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EIGENFUNCTION EXPANSIONS ASSOCIATED WITH A NONLINEAR SCHRÖDINGER EQUATION ON A HALF-LINE

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1 Introduction. Notation. Formulation of results

We consider the equation

$$-u'' + f(u^2)u = \lambda u, \quad u = u(x), \quad x \in (0, +\infty), \tag{1}$$

supplied with boundary conditions of one of the following two kinds:

$$u(0) = p, \ u'(0) = 0, \ \sup_{x>0} |u(x)| < \infty \tag{2}$$

or

$$u(0) = 0, \ u'(0) = p, \ \sup_{x>0} |u(x)| < \infty. \tag{3}$$

Hereafter all quantities are real, $\lambda \in R$ is the spectral parameter, f is a given function such that $f(u^2)u$ is continuously differentiable with respect to $u \in R$, and p is an arbitrary positive parameter fixed throughout the paper. In view of our assumption on the function f, for the Cauchy problem for the equation (1) with arbitrary initial data standard local theorems of existence, uniqueness and continuous dependence on the initial data and the parameter λ take place. We understand the condition (2) (resp. (3)) in the sense that the solution of the Cauchy problem for the equation (1) with the initial data $u(0) = p, \ u'(0) = 0$ (resp. $u(0) = 0, \ u'(0) = p$) can be continued on the whole half-line $x > 0$ and that this continuation is bounded on this half-line. Everywhere further under solutions of various Cauchy problems we mean solutions continued on maximal intervals of their existence. We also note that the case $p < 0$ can obviously be reduced to the considered one by the change of variables $u(x) \to -u(x)$ (if $p = 0$, then $u(x) \equiv 0$ for all $x \in R$ and any $\lambda \in R$ for each of the problems under consideration; so this case is trivial). In addition, since each of the boundary conditions (2) and (3) contains Cauchy data as a part, for each value of the parameter λ at most one function $u(x)$ satisfying the problem (1),(2) or (1),(3) can exists. If a pair (λ, u), consisting of a real number λ and a twice continuously differentiable function $u = u(x)$ of the argument $x \geq 0$, satisfies the problem (1),(2) or (1),(3), then we call λ the *eigenvalue* and $u(x)$ the corresponding *eigenfunction* of this problem. For each of the problems (1),(2) and (1),(3) we call the set Λ of all its eigenvalues the *spectrum* of this problem. We shall also denote by $u(\lambda, x)$ eigenfunctions indicating explicitly their dependence on $\lambda \in \Lambda$.

In the present paper, our assumption about the function f is the following.

(f) *Let $f(s)$ be a real-valued continuous monotonically nondecreasing function of the argument $s \geq 0$ and let $f(u^2)u$ be a continuously differentiable function of the argument $u \in R$.*
The assumption (f) is valid, for example, for $f(s) = |s|^q$ with a nonnegative constant q.

We introduce the following notation. By $L_2(a,b)$, where $-\infty \leq a < b \leq +\infty$, we denote the usual Lebesque space, consisting of real-valued functions of the argument $x \in (a,b)$, square integrable between a and b, with the scalar product $(u,v)_{L_2(a,b)} = \int\limits_a^b u(x)v(x)dx$ and the norm $||u||_{L_2(a,b)} = (u,u)^{\frac{1}{2}}_{L_2(a,b)}$. We set $e(k,x) = p\cos\frac{\pi x}{2k}$ for the problem (1),(2) so that $-e''_{xx}(k,x) = z(k)e(k,x)$, $e(k,0) = p$, $e'_x(k,0) = 0$ where $z(k) = \left(\frac{\pi}{2k}\right)^2$, and $e(k,x) = \pi^{-1}pk\sin\frac{\pi x}{k}$ in the case of the problem (1),(3), so that again $-e''_{xx} = z(k)e(k,x)$, $e(k,0) = 0$, $e'_x(k,0) = p$ and $z(k) = \left(\frac{\pi}{k}\right)^2$ in this case. Also, let $l(x) = 2x+1$ for the problem (1),(2) and $l(x) = x+1$ for the problem (1),(3). For any $g(\cdot) \in L_2(0,\infty)$ let $\hat{g}(\cdot) \in L_2(0,\infty)$ be its renormalized Fourier transform such that

