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LINEAR SYNCHRO-BETATRON RESONANCES, DRIVEN BY BEAM-BEAM INTERACTION

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I. INTRODUCTION

When two bunches pass through one another at the interaction point in a colliding beam storage device the particles of each of them experience a localized force. In the linear approximation when the dispersion at the interaction point is zero this force produces a small tune shift from the unperturbed tune and causes parametric beam-beam resonances. In the case of non-zero dispersion the beam-beam force excites nonlinear resonances of betatron motion (this takes place in the case of zero dispersion as well), as well as synchro-betatron resonances.

In the present paper we analyse the linear synchro-betatron resonance due to a non-zero dispersion at the interaction point, acting together with the standard beam-beam parametric resonance. Using further the single linear synchro-betatron resonance model we calculate the amplitude beating in the stability region. Finally we show that the square of projected emittance on the horizontal phase subspace oscillates around a mean value, proportional to the square of the synchro-betatron coupling strength.

II. HAMILTON FORMULATION OF THE SYNCHROTRON AND BETATRON MOTION, TREATED SIMULTANEOUSLY

In order to be self-explanatory we begin this Section by summarizing some well-known facts. The motion of a particle with rest mass m_o and charge e is governed by the Hamiltonian

$$\mathcal{H}=c\left\{m_o^2c^2+(p_x-eA_x)^2+(p_z-eA_z)^2+\left(\frac{p_s}{1+xK_x+zK_z}-eA_s\right)^2\right\}^{1/2}+e\varphi\ , \qquad (2.1)$$

written in the natural coordinate system, defined by the triple $(\vec{n},\vec{b},\vec{\tau})$ along the design orbit. The quantities φ and $\vec{A}=(A_x,A_z,A_s)$ are as usual the scalar and the vector potentials of electromagnetic field and K_x, K_z being the curvatures of the design orbit in horizontal and vertical plane respectively with $K_xK_z=0$ $[(K_x,K_z)\neq(0,0)]$. A new Hamiltonian can be constructed[1] from (2.1) in a new independent variable s - curve length of design orbit, instead of time t

$$H=-p_s=-(1+xK_x+zK_z)\left\{\frac{(\mathcal{H}-e\varphi)^2}{c^2}-m_o^2c^2-(p_x-eA_x)^2-(p_z-eA_z)^2\right\}^{1/2}-$$
$$-(1+xK_x+zK_z)eA_s\ . \qquad (2.2)$$

Introduction of new variables

$$x \Rightarrow \tilde{p}_x=\frac{p_x}{p_{os}} \quad ; \quad z \Rightarrow \tilde{p}_z=\frac{p_z}{p_{os}} \quad ; \quad -\beta_s ct=\tau \Rightarrow h=\frac{\mathcal{H}}{\beta_s^2 E_s}\ , \qquad (2.3)$$

yields the new Hamiltonian

$$\tilde{H}=\frac{H}{p_{os}}=-(1+xK_x+zK_z)\left\{\left(\beta_s h-\frac{e\varphi}{\beta_s E_s}\right)^2-\frac{1}{\beta_s^2\gamma_s^2}-\left(\tilde{p}_x-\frac{e}{p_{os}}A_x\right)^2-\left(\tilde{p}_z-\frac{e}{p_{os}}A_z\right)^2\right\}^{1/2}-$$
$$-(1+xK_x+zK_z)\frac{e}{p_{os}}A_s\ , \qquad (2.4)$$

where p_{os} and $E_s=m_o\gamma_s c^2$ are the momentum and energy of synchronous particle, and $\beta_s=v_s/c$, $\gamma_s=(1-\beta_s^2)^{-1/2}$.

The electric $\vec{\mathcal{E}}$ and magnetic $\vec{B}$ fields are given by[2]

$$\vec{\mathcal{E}}=-\frac{\partial\vec{A}}{\partial t}=-\frac{\partial\vec{A}}{\partial\tau}(-\beta_s c)=\beta_s c\frac{\partial\vec{A}}{\partial\tau}\ ,$$

$$B_x=\frac{1}{h_o}\left[\frac{\partial(h_oA_s)}{\partial z}-\frac{\partial A_z}{\partial s}\right]\ ,$$

$$B_z=\frac{1}{h_o}\left[\frac{\partial A_x}{\partial s}-\frac{\partial(h_oA_s)}{\partial x}\right]\ ,$$

$$B_s=\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}\ ,$$

where $h_o=1+xK_x+zK_z$. For various lattice structure elements in particular we have:

1. RF-cavity:

$$\mathcal{E}_s=\hat{\mathcal{E}}_o(s)\sin\left(\frac{\omega\tau}{\beta_s c}+\phi_o\right) \quad ; \quad \mathcal{E}_x=\mathcal{E}_z=0$$

and therefore

$$R\frac{e}{p_{os}}A_s=-\frac{e\hat{\mathcal{E}}_o(s)}{\beta_s E_s}\,\frac{cR}{\omega}\cos\left(\frac{\omega\tau}{\beta_s c}+\phi_o\right) \quad ; \quad A_x=A_z=0\ , \qquad (2.5)$$

where R is the mean radius of the ring, $\hat{\mathcal{E}}_o$ is the peak value of electric field, $\omega=k\omega_o$ is the frequency of RF-generator (k being the harmonic acceleration mode, ω_o being the angular frequency of particle revolution) and ϕ_o is the initial phase of RF-field.

