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PROTON BEAM EXTRACTION FROM THE 10 GEV DUBNA SYNCHROPHASOTRON BY EXCITING THE RESONANCE $Q_R = 2/3$ (FURTHER ANALYTICAL CONSIDERATIONS)

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* Instituto di Scienze Fisiche dell'Università, Genova, Italy. Visiting the Dubna JINR Laboratories under the auspices of the CERN Travel Fellowship Scheme.

I. Introduction

The use of the sextupolar second harmonic perturbation has been proposed [1], in order to acquire the slow resonant extraction of the proton beam from the 10 GeV Dubna synchrophasotron, by exciting the resonance

$$Q_R = \frac{2\ radial\ betatron\ oscillations}{3\ revolutions},$$

which seems to be the most suitable [2] for such a kind of accelerator.

The design of the perturbing system, as well as that of the extraction channel, is very well ahead, and its main features are going to be considered in this note.

First of all, the sextupolar 2nd-harmonic pattern is built up involving all four quadrants, i.e. guide field and perturbing field have the same polarities in two opposite quadrants and the opposite polarities in the other two quadrants. This implies that the approximate azimuthal function has to be as follows:

$$F(\theta) \simeq \frac{4}{\pi} \sin 2\theta \quad , \tag{1}$$

which has been successfully employed at Frascati [3] and at Princeton [4,5].

Moreover, the unwanted dipolar component [3] of the perturbing field has been cancelled: the appropriate amount of current is fed into a few wires which are wound around the poles of two opposite quadrants, the same current, but in opposite polarity, is fed into the wires of the other two quadrants [6]. Thus, the perturbing field is a pure sextupolar [4,5] one, namely

$$\Delta B_z (R) = b_2 (R - R_0)^2, \tag{2}$$

where R is the radius of any orbit, R_0 is the equilibrium orbit radius, b_2= 0.0075Gs/cm^2, as it has been assessed after the first results/7/ yielded by computer.

Hence, the radial motion equation takes the simple form

$$\frac{d^2x}{d\theta^2} + Q_R^2 x = -\frac{4}{\pi}\frac{R_0}{B_0} b_2 x^2 \sin 2\theta , \tag{3}$$

(where $x = R - R_0$, B_0 is the guide field on the equilibrium orbit), which is far less sophisticated than equation (4) of ref. 1 and it is quite close to equation (1,2) of ref. 3.

Solutions of equations similar to eq. (3) have been computed by Runge-Kutta programs /7,8/, yielding results which are very similar /8/ to those obtained by matrix programs /4,5,8–10/.

If these results are plotted in the phase plane (x, x') - with $x' = (1/R_0)\, dx/d\theta$ - according to the stroboscopic representation (i.e. at a fixed azimuth), a very typical feature is shown. Namely, there is a particle which behaves as follows:

- at any instant (0-th turn): it passes at a distance x_0 from the equilibrium orbit, with a slope x'_0, defining a point $X_0(x_0, x'_0)$ in the phase plane plot;
- after one turn (1-st turn): it passes at a distance x_1, with a slope x'_1, defining a point $X_1(x_1, x'_1)$;
- after two turns (2-nd turn): it passes at a distance x_2, with a slope x'_2, defining a point $X_2(x_2, x'_2)$;
- after three turns (3-rd turn): it passes at a distance $x_3 = x_0$, with a slope $x'_3 = x'_0$, defining a point $X_3(x_3, x'_3)$ $X_0(x_0, x'_0)$.

Therefore this particle will go on repeating the same behaviour every three turns. Consequently, the three representative points X_0, X_1, X_2 will be transformed into themselves every three revolutions, being for this reason called fixed points.

If an appropriate phase plane plot (x, x^+) is chosen (Appendix I), the triangle defined by the three fixed points is equiangular. Moreover, bearing in mind that the analytical approximation /3/ yields the amplitude

$$a = \frac{4\delta}{K_2} \tag{4}$$

(with $\delta = n - n_{res}$ and $K_2 = (4/\pi)(R_0/B_0).b_2 = 0.5\ (dn/dx)$ of ref. 3) as being the limit between the stable and unstable oscillations, one can state that the fixed points triangle is inscribed (Fig. 1) in the circle of radius a.