$$\lim_{R\to+\infty}\left|\left|g(\cdot) - \int\limits_0^R \hat{g}(r)r^q e(r^{-1},\cdot)dr\right|\right|_{L_2(0,\infty)} = 0,$$

where $q = 0$ for the problem (1),(2) and $q = 1$ for the problem (1),(3). Then, for $\breve{g}(k) = k^{-2}\hat{g}(k^{-1})$ we have

$$\lim_{r\to+0}\left|\left|g(\cdot) - \int\limits_r^\infty \breve{g}(k)k^{-q}e(k,\cdot)dk\right|\right|_{L_2(0,\infty)} = 0.$$

By L we denote the set of all functions $g(\cdot)$ from $L_2(0,\infty)$ for each of which $\breve{g}(\cdot)$ is continuous on $(0,\infty)$ and there exist $a,b:\ 0 < a < b < \infty$ such that $\breve{g}(k) = 0$ for $k \in (0,a) \cup (b,\infty)$. Let l_2 be the space consisting of square summable sequences $a = (a_0, a_1, ..., a_n, ...)$ of real numbers a_n with the scalar product $(a,b)_{l_2} = \sum\limits_{n=0}^\infty a_n b_n$ and the norm $||a||_{l_2} = (a,a)^{\frac{1}{2}}_{l_2}$

(here $b = (b_0, b_1, ..., b_n, ...) \in l_2$) and let, for a Banach space B with a norm $||\cdot||_B$, $\mathcal{L}(B;B)$ be the space of all bounded linear operators $A, D, ...$ from B in B with the norm $||A||_{\mathcal{L}(B;B)} = \sup\limits_{u\in B:\ ||u||_B=1} ||Au||_B$. Finally, by $C, C_1, C_2, C', C'', ...$ we shall denote positive constants.

Questions of expansions on a segment of functions from spaces, containing as a part the set of all continuous functions, over eigenfunctions of nonlinear boundary-value problems with denumerable spectra are considered in a number of papers (see, for example, [1-7]). In the monograph [1], some interesting results in this direction are established. In the first author's paper [2], in which the following eigenvalue problem is considered

$$-y'' + f(y^2)y = \lambda y, \quad y = y(x), \quad x \in (0,1),$$

$$y(0) = y(1) = 0,$$

$$\int\limits_0^1 y^2(x)dx = 1$$

with a function f satisfying the hypothesis (f), there was obtained a result on the possibility of such an expansion of an arbitrary function from $L_2(0,1)$. This result is also reestablished (without proofs) in [3]. However, we have to note that the paper [2] contains some errors which fortunately can be corrected. We carry out these corrections in Appendix of the present paper. Also, problems similar to that mentioned were considered by the author in [4-6]; in these articles the property of being a basis in the space L_2 (the property of being a Riesz basis in some cases) for systems of eigenfunctions of these problems is proved (we especially mention the paper [5] were an independent proof of the above-indicated result from [2] is presented). In [7], also the following boundary-value problem (without a spectral parameter) is considered:

$$u'' + g(u^2)u = 0, \quad u = u(x), \quad u(0) = u(1) = 0$$

with

$$g(0) \leq 0 \quad \text{and} \quad \lim_{r\to+\infty} g(r) = +\infty.$$

It is proved in this paper that there exists $s_0 < 0$ such that an arbitrary system of solutions of this problem, which for any integer $n \geq 0$ possesses a unique solution with precisely n zeros in the interval $(0,1)$, is a basis in the Sobolev space H^s if $s < s_0$.

In the present paper, our main result is the following.

