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E9 - 8811

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# THE MAGNETIC FIELD OF A UNIFORMLY MAGNETIZED POLYHEDRAL IRON

1975

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## S U M M A R Y

Analytical expressions of the magnetic fields, obtained for uniformly magnetized polyhedrons of an arbitrary form, have been described. The computer program has been described used for calculating the field from one or from combination of polyhedrons composing the electromagnet. The expression obtained can be used in calculating three-dimensional configurations of magnets.

In this case the arbitrariness of the polyhedron form allows the magnet body to be described in the best way.


## Introduction

There are a lot of methods of numerical calculation of the magnetic field. Some of them are based on the solution of partial differential equations(PDE),e.g.method of relaxation or finite element techniques.The others are connected with the solution of nonlinear integral equations (IE).

If the magnetization of the iron is known,we can calculate the magnetic field directly from the integral formulae.

Usually it is done under the assumption that the iron region is divided into the constantly magnetized sections.

In this note the method of the direct calculation of the field from constantly magnetized iron is described.

The methods based on integral formulae were used by Turner [1] for the solution of the integral equation describing the magnetic field and by Danilov and Savchenko[2] for calculating the field in magnetic channel. But in those papers the magnetic field was calculated for iron elements of the shape of the right polygonal prisms.

The method described here calculates the field for any polyhedral iron element and so has two advantages over the previous methods:

1) Constantly magnetized iron element can be calculated directly, i.e. iron is not divided into sections.

2) For an unconstantly magnetized element the higher accuracy can be reached because of the arbitrary shape of polyhedron.

## §1. Theoretical basis

The magnetic induction of an iron bar is given in the Gauss system of units by the formula

$$1.1)\quad \bar{B}(a) = \frac{1}{c}\int_V \frac{\bar{j}(q)\times\bar{R}_{qa}}{R_{qa}^3}\,dV_q + \frac{1}{c}\int_S \frac{\bar{i}(q)\times\bar{R}_{qa}}{R_{qa}^3}\,dS_q ,$$

where $\bar{B}(a)$ denotes magnetic induction vector in the point $a=(a_1,a_2,a_3)$, $\bar{R}_{qa}=[a_1-q_1, a_2-q_2, a_3-q_3]$ denotes the radius vector from any point $q=[q_1,q_2,q_3]$ belonging to iron to an observation point a ,

$R_{qa}$ denotes the length of the radius,

$\bar{j}(q)$ denotes volume current density in the point $q$ belonging to the iron,

$\bar{i}(q)$ denotes surface current density in the point $q$ belonging to the surface $S$ of the iron,

$\bar{A}\times\bar{B}$ vector product of vectors $\bar{A}$ and $\bar{B}$,

$c$ velocity of light.

In the lack of conductivity currents (i.e. the currents from external sources) the volume current density is given by the formula

$$1.2)\quad \bar{j}(q) = c\cdot \operatorname{rot}\bar{M}(p)\Big/_{p=q}$$

and the surface current density is given by



$$1.3)\quad \bar{i}(q) = c\cdot\bar{M}(q)\times\bar{n}(q),$$

where $\bar{M}$ denotes the magnetization vector and $\bar{n}$ denotes the vector normal to the surface $S$ , directed outside the region $V$.

If the magnetization is constant, i.e. $\bar{M}(q)=\bar{M}$ , then

$$1.4)\quad \bar{j}(q) \equiv 0$$

and the magnetic induction is given by the second term of formula (1.1) only. Taking (1.3) into account we receive:

$$1.5)\quad \bar{B}(a) = \int_S \frac{(\bar{M}\times\bar{n}(q))\times\bar{R}_{qa}}{R_{qa}^3}\,dS_q .$$

If the iron element has a polyhedral shape, then its surface is the sum of polygons, and the field from such an element will be the sum of fields from those polygons.