Hence, the value of the stable area is

$$S = 12\sqrt{3}\left(\frac{\delta}{K_2}\right)^2 , \qquad (5)$$

which is very near to the one obtained by matrix consideration (eq. (3) of ref. 10).

In passing, the triangle of Fig. 1 rotates in the phase plane, as the azimuth chosen for the stroboscopic representation varies. Numerical computations /7/ of equation (3) show (Fig. 2) how this triangle revolves at eight different azimuths. In the real case, Fig. 2 must be looked at through a mirror, as the guide field is downward directed, still considering $b_2 x^2$ as positive, when both are in the same polarity.

II. Slow Extraction

The usual way of slowly extracting the beam is here considered. As the field index is slowly brought up to resonant value n_{res}, according to the procedure extensively discussed in ref. 1, the separator triangle of Fig. 1 squeezes (bear in mind eqs. (4) and (5)) till collapsing over the closed-orbit representative point.

Numerical computations /7,8/ show that, due to the absence of the 2nd-harmonic dipolar term $b_0 \sin 2\theta$, the perturbed closed orbit coincides with the unperturbed one O(0,0) and - very important - quite no relevant variation of n_{res} results.

The beam spill out begins to take place, when the sides of the separator triangle impinge the circle representing the circulating beam. Indeed the bounded stroboscopic points do not keep their regular motion along circular paths, but they trend to describe the usual pseudo-triangular curves, thus diluiting the phase plane density distribution.

In fact, in the (a, ϕ) representation /3/ the equation of the separatrix curve is

$$\cos 3\phi = -\frac{1}{2}\left(\frac{a}{\bar{a}}\right)^3 + \frac{3}{2}\left(\frac{a}{\bar{a}}\right) \qquad (6)$$

which is nothing but eq. (1.5) of ref. /3/.

Solutions of eq. (6) for $\cos 3\phi = -1$ and $\cos 3\phi = 1$ are $a = \bar{a}/2$ and $a = \bar{a}$, respectively, i.e. the betatron oscillation amplitudes of these particles undergo variations from $\bar{a}/2$ to $\bar{a}$ (see upper part of Fig. 3). This means, that, in terms of $(x, \dot{x})$ plot, the corresponding stroboscopic points are moving around following the shape of the separator triangle, their amplitude being equal to $\bar{a}/2$, everywhere the triangle sides touch the profile of the unperturbed circulating beam (dashed circle in the lower part of Fig. 3) and equal to $\bar{a}$ in correspondence of each vertex.

The practical shrinking of the stable area takes place over an enormous number of turns: one turn means (1/1.5) μsec, hence a 300 ÷ 400 msec spill out means 450000 ÷ 600000 turns. This implies (adiabatic hypothesis):

i) the stable points (i.e. the strob. points belonging to the stable area) succeed in approaching the fixed points, without being turned into unstable (Appendix II) by the inward displacement of the triangle sides;

ii) the fixed points can be considered as such, since δ remains practically constant over a quite big number of turns: $\Delta a/a \approx 0.001$ for 600 turns. Nevertheless, a very slow inward displacement is however taking place, implying:

iii) the stable points near the vertices become little by little fixed points;

iv) concurrently, points, with betatron amplitudes slightly bigger than a, leave the fixed points moving away to infinity, along three of the unstable separatrices (look at the arrows of Fig. 1), with radial jumps faster than exponential.

Hence, the whole extraction can be outlined as in Fig. 4, which shows from left to right: the fixed points triangle at the beginning of the spill ($a/2 = r_{beam} \approx 7.5$ cm), a few triangles at different values of n, the final situation at $n = n_{res}$, and finally the emittance thus obtained beyond the septum. (Notice that the former $r_b \simeq 7.5$ cm corresponds to a half of the radial width of the circulating beam).

3. Reduction of the Emittance of the Extracted Beam

An analysis of Fig. 4 suggests a possible method for minimizing the emittance of the extracted beam. In fact, if a 2nd-harmonic dipolar field is added to the 2nd-harmonic sextupolar perturbation, the closed orbit representative point is upward displaced along the x^+ axis. (Notice that the resulting perturbation is now of the type $(b_0 + b_2 x^2)\sin 2\theta$, instead of the former unwanted $(b_0 - b_2 x^2)\sin 2\theta$.