Theorem *Under the hypothesis* (f)
(a) $\Lambda = [f(p^2), +\infty)$ *for the problem* (1),(2) (*we denote* $\overline{\lambda} = f(p^2)$ *in this case*) *and in the case of the problem* (1),(3) *there exists* $\overline{\lambda} \geq f(0)$ *such that either* $\Lambda = [\overline{\lambda}, +\infty)$ *or* $\Lambda = (\overline{\lambda}, +\infty)$;
(b) *for each* $\lambda \in (\overline{\lambda}, +\infty)$ *there exists a minimal positive zero* $k = k(\lambda)$ *of* $u(\lambda, x)$ *as a function of the argument* $x \in (0, +\infty)$; *the function* $k : (\overline{\lambda}, +\infty) \to (0, +\infty)$ *is continuously differentiable,* $k'(\lambda) < 0$ *for any* $\lambda \in (\overline{\lambda}, +\infty)$, *and* $\lim\limits_{\lambda \to \overline{\lambda}+0} k(\lambda) = +\infty$, $\lim\limits_{\lambda \to +\infty} k(\lambda) = 0$. *By* $\lambda = \lambda(k)$ *we denote the function with the domain* $(0, +\infty)$ *inverse to the function* $k(\lambda)$;
(c) *for any function* $g(\cdot) \in L$ *continuous on* $(0, \infty)$ *there exists a unique function* $\overline{g}(k)$ *of the argument* $k > 0$, *continuous on the half-line* $(0, +\infty)$, *for some* $b > 0$ *satisfying the estimate*

$$|\overline{g}(k)| \leq C k^{-\frac{1}{2}}, \quad k \in (0, b], \tag{4}$$

with a constant $C > 0$ *independent of* $k \in (0, b]$ *and the condition* $\overline{g}(k) = 0$ *for* $k > b$, *such that for any point* $x > 0$ *the following equality takes place:*

$$g(x) = \int_0^b \overline{g}(k) k^{-q} u(\lambda(k), x) dk. \tag{5}$$

Remark 1. As it is proved further (see Lemma 1), for any $\overline{k} > 0$ there exists $C > 0$ such that $\sup\limits_{x \in (0,\infty)} |u(\lambda(k), x)| \leq C k^q$ for all $k \in (0, \overline{k}]$, so that the right-hand side in (5) is well-defined.

Remark 2. The expansion (5) stated by the Theorem is obviously an analog for our nonlinear case of the expansion in the Fourier integral which is associated with a linear self-adjoint eigenvalue problem. We also note that, with our Theorem, we do not strive to obtain a maximally strong result, but we want only to demonstrate a possibility of the expansion of an "arbitrary function" in an integral over eigenfunctions of each of our nonlinear problems.

Remark 3. The author does not know nontrivial examples of the function f for eigenfunctions of the problem (1),(2) or (1),(3) being representable as superpositions of elementary functions.

Remark 4. Since the Fourier transform is a linear one-to-one mapping from $L_2(0, \infty)$ on $L_2(0, \infty)$ continuous with the inverse one, the set L is dense in $L_2(0, \infty)$. Therefore, our Theorem yields a property in the space $L_2(0, \infty)$ of the systems of eigenfunctions $\{u(\lambda(k), x)\}_{k>0}$ of the problems (1),(2) and (1),(3) analogous to the completeness.

Remark 5. Concerning applications of our results, we believe that, with possible further developments of this direction, eigenfunction expansions associated with nonlinear differential equations can be useful, for example, in the Fourier and Galerkin methods for solving various nonlinear equations, in particular, nonlinear Schrödinger, wave and diffusion equations.

In Section 2, auxiliary results used for proving the Theorem are presented. Section 3 contains the proof of our Theorem. In Appendix, as we already noted, we present corrections to the paper [2].

Everywhere in the next sections except Appendix it is accepted that the assumption (f) is valid.

2 Auxiliary results

Consider two Cauchy problems for the equation

$$-y'' + f(y^2)y = \lambda y, \quad y = y(x), \quad x > 0 \tag{6}$$

with the initial conditions

$$y(0) = p, \; y'(0) = 0 \tag{7}$$

or

$$y(0) = 0, \; y'(0) = p. \tag{8}$$

The following result is proved in [4,6] for the problem (6),(7) and in [6] for the problem (6),(8).