Let $P_i$ denotes a polygonal part of the surface, and let $S$ be the sum of n polygons:

$$1.6)\quad S = \bigcup_{i=1}^{n} P_i ,$$

then

$$1.7)\quad \bar{B}(a) = \sum_{i=1}^{n}\int_{P_i} \frac{(\bar{M}\times\bar{n}(q))\times\bar{R}_{qa}}{R_{qa}^3}\,dS_q .$$

Thus, if we can calculate the field from any polygon, we will be able to find the field of the whole constantly magnetized polyhedron.

## §2. The Calculation of the field of the polygon

Let us return to formula(1.5),expressing the vector product by the scalar one, and substituting the surface S by the polygon P (formula is valid for every surface) we get

$$2.1)\quad \bar{B}(a) = \int_P \frac{(\bar{M}\cdot\bar{R}_{qa})\cdot\bar{n}(q) - \bar{M}(\bar{n}(q)\cdot\bar{R}_{qa})}{R_{qa}^3}\cdot dS_q .$$

Let the vectors $\bar{B}$, $\bar{M}$, $\bar{R}$, $\bar{n}$ have the coordinates:

$$(B_1,B_2,B_3),\ (M_1,M_2,M_3),\ (R_1,R_2,R_3),\ (n_1,n_2,n_3)$$

then the equation(2.1)we can write as a system of scalar equations:

$$2.2)\quad B_K(a_1,a_2,a_3) = \int_P \frac{1}{(\sqrt{\sum R_i^2})^3}\left(n_K \sum_{i=1}^{3} M_i R_i - M_K \sum_{i=1}^{3} n_i R_i\right) dS,$$

$K=1,2,3.$

Without loosing the generality we can assume that our polygon P lies in $z=0$ plane, and the observation point (point a ) lies on z axis. That means that the first two coordinates of observation point and the third coordinate of points of polygon P are equal to zero $(a=(0,0,z))$, $(q=(x,y,0))$.

In this case $\bar{n}(q)=(0,0,1)$ ,



hence equations (2.2) get the form

$$B_1(0,0,z) = -M_1 \int_P \frac{z}{(\sqrt{x^2+y^2+z^2})^3}\,dx\,dy,$$

$$2.3)\quad B_2(0,0,z) = -M_2 \int_P \frac{z}{(\sqrt{x^2+y^2+z^2})^3}\,dx\,dy,$$

$$B_3(0,0,z) = -\int_P \frac{M_1 x + M_2 y}{(\sqrt{x^2+y^2+z^2})^3}\,dx\,dy.$$

Let us denote

$$C_1 = -\int_P \frac{x}{(\sqrt{x^2+y^2+z^2})^3}\,dx\,dy,$$

$$2.4)\quad C_2 = -\int_P \frac{y}{(\sqrt{x^2+y^2+z^2})^3}\,dx\,dy,$$

$$C_3 = -\int_P \frac{z\,dx\,dy}{(\sqrt{x^2+y^2+z^2})^3}$$

then equations(2.3)yield:

$$2.5)\quad \begin{aligned} B_1 &= M_1 C_3 ,\\ B_2 &= M_2 C_3 ,\\ B_3 &= (M_1 C_1 + M_2 C_2 ). \end{aligned}$$

We shall calculate now plane integral(2.4).

According to Stokes formula

$$\int_S dx\,dy\left(\frac{\partial Q(x,y)}{\partial x} + \frac{\partial P(x,y)}{\partial y}\right) = \int_{\partial S} Q(x,y)\,dy - P(x,y)\,dx ,$$

where $\partial S$ denotes the contour of the plane region S , we obtain[x]:

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x) We can use the Stokes formula only if the observation point does not lie on the contour.

$$C_1 = -\int_P \frac{x}{(\sqrt{x^2+y^2+z^2})^3}\,dx\,dy = \int_{\partial P} \frac{dy}{\sqrt{x^2+y^2+z^2}},$$

2.6) $$C_2 = -\int_P \frac{x}{(\sqrt{x^2+y^2+z^2})^3}\,dx\,dy = -\int_{\partial P} \frac{dx}{\sqrt{x^2+y^2+z^2}},$$

$$C_3 = -z\int_P \frac{dx\,dy}{(\sqrt{x^2+y^2+z^2})^3} = -z\int_{\partial P} \frac{x\,dy}{(y^2+z^2)\sqrt{x^2+y^2+z^2}}.$$

