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DISSIPATIVE PARTICLE DYNAMICS IN e^+e^- STORAGE RINGS IN THE PRESENCE OF LINEAR SYNCHRO-BETATRON COUPLING

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Диссипативная динамика частиц в $e\bar{e}$ накопительных кольцах в присутствии линейной синхро-бетатронной связи

Исследовано влияние синхротронного излучения на динамику частиц в $e\bar{e}$ накопительных кольцах в присутствии линейной синхро-бетатронной связи. Представлена систематическая процедура адиабатического исключения угловых переменных при помощи метода проекционного оператора. Получено выражение для времени жизни пучка в горизонтальном направлении.

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Tzenov S.I. E9-91-293

Dissipative Particle Dynamics in $e\bar{e}$ Storage Rings in the Presence of Linear Synchro-Betatron Coupling

The influence of synchrotron radiation on particle dynamics in $e\bar{e}$ storage rings in the presence of linear synchro-betatron coupling has been investigated. A systematic method for adiabatic elimination of angle variables, using the projection operator technique has been presented. An expression for the horizontal beam lifetime has been derived.

The investigation has been performed at the Laboratory of Nuclear Reaction, JINR.

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I. INTRODUCTION

The most commonly used approximation in the study of beam-beam interaction is the "strong-weak" case, namely, the case where the particles in one of the beams (called the weak beam) are perturbed by the beam-beam force. This force usually causes blowup of the transverse beam size with consequent loss of luminosity, and imposes severe limitations on the performance of a colliding beam storage device. The synchrotron radiation in $e\bar{e}$ storage rings may, however, improve the situation, for particle distribution is determined by the balance between the quantum fluctuations and radiation damping on the one hand and the beam resonant terms on the other hand.

In the present paper we study the dissipative particle dynamics in $e\bar{e}$ storage rings in the presence of linear synchro-betatron coupling, via nonzero dispersion at the interaction point. In a real machine there is an absorbing boundary, such as the vacuum chamber. Mathematically such a boundary is expressed by the equality between the beam size and the aperture of the vacuum chamber. Particles diffusing as a result of the stochastic photon emission would be absorbed at the walls. Much attention is paid in Sec. IV to the exit (escape) time, needed for the above process, called the beam lifetime.

II. CANONICAL TRANSFORMATIONS AND THE FOKKER-PLANCK EQUATION

Our starting equations are the following stochastic equations of motion

$$\frac{dU}{ds} = \frac{\partial H}{\partial p_U} \quad ; \quad \frac{dp_U}{ds} = -\frac{\partial H}{\partial U} + \langle \Pi_U \rangle + \Pi_U \xi(s) \quad ; \quad U=(x,z), \tag{2.1a}$$

$$-\frac{dt}{ds} = \frac{\partial H}{\partial \mathcal{H}} \quad ; \quad \frac{d\mathcal{H}}{ds} = -\frac{\partial H}{\partial (-t)} + \langle \Pi_H \rangle + \Pi_H \xi(s), \tag{2.1b}$$

where the Hamiltonian of an electron with rest mass m_e and charge e is taken in the form [1]:

$$H = -(1+xK)\left[\frac{(\mathcal{H}-e\varphi)^2}{c^2} - m_e^2c^2 - \left(p_x - eA_x\right)^2 - \left(p_z - eA_z\right)^2\right]^{1/2} - (1+xK)eA_s \tag{2.2}$$

with the curve length s as an independent variable. The curvature of the design orbit is denoted by K; $\mathcal{H}$ being the total energy and $\vec{A} = \left(A_x, A_z, A_s\right)$ is the vector potential of electromagnetic field. The quantities $\langle \Pi_U \rangle$, $\langle \Pi_H \rangle$, Π_U and Π_H in formulae (2.1) are defined in terms of the original Hamiltonian H and the mean and fluctuating parts of the radiated power as follows [2]:

$$\langle \Pi_U \rangle = -\frac{c_1 p^4}{c^2}\frac{\partial H}{\partial p_U} \quad ; \quad \langle \Pi_H \rangle = c_1 p^4 \frac{\partial H}{\partial \mathcal{H}}, \tag{2.3a}$$

$$\Pi_U = -\frac{p^2\sqrt{c_2\gamma^3}}{c^2}\frac{\partial H}{\partial p_U} \quad ; \quad \Pi_H = p^2\sqrt{c_2\gamma^3}\,\frac{\partial H}{\partial \mathcal{H}}, \tag{2.3b}$$

where p is the total momentum, γ is the Lorentz factor and [3]

$$c_1 = \frac{2r_e}{3m_e^3c}K^2(s) \quad ; \quad c_2 = \frac{55 r_e \hbar c}{24\sqrt{3}\, m_e^3}K^3(s), \tag{2.4a}$$

$$r_e = \frac{e^2}{4\pi\varepsilon_0 m_e c^2}. \tag{2.4b}$$

Furthermore $\xi(s)$ is a centred, Gaussian Markov process having the formal properties:

$$\langle \xi(s) \rangle = 0 \quad ; \quad \langle \xi(s)\xi(s') \rangle = \delta(s-s'). \tag{2.5}$$

The bracket <...> means the expectation value of the expression entering it.

Here we consider the horizontal motion of an electron, where the contribution from synchrotron motion is realized via non-zero dispersion. Applying the non-canonical scaling transformation

$$x \Rightarrow \tilde{p}_x = \frac{p_x}{p_{os}} \quad ; \quad -\beta_s ct = \tau \Rightarrow h = \frac{\mathcal{H}}{\beta_s^2 E_s} \tag{2.6}$$

one can write eqs. (2.1), taking the azimuthal angle ϑ as a new independent variable instead of s, in the form:

$$\dot{x}=\frac{\partial\tilde H}{\partial\tilde p_x}\quad;\qquad \dot{\tilde p}_x=-\frac{\partial\tilde H}{\partial x}+\langle\Pi_p\rangle+\Pi_p\xi(\vartheta), \tag{2.7a}$$

$$\dot\tau=\frac{\partial\tilde H}{\partial h}\quad;\qquad \dot h=-\frac{\partial\tilde H}{\partial\tau}+\langle\Pi_h\rangle+\Pi_h\xi(\vartheta), \tag{2.7b}$$

where β_s, p_{os} and E_s are the relative velocity, momentum and energy of the synchronous particle respectively, and

$$\langle\Pi_p\rangle=-\frac{c_1p^4}{c^2p_{os}^2}\frac{\partial\tilde H}{\partial\tilde p_x}\quad;\qquad \langle\Pi_h\rangle=\frac{c_1p^4p_{os}}{\beta_s^4E_s^2}\frac{\partial\tilde H}{\partial h}, \tag{2.8a}$$

$$\Pi_p=-\frac{\sqrt{c_2\gamma^3}}{\sqrt R}\frac{p^2}{c^2p_{os}^2}\frac{\partial\tilde H}{\partial\tilde p_x}\quad;\qquad \Pi_h=\frac{\sqrt{c_2\gamma^3}}{\sqrt R}\frac{p^2p_{os}}{\beta_s^4E_s^2}\frac{\partial\tilde H}{\partial h}, \tag{2.8b}$$

$$\tilde H=\frac{RH}{p_{os}}. \tag{2.9}$$

The dot in eqs. (2.7) means differentiation with respect to ϑ and R stands for the mean machine radius.