Proposition 1. *There exists $\overline{\lambda} \geq f(p^2)$ (precisely $\overline{\lambda} = f(p^2)$ for the problem (6),(7)) such that*
(a) for any $\lambda > \overline{\lambda}$ the corresponding solution of the problem (6),(7) (or (6),(8)) can be continued on the whole real line and there exists a continuously differentiable function $x_0(\lambda) > 0$ of the argument $\lambda > \overline{\lambda}$ such that $x_0'(\lambda) < 0$ for all $\lambda > \overline{\lambda}$, the solution $y(x)$ of the problem (6),(7) (resp. (6),(8)) is positive on $(0, x_0(\lambda))$ and $y(x_0(\lambda)) = 0$;
(b) if $\lambda = \overline{\lambda}$, then $y(x) \equiv p$ for the problem (6),(7) and if $\lambda < \overline{\lambda}$, then the solution $y(x)$ of the problem (6),(7) (or (6),(8)) is unbounded in the maximal interval of its existence from $(0, +\infty)$;
(c) for any $a, b: \overline{\lambda} < a < b$ for the corresponding solutions $y_a(x)$ and $y_b(x)$ of the problem (6),(7) (or (6),(8)) taken with $\lambda = a$ and $\lambda = b$, respectively, for all $x \in [0, k(b)]$ the following inequality takes place: $y_a(x) \geq y_b(x)$. In addition, in the case of the problem (6),(7) $|y(x)| \leq p$ for all $\lambda > \overline{\lambda}$ and all $x \in R$;
(d) any solution $y(x)$ of the equation (6) is odd with respect to an arbitrary its zero $\overline{x}$ and is even with respect to any point $\tilde{x}$ such that $y'(\tilde{x}) = 0$, i. e. $y(\tilde{x} - x) = y(\tilde{x} + x)$ and $y(\overline{x} - x) = -y(\overline{x} + x)$ for all $x \in R$;
(e) $\lim_{\lambda \to +\infty} x_0(\lambda) = 0$ and $\lim_{\lambda \to \overline{\lambda}+0} x_0(\lambda) = +\infty$;
(g) for any $\lambda > \overline{\lambda}$ the zeros of the corresponding solution $y(x)$ of the problem (6),(7) (resp. (6),(8)) are precisely the points $l(m)x_0(\lambda)$ where $m = 0, \pm 1, \pm 2, ...$.

Remark 6. Proposition 1 immediately implies statements (a) and (b) of our Theorem with $k(\lambda) = x_0(\lambda)$.

In what follows, for the simplicity of the notation we rename by $u(k, x)$ the eigenfunction $u(\lambda(k), x)$. Further, in view of Proposition 1, for any $k > 0$

$$\lambda\left(\frac{k}{l(0)}\right) < ... < \lambda\left(\frac{k}{l(n)}\right) < ...$$

are all values of the parameter λ which are greater than $\overline{\lambda}$ and for which the corresponding solutions $y_n(k, x)$ of the problem (6),(7) (or (6),(8)) become zero at $x = k$. In addition, for any $n = 0, 1, 2, ...$ and a fixed $k > 0$ the function $y_n(k, x)$ has precisely n zeros in the interval $(0, k)$. The following statement is proved in [4,6] for the problem (6),(7) and in [6] for the problem (6),(8).

Proposition 2. *For any fixed $k > 0$ the system of functions $\{y_n(k, x)\}_{n=0,1,2,...}$ is a basis in the space $L_2(0, k)$, i. e. for any $g \in L_2(0, k)$ there exists a unique sequence $\{a_n\}_{n=0,1,2,...}$ of real numbers a_n such that $g(\cdot) = \sum_{n=0}^{\infty} a_n y_n(k, \cdot)$ in the sense of the space $L_2(0, k)$. In addition, clearly $y_n(k, x) = u\left(\frac{k}{l(n)}, x\right)$.*
In what follows, we exploit some ideas used earlier for proving this statement in the above-indicated papers. For this reason, though in further considerations we do not prove Proposition 2, we believe that these considerations clarify the main idea of its proof.

According to Proposition 2, for any $k > 0$ there exists a unique sequence of real numbers $\{d_n(k)\}_{n=0,1,2,...}$ such that

$$e(k, \cdot) = \sum_{n=0}^{\infty} d_n(k) u\left(\frac{k}{l(n)}, \cdot\right) \tag{9}$$

in the space $L_2(0, k)$. Further, since $e(k, x)$ as a function of the argument x with a fixed $k > 0$ is odd with respect to any its zero and is even with respect to any zero of its first derivative and for the functions $u\left(\frac{k}{l(n)}, x\right)$ this property is valid due to Proposition 1 and since in addition in view of Proposition 1 each zero of the function $e(k, x)$ is also a zero of each function $u\left(\frac{k}{l(n)}, x\right)$, the equality (9) holds in each space $L_2(x_1, x_2)$ where $x_1 < x_2$ are two arbitrary nearest zeros of the function $e(k, x)$. Thus, the equality (9) is also valid in the space $L_2(a, b)$ with arbitrary $a, b: -\infty < a < b < +\infty$.