2. Quadrupole:

$$A_s=\left(\frac{\partial B_z}{\partial x}\right)_{x=z=o}\frac{z^2-x^2}{2} \qquad ; \qquad A_x=A_z=0 \ ,$$

so that

$$R\frac{e}{p_{os}}A_s=\frac{g_o}{2R}(z^2-x^2) \qquad ; \qquad g_o=R^2\ \frac{e}{p_{os}}\left(\frac{\partial B_z}{\partial x}\right)_{x=z=o}. \tag{2.6}$$

3. Skew quadrupole:

$$A_s=\left(\frac{\partial B_x}{\partial x}-\frac{\partial B_z}{\partial z}\right)_{x=z=o}\frac{xz}{2} \qquad ; \qquad A_x=A_z=0 \ ,$$

$$R\frac{e}{p_{os}}A_s=\frac{N_o}{R}xz \qquad ; \qquad N_o=R^2\ \frac{e}{2p_{os}}\left(\frac{\partial B_x}{\partial x}-\frac{\partial B_z}{\partial z}\right)_{x=z=o}. \tag{2.7}$$

4. Sextupole:

$$R\frac{e}{p_{os}}A_s=-\frac{\lambda_o}{6R^2}(x^3-3xz^2) \qquad ; \qquad \lambda_o=R^3\ \frac{e}{p_{os}}\left(\frac{\partial^2 B_z}{\partial x^2}\right)_{x=z=o}. \tag{2.8}$$

5. Octupole:

$$R\frac{e}{p_{os}}A_s=\frac{\mu_o}{24R^3}(x^4-6x^2z^2+z^4) \qquad ; \qquad \mu_o=R^4\ \frac{e}{p_{os}}\left(\frac{\partial^3 B_z}{\partial x^3}\right)_{x=z=o}. \tag{2.9}$$

6. Synchrotron magnet:

$$R\frac{e}{p_{os}}A_s=-\frac{R}{2}(1+xK_x+zK_z)+\frac{g_o}{2R}(z^2-x^2)+\dots \ , \tag{2.10}$$

where

$$K_x=\frac{e}{p_{os}}B_z^{(o)} \qquad ; \qquad K_z=-\frac{e}{p_{os}}B_x^{(o)} \quad \text{and} \quad K_xK_z=0 \ . \tag{2.11}$$

Next we substitute expressions (2.5)-(2.10) into the Hamiltonian (2.4) and expand the result in a power series in the variables $\tilde{p}_x/\Gamma$, $\tilde{p}_z/\Gamma$ and $e\varphi/E_s$ to obtain

$$\tilde{H}=\tilde{H}_o+\tilde{H}_1+\tilde{H}_2+\tilde{H}_3+\dots+\tilde{H}_{bb} \ , \tag{2.12}$$

where

$$\tilde{H}_o=-\Gamma+\frac{e\hat{\mathcal{E}}_o(s)}{\beta_s E_s}\ \frac{c}{\omega}\ \cos\left(\frac{\omega\tau}{\beta_s c}+\phi_o\right) \ , \tag{2.13a}$$

$$\tilde{H}=-(\Gamma-1)\ (xK_x+zK_z) \ , \tag{2.13b}$$

$$\tilde{H}_2 = \frac{\tilde{p}_x^2+\tilde{p}_z^2}{2\Gamma} + \frac{G_x x^2 + G_z z^2}{2R^2} , \qquad (2.13c)$$

$$\tilde{H}_3 = (xK_x+zK_z)\left[\frac{\tilde{p}_x^2+\tilde{p}_z^2}{2\Gamma} + \frac{g_o}{2R^2}(x^2-z^2)\right] + \frac{\lambda_o}{6R^3}(x^3-3xz^2)+\ldots , \qquad (2.13d)$$

....

$$\tilde{H}_{bb} = (1+xK_x+zK_z)\ \frac{1+\beta_s^2}{\beta_s^2}\ \frac{e\varphi_b}{E_s}\ \delta_b(s) \ . \qquad (2.13e)$$

Here we have used the notations:

$$\Gamma = \left(\beta_s^2 h^2 - \frac{1}{\beta_s^2\gamma_s^2}\right)^{1/2} , \qquad (2.14a)$$

$$G_x = g_o + R^2K_x^2 \quad ; \quad G_z = -g_o + R^2K_z^2 , \qquad (2.14b)$$

$$\delta_b(s) = \sum_{k=1}^{M} \delta_p(\vartheta-\vartheta_k) = \frac{1}{2\pi} \sum_{k=1}^{M} \sum_{n=-\infty}^{\infty} e^{in(\vartheta-\vartheta_k)} , \qquad (2.14c)$$

assuming that there are M interaction points, placed at azimuth ϑ_k ($k=1,2,\ldots,M$). The quantity φ_b in eq. (2.13e) is the beam-beam interaction potential for a Gaussian bunch, which reads as [3]

$$\varphi_b(x,z) = -\frac{Ne}{4\pi\varepsilon_o R} \int_0^\infty \frac{\exp\left(-\frac{x^2}{2\sigma_x^2+q} - \frac{z^2}{2\sigma_z^2+q}\right)}{\left[(2\sigma_x^2+q)\,(2\sigma_z^2+q)\right]^{1/2}}\, dq , \qquad (2.15)$$