We are ready now to perform four canonical transformations successively, from the variables (2.6) to the action-angle variables (J_x,α_x) and (J_s,α_s). Three of them are linear in the old and new canonical variables, so that there are no serious difficulties in the utilization of the basic rules of Ito calculus. We first write down [1]:

(i) The first canonical transformation with a generating function

$$F_2^{(1)}\left(x,\tilde{\tilde p}_x,\tau,\eta;\vartheta\right)=x\tilde{\tilde p}_x+\tau\left(\eta+\frac{1}{\beta_s^2}\right)+\eta R\vartheta, \tag{2.10}$$

where

$$\tilde x=x\quad;\quad \tilde{\tilde p}_x=\tilde p_x\quad;\quad \sigma=\tau+R\vartheta\quad;\quad \eta=h-\frac{1}{\beta_s^2}, \tag{2.10a}$$

$$\tilde{\tilde H}=\tilde H+\frac{\partial F_2^{(1)}}{\partial\vartheta}. \tag{2.10b}$$

(ii) The second canonical transformation given by the generating function

$$F_2^{(2)}\left(\tilde x,\hat p_x,\sigma,\hat\eta;\vartheta\right)=\hat p_x\left(\tilde x-\psi\hat\eta\right)+\frac{\tilde x\hat\eta\dot\psi}{R}-\frac{\psi\dot\psi}{2R}\hat\eta^2+\sigma\hat\eta \tag{2.11}$$

with

$$\hat x=\tilde x-\psi\eta\quad;\quad \hat p_x=\tilde p_x-\frac{\dot\psi\eta}{R}\quad;\quad \hat\sigma=\sigma+\frac{\dot\psi\tilde x}{R}-\psi\tilde p_x\quad;\quad \hat\eta=\eta, \tag{2.11a}$$

$$\hat H=\tilde{\tilde H}+\frac{\partial F_2^{(2)}}{\partial\vartheta}, \tag{2.11b}$$

where ψ is the dispersion, satisfying the differential equation

$$\frac{d^2\psi}{d\vartheta^2}+G\psi=KR^2\quad;\qquad G=K^2R^2+R^2\frac{e}{p_{os}}\left(\frac{\partial B_z}{\partial x}\right)_{x=z=0}.$$

(iii) The third canonical transformation, whose generating function is

$$F_2^{(3)}\left(\hat x,P,\hat\sigma,\tilde\eta;\vartheta\right)=\frac{\hat xP}{\sqrt\beta}+\frac{\dot\beta\hat x^2}{4\beta R}+\tilde\sigma\tilde\eta \tag{2.12}$$

with

$$X=\frac{\hat x}{\sqrt\beta}\quad;\quad P=\hat p_x\sqrt\beta-\frac{\dot\beta\hat x}{2R\sqrt\beta}\quad;\quad \tilde\sigma=\hat\sigma\quad;\quad \tilde\eta=\hat\eta, \tag{2.12a}$$

$$\bar{\bar H}=\hat H+\frac{\partial F_2^{(3)}}{\partial\vartheta}. \tag{2.12b}$$

The well-known β-function is a solution of the equation [1,3]:

$$\frac{\beta\ddot\beta}{2}-\frac{\dot\beta^2}{4}+G\beta^2=R^2$$

Equations (2.7) are transformed straightforwardly to give

$$\dot X=\frac{\partial\bar{\bar H}}{\partial P}-\frac{\psi}{\sqrt\beta}\langle\Pi_h\rangle-\frac{\psi}{\sqrt\beta}\Pi_h\xi(\vartheta), \tag{2.13a}$$

$$\dot P=-\frac{\partial\bar{\bar H}}{\partial X}+\sqrt\beta\left[\langle\Pi_p\rangle-\left(\frac{\dot\psi}{R}+\frac{\alpha\psi}{\beta}\right)\langle\Pi_h\rangle\right]+\sqrt\beta\left[\Pi_p-\left(\frac{\dot\psi}{R}+\frac{\alpha\psi}{\beta}\right)\Pi_h\right]\xi(\vartheta), \tag{2.13b}$$

$$\dot{\hat\sigma}=\frac{\partial\bar{\bar H}}{\partial\hat\eta}-\psi\langle\Pi_p\rangle-\psi\Pi_p\xi(\vartheta), \tag{2.13c}$$

$$\dot{\hat{\eta}} = -\frac{\partial \bar{\bar{H}}}{\partial \hat{\sigma}} + \langle \Pi_h \rangle + \Pi_h \xi(\vartheta) \quad , \qquad (2.13d)$$

where

$$\alpha = -\frac{\dot{\beta}}{2R}$$

The last canonical transformation needs some more work for it is nonlinear in the canonical variables. From the generating function [1]

$$S_1\left(X, \alpha_x, \Delta\hat{\sigma}, \alpha_s; \vartheta\right) = -\frac{X^2}{2}\mathrm{tg}\phi - \lambda\frac{(\Delta\hat{\sigma})^2}{2}\mathrm{tg}\alpha_s \qquad (2.14)$$

we find

$$X = \sqrt{2J_x}\cos\phi \quad ; \quad P = -\sqrt{2J_x}\sin\phi \quad , \qquad (2.15a)$$

$$\Delta\hat{\sigma} = \sqrt{\frac{2J_s}{\lambda}}\cos\alpha_s \quad ; \quad \hat{\eta} = -\sqrt{2\lambda J_s}\sin\alpha_s \quad , \qquad (2.15b)$$

$$\hat{\mathcal{H}} = \bar{\bar{H}} + \frac{\partial S_1}{\partial \vartheta} \quad , \qquad (2.15c)$$

where

$$\phi = \alpha_x + \chi_x - \nu_x\vartheta \quad ; \quad \dot{\chi}_x = \frac{R}{\beta} \quad ; \quad \nu_x = \frac{R}{2\pi}\int_0^{2\pi}\frac{d\vartheta}{\beta(\vartheta)} \quad ,$$

$$\lambda^2 = \frac{\Delta E_o}{E_s}\frac{k}{2\pi\beta_s^2 R^2 \mathcal{K}}\cos\phi_s \quad ; \quad \Delta\hat{\sigma} = \hat{\sigma} - \sigma_s \quad .$$