The contour $\partial P$ of the polygon P is the sum of the line segments $\partial P = \bigcup_{K=1}^{m} I_K$ and thus coefficients $C_1$, $C_2$, $C_3$ are the sums of integrals:

$$C_1 = \sum_{K=1}^{m} \int_{I_K} \frac{dy}{\sqrt{x^2+y^2+z^2}},$$

2.7) $$C_2 = \sum_{K=1}^{m} \int_{I_K} \frac{dx}{\sqrt{x^2+y^2+z^2}},$$

$$C_3 = -z\sum_{K=1}^{m} \int_{I_K} \frac{x\,dy}{\sqrt{x^2+y^2+z^2}\,(y^2+z^2)}.$$

Let (XA, YA, 0) denotes the coordinates of the beginning of $I_K$ segment and (XB, YB, 0) the coordinates of its end. Then the integrals over the $I_K$ segment are given by the following formulae:

if XA=XB and YA≠YB then $\int_{I_K} \frac{dx}{\sqrt{x^2+y^2+z^2}} = 0$

and the other integrals like in(2.8) ,

if YA=YB and XA≠XB then $\int_{I_K} \frac{x\,dy}{(y^2+x^2)\sqrt{x^2+y^2+z^2}} = 0,$

and the third integral like in(2.8) , $\int_{I_K} \frac{dy}{\sqrt{x^2+y^2+z^2}} = 0$



if YA=YB and XA=XB , then all the imtegrals are equal to zero,

if XA≠XB and YA≠YB , then

$$\sqrt{a^2+1}\int_{I_K} \frac{dx}{\sqrt{x^2+y^2+z^2}} = \ln\left[(a^2+1)x + ab + \sqrt{(a^2+1)((a^2+1)x^2+2abx+b^2+z^2)}\right]\Bigg|_{XA}^{XB},$$

2.8) $$\sqrt{\tilde a^2+1}\int_{I_K} \frac{dy}{\sqrt{x^2+y^2+z^2}} = \ln\left[(\tilde a^2+1)y + \tilde a\tilde b + \sqrt{(\tilde a^2+1)((\tilde a^2+1)y^2+2\tilde a\tilde b y+\tilde b^2+z^2)}\right]\Bigg|_{YA}^{YB},$$

$$-z\int_{I_K} \frac{x\,dy}{(y^2+z^2)\sqrt{x^2+y^2+z^2}} = \begin{cases} 0 & \text{when } z=0 \text{ or } a=b=0, \\ \operatorname{arctg} \dfrac{\tilde a z^2 - \tilde b y}{z\sqrt{(\tilde a^2+1)y^2+2\tilde a\tilde b y+\tilde b^2+z^2}}\Bigg|_{y=YA}^{y=YB} & \text{in other cases.} \end{cases}$$

In above formulae

$a$ denotes $\frac{YB-YA}{XB-XA}$ , $b$ denotes $\frac{XB\cdot YA - XA\cdot YB}{XB-XA}$,

$\tilde a$ denotes $\frac{XB-XA}{YB-YA}$ and $\tilde b$ denotes $\frac{YB\cdot XA - XB\cdot YA}{YB-YA}$.

Formulae(2.8) were used for numerical calculation of the coefficients $C_1$, $C_2$, $C_3$ and then a magnetic induction $\bar B$ for a polygon was found.

## § 3. Field calculation for the whole polyhedron

On the basis of method described in § 2 we can calculate now the magnetic induction of any polygon in any point (except the boundary points). Indeed for any given observation point a and any given polygon P we can transform the coordinates in order to obtain the Z axis passing through the point a and the Z=0 plane through the polygon.

As an example let us calculate the magnetic induction from the face P of the polyhedron shown in the fig. 2.