where N is the total number of particles contained in the partner bunch and σ_x, σ_z are the horizontal and vertical standard deviations respectively. The definition of the classical charged particle radius

$$r_e = \frac{e^2}{4\pi\varepsilon_o m_o c^2}$$

helps us to rewrite the Hamiltonian $\tilde{H}_{bb}$ as

$$\tilde{H}_{bb} = -\frac{Nr_e}{R\gamma_s}\ \frac{1+\beta_s^2}{\beta_s^2}\ (1+xK_x+zK_z)\delta_b(s) \int_0^\infty \frac{\exp\left(-\frac{x^2}{2\sigma_x^2+q} - \frac{z^2}{2\sigma_z^2+q}\right)}{\left[(2\sigma_x^2+q)\,(2\sigma_z^2+q)\right]^{1/2}}\, dq . \qquad (2.16)$$

The description of particle dynamics in terms of a new independent variable azimuth ϑ instead of s means a simple multiplication of the Hamiltonian (2.12) by R.

We perform a canonical transformation with a generating function

$$F_2(u,\tilde{\tilde{p}}_u,\tau,\eta;\vartheta)=x\tilde{\tilde{p}}_x+z\tilde{\tilde{p}}_z+\tau\left(\eta+\beta_s^{-2}\right)+\eta R\vartheta \ , \tag{2.17}$$

defining new canonical variables

$$\tilde{u}=u \quad ; \quad \tilde{\tilde{p}}_u=\tilde{p}_u \quad ; \quad \eta=h-\beta_s^{-2} \quad ; \quad \sigma=R\vartheta+\tau=s+\tau \tag{2.18}$$

and the new Hamiltonian

$$\bar{H}_o=\frac{R\eta^2}{2\gamma_s^2}+\frac{e\hat{\mathcal{E}}_o(s)}{\beta_s E_s}\frac{cR}{\omega}\cos\left(\frac{\omega\sigma}{\beta_s c}-k\vartheta+\phi_o\right) \ , \tag{2.19a}$$

$$\bar{H}_1=-R\left(\tilde{x}K_x+\tilde{z}K_z\right)\eta \ , \tag{2.19b}$$

$$\bar{H}_2=\frac{R}{2}\left(\tilde{p}_x^2++\tilde{p}_z^2\right)+\frac{G_x\tilde{x}^2+G_z\tilde{z}^2}{2R} \ , \tag{2.19c}$$

$$\bar{H}_3=R\left(K_x\tilde{x}+K_z\tilde{z}\right)\left[\frac{\tilde{p}_x^2+\tilde{p}_z^2}{2}+\frac{g_o}{2R^2}\left(\tilde{x}^2-\tilde{z}^2\right)\right]+\frac{\lambda_o}{6R^2}\left(\tilde{x}^3-3\tilde{x}\tilde{z}^2\right)+\dots \ , \tag{2.19d}$$

....

$$\bar{H}_{bb}=-\frac{Nr_e}{\gamma_s}\frac{1+\beta_s^2}{\beta_s^2}\delta_b(s)\int_0^\infty\frac{\exp\left(-\frac{\tilde{x}^2}{2\sigma_x^2+q}-\frac{\tilde{z}^2}{2\sigma_z^2+q}\right)}{\left[\left(2\sigma_x^2+q\right)\left(2\sigma_z^2+q\right)\right]^{1/2}}dq \ , \tag{2.19e}$$

where we have represented the quantity Γ as a power series in η, and assumed that $K_x=K_z=0$ at the interaction point.

The next step is to define the dispersions ψ_x and ψ_z by the canonical transformation

$$G_2\left(\tilde{u},\hat{p}_u,\sigma,\hat{\eta};\vartheta\right)=\sum_{u=(x,z)}\left[\hat{p}_u\left(\tilde{u}-\hat{\eta}\psi_u\right)+\frac{\tilde{u}\hat{\eta}\psi_u'}{R}-\frac{\psi_u\psi_u'}{2R}\hat{\eta}^2\right]+\sigma\hat{\eta} \ , \tag{2.20}$$

which cancels $\bar{H}_1$. Furthermore we have

$$\tilde{u}=\hat{u}+\hat{\eta}\psi_u \quad ; \quad \tilde{p}_u=\hat{p}_u+\frac{\hat{\eta}\psi_u'}{R} \quad ; \quad u=(x,z) \ , \tag{2.20a}$$

$$\sigma=\sum_{u=(x,z)}\left[\hat{p}_u\psi_u-\frac{\hat{u}\psi_u'}{R}\right]+\hat{\sigma} \quad ; \quad \eta=\hat{\eta} \ , \tag{2.20b}$$

where the dot means differentiation with respect to ϑ.