More exactly, at the beginning of the spill b_0 must be null, at the end of the spill b_0 must have such a value that the closed orbit displacement is

$$x_{c.o.} = 0 \tag{7a}$$

$$x^+_{c.o.} = (1/2)\, a_{spill} = r_b \, . \tag{7b}$$

Besides, it is necessary that b_0 increases and δ decreases both according to the same law, i.e.

$$b_0(\theta) = (b_0)_{final} \left[1 - \frac{\delta(\theta)}{\delta_{spill}}\right], \tag{8}$$

where θ is null at the beginning of the spill out, and $\delta(\theta) = n(\theta) - n_{res}$.

Under these conditions, all the upper sides of the successive fixed-point triangles should coincide (Fig. 5) during the whole extraction process, yielding an external beam with a vanishingly small emittance, in principle at least.

The analytically predicted result, illustrated in Fig. 5, shows features in common with other results (obtained by numerical computations) regarding weak focusing synchrotrons perturbed by a radial magnetic bump /11–13/ located at a single azimuth (lumped perturbation).

More quantitatively, as the closed orbit equations are:

$$x_{c.o.} = \frac{K_0}{4 - Q_R^2} \sin 2\theta , \tag{9a}$$

$$x^+_{c.o.} = \frac{2K_0}{4 - Q_R^2} \cos 2\theta . \tag{9b}$$

One has, over the resonance $Q_R = 2/3$ and at the azimuth $\theta = 0°$:

$$x_{c.o.} = 0 , \tag{10a}$$

$$x^+_{c.o.} = (27/32) K_0 . \tag{10b}$$

Combining eqs. (4), (7b) and (10b), the following set of relevant equations is obtained:

$$K_2 \, (K_0)_{final} = \frac{64}{27} \delta_{spill} , \tag{11}$$

$$(K_0)_{final} = \frac{32}{27} r_B , \tag{12}$$

$$\delta_{spill} = \frac{1}{2} K_2 \, r_B \tag{13}$$

Equations (12) and (13) can be transformed into the following ones, more suitable for practical purposes:

$$(b_0)_{final} = \frac{8}{27} \pi B_0 \frac{r_B}{R_0} , \tag{14}$$

$$\delta_{spill} = \frac{2}{\pi} \frac{b_2 \, r_B \, R_0}{B_0} . \tag{15}$$

which yields $(b_0)_{final} \approx 24.2$ Gauss and $\delta_{spill} \approx 0.0103$, with the data considered in this note.

In the previous considerations, the difference between any general Q_R^2 and $Q_{res}^2 = (4/9)$ has been disregarded, as being a small quantity with respect to 4 (look at eqs. (9)).

This kind of dipolar component has to lift the perturbed n_{res}: just the opposite of what happened /8,12/ when b_0 and $b_2 x^2$ were in opposite polarity. First results obtained by numerical computations /7/ show that the n_{res} -displacement is not very big: from 0.626$(b_0 = 0)$ to 0.633$(b_0 = 30\ Gs)$. Incidentally, $n_{res}(b_0 = 0)$ coincides with the value of n, evaluated by setting $Q_R = 2/3$ in equation (AI.5).

4. Conclusions

The resonant extraction via 2/3 non-linear resonance seems more and more promising. The suggested method of distorting the closed orbit, by means of an appropriate 2nd-harmonic dipolar component, looks very attractive. In fact, the proposed procedure succeeds in minimizing the emittance of the extracted beam and gives the possibility of having a double control - on both $b_0(\theta)$ and $\delta(\theta)$ - over the whole process of the slow spill out.