The polyhedron is shown in $(x'', y'', z'')$ coordinate system which we shall call the first coordinate system. When the vertices A, B, C are given we can calculate the unit vectors i, j, k as follows:

$$i = \frac{1}{|\overline{AB}|} \cdot \overline{AB}, \quad K = \frac{\overline{AB} \times \overline{AC}}{|\overline{AB} \times \overline{AC}|}, \quad j = K \times i,$$

or $$K = \frac{\overline{AC} \times \overline{AB}}{|\overline{AC} \times \overline{AB}|}$$ if the angle ABC is not convex.

In the coordinate system $(x', y', z')$ given by the unit vectors i, j, k (the second coordinate system) the polygon P is parallel to the $Z' = 0$ plane.

After the displacement (3.3) we obtain the third coordinate system $(x, y, z)$ in which the observation point a lies on the Z axis and the polygon P on the Z =0 plane (see fig. 1).

When the new coordinates of the vertices of the polygon $P$ and the new coordinates of the magnetization vector $\bar{M}$ are given, we can calculate the vector $\bar{B}$ in this (i.e. the third) coordinate system due to procedure described in § 2. Then we transform the vector $\bar{B}$ from the third into the first coordinate system. Finally we obtain the magnetic induction for every polygon

3.1) $$B''_p = \sum_{q=1}^{3} a_{pq} M''_q, \qquad p = 1, 2, 3,$$



where $a_{pq} = C_3 (i_p i_q + j_p j_k) + K_p (C_1 i_q + C_2 j_q)$, $p = 1,2,3$, $q = 1,2,3$

and $(B''_1, B''_2, B''_3), (M''_1, M''_2, M''_3), (i_1, i_2, i_3), (j_1, j_2, j_3), (K_1, K_2, K_3)$ denote the coordinates of magnetic induction vector $\bar{B}$ magnetization vector $\bar{M}$ and unit vectors i, j, k all in the first coordinate system. The coefficients $C_1$, $C_2$, $C_3$ are obtained from formula(2.4).

Continuing such a procedure for every face of the polyhedron we obtain the value of the magnetic induction of the whole constantly magnetized volume.

In the procedure described above the following formulae were used :

a) transformation from the first to the second coordinate system:

3.2) $$\begin{aligned} x' &= i_1 x'' + i_2 y'' + i_3 z'', \\ y' &= j_1 x'' + j_2 y'' + j_3 z'', \\ z' &= K_1 x'' + K_2 y'' + K_3 z'', \end{aligned}$$

where $(x'', y'', z'')$ denote the coordinates of an arbitrary point or vector in the first and $(x', y', z')$ in the second coordinate system,

b) the displacement from the second to the third coordinate system

3.3) $$\begin{aligned} x &= x' - a'_1, \\ y &= y' - a'_2, \\ z &= z' - A'_3, \end{aligned}$$

where $(a'_1, a'_2, a'_3)$ denote the coordinates of the observation point and $(A'_1, A'_2, A'_3)$ the coordinates of the vertex A both in the second coordinate system,

c) the inverse transformation of the magnetic induction vector

$$
3.4) \quad \begin{aligned}
B_1'' &= i_1 B_1 + j_1 B_2 + k_1 B_3, \\
B_2'' &= i_2 B_1 + j_2 B_2 + k_2 B_3, \\
B_3'' &= i_3 B_1 + j_3 B_2 + k_3 B_3,
\end{aligned}
$$

where $\bar{B} = (B_1, B_2, B_3)$ denotes the field in the third and $\bar{B}'' = (B_1'', B_2'', B_3'')$ in the first coordinate system. The inverse displacement is not necessary for the magnetic induction depends on the location of the observation point due to the polygon only.

§ 4. Applications

1) In the IE method of magnetic field calculation the iron region is divided into the sections of a constant magnetization (in general those regions are spheres, parallelepipeds or right prisms).

Thus from the integral equation we obtain the system of algebraic equations and the number of the equations depends on the number of constantly magnetized regions.

The large number of the algebraic equations is a great disadvantage when the numerical methods are used. But if the constantly magnetized regions have the arbitrary polyhedral shape, then the less number of regions and hence the less number of equations can be used for the same accuracy of the calculations.



