=-0,208 \tag{37}$$

Under the assumption that $a_{13}=a_{31}$ from (37) follows that

$$a_{11}=0,09\,(2f^2=0,16)\ ,\quad a_{11}=-0,05\,(2f^2=0,19)\ . \tag{38}$$

It is to be emphasized that the errors in the numerical values of a_{31} and a_{11} are very difficult to be determined. Though the idea about the small and negative scattering length a_{31} and the small a_{11} is roughly in consistence with the experimental data about πN -scattering the value of D'_{-0} does not appreciably correspond to the experimental data.

For the values of the derivatives of the dispersion integrals as a result of the numerical integration we obtain

$$2J'_+(\mu)=0,08\lambda_c \tag{39}$$

$$2J'_-(\mu)=0,04\lambda_c\ . \tag{39'}$$

To obtain the consequences analogous to (36)-(38) is very difficult at present since more exact data are necessary.

Thus, the analysis which made use of the data about the scattering phases only at low energies supports Puppi-Stanghellini's result. While for the π^+p scattering the dispersion relations make it possible to determine a_{31} from the data about a_1, a_3 and a_{33} in accordance with the experimental data, the consequences of the dispersion relations for π^-p -scattering being not compatible with the experiment. Then discrepancy itself at low energies is not so great as at high energies. Strictly speaking, it is not clear to what extend it exceeds the limits of the conventional errors.

The causes of such a discrepancy remain obscure. The isotopic noninvariant corrections of any kind are not great. Due to the considerable role which the isotopic invariance plays in analysis of scattering data the possibility of additional verification of the isotopic invariance in the scattering is discussed in the Appendix.

In such a situation it would become very desirable to arrange some additional experiments as well as more precise treatment of the experimental data. The experiments on the study of the recoil nucleon polarization as well as of the interference of the nuclear scattering with the Coulomb one appear to be especially important. The accuracy both of the experimental data and the calculations of the dispersion integrals is very likely to be over-estimated.

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Appendix

The unitarity condition of the S-matrix together with the invariance under time reversal is known to reduce considerably the number of the independent parameters entering into the S-matrix. It was recently shown [6] how these conditions make it possible to reconstruct the scattering matrix in case when only the elastic scattering occurs which is not accompanied by the inelastic processes.

Let us consider now the unitarity conditions for $\pi^- - p$ - scattering when the inelastic process - the transformation of the charged meson into a neutral one occurs parallel to the elastic scattering. We also intend to show in what way they may contribute to the verification of the isotopic invariance.

Let us indroduce the S-matrix elements $S_{ik}^{J,\ell} = \rho_{ik}^{J,\ell} e^{2i\alpha_{ik}^{J,\ell}}$ (α and ρ are the real numbers) for every state characterized by the momentum J and $\ell = J \pm 1/2$ (the indices J, ℓ are omitted further). S_{11} - corresponds to the elastic scattering of π^- -meson by a proton $\pi^- + p \to \pi^-$, $S_{12} = S_{21}$ corresponds to the charge exchange scattering $\pi^- + p \to \pi^0$, whereas S_{22} to the elastic scattering $\pi^0 + n \to \pi^0$.

It can be easily seen that the unitary condition written in the form

$$\sum_k S_{ik}^* S_{ek} = \delta_{ie} \tag{A.I}$$

gives three independent conditions

$$|S_{11}|^2 + |S_{12}|^2 = 1 \tag{A.2}$$

$$|S_{22}|^2 + |S_{12}|^2 = 1 \tag{A.3}$$

$$S_{11}^{*} S_{12} + S_{12}^{*} S_{22} = 0 \qquad (A.4)$$

A condition which gives the possibility of introducing the real phases follows from (1) for $\pi^{+}-p$ -scattering instead of (2)-(3).

From (2) and (3) one may get the relations between the moduli

$$|S_{11}|^2 = |S_{22}|^2 = \rho_{11}^2 = \rho_{22}^2 = \rho^2 ; \quad |S_{12}|^2 = \rho_{12}^2 = 1 - \rho^2 \qquad (A.5)$$

and from (4) and (5) - the relation between the phases.

$$4\alpha_{12} = 2(\alpha_{11} + \alpha_{22}) + n\pi \qquad (A.6)$$

Therefore, in this case the amplitudes of three processes involving both elastic and inelastic scattering may be in virtue of three relations (2)-(4) expressed in terms of three real numbers which are necessary to determine from the experiment.

Note that when passing to the great number of the channels relation (6) remains as approximate if the transition probabilities between the channels are small in comparison with the transitions inside the channels [7].