Lemma 1. *For any $\overline{k} > 0$ there exists $C > 0$ such that for all $k \in (0, \overline{k}]$ one has*

$$|e(k, x) - u(k, x)| \leq Ck, \quad x \in R,$$

for the problem (1),(2) and

$$|e(k, x) - u(k, x)| \leq Ck^2, \quad x \in R,$$

for the problem (1),(3).

Proof. The proof of Lemma 1 in fact repeats the proof of an analogous statement for a linear problem from [8] (see Lemma 1.7 from [8]). We sketch this proof for the convenience of readers.

Take an arbitrary $\overline{k} > 0$. The uniform boundedness of the family of functions $\{u(k,x)\}_{k\in(0,\overline{k}]}$ with respect to $x \in R$ following from Proposition 1 and the standard comparison theorem imply the existence of $D > 0$ such that

$$|\lambda(k) - z(k)| \le D \tag{10}$$

for all $k \in (0, \overline{k}]$. We consider only the problem (1),(3) because for the problem (1),(2) the proof can be made by analogy. So, one can easily verify that any solution $u(k,x)$ of the problem (1),(3) with a sufficiently small $k > 0$ for any $x \ge 0$ satisfies the equation

$$u(k,x) = p\lambda^{-\frac{1}{2}}(k)\sin(\lambda^{\frac{1}{2}}(k)x) +$$

$$+\lambda^{-\frac{1}{2}}(k)\int_0^x \sin\{\lambda^{\frac{1}{2}}(k)(x-t)\}f(u^2(k,t))u(k,t)dt \tag{11}$$

(since due to Proposition 1 $\lim\limits_{k\to+0}\lambda(k) = +\infty$, the right-hand side of (11) is well-defined for all sufficiently small $k > 0$). Therefore, due to the uniform boundedness of the family of functions $\{u(k,x)\}_{k\in(0,\overline{k}]}$ we get from (11) for all sufficiently small $k > 0$ and all $x \in [0,k]$:

$$|u(k,x) - p\lambda^{-\frac{1}{2}}(k)\sin(\lambda^{\frac{1}{2}}(k)x)| \le C_1\lambda^{-\frac{1}{2}}(k)$$

with a constant $C_1 > 0$ independent of the above x and k. Applying this estimate to the integrand from the right-hand side of (11), we obtain

$$|u(k,x) - p\lambda^{-\frac{1}{2}}(k)\sin(\lambda^{\frac{1}{2}}(k)x)| \le C_2\lambda^{-1}(k)$$

with a constant $C_2 > 0$ independent of sufficiently small $k > 0$ and $x \in [0,k]$. In addition, since for the problem (1),(3) due to (10) $\lambda(k) \ge C_3k^{-2}$ for all sufficiently small $k > 0$ and

$$|e(k,x) - p\lambda^{-\frac{1}{2}}(k)\sin(\lambda^{\frac{1}{2}}(k)x)| \le C_4\lambda^{-1}(k)$$

with positive constants C_3 and C_4 independent of sufficiently small $k > 0$ and $x \in [0,k]$, we get

$$|e(k,x) - u(k,x)| \le C_5k^2$$

with a constant $C_5 > 0$ independent of sufficiently small $k > 0$ and $x \in [0,k]$. Finally, in view of Proposition 1 the latter estimate holds

for all $x \in R$. Thus, Lemma 1 is proven.□

One can easily see that for arbitrary $k > 0$ and integer $n \ge 0$ the functions $\left\{e(\frac{k}{l(n)l(m)}, x\right\}_{m=0,1,2,\dots}$ form an orthogonal basis in the space $L_2\left(0, \frac{k}{l(n)}\right)$. Hence, we have