A few words are going to be spoken about the most appropriate time dependence of $\delta(\theta)$, and consequently of $b_0(\theta)$ according to eq. (8). (Notice that $\theta = 2\pi f t$, where f is the proton frequency at 10 GeV and t is the extraction time). The time dependence of the Q_R -value for obtaining uniform spill-out has been evaluated elsewhere (see Appendix IV of ref. 10). Taking into account eq. (5) and eq. (IV, 5) of ref. 10, assuming that 99% of the particles are contained in the triangle, circumscribing the circle of radius $r_b = 7.5$ cm, one has:

$$\delta(\theta) \approx 0.466\, \delta_{spill} \sqrt{ln \frac{\Delta\theta}{\theta}}\ , \qquad (16)$$

where $\delta_{spill} = 0.0103$, $\Delta\theta \simeq 2\pi\, 500000$ radians and θ is counted from the beginning of the spill. Fig. 6 shows $\delta(\theta)/0.466\,\delta_{spill}$, which is nothing but Fig. 15 of ref. 10.

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Appendix I

Normalized Units

Any sector machine, with N straight sections, can be considered as a circular machine, with the same field index, but fulfilling the following requirements:

$$2\pi R_0 = 2\pi R_{00} + NL$$

(particles travel the same length per revolution), where R_0 is the equilibrium orbit radius of the equivalent circular machine; R_{00} is the equilibrium orbit radius in the sector magnet; L is the length of each straight section;
hence

$$R_0 = R_{00}\left(1 + \frac{NL}{2\pi R_{00}}\right)$$

and if $N = 4$

$$R_0 = R_{00}\left(1 + 2\frac{L}{\pi R_{00}}\right). \qquad \text{(AI.1)}$$

As

$$B_0 R_0 = B_{00} R_{00} \qquad \text{(same final energy)}$$

one has, for $2L/\pi R_{00} \ll 1$

$$B_0 \approx B_{00}\left(1 - 2\frac{L}{\pi R_{00}}\right). \qquad \text{(AI.2)}$$

If the actual values $L = 8$ m, $R_{00} = 28$ m and $B_{00} = 12.620$ Gs are considered, eqs. (AI.1) and (AI.2) yield:

$$R_0 \approx 33.1 \ m \qquad \text{(AI.3)}$$

$$B_0 \approx 10325 \ Gs \qquad \text{(AI.4)}$$

which should be used instead of R_{00}, B_{00} whenever they appear.

Moreover, the simplest periodic structure (one straight section one quadrant) is described by a matrix, which half-trace is

$$\frac{1}{2} T_r (M) \cong \cos \frac{\pi}{2} \sqrt{1-n} - \frac{L}{2 R_{00}} \sqrt{1-n} \sin \frac{\pi}{2} \sqrt{1-n}$$

which becomes, for $L/2R_{00} \ll 1$

$$\frac{1}{2} T_r (M) = \cos \frac{\pi}{2} Q_R \approx \cos \frac{\pi}{2} \sqrt{1-n} (1 + \frac{L}{\pi R_{00}}),$$

i.e.

$$Q_R \approx \sqrt{1-n} (1 + \frac{L}{\pi R_{00}}). \tag{AI.5}$$

Notice that equations (AI.2) and (AI.5) are given with an approximation of about 3%.

Finally, if such units are required that unperturbed trajectories in the phase plane have to be circles instead of ellipses, it suffices to choose, as coordinates, the same displacement x but a slightly different divergence, i.e.

$$x^+ = \beta x' = \frac{R_0}{Q_R} \frac{1}{R_0} \frac{dx}{d\theta} = \frac{1}{Q_R} \frac{dx}{d\theta}$$

or, according to (AI.5):

$$x^+ \approx \frac{1}{\sqrt{1-n}} (1 - \frac{L}{\pi R_{00}}) \frac{dx}{d\theta}; \tag{AI.6}$$

All over this note, the following equation has been used:

$$K_m = \frac{4}{\pi} \frac{R_0}{B_0} b_m, \tag{AI.7}$$

where $m = 0$ - dipole, $m = 1$ - quadrupole, $m = 2$ - sextupole and so on.