The relations (5) and (6) may be given in another form. The situation here reminds slightly that which occurs in the nucleon-nucleon scattering when considering the transitions $^3P_2 \to {}^3P_2$, $^3F_2 \to {}^3F_2$, $^3P_2 \to {}^3F_2$ one may introduce the real phases and the mixing parameters.[8] In accordance with this let us introduce two real phases δ_J^{I} and δ_J^{II} and the mixing parameter ε_J

$$S_{11}^{Je} = \exp[2i\delta_J^{I}]\cos^2\varepsilon_J + \exp[2i\delta_J^{II}]\sin^2\varepsilon_J$$

$$2S_{12}^{Je} = \{\exp[2i\delta_J^{I}] - \exp[2i\delta_J^{II}]\}\sin 2\varepsilon_J$$

$$S_{22}^{Je} = \exp[2i\delta_J^{I}]\sin^2\varepsilon_J + \exp[2i\delta_J^{II}]\cos^2\varepsilon_J$$

(A.7)

(in contrast to the case of N.N -scattering, here both $\delta_J^{I,II}$ and $\varepsilon_J^{I,II}$ depend both upon e and J whereas ε_J do not drop out from the expressions for the integrated cross sections.
The relation between the quantities $\rho, \alpha_{11}, \alpha_{22}$ and δ_J^{I}, δ_J^{II}, ε_J follows from (5), (6) and (7). In such a manner the S-matrix elements are expressed precisely when the existence of the isotopic invariance is not suggested. If the latter is valid it is known that

$$3S_{11}^{Je} = 2b_1 + b_3, \quad 3S_{12}^{Je} = \sqrt{2}(b_3 - b_1), \quad 3S_{22}^{Je} = b_1 + 2b_3 \tag{A.8}$$

$b_{2T}^{Je} = \exp[2i\delta_{2T}^{Je}]$ characterized the scattering in the state with the given J and the isotopic spin T). It can be seen from the comparison of (8) with (7) that the isotopic invariance corresponds to the case when ε_J being independent of J and e (there remain only two real parameters) assumes the constant value which is equal to arctg $\sqrt{2} \cong 55^0$ if δ_J^{I} corresponds to δ_3, whereas $\delta_J^{II} - \delta_1$.

The verification of the latter statements (the independence of the quantity ε_J of J and e and its definite value) gives the possibility of checking the isotopic invariance in the scattering of the charged mesons. It might have seemed that since the cross section of the π^0-meson elastic scattering enters into the Heitler relation

$$d\sigma(\pi^+ + p \to \pi^+) + d\sigma(\pi^- + p \to \pi^-) = 2d\sigma(\pi^0 + p \to \pi^0) + d\sigma(\pi^- + p \to \pi^0)$$

then πN scattering cannot be used for checking the isotopic invariance. Fermi has indicated [7] that if, at least, only S and P waves are taken into account such a verification may be carried on in the experiments with the charged mesons. It follows from the Appendix that this may be carried on if any number of states are taken into account. The verification suggested here is still deeper. It makes possible to watch with what accuracy the isotopic invariance is satisfied in every state of πN system.

Note the generalization of Minami ambiguities [9,10] for this case when the inelastic process occurs parallely to the elastic one. By a direct check it is easy to see that the unpolarized cross sections of the considered processes remain unchangable if the simultaneous substituion

$$\delta^{I,II}_{J,J-1/2} \rightleftarrows \delta^{I,II}_{J,J+1/2},$$

$$\varepsilon^{J}_{J-1/2} \rightleftarrows \varepsilon^{J}_{J+1/2}$$

is performed.

References

1. F.E. Low Phys. Rev. 87, 1392, 1955.
   G.C. Wick Rev.Mod.Phys. 27, 339, 1955.
   G.F. Chew and F.E. Low Phys. Rev. 101, 1570, 1956.
   See also
   L. Castillejo, R.H. Dalitz, and F.Y. Dyson Phys.Rev., 101, 453, 1956.
   F.Y. Dyson Phys.Rev., 106, 157, 1957.
2. M.L. Goldberger Phys. Rev. 99, 508, 1955.
   M.L. Goldberger, H. Mizawa, and R. Oehme Phys. Rev., 99, 986, 1955.
   B.L. Ioffe JETP, 31, 583, 1956.
   N.N. Bogoliubov, B.M. Medvedev, M.K. Polivanov "Problems of Dispersion Relations Theory". (In print).
3. L. Landau and J.A. Smorodinsky JETP, 14, 269, 1954.
   H. Bethe Phys. Rev. 76, 38, 1949.
4. T. Orear Phys. Rev. 96, 176, 1954.
   A.I. Mukhin, B.M. Pontecorvo JETP, 31, 550, 1956.
5. G. Puppi and A. Stanghellini Nuovo Cimento, 5, 1303, 1957.
6. L. Puzikov, R. Ryndin, J. Smorodinsky JETP, 32, 592, 1957.
7. See, e.g., Suppl. Nuovo Cimento, 2, 17, 1955.
8. T.M. Blatt and L.C. Bidenharn Rev. Mod. Phys., 24, 282, 1952.
9. S. Minami Prog. Theor. Phys., 11, 213, 1954.
10. R. Ryndin, J. Smorodinsky Dokl. Akad. Nauk SSSR 102, 69, 1955.