The quantity ΔE_o is the total energy gain per revolution, k is the harmonic acceleration mode and $\mathcal{K} = \alpha_M - \gamma_s^{-2}$ (α_M being the momentum compaction factor). Next we represent the action and angle variables as functions of X, P, $\Delta\hat{\sigma}$, $\hat{\eta}$ and utilize Îto formula for the change of variables (see e.g. [4]). Note that

$$J_x = \frac{1}{2}\left(X^2 + P^2\right) \quad ; \quad \alpha_x = \nu_x\vartheta - \chi_x - \mathrm{arctg}\left(\frac{P}{X}\right)$$

and therefore

$$dJ_x = XdX + PdP + \frac{1}{2}(dX)^2 + \frac{1}{2}(dP)^2 \qquad (2.16a)$$

$$d\alpha_x = \left(\nu_x - \dot{\chi}_x\right)d\vartheta + \frac{P}{2J_x}dX - \frac{X}{2J_x}dP - \frac{XP}{4J_x^2}(dX)^2 + \frac{XP}{4J_x^2}(dP)^2 + \frac{X^2 - P^2}{4J_x^2}dXdP \quad . \qquad (2.16b)$$

Substitute now eqs. (2.13a,b) into the last two equations and use the basic rules of Îto stochastic differential calculus [4]:

$$\xi d\vartheta^2 = 0 \quad ; \quad \xi^2 d\vartheta^2 = d\vartheta \quad . \qquad (2.17)$$

The calculations are straightforward but tedious, giving the following final result:

$$\dot{J}_x = -\frac{\partial\hat{\mathcal{H}}}{\partial\alpha_x} + \langle\Pi_j^{(x)}\rangle + \Pi_j^{(x)}\xi(\vartheta) \quad , \qquad (2.18a)$$

$$\dot{\alpha}_x = \frac{\partial\hat{\mathcal{H}}}{\partial J_x} + \langle\Pi_\alpha^{(x)}\rangle + \Pi_\alpha^{(x)}\xi(\vartheta) \quad , \qquad (2.18b)$$

where

$$\langle\Pi_j^{(x)}\rangle = -\left[\psi\sqrt{\frac{2J_x}{\beta}}\cos\phi - \sqrt{2\beta J_x}\left(\frac{\dot{\psi}}{R} + \frac{\alpha\psi}{\beta}\right)\sin\phi\right]\langle\Pi_h\rangle - \sqrt{2\beta J_x}\sin\phi\langle\Pi_p\rangle +$$
$$+ \frac{\psi^2}{2\beta}\Pi_h^2 + \frac{\beta}{2}\left[\Pi_p - \left(\frac{\dot{\psi}}{R} + \frac{\alpha\psi}{\beta}\right)\Pi_h\right]^2 \quad , \qquad (2.19a)$$

$$\Pi_j^{(x)} = -\left[\psi\sqrt{\frac{2J_x}{\beta}}\cos\phi - \sqrt{2\beta J_x}\left(\frac{\dot{\psi}}{R} + \frac{\alpha\psi}{\beta}\right)\sin\phi\right]\Pi_h - \sqrt{2\beta J_x}\sin\phi\ \Pi_p, \qquad (2.19b)$$

$$\langle\Pi_\alpha^{(x)}\rangle = \left[\frac{\psi\sin\phi}{\sqrt{2\beta J_x}} + \sqrt{\frac{\beta}{2J_x}}\left(\frac{\dot{\psi}}{R} + \frac{\alpha\psi}{\beta}\right)\cos\phi\right]\langle\Pi_h\rangle - \sqrt{\frac{\beta}{2J_x}}\cos\phi\langle\Pi_p\rangle + \frac{\sin 2\phi}{4J_x} *$$
$$* \left\{\frac{\psi^2\Pi_h^2}{\beta} - \beta\left[\Pi_p - \left(\frac{\dot{\psi}}{R} + \frac{\alpha\psi}{\beta}\right)\Pi_h\right]^2\right\} - \frac{\psi\cos 2\phi}{2J_x}\left[\Pi_p - \left(\frac{\dot{\psi}}{R} + \frac{\alpha\psi}{\beta}\right)\Pi_h\right]\Pi_h \quad , \qquad (2.19c)$$

$$\Pi_\alpha^{(x)} = \left[\frac{\psi\sin\phi}{\sqrt{2\beta J_x}} + \sqrt{\frac{\beta}{2J_x}}\left(\frac{\dot{\psi}}{R} + \frac{\alpha\psi}{\beta}\right)\cos\phi\right]\Pi_h - \sqrt{\frac{\beta}{2J_x}}\cos\phi\ \Pi_p \quad . \qquad (2.19d)$$

With the equations

$$J_s = \frac{\lambda^2(\Delta\hat{\sigma})^2 + \hat{\eta}^2}{2\lambda} \quad ; \quad \alpha_s = -\mathrm{arctg}\left[\frac{\hat{\eta}}{\lambda(\Delta\hat{\sigma})}\right] \qquad (2.20)$$

in hand one can obtain exactly in the same manner similar stochastic differential equations for the longitudinal action-angle variables. They are:

$$\dot{J}_s = -\frac{\partial\hat{\mathcal{H}}}{\partial\alpha_s} + \langle\Pi_j^{(s)}\rangle + \Pi_j^{(s)}\xi(\vartheta)\,, \qquad (2.21a)$$

$$\dot{\alpha}_s = \frac{\partial\hat{\mathcal{H}}}{\partial J_s} + \langle\Pi_\alpha^{(s)}\rangle + \Pi_\alpha^{(s)}\xi(\vartheta)\,, \qquad (2.21b)$$

where

$$\langle\Pi_j^{(s)}\rangle = -\psi\sqrt{2\lambda J_s}\cos\alpha_s\langle\Pi_p\rangle - \sqrt{\frac{2J_s}{\lambda}}\sin\alpha_s\langle\Pi_h\rangle + \frac{\lambda\psi^2\Pi_p^2}{2} + \frac{\Pi_h^2}{2\lambda}\,, \qquad (2.22a)$$