2) In the case when the magnetization vector is approximately uniform inside the iron volume, while it is arbitrary on the surface of the iron[x]),then approximation of the surface by the polygons can be useful for the IE calculations.

3) When we deal with the magnets with the narrow air gap,then IE method cannot be used because of limited number of partitioning, effecting on the number of algebraic equations.

But if the vector $\bar{M}$ can be found in the whole iron volume either from the field measurements or by the grid method then after the partitioning of iron volume[xx]) the magnetic induction can be found directly due to the method described in §§ 2,3.

4) The method described in §§ 2,3 can be used also for a direct calculation of the magnetic field of polygonal current sheets.

Speaking more accurately the magnetic induction of an arbitrary set of polygonal sheets (see fig. 3) can be found.

Then the final formulae for calculation are obtained from the equation(1.1)where $\bar{i}$ is given and $\bar{j}=0$ instead of from the equation(1.5)like in the calculation for the uniformly magnetized iron.

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x) Magnets of such a type are described in [ 3].

xx) The method of automatical partitioning is described in [4].

§ 5. The vertification of the method

On the base of the theory described in §§ 2, 3 a computer program was written. The program calculates the magnetic induction of an arbitrary and constantly magnetized polyhedral iron (or of a set of constantly magnetized polyhedrons).

As a verification of the program, a magnetic induction for a right parallelpipedal bar was calculated and the results coincide with those of Danilov and Savchenko[2]. For the verification the both methods of date input (see § 6) were used.

Then the program (with the modified method of data input) was used for the field calculation of a constantly magnetized ball.

The ball was approximated by the polyhedron and for sufficiently large number of its vertices the coincidence with analytically calculated field was reached.



§ 6. Data input

The vertices of the polyhedron should be put in as follows:

For every polygon the sequence of its vertices is ordered due to the orientation of the polygon, and the second point in the sequence should be the vertex of the convex angle of the polygon.

The vertices are put in one after another, i.e. three coordinates of the first, then the three of the second etc.

An example of ordering the vertices of the polygon P is given in fig. 4.

The proper orders are following:

$a_1, a_2, a_3, \; b_1, b_2, b_3, \; c_1, c_2, c_3, \; d_1, d_2, d_3,$

or

$b_1, b_2, b_3, \; c_1, c_2, c_3, \; d_1, d_2, d_3, \; a_1, a_2, a_3,$

or

$d_1, d_2, d_3, \; a_1, a_2, a_3, \; b_1, b_2, b_3, \; c_1, c_2, c_3,$

where

$a_1, a_2, a_3, \; b_1, b_2, b_3, \; c_1, c_2, c_3, \; d_1, d_2, d_3$

denotes the coordinates of the points A, B, C, D , respectively.

The following sequences are orinnted improperly:

$c_1, c_2, c_3, d_1, d_2, d_3, \; a_1, a_2, a_3, \; b_1, b_2, b_3$

for the second vertex is the vertex of not convex angle ,

$d_1, d_2, d_3, \; c_1, c_2, c_3, \; b_1, b_2, b_3, \; a_1, a_2, a_3$

for the orientation is improper.

The faces of the regions are also put in one after another, i.e. all vertices of first face (in the order described above) the all vertices of the second a.s.o.

In this procedure the coordinates of every vertex are read in many times (three times for one region at least) for the vertex of any polyhedron belongs to three faces at least.

In order to omit this disadvantage a slight modification was made which can be useful for the polyhedron vertices of which belongs to two planes only (i.e. there exist such two planes that all the vertices of the polyhedron belongs to them).

If those faces of the polyhedron which don't belong to the planes are all quadrangles, then every vertex will be introduced once. If there are triangles among those faces, then one vertex of a triangle will be introduced twice (we treat the triangle as a degenerate quadrangle two vertices of which belong to the first plane, and they coincide and two vertices (which do not coincide) belong to the second one).