The new Hamiltonian

$$\hat{H}_o=-\frac{R}{2}\left(K_x\psi_x+K_z\psi_z-\frac{1}{\gamma_s^2}\right)\hat{\eta}^2+\frac{e\hat{\mathcal{E}}_o(s)}{\beta_s E_s}\frac{cR}{\omega}\cos\left(\frac{\omega\sigma}{\beta_s c}-k\vartheta+\phi_o\right), \quad (2.21a)$$

$$\hat{H}_2=\frac{R}{2}\left(\hat{p}_x^2+\hat{p}_z^2\right)+\frac{G_x\hat{x}^2+G_z\hat{z}^2}{2R}, \quad (2.21b)$$

....

has been obtained by the utilization of the definition of dispersions ψ_u, satisfying equations

$$\frac{d^2\psi_u}{d\vartheta^2}+G_u\psi_u=R^2K_u \quad ; \quad u=(x,z) . \quad (2.22)$$

We replace now the coefficient of $\hat{\eta}^2$ in (2.21a) by its average over one revolution and note that $\langle K_x\psi_x+K_z\psi_z\rangle=\alpha_M$, where α_M is the momentum compaction factor. Thus we have

$$\hat{H}_o=-\frac{R\mathcal{K}}{2}\hat{\eta}^2+\frac{\Delta E_o}{\beta_s E_s}\frac{c}{2\pi\omega}\cos\left(\frac{\omega\sigma}{\beta_s c}+\Phi_o\right), \quad (2.23)$$

where

$$\mathcal{K}=\alpha_M-\frac{1}{\gamma_s^2} \quad ; \quad \Phi_o=\phi_o-k\vartheta_o , \quad (2.23a)$$

$$\vartheta_o=\text{arctg}\,\frac{\langle eR\hat{\mathcal{E}}_o(\vartheta)\sin k\vartheta\rangle}{\langle eR\hat{\mathcal{E}}_o(\vartheta)\cos k\vartheta\rangle} . \quad (2.23b)$$

The quantity $\Delta E_o=\langle eR\hat{\mathcal{E}}_o(\vartheta)\rangle$ is the total energy gain for one revolution. The particle's phase $\omega\sigma/\beta_s c$ is usually quite close to the synchronous phase $\omega\sigma_s/\beta_s c$, thus one is allowed to expand the cosine term of (2.23) in a power series in $\Delta\sigma=\sigma-\sigma_s$. Supposing that there is no energy uptake in the cavities (the energy gain provided by the RF compensates for the radiation loss) we drop the term proportional to $\Delta\sigma$ and get

$$\hat{H}_o=-\frac{R\mathcal{K}}{2}\hat{\eta}^2-\left[\frac{\Delta E_o}{\beta_s E_s}\frac{\omega}{2\pi c\beta_s^2}\cos\Phi_s\right]\frac{(\Delta\sigma)^2}{2}, \quad (2.24)$$

where $\Phi_s=\dfrac{\omega\sigma_s}{\beta_s c}+\Phi_o$.

It should be mentioned here that $\psi_u = \dot{\psi}_u = 0$ for RF-cavities, in order to avoid synchro-betatron resonances in them. This fact has already been used in the derivation of eq. (2.23) as $\sigma = \hat{\sigma}$ (equivalently $\Delta\sigma = \Delta\hat{\sigma}$) for $\psi_u = \dot{\psi}_u = 0$.

The final step of our synchro-betatron formalism is to cast the Hamiltonian $\hat{H}$ to action-angle variables. Let us introduce the last canonical transformation, given by the generating function

$$S_1(\hat{u}, \alpha_u, \Delta\hat{\sigma}, \alpha_s; \vartheta) = \sum_{u=(x,z)} \frac{\hat{u}^2}{2\beta_u}\left[\frac{\dot{\beta}_u}{2R} - tg\,(\alpha_u + \chi_u - \nu_u\vartheta)\right] - \lambda\frac{(\Delta\hat{\sigma})^2}{2} tg\,\alpha_s, \quad (2.25)$$

with new canonical variables α_u, J_u determined from the expressions

$$\hat{u} = \left(2\beta_u J_u\right)^{1/2} \cos\,(\alpha_u + \chi_u - \nu_u\vartheta) \ , \quad (2.26a)$$

$$\hat{p}_u = \left(\frac{2J_u}{\beta_u}\right)^{1/2} \left[\frac{\dot{\beta}_u}{2R}\cos\,(\alpha_u + \chi_u - \nu_u\vartheta) - \sin\,(\alpha_u + \chi_u - \nu_u\vartheta)\right] \ , \quad (2.26c)$$

where the well-known β_u-functions and the phase advances χ_u satisfy the equations

$$\frac{\beta_u\ddot{\beta}_u}{2} - \frac{\dot{\beta}_u^2}{4} + G_u\beta_u^2 = R^2 \qquad ; \qquad \dot{\chi}_u = \frac{R}{\beta_u} \ . \quad (2.27)$$

For the longitudinal degree of freedom one is ready to obtain

$$\Delta\hat{\sigma} = \left(\frac{2J_s}{\lambda}\right)^{1/2} \cos\alpha_s \qquad ; \qquad \hat{\eta} = -\left(2\lambda J_s\right)^{1/2} \sin\alpha_s \ , \quad (2.28)$$

$$\lambda^2 = \frac{\Delta E_o}{E_s} \frac{k}{2\pi\beta_s^2 R^2 \mathcal{X}} \cos\phi_s \ . \quad (2.29)$$

The unperturbed by the beam-beam interaction Hamiltonian is transformed now as follows:

$$\hat{H}_{oo} = \hat{H}_o + \hat{H}_2 = \nu_x J_x + \nu_z J_z - \nu_s J_s \ , \quad (2.30)$$

where

$$\nu_s^2 = \frac{\Delta E_o}{E_s} \frac{k\mathcal{X}}{2\pi\beta_s^2} \cos\phi_s \ . \quad (2.31)$$

In what follows we examine the case of $\psi_z = \dot{\psi}_z = 0$ at the interaction point and study the motion in horizontal and longitudinal directions only.