$$\Pi_j^{(s)} = -\psi\sqrt{2\lambda J_s}\cos\alpha_s\Pi_p - \sqrt{\frac{2J_s}{\lambda}}\sin\alpha_s\Pi_h\,, \qquad (2.22b)$$

$$\langle\Pi_\alpha^{(s)}\rangle = \psi\sqrt{\frac{\lambda}{2J_s}}\sin\alpha_s\langle\Pi_p\rangle - \frac{\cos\alpha_s}{\sqrt{2\lambda J_s}}\langle\Pi_h\rangle + \frac{\lambda\psi^2\Pi_p^2\sin2\alpha_s}{4J_s} - \frac{\Pi_h^2\sin2\alpha_s}{4\lambda J_s} - \frac{\psi\Pi_p\Pi_h\cos2\alpha_s}{2J_s}\,, \qquad (2.22c)$$

$$\Pi_\alpha^{(s)} = \psi\sqrt{\frac{\lambda}{2J_s}}\sin\alpha_s\Pi_p - \frac{\cos\alpha_s}{\sqrt{2\lambda J_s}}\Pi_h\,. \qquad (2.22d)$$

It is well-known [4] that the stochastic process, governed by the equations (2.18) and (2.21) may be equivalently described by a corresponding Fokker-Planck equation for the transition probability $P\left(\alpha_x, J_x, \alpha_s, J_s; \vartheta\right)$, which reads as

$$\frac{\partial P}{\partial\vartheta} = -\frac{\partial}{\partial\alpha_i}\left[\left(\frac{\partial\hat{\mathcal{H}}}{\partial J_i} + \langle\Pi_\alpha^{(i)}\rangle\right)P\right] - \frac{\partial}{\partial J_i}\left[\left(-\frac{\partial\hat{\mathcal{H}}}{\partial\alpha_i} + \langle\Pi_j^{(i)}\rangle\right)P\right] + \frac{1}{2}\frac{\partial^2}{\partial\alpha_i\partial\alpha_k}\left(\Pi_\alpha^{(i)}\Pi_\alpha^{(k)}P\right) +$$
$$+ \frac{\partial^2}{\partial\alpha_i\partial J_k}\left(\Pi_\alpha^{(i)}\Pi_j^{(k)}P\right) + \frac{1}{2}\frac{\partial^2}{\partial J_i\partial J_k}\left(\Pi_j^{(i)}\Pi_j^{(k)}P\right)\,, \qquad (2.23)$$

where $i=(x,s)$, $k=(x,s)$ and summation over repeated indices is implied. We recall that each of $\langle\Pi_p\rangle$, $\langle\Pi_h\rangle$, Π_p and Π_h may be found explicitely from (2.8), (2.9) and (2.2), using the canonical transformations (2.10-12) and (2.14) with the result:

$$\langle\Pi_p\rangle = \frac{c_1 p_{os}^3 R}{c^2}\left[\frac{\psi}{R}\sqrt{2\lambda J_s}\sin\alpha_s + \sqrt{\frac{2J_x}{\beta}}\left(\sin\phi + \alpha\cos\phi\right)\right]\,, \qquad (2.24a)$$

$$\langle\Pi_h\rangle = \frac{c_1 p_{os}^5 R}{E_s^2}\left(\sqrt{2\lambda J_s}\sin\alpha_s - 1\right)\,, \qquad (2.24b)$$

$$\Pi_p = \frac{\sqrt{c_2\gamma_s^3 R}\,p_{os}}{c^2}\left[\frac{\psi}{R}\sqrt{2\lambda J_s}\sin\alpha_s + \sqrt{\frac{2J_x}{\beta}}\left(\sin\phi + \alpha\cos\phi\right)\right]\,, \qquad (2.24c)$$

$$\Pi_h = \frac{\sqrt{c_2\gamma_s^3 R}\,p_{os}^3}{E_s^2}\left(\sqrt{2\lambda J_s}\sin\alpha_s - 1\right)\,. \qquad (2.24d)$$

Since we are interested mainly in the establishment of equilibrium on the scale of the damping times, we may average over the relatively short revolution time scale as well as the phase of synchrotron and betatron oscillations. In the averaging procedure we retain terms with slow variation in the angle variables (the resonance terms) in the original Hamiltonian $\hat{\mathcal{H}}$ and drop such terms arising from the diffusion part of the Fokker-Planck equation (2.23). After some simple algebraic manipulations one obtains the result:

$$\frac{\partial P}{\partial\vartheta} = -\frac{\partial}{\partial\alpha_i}\left[\left(\frac{\partial\hat{\mathcal{H}}}{\partial J_i} + \delta_i\right)P\right] - \frac{\partial}{\partial J_i}\left[\left(-\frac{\partial\hat{\mathcal{H}}}{\partial\alpha_i} - 2\alpha_o J_i + 2q_i\right)P\right] +$$
$$+ \frac{1}{2}\frac{\partial^2}{\partial\alpha_i^2}\left(\frac{q_i P}{J_i}\right) + \frac{1}{2}\frac{\partial^2}{\partial J_i^2}\left(4q_i J_i P\right)\,, \qquad (2.25)$$

where the approximation $\Pi_p \cong 0$ and $\Pi_h \cong -\frac{\sqrt{c_2\gamma_s^3 R}\,p_{os}^3}{E_s^2}$ has been used and the following notations:

$$\left\|\frac{c_1 R p_{os}^3}{c^2}\alpha\right\|_\vartheta = -2\delta_x\,; \qquad \lambda\left\|\frac{c_1 R p_{os}^3}{c^2}\frac{\psi\psi}{R}\right\|_\vartheta = 2\delta_s\,, \qquad (2.26a)$$

$$\left\|\frac{c_2\gamma_s^3 R p_{os}^6}{E_s^4}\mathcal{H}_{cs}\right\|_\vartheta = 4q_x\,; \qquad \lambda^{-1}\left\|\frac{c_2\gamma_s^3 R p_{os}^6}{E_s^4}\right\|_\vartheta = 4q_s\,, \qquad (2.26b)$$

$$\mathcal{H}_{cs} = \frac{\psi^2}{\beta} + \beta\left(\frac{\psi}{R} + \frac{\alpha\psi}{\beta}\right)^2\,, \qquad (2.26c)$$

$$\left\|\frac{c_1 R p_{os}^3}{c^2}\right\|_{\vartheta} = 2\alpha_o \qquad (2.26d)$$