Examples of the both methods of data input for the polyhedron are shown in fig. 4. We put in the following vertices:

a) in the modified system:

$e_1, e_2, e_3, \; f_1, f_2, f_3, \; g_1, g_2, g_3, \; h_1, h_2, h_3,$

$a_1, a_2, a_3, \; b_1, b_2, b_3, \; c_1, c_2, c_3, \; d_1, d_2, d_3$

b) in the old system:

$h_1, h_2, h_3, \; g_1, g_2, g_3, \; f_1, f_2, f_3, \; e_1, e_2, e_3,$

$a_1, a_2, a_3, \; b_1, b_2, b_3, \; c_1, c_2, c_3, \; d_1, d_2, d_3,$

$e_1, e_2, e_3, \; f_1, f_2, f_3, \; b_1, b_2, b_3, \; a_1, a_2, a_3,$



$f_1, f_2, f_3, \; g_1, g_2, g_3, \; c_1, c_2, c_3, \; b_1, b_2, b_3,$

$g_1, g_2, g_3, \; h_1, h_2, h_3, \; d_1, d_2, d_3, \; c_1, c_2, c_3,$

$h_1, h_2, h_3, \; e_1, e_2, e_3, \; a_1, a_2, a_3, \; d_1, d_2, d_3,$

For the polyhedron shown in fig. 5 we put in the following vertices:

a) in the modified system

$a_1, a_2, a_3, \; b_1, b_2, b_3, \; c_1, c_2, c_3,$

$d_1, d_2, d_3, \; b_1, b_2, b_3, \; c_1, c_2, c_3,$

or $a_1, a_2, a_3, \; b_1, b_2, b_3, \; c_1, c_2, c_3,$

$d_1, d_2, d_3, \; d_1, d_2, d_3, \; d_1, d_2, d_3,$

b) and in the old system:

$c_1, c_2, c_3, \; b_1, b_2, b_3, \; a_1, a_2, a_3,$

$a_1, a_2, a_3, \; b_1, b_2, b_3, \; d_1, d_2, d_3,$

$b_1, b_2, b_3, \; c_1, c_2, c_3, \; d_1, d_2, d_3,$

$c_1, c_2, c_3, \; a_1, a_2, a_3, \; d_1, d_2, d_3.$

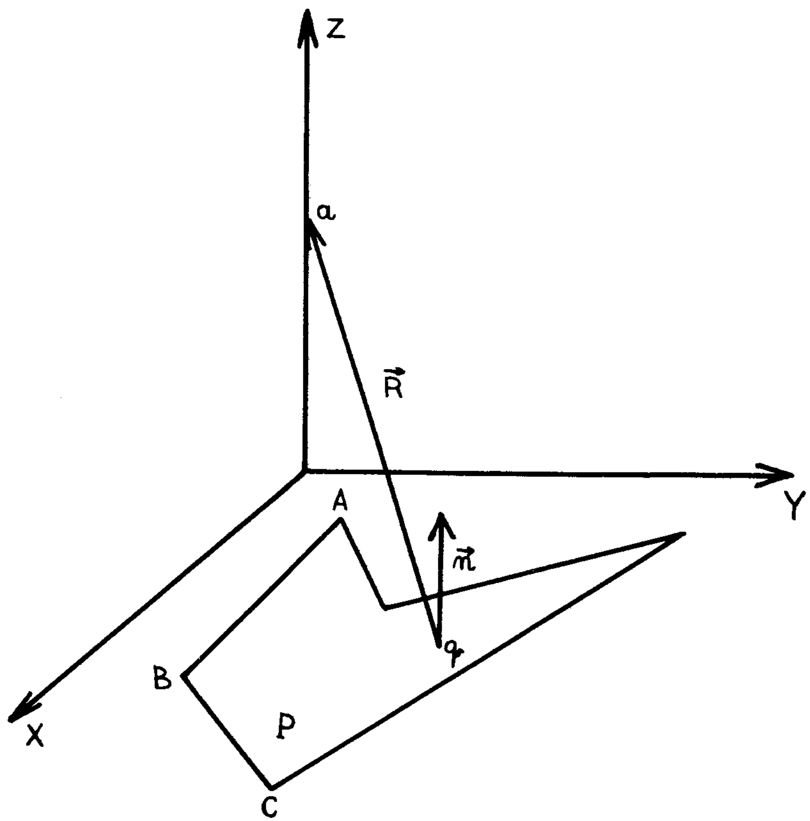


Fig.1



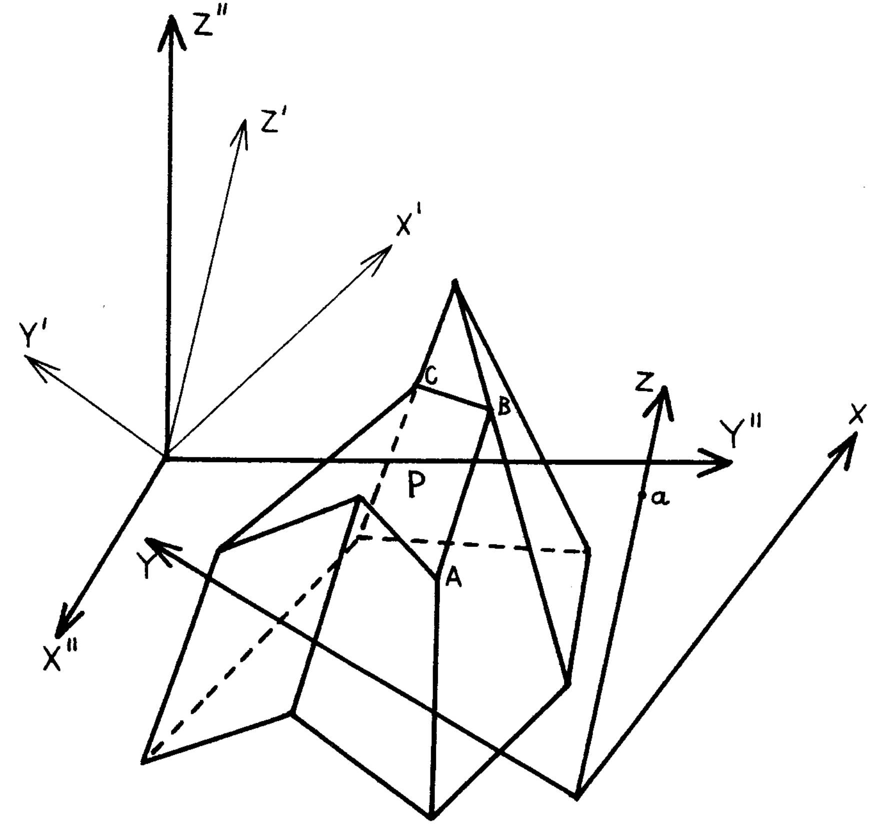