III. LINEAR SYNCHRO-BETATRON RESONANCES

In order to proceed further with our study of synchro-betatron resonances due to beam-beam interaction we must first represent the Hamiltonian (2.19e) as a series of homogeneous polynomials in the variables $\tilde{x}$ and $\tilde{z}$. Doing so up to the fourth order in $\tilde{x}$ and $\tilde{z}$ we have

$$\bar{H}_{bb} = \frac{1+\beta_s^2}{\beta_s^2} \sum_{k=1}^{M} \delta_p(\vartheta-\vartheta_k)\Bigg\{2\pi\xi_x^{(k)}\frac{\tilde{x}^2}{\beta_x^{(k)}} + 2\pi\xi_z^{(k)}\frac{\tilde{z}^2}{\beta_z^{(k)}} - \frac{\pi^2\gamma_s}{Nr_e}\Bigg[\xi_x^{(k)2}\frac{2\sigma_x^2-\sigma_x\sigma_z-\sigma_z^2}{3\sigma_x(\sigma_x-\sigma_z)} *$$

$$* \frac{\tilde{x}^4}{\beta_x^{(k)2}} + 2\xi_x^{(k)}\xi_z^{(k)}\frac{\tilde{x}^2\tilde{z}^2}{\beta_x^{(k)}\beta_z^{(k)}} - \xi_z^{(k)2}\frac{2\sigma_z^2-\sigma_x\sigma_z-\sigma_x^2}{3\sigma_z(\sigma_x-\sigma_z)}\frac{\tilde{z}^4}{\beta_z^{(k)2}}\Bigg] + \dots\Bigg\}, \qquad (3.1)$$

where the index k denotes k-th interaction point and $\xi_{x,z}^{(k)}$ are the horizontal and vertical beam-beam parameters, respectively:

$$\xi_{x,z}^{(k)} = \frac{1}{2\pi}\frac{Nr_e}{\gamma_s}\frac{1}{\sigma_x+\sigma_z}\left(\frac{\beta^{(k)}}{\sigma}\right)_{x,z} . \qquad (3.2)$$

Insert now eqs.(2.20a), (2.26a) and (2.28) into the equation (3.1). After some simple algebra one can pick out the resonance Hamiltonian, which is the sum of $\hat{H}_{oo}$ and the quadratic in $\tilde{x}$ and $\tilde{z}$ terms of $\bar{H}_{bb}$ in the form:

$$\hat{H} = (\nu_x+\Delta\nu_x)J_x + (\nu_z+\Delta\nu_z)J_z - (\nu_s+\Delta\nu_s)J_s + \frac{1+\beta_s^2}{\beta_s^2}\sum_{k=1}^{M}\xi_x^{(k)}J_x\cos(2\alpha_x - n\vartheta + \phi_{kn}) +$$

$$+2\sqrt{\lambda}\,\frac{1+\beta_s^2}{\beta_s^2}\sum_{k=1}^{M}\xi_x^{(k)}\frac{\psi_k}{\left(\beta_x^{(k)}\right)^{1/2}}\left(J_xJ_s\right)^{1/2}\cos(\alpha_x+\alpha_s - m\vartheta + \phi_{km}) , \qquad (3.3)$$

where

$$\Delta\nu_{x,z} = \frac{1+\beta_s^2}{\beta_s^2}\sum_{k=1}^{M}\xi_{x,z}^{(k)} \quad ; \quad \Delta\nu_s = -\lambda\frac{1+\beta_s^2}{\beta_s^2}\sum_{k=1}^{M}\xi_x^{(k)}\frac{\psi_k^2}{\beta_x^{(k)}} , \qquad (3.4)$$

$$\phi_{km} = \chi_x^{(k)} + (m-\nu_x)\vartheta_k + \frac{\pi}{2} \quad ; \quad \phi_{kn} = 2\chi_x^{(k)} + (n-2\nu_x)\vartheta_k . \qquad (3.5)$$

The resonance conditions are written as

$$\nu_x+\Delta\nu_x-\nu_s-\Delta\nu_s=m+\varepsilon_1 \quad ; \quad 2(\nu_x+\Delta\nu_x)=n+\varepsilon_2 \; . \qquad (3.6)$$

The quantity ε_1 is the resonance detuning for the linear synchro-betatron resonance, while ε_2 refers to the standard beam-beam parametric resonance.

For the sake of simplicity we consider one interaction point and omit terms, responsible for the motion in z-direction.