have been introduced $\left(\|\ldots\|_{\vartheta} = \frac{1}{2\pi}\int_0^{2\pi} d\vartheta \ldots\right)$. The Hamiltonian $\hat{\mathcal{H}}$ with account of the linear synchro-betatron resonance via nonzero dispersion ψ_1 at the interaction point is well-known to be [1]

$$\hat{\mathcal{H}} = (\nu_x+\Delta\nu_x)J_x - (\nu_s+\Delta\nu_s)J_s + 2\mathcal{D}_s\sqrt{J_x J_s}\cos(\alpha_x+\alpha_s-m\vartheta+\phi_{1m}) \ , \qquad (2.27)$$

where

$$\Delta\nu_x = \frac{1+\beta_s^2}{\beta_s^2}\xi_x \quad ; \quad \Delta\nu_s = -\frac{1+\beta_s^2}{\beta_s^2}\frac{\lambda\psi_1^2}{\beta_{x1}}\xi_x \ , \qquad (2.28a)$$

$$\mathcal{D}_s = \frac{1+\beta_s^2}{\beta_s^2}\frac{\sqrt{\lambda}\,\psi_1}{\sqrt{\beta_{x1}}}\xi_x \ , \qquad (2.28b)$$

and ξ_x is the well-known beam-beam parameter [10]. The resonance condition is written as

$$\nu_x+\Delta\nu_x-\nu_s-\Delta\nu_s = m+\varepsilon_1 \ . \qquad (2.29)$$

One can remove the explicite ϑ-dependence in eq. (2.25) applying the canonical transformation [1]

$$E_2(\alpha_x,\alpha_s,\hat J_x,\hat J_s;\vartheta) = (\alpha_x+\alpha_s-m\vartheta+\phi_{1m})\hat J_x - \alpha_s\hat J_s \qquad (2.30)$$

with

$$\hat J_x = J_x \quad ; \quad \hat J_s = J_x - J_s \quad ; \quad a = \alpha_x+\alpha_s-m\vartheta+\phi_{1m} \quad ; \quad a_s = -\alpha_s \ . \qquad (2.30a)$$

The result is

$$\frac{\partial P}{\partial\vartheta} = -\frac{\partial}{\partial a}\left[\left(\omega+\frac{\partial F}{\partial \hat J_x}\cos a\right)P\right] - \frac{\partial}{\partial\hat J_x}\left[(G_x+F\sin a)P\right] - \frac{\partial}{\partial \hat J_s}(G_s P) + B_a\frac{\partial^2 P}{\partial a^2} +$$

$$+ \frac{\partial^2}{\partial\hat J_x^2}(2q_x\hat J_x P) + \frac{\partial^2}{\partial\hat J_x\partial\hat J_s}(4q_x\hat J_x P) + \frac{\partial^2}{\partial \hat J_s^2}(B_J P) \ , \qquad (2.31)$$

where

$$\omega = \varepsilon_1+\delta_x+\delta_s \quad ; \quad F(\hat J_x,\hat J_s) = 2\mathcal{D}_s\sqrt{\hat J_x(\hat J_x-\hat J_s)} \ , \qquad (2.32a)$$

$$G_x(\hat J_x) = -2\alpha_o\hat J_x+2q_x \quad ; \quad G_s(\hat J_s) = -2\alpha_o\hat J_s + 2(q_x-q_s) \ , \qquad (2.32b)$$

$$B_a(\hat J_x,\hat J_s) = \frac{1}{2}\left(\frac{q_x}{\hat J_x}+\frac{q_s}{\hat J_x-\hat J_s}\right) \quad ; \quad B_J(\hat J_x,\hat J_s) = 2\left[(q_x+q_s)\hat J_x - q_s\hat J_s\right] \ . \qquad (2.32c)$$

We really want to obtain an equation for the particle distribution function

$$\mathcal{P}(\hat J_x,\hat J_s;\vartheta) = \frac{1}{2\pi}\int_0^{2\pi} da\, P(a,\hat J_x,\hat J_s;\vartheta) \ , \qquad (2.33)$$

holding in the limit, when the machine working point (ν_x,ν_s) is far enough from the linear synchro-betatron resonance examined here. Recently a brilliant renormalization method has been developed by Y.H.Chin [5] in which the problem of small denominators does not exist. For our purposes, however, it will be sufficient to utilize the projection operator technique [4], which is just an abstract formulation of Haken's principle of adiabatic elimination [6].

III. THE PROJECTION OPERATOR TECHNIQUE

Let us now rewrite eq. (2.31) in the form

$$\frac{\partial P}{\partial\vartheta} = \left(\hat L_1+\hat L_2+\hat L_3\right)P \ , \qquad (3.1)$$

where the differential operators $\hat L_1$, $\hat L_2$ and $\hat L_3$ are defined as follows:

$$\hat L_1 = -\omega\frac{\partial}{\partial a} + B_a\frac{\partial^2}{\partial a^2} \ , \qquad (3.2a)$$

$$\hat L_2 = -\frac{\partial F}{\partial\hat J_x}\cos a\frac{\partial}{\partial a} - F\sin a\frac{\partial}{\partial \hat J_x} \ , \qquad (3.2b)$$

$$\hat L_3 = -\frac{\partial}{\partial\hat J_x}G_x - \frac{\partial}{\partial\hat J_s}G_s + \frac{\partial^2}{\partial\hat J_x^2}2q_x\hat J_x + \frac{\partial^2}{\partial\hat J_x\partial\hat J_s}4q_x\hat J_x + \frac{\partial^2}{\partial\hat J_s^2}B_J \ . \qquad (3.2c)$$

Given an arbitrary function of the angle a $f(a;\vartheta)$ the projection operator $\hat{\mathcal{P}}_a$ may be introduced according to the equation

$$\hat{\mathcal{P}}_a f(a;\vartheta) = \frac{1}{2\pi}\int_0^{2\pi} da\; f(a;\vartheta) \; . \qquad (3.3)$$

Next we briefly sketch out some useful properties of the operators, defined above

$$\hat{\mathcal{P}}_a\hat{L}_1 = \hat{L}_1\hat{\mathcal{P}}_a = 0 \quad ; \quad \hat{\mathcal{P}}_a\hat{L}_2\hat{\mathcal{P}}_a = 0 \quad ; \quad \hat{\mathcal{P}}_a\hat{L}_3 = \hat{L}_3\hat{\mathcal{P}}_a \; . \qquad (3.4)$$