Appendix II

Bounded Orbits near the Separatrix

The simplest function which fulfils the requirements $a(0) = a/2$ and $a(\infty) = a$ is

$$a(\theta) = a \frac{3 - e^{-B\theta}}{3 + e^{-B\theta}}. \tag{AII.1}$$

If foot-page equations (i) of ref. 3 are here rewritten in a slightly different form:

$$\frac{d\,a}{d\,\theta} = \frac{3}{16} K_2\, a^2 \sin 3\phi \quad , \qquad \text{(AII.2a)}$$

$$\frac{d\,\phi}{d\,\theta} = \frac{3}{16} K_2\, a \cos 3\phi - \frac{3}{4}\delta \qquad \text{(AII.2b)}$$

equation (AII.2b) can be written, taking into account equation (6), since only particles close to the separatrix are considered:

$$\frac{d\,\phi}{d\,\theta} = -\frac{3}{8}\delta\,\frac{\alpha^2 - a^2}{a^2} \quad . \qquad \text{(AII.3)}$$

Combining (AII.1) and (AII.3) and making the approximations

$$1 - \frac{2}{3} e^{-B\theta} \approx (1 - e^{-B\theta})^{2/3} \quad ,$$

$$1 \approx 1 - \frac{1}{3} e^{-B\theta} \approx (e^{-B\theta})^{1/3}$$

(being $B\theta \gg 1$),

one has:

$$\phi - \frac{\pi}{3} = \frac{3}{2}\frac{\delta}{B}\int_0^\theta \frac{d(3\,e^{B\theta})}{1 + 9e^{2B\theta}(1 - \frac{2}{3}e^{-B\theta})} \approx$$

$$\approx \frac{3}{2}\frac{\delta}{B}\int_0^\theta \frac{d[3\,e^{B\theta}(1 - e^{-B\theta})^{1/3}]}{1 + [3\,e^{B\theta}(1 - e^{-B\theta})^{1/3}]^2}$$

i.e.

$$\phi(\theta) \approx \frac{\pi}{3} - \frac{3}{2}\frac{\delta}{B}\,\mathrm{arc\,tan}\, 3\,e^{B\theta}(1 - e^{-B\theta})^{1/3} . \qquad \text{(AII.4)}$$

Equation (AII.4) yields $\phi(\theta = 0) = \pi/3$, as it is due, and

$$\phi(\infty) = \frac{\pi}{3} - \frac{3}{2}\frac{\delta}{B}\frac{\pi}{2} = \frac{\pi}{3}(1 - \frac{9}{4}\frac{\delta}{B})$$

whihch gives, as $\phi(\infty)$ must be equal to $\pi/6$:

$$B = (9/2)\,\delta \,,$$

i.e.

$$a(\theta) = \alpha\,\frac{3 - e^{-\frac{9}{2}\delta\theta}}{3 + e^{-\frac{9}{2}\delta\theta}} \quad , \qquad \text{(AII.5)}$$

$$\phi(\theta) = \frac{\pi}{3} - \frac{1}{3}\,\mathrm{arc\,tan}\, B\,e^{\frac{9}{2}\delta\theta}(1 - e^{-\frac{9}{2}\delta\theta})^{1/3} . \qquad \text{(AII.6)}$$

Taking into consideration the amplitude variation after the first turns, starting from $a(0) = a/2$, it is possible to write by means of equation (AII.5):

$$\Delta a = a(6\pi) - a(0) \approx \frac{81}{8}\pi a \delta \qquad \text{(AII.7)}$$

having considered the quantity $27\pi\delta$ as being small with respect to unity.

Concurrently, each side of the fixed-point triangle undergoes an inward displacement, perpendicular to itself, given by

$$\Delta S = \frac{1}{2}\Delta a = 3\pi \frac{d\,a}{d\,\theta}\,, \qquad \text{(AII.8)}$$

as it can be found by simple geometrical considerations and by bearing in mind that, over three turns, the time variation of ΔS is obviously constant.

As the adiabatic hypothesis implies that

$$\Delta S \ll \Delta a \qquad \text{(AII.9)}$$

equations (AII.5), (AII.8), (AII.9), together with equations (4), yield:

$$\frac{d\,\delta}{d\,\theta} \ll \frac{27}{8}\,\delta^2_{spill} \qquad \text{(AII.10)}$$

meaning that the 3-turns variation of the field index must be much smaller than 0.01, roughly, if all the particles are due to spill out "through" the fixed points.

Notice that the bounded points slow deadly down (see eq. (AII.5)) when they approach the vertices of the triangle.At this stage, equation (AII.9) is no longer variable,as Δa is almost null.