Our aim now is to remove the explicit ϑ-dependence in the Hamiltonian (3.3) by the canonical transformation

$$\tilde{J}_x=J_x \; ; \quad \tilde{J}_s=J_s \; ; \quad \alpha_x=\tilde{\alpha}_x+(\nu_x+\Delta\nu_x-\lambda_x)\vartheta-\zeta_x \; ; \quad \alpha_s=\tilde{\alpha}_s-(\nu_s+\Delta\nu_s+\lambda_s)\vartheta-\zeta_s \quad (3.7)$$

with a generating function

$$\tilde{F}_2(\alpha_x,\alpha_s,\tilde{J}_x,\tilde{J}_s;\vartheta)=\left[\alpha_x+(\lambda_x-\nu_x-\Delta\nu_x)\vartheta+\zeta_x\right]\tilde{J}_x+\left[\alpha_s+(\lambda_s+\nu_s+\Delta\nu_s)\vartheta+\zeta_s\right]\tilde{J}_s , \quad (3.8)$$

where

$$\lambda_x=\frac{\varepsilon_2}{2} \quad ; \quad \lambda_s=\varepsilon_1-\frac{\varepsilon_2}{2} , \qquad (3.9)$$

$$\zeta_x=\frac{\Phi_{1n}}{2} \quad ; \quad \zeta_s=\Phi_{1m}-\frac{\Phi_{1n}}{2} . \qquad (3.10)$$

The transformed Hamiltonian reads as

$$\hat{K}=\lambda_x J_x+\lambda_s J_s+2\mathcal{D}_s\sqrt{J_x J_s}\,\cos(\tilde{\alpha}_x+\tilde{\alpha}_s)+\mathcal{D}_p J_x\cos 2\tilde{\alpha}_x \qquad (3.11)$$

with the notations

$$\mathcal{D}_s=\xi_x\frac{1+\beta_s^2}{\beta_s^2}\,\frac{\sqrt{\lambda}\,\psi_1}{\sqrt{\beta_{x1}}} \quad ; \quad \mathcal{D}_p=\xi_x\frac{1+\beta_s^2}{\beta_s^2} . \qquad (3.12)$$

Next we utilize the more suitable for investigation of the case of two resonances combined action rectangular canonical variables

$$X=\sqrt{2J_x}\,\cos\tilde{\alpha}_x \quad ; \quad P_x=-\sqrt{2J_x}\,\sin\tilde{\alpha}_x , \qquad (3.13a)$$

$$S=\sqrt{2J_s}\,\cos\tilde{\alpha}_s \quad ; \quad P_s=-\sqrt{2J_s}\,\sin\tilde{\alpha}_s , \qquad (3.13b)$$

defined by a new canonical transformation with generating function

$$\tilde{S}_1(X,\tilde{\alpha}_x,S,\tilde{\alpha}_s)=\frac{X^2}{2}\,\mathrm{tg}\tilde{\alpha}_x+\frac{S^2}{2}\,\mathrm{tg}\tilde{\alpha}_s . \qquad (3.14)$$

The Hamiltonian (3.11) is now cast to the form

$$\hat{K}=\frac{\lambda_x}{2}(P_x^2+X^2)+\frac{\lambda_s}{2}(P_s^2+S^2)+\mathcal{D}_s(XS-P_xP_s)+\frac{\mathcal{D}_p}{2}(X^2-P_x^2) . \qquad (3.15)$$

The characteristic equation of Hamilton equations of motion

$$\dot{X}=(\lambda_x-\mathcal{D}_p)P_x-\mathcal{D}_sP_s , \qquad (3.16a)$$

$$\dot{P}_x=-(\lambda_x+\mathcal{D}_p)X-\mathcal{D}_s S \ , \tag{3.16b}$$

$$\dot{S}=-\mathcal{D}_s P_x+\lambda_s P_s \ , \tag{3.16c}$$

$$\dot{P}_s=-\mathcal{D}_s X-\lambda_s S \tag{3.16d}$$

governed by (3.15) is easily calculated to give

$$\varkappa^4+(\lambda_x^2+\lambda_s^2-\mathcal{D}_p^2-2\mathcal{D}_s^2)\varkappa^2+\mathcal{D}_s^4-2\lambda_x\lambda_s\mathcal{D}_s^2+\lambda_s^2(\lambda_x^2-\mathcal{D}_p^2) = 0 \ . \tag{3.17}$$

The motion, described by the system (3.16) is stable, provided all the roots of eq. (3.17) are pure imaginary. This means that the following inequalities

$$\mathcal{D}_s^4-2\lambda_x\lambda_s\mathcal{D}_s^2+\lambda_s^2(\lambda_x^2-\mathcal{D}_p^2) > 0 \ , \tag{3.18a}$$

$$\lambda_x^2+\lambda_s^2-\mathcal{D}_p^2-2\mathcal{D}_s^2 > 0 \tag{3.18b}$$

must hold. Note that the quantities λ_x and λ_s are linear combinations [see eqs. (3.9)] of the two resonance detunings ε_1 and ε_2. Thus conditions (3.18) determine the stability region in the ε_1- ε_2 plane.

Further we assume that the horizontal betatron frequency is sufficiently far from the parametric resonance value. This means that the standard beam-beam parametric resonance term in eq. (3.3) may be neglected.