The eigenfunctions and the eigenvalues of $\hat{L}_1$ are $g\left(\hat{J}_x,\hat{J}_s\right)e^{i\nu(\hat{J}_x,\hat{J}_s)a}$ and $-\nu\left(i\omega+\nu B_a\right)$ respectively, where g and ν are arbitrary functions of the actions $\hat{J}_x$ and $\hat{J}_s$, so that

$$\hat{L}_1 ge^{i\nu a} = -\nu\left(i\omega+\nu B_a\right)ge^{i\nu a} \; . \qquad (3.5)$$

The distribution function $\mathcal{W}$ [see eq. (2.33)] with account of (3.3) may be written as

$$\mathcal{W} = \hat{\mathcal{P}}_a P \quad ; \quad \mathcal{V} = \left(1-\hat{\mathcal{P}}_a\right)P = P-\mathcal{W} \; . \qquad (3.6)$$

The projection of the Fokker-Planck equation (3.1), using eqs. (3.4) gives

$$\frac{\partial\mathcal{W}}{\partial\vartheta} = \hat{\mathcal{P}}_a\hat{L}_2\mathcal{V} + \hat{L}_3\mathcal{W} \; , \qquad (3.7a)$$

$$\frac{\partial\mathcal{V}}{\partial\vartheta} = \left[\hat{L}_1 + \left(1-\hat{\mathcal{P}}_a\right)\hat{L}_2 + \hat{L}_3\right]\mathcal{V} + \hat{L}_2\mathcal{W} \; . \qquad (3.7b)$$

The above equations are linear in $\mathcal{W}$ and $\mathcal{V}$ and their solution may be found in a convenient form, applying the Laplace transform, which easily lends itself to perturbation expansion.

The Laplace transform for an arbitrary function $Z(\vartheta)$, defined according to

$$\tilde{Z}(S) = \int_0^{\infty} Z(\vartheta)e^{-S\vartheta}d\vartheta \qquad (3.8)$$

may be disseminated without any effort to operators and abstract vectors. Thus eqs. (3.7) take the form

$$S\tilde{\mathcal{W}}(S) = \hat{\mathcal{P}}_a\hat{L}_2\tilde{\mathcal{V}}(S) + \hat{L}_3\tilde{\mathcal{W}}(S) + \mathcal{W}(0) \; , \qquad (3.9a)$$

$$S\tilde{\mathcal{V}}(S) = \left[\hat{L}_1 + \left(1-\hat{\mathcal{P}}_a\right)\hat{L}_2 + \hat{L}_3\right]\tilde{\mathcal{V}}(S) + \hat{L}_2\tilde{\mathcal{W}}(S) + \mathcal{V}(0) \; . \qquad (3.9b)$$

For the sake of simplicity we presume that

$$\mathcal{V}(0) = 0 \; . \qquad (3.10)$$

The meaning of the latter equality is that the initial distribution does not depend on the angle variable a. One can immediately obtain the formal solution of eq. (3.9a), which reads as

$$S\tilde{\mathcal{W}}(S) = -\hat{\mathcal{P}}_a\hat{L}_2\left[\hat{L}_1 + \left(1-\hat{\mathcal{P}}_a\right)\hat{L}_2 + \hat{L}_3 - S\right]^{-1}\hat{L}_2\tilde{\mathcal{W}}(S) + \hat{L}_3\tilde{\mathcal{W}}(S) + \mathcal{W}(0) \; . \qquad (3.11)$$

It was pointed out at the end of Sec. II, that the working point is taken to be far enough from the exact resonance, i.e.

$$\sqrt{\omega^2+B_a^2} > \left|\frac{\partial F}{\partial\hat{J}_x}\right| \; . \qquad (3.12)$$

Therefore we are allowed to expand the expression $\left[\ldots\ldots\right]^{-1}$ in eq. (3.11) in a power series in $\nu^{-1}\left(\omega^2+\nu^2 B_a^2\right)^{-1}$. Up to the first order we have

$$S\tilde{\mathcal{W}}(S) = -\hat{\mathcal{P}}_a\hat{L}_2\hat{L}_1^{-1}\hat{L}_2\tilde{\mathcal{W}}(S) + \hat{L}_3\tilde{\mathcal{W}}(S) + \mathcal{W}(0) \; . \qquad (3.13)$$

The straightforward calculation gives

$$-\hat{\mathcal{P}}_a\hat{L}_2\hat{L}_1^{-1}\hat{L}_2\tilde{\mathcal{W}}(S) = -\frac{\partial}{\partial\hat{J}_x}\left[\mathcal{A}\tilde{\mathcal{W}}(S)\right] + \frac{1}{2}\frac{\partial^2}{\partial\hat{J}_x^2}\left[\mathcal{B}\tilde{\mathcal{W}}(S)\right] \; , \qquad (3.14)$$

where

$$\mathcal{A}\left(\hat{J}_x,\hat{J}_s\right) = \frac{1}{2}\frac{\partial\mathcal{B}}{\partial\hat{J}_x} \quad ; \quad \mathcal{B}\left(\hat{J}_x,\hat{J}_s\right) = \frac{B_a F^2}{\omega^2+B_a^2} \qquad (3.15)$$

Substituting (3.14) into (3.13) and carrying out the inverse Laplace transform one finally obtains

$$\frac{\partial\mathcal{W}}{\partial\vartheta} = -\frac{\partial}{\partial\hat{J}_x}\left[\left(\mathcal{A}+G_x\right)\mathcal{W}\right] - \frac{\partial}{\partial\hat{J}_s}\left(G_s\mathcal{W}\right) + \frac{1}{2}\frac{\partial^2}{\partial\hat{J}_x^2}\left[\left(4q_x\hat{J}_x+\mathcal{B}\right)\mathcal{W}\right] + \frac{\partial^2}{\partial\hat{J}_x\partial\hat{J}_s}\left(4q_s\hat{J}_x\mathcal{W}\right) + \frac{1}{2}\frac{\partial^2}{\partial\hat{J}_s^2}\left(2B_J\mathcal{W}\right) \; . \qquad (3.16)$$

Note that the Fokker-Planck equation (3.16) describes the long-time behaviour of the dissipative dynamical system under consideration.