$$u\left(\frac{k}{l(n)},\cdot\right) = \sum_{m=0}^{\infty} b'_{n,m}(k)e\left(\frac{k}{l(n)l(m)},\cdot\right)$$

in the sense of the space $L_2\left(0, \frac{k}{l(n)}\right)$. Also, as in the case of the expansion (9), the latter expansion holds in the space $L_2(a,b)$ with arbitrary $-\infty < a < b < +\infty$. Therefore, for any $k > 0$ we have the sequence of expansions

$$u\left(\frac{k}{l(n)},\cdot\right) = \sum_{m=0}^{\infty} b_{n,m}(k)e\left(\frac{k}{l(m)},\cdot\right), \quad n = 0,1,2,..., \tag{12}$$

held in the sense of the spaces $L_2(0,k)$ and $L_2(a,b)$ with arbitrary $a,b:$ $-\infty < a < b < \infty$ where $b_{n,m}(k) = b'_{n,r}(k)$ if $l(m) = l(n)l(r)$ for some $r = 0,1,2,...$ and $b_{n,m}(k) = 0$ otherwise. Thus $B(k) = (b_{n,m}(k))_{n,m=0,1,2,\dots}$ is an upper triangular matrix and, for any n,

$$b_{n,n}(k) = \left\|e\left(\frac{k}{l(n)},\cdot\right)\right\|_{L_2(0,k)}^{-2}\left(u\left(\frac{k}{l(n)},\cdot\right), e\left(\frac{k}{l(n)},\cdot\right)\right)_{L_2(0,k)} > 0$$

because the functions $u\left(\frac{k}{l(n)}, x\right)$ and $e\left(\frac{k}{l(n)}, x\right)$ are of the same sign. So, in (12)

$$b_{n,m}(k) = 0 \text{ if } l(m) \ne l(n)l(s) \text{ for all } s = 0,1,2,...$$
$$\text{and } b_{n,n}(k) > 0 \text{ for all } n = 0,1,2,... \tag{13}$$

We also remark that generally speaking the properties (13) do not yield the completeness of the system of functions $\left\{u\left(\frac{k}{l(n)},\cdot\right)\right\}_{n=0,1,2,\dots}$ in the space $L_2(0,k)$ (see a counterexample in [7]).

Let for any $k > 0$ and numbers $n \ge 0$ and $m \ge 0$

$$\overline{u}\left(\frac{k}{l(n)}, x\right) = u\left(\frac{k}{l(n)}, x\right)\left\|u\left(\frac{k}{l(n)},\cdot\right)\right\|_{L_2(0,k)}^{-1},$$

$$\overline{e}\left(\frac{k}{l(n)},x\right)=e\left(\frac{k}{l(n)},x\right)\left\|e\left(\frac{k}{l(n)},\cdot\right)\right\|^{-1}_{L_2(0,k)}$$

and $$\overline{b}_{n,m}(k)=b_{n,m}(k)\left\|u\left(\frac{k}{l(n)},\cdot\right)\right\|^{-1}_{L_2(0,k)}\left\|e\left(\frac{k}{l(m)},\cdot\right)\right\|_{L_2(0,k)}. \quad (14)$$

Then, due to (12)-(14)

$$\overline{u}\left(\frac{k}{l(n)},\cdot\right)=\sum_{m=0}^{\infty}\overline{b}_{n,m}(k)\overline{e}\left(\frac{k}{l(m)},\cdot\right),\quad n=0,1,2,... \quad (15)$$

in the spaces $L_2(0,k)$ and $L_2(a,b)$ with arbitrary $a,b: -\infty<a<b<+\infty$ where

$$\overline{b}_{n,m}(k)=0\ \ (n>m)\ \ \text{and}\ \ \overline{b}_{n,n}(k)>0\ \ (n=0,1,2,...). \quad (16)$$

Lemma 2. *For each $k>0$ $u(k,x)$ as a function of the argument $x\in[0,k]$ is concave. For any $k>0$ $\max\limits_{x\in[0,k]}\overline{u}(k,x)\in[k^{-\frac{1}{2}},\sqrt{3}k^{-\frac{1}{2}}]$ and for any $\overline{k}>0$ there exist $c,C:\ 0<c<C$ such that $ck^{\frac{1}{2}+q}\leq\|u(k,\cdot)\|_{L_2(0,k)}\leq Ck^{\frac{1}{2}+q}$ for all $k\in(0,\overline{k}]$. In addition, $b_{n,n}(k)\geq\hat{b}$ for all $k>0$ where $\hat{b}=4\sqrt{2}\pi^{-2}$.*