Resonantly Blasting Orbits

Over a resonance $(\delta = 0)$, equation (6) becomes just $\cos 3\phi = 0$, if equation (4) is considered. This implies that $\sin 3\phi = 1$; hence equation (AII.2a) becomes

$$\frac{d\,a}{d\,\theta} = \frac{3}{16} K_2\, a^2 \qquad \text{(AII.11)}$$

being very similar to equation (I, 15) of ref. 10 where the Rayleigh frequency distribution has been considered.

Integrating equation (AII.7), one has:

$$a(\theta) = \frac{a_0}{1 - \frac{3}{16} K_2 a_0\, \theta} \qquad \text{(AII.12)}$$

where a_0 is a betatron amplitude slightly outside the collapsed stable area.

Equation (AII.8) shows that a (θ) becomes divergent for approaching

$$\theta_{\infty} = \frac{16}{3 K_2 a_0} \tag{AII.13}$$

Notice that the weaker the perturbation, the bigger the number of revolutions required for reaching a given amplitude.

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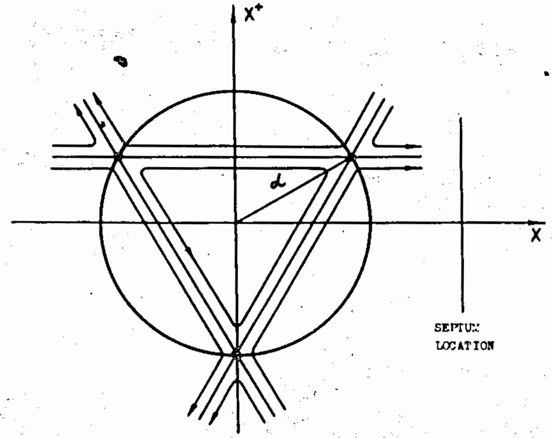

Fig. 1. Phase plane plot for 2/3 resonance. Arrows indicate the directions of the representative points displacements. Notice the three fixed points, as vertices of the separator triangle which contains the stable area. The outer part - unstable area - is divided in further six portions, by six unstable separatrix-lines. Each of these portions defines a different way of resonantly increasing the betatron amplitudes.

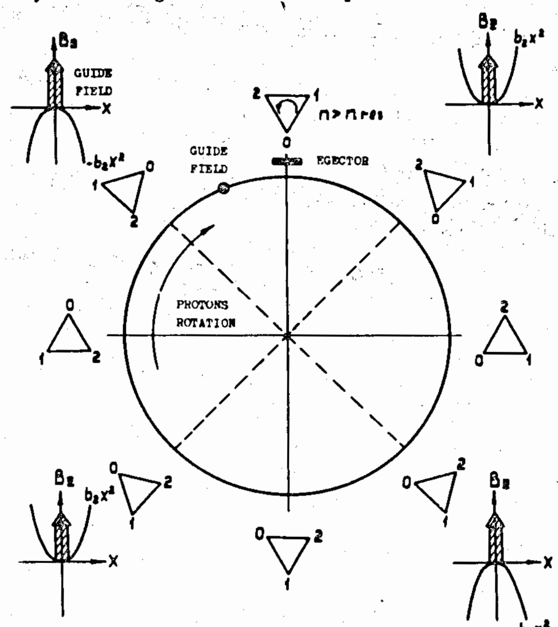

Fig. 2. Rotation of the fixed points triangle at eight azimuths. The perturbing field is considered positive when it has the same polarity of the guide field.