Choose now a generating function of the form[4]

$$E_2(\alpha_x,\alpha_s,\hat{J}_x,\hat{J}_s;\vartheta)=(\alpha_x+\alpha_s-m\vartheta+\phi_{1n})\hat{J}_x-\alpha_s\hat{J}_s \ , \tag{3.19}$$

defining new canonical variables

$$J_x=\hat{J}_x \quad ; \quad J_s=\hat{J}_x-\hat{J}_s \quad ; \quad \hat{\alpha}_x=\alpha_x+\alpha_s-m\vartheta+\phi_{1n} \quad , \quad \hat{\alpha}_s=-\alpha_s \tag{3.20}$$

and the new Hamiltonian

$$\hat{H}=\varepsilon_1\hat{J}_x+(\nu_s+\Delta\nu_s)\hat{J}_s+2\mathcal{D}_s\sqrt{\hat{J}_x(\hat{J}_x-\hat{J}_s)}\ \cos\hat{\alpha}_x \ . \tag{3.21}$$

Note that the Hamiltonian (3.21) does not depend on the new angle $\hat{\alpha}_s$. It means that $\hat{J}_s=J_x-J_s=C$ must be constant. From the Hamilton equations for $\hat{J}_x$ and $\hat{\alpha}_x$ one easily obtains[5]

$$\frac{d^2J_x}{d\vartheta^2}+\varepsilon^2 J_x=\hat{\mathcal{D}} \ , \tag{3.22}$$

where

$$\varepsilon^2=\varepsilon_1^2-4\mathcal{D}_s^2 \quad ; \quad \hat{\mathcal{D}}=\varepsilon_1\mathcal{H}-2\mathcal{D}_s^2 C \ , \tag{3.22a}$$

$$\mathcal{K}=\hat{H}-(\nu_s+\Delta\nu_s)C \ . \qquad (3.22b)$$

The solution of eq. (3.22) is stable, provided the following inequality

$$\varepsilon^2 > 0 \qquad (3.23)$$

holds. Note that eq. (3.22) follows from an effective Hamiltonian[5]

$$\mathcal{H}_e=\frac{p_0^2}{2}+V_e(J_x) \ , \qquad (3.24)$$

where

$$V_e(J_x)=\frac{\varepsilon^2 J_x^2}{2}-\hat{\mathcal{D}}J_x \ . \qquad (3.25)$$

Using the relation

$$p_0=\dot{J}_x=-\frac{\partial\hat{H}}{\partial\hat{\alpha}_x}=2\mathcal{D}_s\sqrt{J_x(J_x-C)}\,\sin\hat{\alpha}_x \qquad (3.26)$$

it is easily checked that

$$\mathcal{H}_e=-\frac{\mathcal{K}^2}{2}\leq 0 \ . \qquad (3.27)$$

The motion takes place in the potential well, described by the effective potential (3.25), provided $\hat{\mathcal{D}}>0$. Moreover the line of constant energy $\mathcal{H}_e$ [see eq. (3.27)] intersects the curve $V_e(J_x)$ if $\hat{\mathcal{D}}^2-\varepsilon^2\mathcal{K}^2>0$. One can check that these two inequalities are automatically satisfied, provided (3.23) holds. The oscillation amplitude of J_x is given by[5]

$$\mathrm{Amp}\,(J_x)=\frac{J_{x2}-J_{x1}}{2}=\frac{1}{\varepsilon^2}\sqrt{\hat{\mathcal{D}}^2-\varepsilon^2\mathcal{K}^2} \ , \qquad (3.28)$$

where J_{x1} and J_{x2} are the minimum and the maximum values of J_x respectively, so that

$$V_e(J_{x1,2})=-\frac{\mathcal{K}^2}{2} \ . \qquad (3.29)$$

The solution of eq. (3.22) can be obtained directly from (3.24) using (3.26). We have:

$$\vartheta-\vartheta_0=\int_{J_{x0}}^{J_x}\frac{d\mathcal{J}_x}{\sqrt{2\left[\mathcal{H}_e-V_e(\mathcal{J}_x)\right]}} \ ,$$

or

$$J_x = \mathrm{Amp}\,(J_x)\sin\left[\varepsilon(\vartheta-\vartheta_o)+\Delta\varphi_o\right] + \frac{\hat{\mathcal{D}}}{\varepsilon^2}\,, \tag{3.30}$$

where

$$\Delta\varphi_o = \arcsin\frac{J_{xo} - \dfrac{\hat{\mathcal{D}}}{\varepsilon^2}}{\mathrm{Amp}(J_x)}\,. \tag{3.31}$$

IV. THE EFFECT OF SYNCHRO-BETATRON COUPLING ON PROJECTED EMITTANCES

The transport matrix of a lattice structure, defined by the equations of motion, following from a Hamiltonian $\hat{H}=\hat{H}_2+\hat{H}_o$ [see eqs. (2.21b) and (2.24)] is given by

$$(LS)_{ij} = \begin{Vmatrix} \cos\mu_x & \beta_{x1}\sin\mu_x & 0 & 0 \\ -\dfrac{\sin\mu_x}{\beta_{x1}} & \cos\mu_x & 0 & 0 \\ 0 & 0 & \cos\mu_s & -\varrho\sin\mu_s \\ 0 & 0 & \dfrac{\sin\mu_s}{\varrho} & \cos\mu_s \end{Vmatrix}, \tag{4.1}$$

where

$$\mu_x = R\int_{\vartheta_o}^{\vartheta}\frac{d\vartheta'}{\beta(\vartheta')} \quad ; \quad \mu_s = \nu_s(\vartheta-\vartheta_o) \quad ; \quad \varrho = \frac{R\mathcal{K}}{\nu_s} \tag{4.1a}$$

and $\dot{\beta}_{x1} = 0$. The transport matrix for the interaction point $(\beta_s=1)$ is [see eq. (3.1)]

$$(IP)_{ij} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ -\dfrac{8\pi\xi_x}{\beta_{x1}} & 1 & 0 & -\dfrac{8\pi\xi_x\psi_1}{\beta_{x1}} \\ \dfrac{8\pi\xi_x\psi_1}{\beta_{x1}} & 0 & 1 & \dfrac{8\pi\xi_x\psi_1^2}{\beta_{x1}} \\ 0 & 0 & 0 & 1 \end{Vmatrix}. \tag{4.2}$$