Proof. First of all, obviously $\overline{b}_{n,n}(k)=\overline{b}_{0,0}\left(\frac{k}{l(n)}\right)$. We state that for any $k>0$ $u(k,x)$ as a function of the argument $x\in[0,k]$ is concave. To prove this fact, it suffices to show that $u''_{xx}(k,x)\leq0$ for all $x\in[0,k]$. In the case of the problem (1),(2) the latter fact follows from the equation (1) because due to Proposition 1 $\lambda>f(p^2)$ and $u(k,x)>0$ for $x\in[0,k)$. Let us show this inequality for the problem (1),(3). Suppose this is not right and there exist $k>0$ and $x_0\in(0,k)$ such that $u''_{xx}(k,x_0)>0$. Since due to Proposition 1 $u\left(k,\frac{k}{2}+x\right)=u\left(k,\frac{k}{2}-x\right)$ for all $x\in R$, we can accept that $x_0\in\left(0,\frac{k}{2}\right)$. But then, we have $u(k,x_0)>0$, $u'_x(k,x_0)>0$ and $u''_{xx}(k,x_0)>0$, therefore, in view of the equation (1) and due to the fact that $f(s)$ is a nondecreasing function of the argument $s\geq0$, we immediately derive that $u(k,x)>u(k,x_0)$, $u'_x(k,x)>0$ and $u''_{xx}(k,x)>0$ for all $x>x_0$ for which this solution exists. Thus, we get the contradiction, and the concavity of the function $u(k,x)$ for $x\in[0,k]$ is proved.

The second and third statements of Lemma 2 easily follow from this concavity. Indeed, consider, for example, the problem (1),(3). Then, we have for $R(k,x)=2k^{-1}x\overline{u}\left(k,\frac{k}{2}\right)$ (here, due to Proposition 1, $x=\frac{k}{2}$ is the

unique point of the maximum of the function $u(k,x)$ on $[0,k]$):

$$\frac{k}{3}\overline{u}^2\left(k,\frac{k}{2}\right)=2\int_0^{\frac{k}{2}}R^2(k,x)dx\leq\|\overline{u}(k,\cdot)\|^2_{L_2(0,k)}=1\leq k\overline{u}^2\left(k,\frac{k}{2}\right)$$

and therefore $\overline{u}\left(k,\frac{k}{2}\right)=\max\limits_{x\in[0,k]}\overline{u}(k,x)\in\left[k^{-\frac{1}{2}},\sqrt{3}k^{-\frac{1}{2}}\right]$. The estimate for $\|u(k,\cdot)\|_{L_2(0,k)}$ can be obtained by analogy with the use of Lemma 1 and the uniform boundedness of the family of functions $\{u(k,x)\}_{k\in(0,\overline{k}]}$ following from Proposition 1. Finally,

$$\overline{b}_{0,0}(k)\geq2(R(k,x),\overline{e}(k,x))_{L_2(0,\frac{k}{2})}\geq\hat{b}$$

for all $k>0$. For the problem (1),(2) the second and third statements of Lemma 2 can be proved by analogy. Lemma 2 is proven.□

For a matrix $A=(a_{i,j})_{i,j=0,1,2,...}$ we set $t(A)=\left\{\sum\limits_{i,j=0}^{\infty}a^2_{i,j}\right\}^{\frac{1}{2}}$. Also, for two infinite matrices $A_1=(a^1_{i,j})_{i,j=0,1,2,...}$ and $A^2=(a^2_{i,j})_{i,j=0,1,2,...}$ we define their sum A_1+A_2 and product A_1A_2 in the usual way, so that, for example, $(A_1A_2)_{i,j}=\sum\limits_{m=0}^{\infty}a^1_{i,m}a^2_{m,j}$ assuming these infinite sums to be converging (otherwise the product A_1A_2 is not determined).