IV. THE HORIZONTAL BEAM LIFETIME

To proceed further in our study we revert to the old action variables J_x and J_s and represent the Fokker-Planck equation (3.16) as

$$\frac{\partial \mathcal{V}}{\partial \vartheta} = -\frac{\partial}{\partial J_x}\left\{\left[\frac{1}{2}\left(\frac{\partial \mathcal{B}}{\partial J_x}+\frac{\partial \mathcal{B}}{\partial J_s}\right)-2\alpha_o J_x+2q_x\right]\mathcal{V}\right\}-\frac{\partial}{\partial J_s}\left\{\left[\frac{1}{2}\left(\frac{\partial \mathcal{B}}{\partial J_x}+\frac{\partial \mathcal{B}}{\partial J_s}\right)-2\alpha_o J_s+2q_s\right]\mathcal{V}\right\}+$$
$$+\frac{1}{2}\frac{\partial^2}{\partial J_x^2}\left[\left(4q_x J_x+\mathcal{B}\right)\mathcal{V}\right]+\frac{\partial^2}{\partial J_x \partial J_s}\left(\mathcal{B}\mathcal{V}\right)+\frac{1}{2}\frac{\partial^2}{\partial J_s^2}\left[\left(4q_s J_s+\mathcal{B}\right)\mathcal{V}\right]. \qquad (4.1)$$

Next we note that

$$\dot{\mathcal{B}}\left(J_x,J_s\right) \approx 2\mu_o\left(q_x J_s+q_s J_x\right), \qquad (4.2)$$

where

$$\mu_o=\frac{\mathcal{D}_s^2}{\omega^2+\alpha_o^2}. \qquad (4.3)$$

Moreover the stationary point of the deterministic part of eq. (4.1) is given by

$$J_{xd}=\frac{q_x}{\alpha_o}+\frac{\mu_o}{2\alpha_o}\left(q_x+q_s\right) \quad ; \quad J_{sd}=\frac{q_s}{\alpha_o}+\frac{\mu_o}{2\alpha_o}\left(q_x+q_s\right). \qquad (4.4)$$

Substituting the expressions (4.4) into the diffusion tensor of eq. (4.1) and introducing the new variables

$$u=J_x-J_{xd} \quad ; \quad v=J_s-J_{sd} \qquad (4.5)$$

we find that the standard two-dimensional Ornstein-Uhlenbeck process, defined by the equation

$$\frac{\partial \mathcal{V}}{\partial \vartheta}=-\frac{\partial}{\partial u}\left(-2\alpha_o u\mathcal{V}\right)-\frac{\partial}{\partial v}\left(-2\alpha_o v\mathcal{V}\right)+\frac{\mathcal{B}_{11}}{2}\frac{\partial^2 \mathcal{V}}{\partial u^2}+\mathcal{B}_{12}\frac{\partial^2 \mathcal{V}}{\partial u \partial v}+\frac{\mathcal{B}_{22}}{2}\frac{\partial^2 \mathcal{V}}{\partial v^2} \qquad (4.6)$$

is the first approximation to our problem. Here we have used the following notations:

$$\mathcal{B}_{11}=4q_x J_{xd}+\mathcal{B}\left(J_{xd},J_{sd}\right)=\frac{1}{\alpha_o}\left[4q_x^2+2\mu_o q_x\left(q_x+3q_s\right)+\mu_o^2\left(q_x+q_s\right)^2\right], \qquad (4.7a)$$

$$\mathcal{B}_{12}=\mathcal{B}_{21}=\mathcal{B}\left(J_{xd},J_{sd}\right)=\frac{\mu_o}{\alpha_o}\left[4q_x q_s+\mu_o\left(q_x+q_s\right)^2\right], \qquad (4.7b)$$

$$\mathcal{B}_{22}=4q_s J_{sd}+\mathcal{B}\left(J_{xd},J_{sd}\right)=\frac{1}{\alpha_o}\left[4q_s^2+2\mu_o q_s\left(3q_x+q_s\right)+\mu_o^2\left(q_x+q_s\right)^2\right]. \qquad (4.7c)$$

The search for an orthogonal transformation

$$\begin{Vmatrix} u_1 \\ v_1 \end{Vmatrix} = \hat{O}*\begin{Vmatrix} u \\ v \end{Vmatrix} = \begin{Vmatrix} \cos\varkappa & \sin\varkappa \\ -\sin\varkappa & \cos\varkappa \end{Vmatrix}*\begin{Vmatrix} u \\ v \end{Vmatrix}, \qquad (4.8)$$

diagonalizing the diffusion tensor of eq. (4.6) gives the result:

$$\mathrm{tg}2\varkappa=\frac{2\mathcal{B}_{12}}{\mathcal{B}_{11}-\mathcal{B}_{22}}=\frac{\mu_o\left[4q_x q_s+\mu_o\left(q_x+q_s\right)^2\right]}{\left(\mu_o+2\right)\left(q_x^2-q_s^2\right)}. \qquad (4.9)$$

The transformed Fokker-Planck equation reads as

$$\frac{\partial \mathcal{V}}{\partial \vartheta}=-\frac{\partial}{\partial u_1}\left(-2\alpha_o u_1\mathcal{V}\right)-\frac{\partial}{\partial v_1}\left(-2\alpha_o v_1\mathcal{V}\right)+\frac{B_{11}}{2}\frac{\partial^2 \mathcal{V}}{\partial u_1^2}+\frac{B_{22}}{2}\frac{\partial^2 \mathcal{V}}{\partial v_1^2}, \qquad (4.10)$$

where

$$B_{11}=\frac{1}{2}\left[\mathcal{B}_{11}+\mathcal{B}_{22}+\sqrt{4\mathcal{B}_{12}^2+\left(\mathcal{B}_{11}-\mathcal{B}_{22}\right)^2}\right], \qquad (4.11a)$$

$$B_{22}=\frac{1}{2}\left[\mathcal{B}_{11}+\mathcal{B}_{22}-\sqrt{4\mathcal{B}_{12}^2+\left(\mathcal{B}_{11}-\mathcal{B}_{22}\right)^2}\right]. \qquad (4.11b)$$

One is given the opportunity to learn almost everything about the particle distribution, governed by eq. (4.10) for the theory of Ornstein-Uhlenbeck process is well developed. Details may be found in the excellent guides on stochastic methods [4,7] available.