The total transport matrix M_{ij} of a system consisting of a lattice structure and an interaction point will be the product of (4.2) and (4.1). Thus we have

$$M_{ij} = \sum_{k=1}^{4} (IP)_{ik}(LS)_{kj} . \qquad (4.3)$$

Following K.L.Brown and R.V.Servranckx /6/ we write the matrix M in the form

$$M = \left\| \begin{matrix} A & B \\ C & D \end{matrix} \right\| , \qquad (4.4)$$

where A, B, C and D are 2×2 matrices. Let Σ be a positive definite matrix called the beam envelope matrix /6/

$$\Sigma = \left\| \begin{matrix} \Sigma_x & T \\ \tilde{T} & \Sigma_s \end{matrix} \right\| , \qquad (4.5)$$

and $\tilde{T}$ is the transpose of T. The relation between initial Σ_o and final Σ_1 is given by the equation

$$\Sigma_1 = M \, \Sigma_o \tilde{M} . \qquad (4.6)$$

If we assume that the initial beam is uncoupled $T_o=0$ (i.e. there is no synchro-betatron coupling) a simple matrix manipulation yields the result /6/

$$\Sigma_{x1} = A\Sigma_{xo}\tilde{A} + B\Sigma_{so}\tilde{B} , \qquad (4.7a)$$

$$\Sigma_{s1} = C\Sigma_{xo}\tilde{C} + D\Sigma_{so}\tilde{D} . \qquad (4.7b)$$

Using the definition of beam emittance projections on $\hat{x} - \hat{p}_x$ and $\Delta\hat{\sigma} - \hat{\eta}$ subspaces

$$\varepsilon_x^2 = \det\left(\Sigma_x\right) \quad ; \quad \varepsilon_s^2 = \det\left(\Sigma_s\right) \qquad (4.8)$$

and the symplecticity of the transport matrix M one obtains

$$\varepsilon_{x1}^2 - \varepsilon_{s1}^2 = \varepsilon_{xo}^2 - \varepsilon_{so}^2 , \qquad (4.9a)$$

$$\varepsilon_{x1}^2 = \varepsilon_{xo}^2 + \frac{\left(8\pi\xi_x\Psi_1\right)^2}{\beta_{x1}} \varepsilon_{xo} \left(\frac{\sigma_{so}^2}{\rho^2} \sin^2\mu_s + \sigma_{Eo}^2 \cos^2\mu_s \right) , \qquad (4.9b)$$

where σ_s is the bunch length and σ_E is the relative beam energy spread. In deriving eqs. (4.9) we have assumed that the initial beam ellipses are upright because of $\dot{\beta}_{x1} = 0$.

The first formula (4.9) expresses the fact that $\varepsilon_x^2 - \varepsilon_s^2$ is an invariant. The most remarkable feature of formula (4.9b) is that the $\hat{x} - \hat{p}_x$ projection of the beam emittance does not depend on the betatron phase advance. It shows that ε_x^2 experiences slow oscillations around the mean value

$$\varepsilon_{xm}^2 = \varepsilon_{xo}^2 + \frac{32\left(\pi\xi_x\psi_1\right)^2}{\beta_{x1}}\,\varepsilon_{xo}\left(\frac{\sigma_{so}^2}{\varrho^2} + \sigma_{Eo}^2\right) . \tag{4.10}$$

V. CONCLUDING REMARKS

We have analysed linear synchro-betatron resonances due to a non-zero dispersion at the interaction point and obtained the stability conditions for the combined action of a synchro-betatron resonance and parametric beam-beam resonance. The beating amplitude for an isolated linear synchro-betatron resonance is calculated in invariant form.

Note that the invariant $C = J_x - J_s$, derived in Sec.III may be written in the form $C^* = \varepsilon_x - \varepsilon_s$ for the action variable is proportional to the beam emittance. One can see the consent in principle between C^* and the exact invariant (4.9a). From eq. (4.9b) it is evident that if there are no betatron oscillations initially ($\varepsilon_{xo} = 0$) the projected emittance ε_x will be zero all the time. This means that the projection of the 4-dimensional beam ellipsoid (see e.g. Ref.7) on $\hat{x} - \hat{p}_x$ plane will be a rotating segment and there will be no longitudinal emittance pumping over into transversal emittance.

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REFERENCES:

1. A.J.Dragt, In AIP Conf. Proc., No.87., R.A.Carrigan et al. eds., 1982.

2. D.P.Barber et al., DESY 86-147., Hamburg, 1986.

3. K.Takayama, Lett. al Nuovo Cimento., Vol.34, No.7 (1982) p. 190.

4. A.J.Lichtenberg, M.A.Lieberman, Regular and Stochastic Motion., Springer-Verlag, Berlin, 1982.

5. V.A.Mikhailov, S.I.Tzenov, JINR Report 9-87-438., Dubna, 1987. (in Russian)

6. K.L.Brown, R.V.Servranckx, SLAC-PUB-4679., Stanford, August 1989 (A).

7. I.Borchardt et al., DESY 87-161., Hamburg, 1987.

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