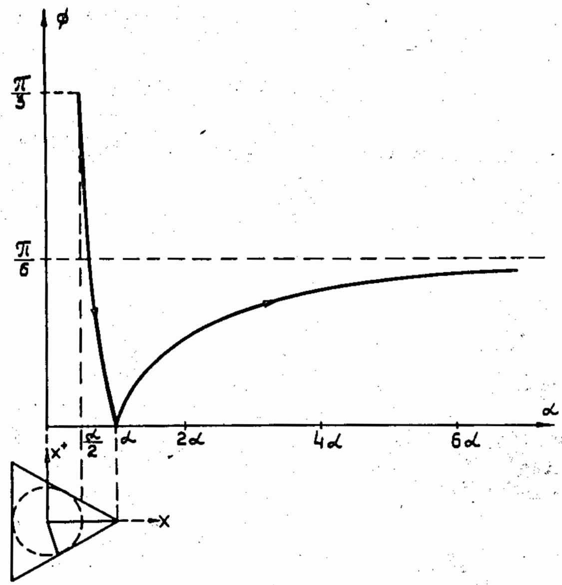

Fig. 3. Behaviours of stable stroboscopic points very near to the separatrix in both (a, ϕ) and (x, x^{+}) plots. The broken circle is the profile of the unperturbed circulating beam, shown as an useful reference.

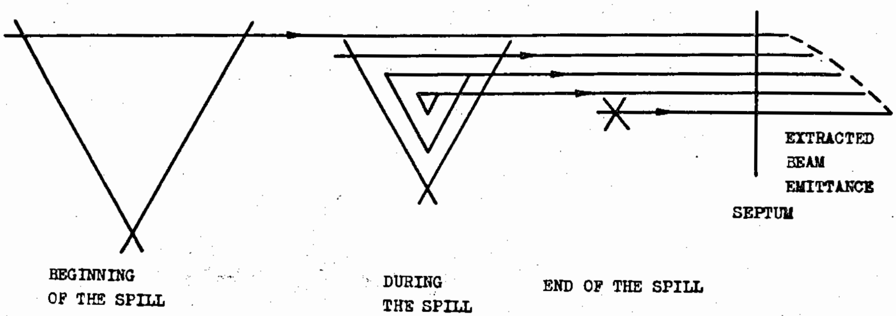

Fig. 4. Sketch of the resonant extraction procedure. Triangle on the left has its height equal to (3/2) r_b where $2r_b \approx 15$ cm is the radial width of the unperturbed circulating beam.

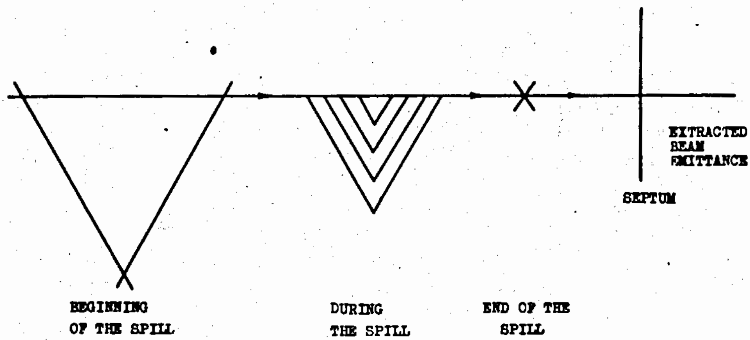

Fig. 5. Sketch of the proposed procedure for minimizing the ejected beam emittance. The closed orbit is continuously lifted during the spill-out, in order to keep up always the same divergence at the septum abscissa.

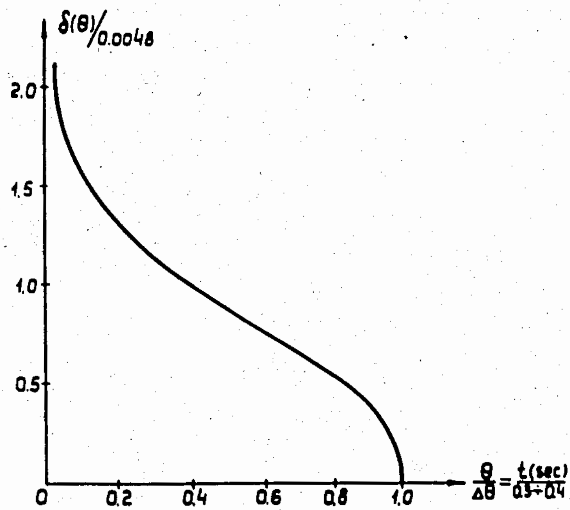

Fig. 6. Time dependence of $\delta(\theta) = n(\theta) - \bar{n}_{res}$ for obtaining uniform spill out of the beam.