Fig.2

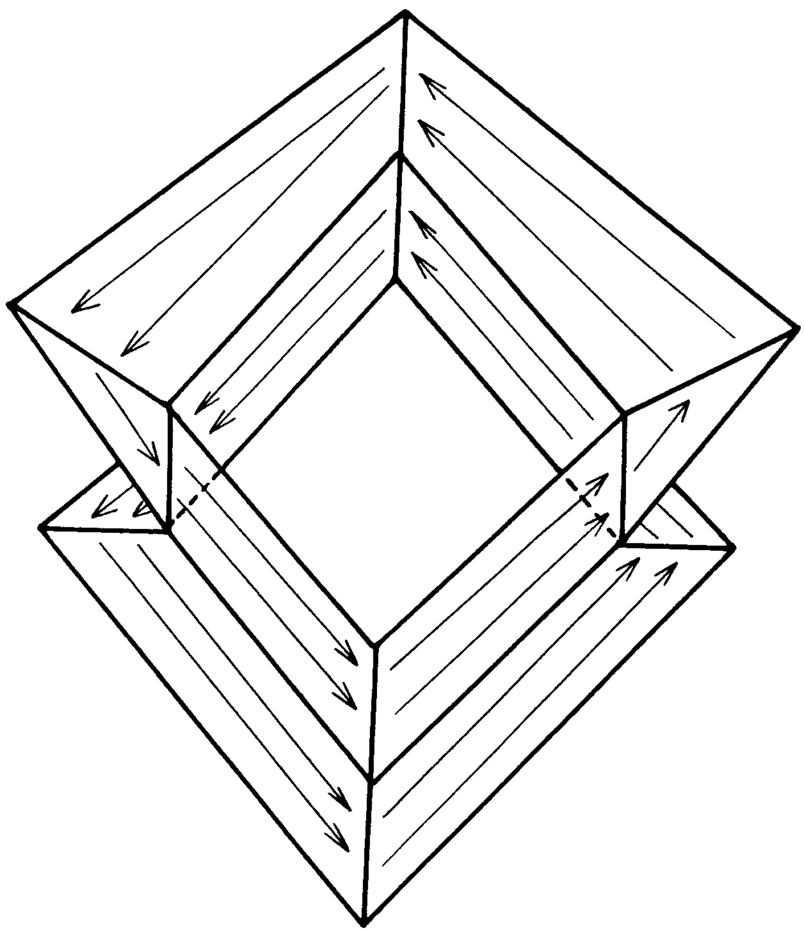

Fig.3. An example of current sheets.



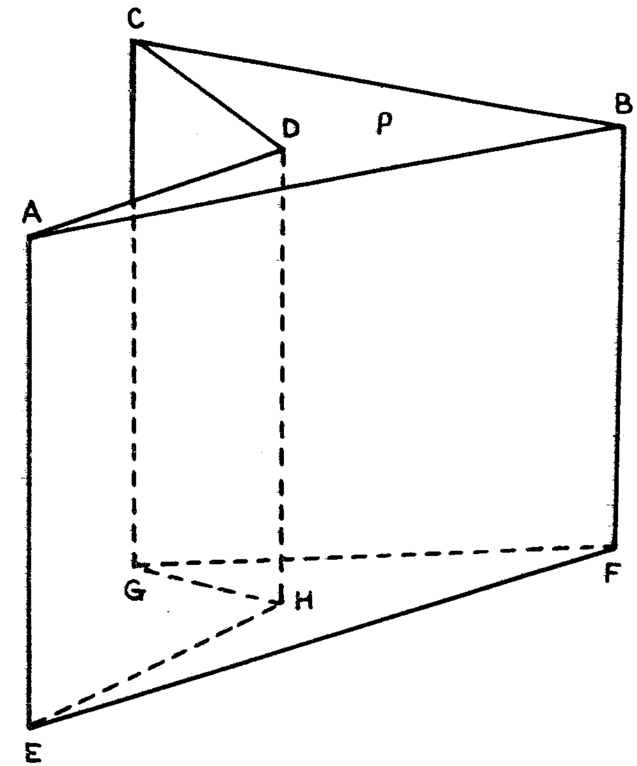


Fig.4

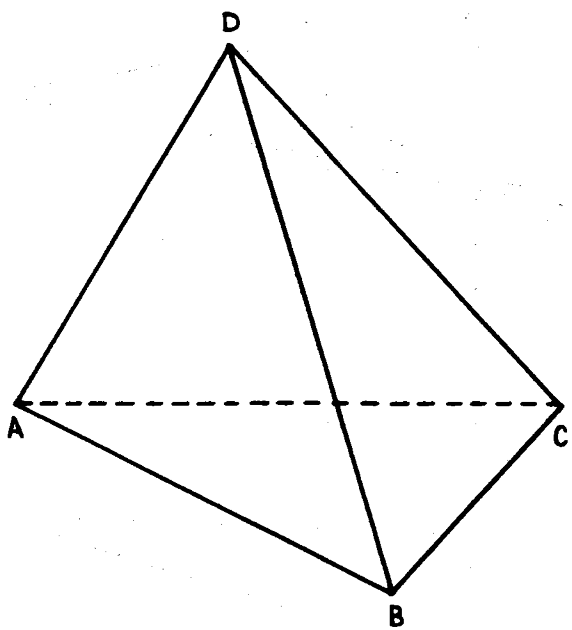


Fig.5



R E F E R E N C E S

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Received by Publishing Department in April 18, 1975.