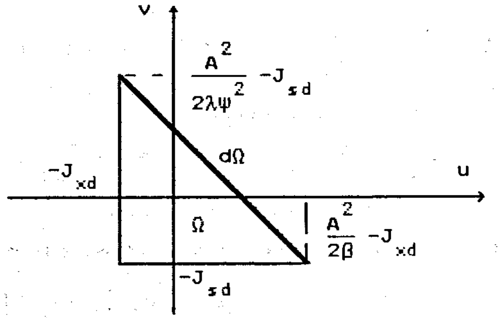
Here we concentrate our attention on the beam lifetime problem. The separation ansatz for $\mathcal{V}$ in eq. (4.6):

$$\mathcal{V}(u,v;\vartheta)=\upsilon(u,v)e^{-\Lambda\vartheta} \qquad (4.12)$$

leads to

$$-\Lambda\upsilon=-\frac{\partial}{\partial u}\left(-2\alpha_o u\upsilon\right)-\frac{\partial}{\partial v}\left(-2\alpha_o v\upsilon\right)+\frac{\mathcal{B}_{11}}{2}\frac{\partial^2 \upsilon}{\partial u^2}+\mathcal{B}_{12}\frac{\partial^2 \upsilon}{\partial u \partial v}+\frac{\mathcal{B}_{22}}{2}\frac{\partial^2 \upsilon}{\partial v^2}. \qquad (4.13)$$

Obviously Λ is inversely proportional to the lifetime Θ for a certain eigenvalue problem of eq. (4.13). The boundary conditions and the geometry of the domain Ω are specified as follows (see the Figure below):



1. The probability currents:

$$S_u=-2\alpha_o u\vartheta-\frac{1}{2}\left(\mathcal{B}_{11}\frac{\partial\vartheta}{\partial u}+\mathcal{B}_{12}\frac{\partial\vartheta}{\partial v}\right), \qquad (4.14a)$$

$$S_v=-2\alpha_o v\vartheta-\frac{1}{2}\left(\mathcal{B}_{12}\frac{\partial\vartheta}{\partial u}+\mathcal{B}_{22}\frac{\partial\vartheta}{\partial v}\right) \qquad (4.14b)$$

should vanish at the boundaries $u=-J_{xd}$ and $v=-J_{sd}$ (reflecting boundaries).

2. The probability density ϑ vanishes at the boundary

$$u+\frac{\lambda\psi^2}{\beta}v=\Sigma \quad ; \quad \Sigma=\frac{A^2}{2\beta}-J_{xd}-\frac{\lambda\psi^2}{\beta}J_{sd} \qquad (4.15)$$

(absorbing boundary), where A is the horizontal aperture of the vacuum chamber.

Integrating eq. (4.13) over the domain Ω, having in mind the boundary conditions (4.14) and (4.15) we obtain

$$\Lambda=-\frac{1}{2}\,\frac{\displaystyle\int_{d\Omega}\left[\left(\mathcal{B}_{11}\frac{\partial\vartheta}{\partial u}+\mathcal{B}_{12}\frac{\partial\vartheta}{\partial v}\right)du+\left(\mathcal{B}_{12}\frac{\partial\vartheta}{\partial u}+\mathcal{B}_{22}\frac{\partial\vartheta}{\partial v}\right)dv\right]}{\displaystyle\int_{\Omega}\vartheta\,du\,dv}. \qquad (4.16)$$

It has been proved by Matkowsky and Schuss [8], that the stationary distribution

$$\mathcal{W}_{st}(u,v)=\mathcal{N}_o\exp\left(-2\alpha_o\vec{u}^{T}\mathcal{B}^{-1}\vec{u}\right), \qquad (4.17)$$

where

$$\vec{u}=\begin{Vmatrix}u\\ v\end{Vmatrix} \quad ; \quad \mathcal{B}^{-1}=\frac{1}{\Delta}\begin{Vmatrix}\mathcal{B}_{22} & -\mathcal{B}_{12}\\ -\mathcal{B}_{12} & \mathcal{B}_{11}\end{Vmatrix} \quad ; \quad \Delta=\mathcal{B}_{11}\mathcal{B}_{22}-\mathcal{B}_{12}^2 \qquad (4.18)$$

may be used as an approximation to ϑ in formula (4.16), nevertheless it does not satisfy the boundary condition $\vartheta=0$ at $d\Omega$. Moreover Wentzel and Freidlin have shown [9] that if there is a point on the boundary $d\Omega$, where the maximum of the stationary distribution $\mathcal{W}_{st}$ is attained, particles escape from that point with almost unit probability. In our case such a point does exist and it is easily checked that its coordinates are:

$$u_o=\frac{\mathcal{B}_{11}+r\mathcal{B}_{12}}{\mathcal{B}_r}\Sigma \quad ; \quad v_o=\frac{\mathcal{B}_{12}+r\mathcal{B}_{22}}{\mathcal{B}_r}\Sigma, \qquad (4.19)$$

where

$$r=\frac{\lambda\psi^2}{\beta} \quad ; \quad \mathcal{B}_r=\mathcal{B}_{11}+2r\mathcal{B}_{12}+r^2\mathcal{B}_{22}, \qquad (4.20)$$

and obviously $u_o+rv_o=\Sigma$. This fact allows us to carry out the integration of the numerator of the r.h.s. of expression (4.16) by Laplace's method. Noting that the denominator of (4.16) is proportional to $\pi\sqrt{\Delta}/8\alpha_o$ by a factor quite close to unity, we obtain

$$\Theta^{-1}=\Lambda\cong\frac{8\alpha_o}{\pi\mathcal{B}_r}\left\{\left(1+r^2\right)\sqrt{\Delta}+\sqrt{\frac{2\pi\alpha_o}{\mathcal{B}_r}}\left[\left(\mathcal{B}_{11}+\mathcal{B}_{22}\right)r+\mathcal{B}_{12}\left(1+r^2\right)\Sigma\right]\right\}*$$

$$*\exp\left(-\frac{2\alpha_o\Sigma^2}{\mathcal{B}_r}\right). \qquad (4.21)$$

V. CONCLUDING REMARKS

We have studied the dissipative particle dynamics in $e\bar{e}$ storage rings in the presence of linear synchro-betatron coupling. A systematic method for adiabatic elimination of the angle variables, using the projection operator technique has been presented.

The effect of the linear synchro-betatron coupling is quite apparent from eq. (4.4), showing that the stationary transverse and longitudinal emittances are enlarged by a value of $\mu_0\left(q_x+q_s\right)/2\alpha_0$. Moreover, the synchrotron radiation has a stabilizing action on the resonance, for the effective resonance detuning is $\omega=\varepsilon_1+\delta_x+\delta_s$ [see eq. (2.32a)].

Finally we have derived an expression [see eq. (4.21)] for the horizontal beam lifetime. It is worth noting, that an exact formula for the beam lifetime may be obtained, using the eigenfunctions for the boundary value problem (4.13), which are expressed in terms of Hermite polynomials [details may be found in Ref. 4].

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