Let $\delta_{n,n}=1$ and $\delta_{n,m}=0$ for $m\neq n$.We introduce the following matrices:

$$B_0(k)=(\overline{b}_{n,m}(k)\delta_{n,m})_{n,m=0,1,2,...},\quad \Lambda=(\delta_{n,m})_{n,m=0,1,2,...},$$

$$B_1(k)=((1-\delta_{n,m})(\overline{b}_{n,n}(k))^{-1}\overline{b}_{n,m}(k))_{n,m=0,1,2,...}.$$

Then, $\overline{B}(k)=(\overline{b}_{n,m}(k))_{n,m=0,1,2,...}=B_0(k)(\Lambda+B_1(k))$.

Lemma 3. *For two arbitrary matrices $A_m=(a^m_{i,j})_{i,j=0,1,2,...}$, $m=1,2$, satisfying the condition $t(A_m)<\infty$, $m=1,2$, the product A_1A_2 is determined, $t(A_1A_2)\leq t(A_1)t(A_2)$ and $t(A_1+A_2)\leq t(A_1)+t(A_2)$.*

Proof. Let us take arbitrary numbers i and j. Then,

$$\left|\sum_{m=M}^{N}a^1_{i,m}a^2_{m,j}\right|\leq\left\{\sum_{m=M}^{N}(a^1_{i,m})^2\right\}^{\frac{1}{2}}\left\{\sum_{m=M}^{N}(a^2_{m,j})^2\right\}^{\frac{1}{2}}\to0$$

as $M, N \to \infty$, therefore the matrix A_1A_2 is determined. Further,

$$t(A_1A_2) = \left\{ \sum_{i,j=0}^{\infty} \left(\sum_{m=0}^{\infty} a^1_{i,m} a^2_{m,j} \right)^2 \right\}^{\frac{1}{2}} \leq$$

$$\leq \left\{ \sum_{i,j=0}^{\infty} \left(\sum_{m=0}^{\infty} (a^1_{i,m})^2 \right) \left(\sum_{m=0}^{\infty} (a^2_{m,j})^2 \right) \right\}^{\frac{1}{2}} = t(A_1)t(A_2).$$

The inequality $t(A_1 + A_2) \leq t(A_1) + t(A_2)$ can be obtained by analogy with the use of the Minkowski's inequality. Thus, Lemma 3 is proven.□

Lemma 4. *Let $A = (a_{i,j})_{i,j=0,1,2,...}$ and $t(A) < \infty$. Then, $A \in \mathcal{L}(l_2; l_2)$ and $||A||_{\mathcal{L}(l_2;l_2)} \leq t(A)$.*

Proof. For an arbitrary $u = (u_0, u_1, ..., u_n, ...) \in l_2$ in view of the Hölder's inequality the product Au is determined and we have

$$||Au||_{l_2} = \left\{ \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} a_{i,j} u_j \right)^2 \right\}^{\frac{1}{2}} \leq$$

$$\leq \left\{ \sum_{i=0}^{\infty} \left(\sum_{j=0}^{\infty} a^2_{i,j} \right) \left(\sum_{j=0}^{\infty} u_j^2 \right) \right\}^{\frac{1}{2}} = ||u||_{l_2} t(A),$$

and Lemma 4 is proven.□

Remark 7. One can prove by analogy with Lemmas 3 and 4 that for any two matrices A and B, satisfying the conditions $t(A) < \infty$ and $t(B) < \infty$, and a vector $u \in l_2$ one has $A(Bu) = (AB)u$. This implies in particular that for any two operators A and B from $\mathcal{L}(l_2; l_2)$, being matrices which satisfy the conditions $t(A) < \infty$ and $t(B) < \infty$, the product of these matrices corresponds to the product of operators.

Lemma 5. *There exists $k_1 > 0$ such that the matrix $\overline{B}(k)$ with an arbitrary $k \in (0, k_1]$ belongs to $\mathcal{L}(l_2; l_2)$, $t(B_1(k)) < \frac{1}{4}$ and, as an operator from $\mathcal{L}(l_2; l_2)$, $\overline{B}(k)$ has an inverse operator $T(k) = [\overline{B}(k)]^{-1} \in \mathcal{L}(l_2; l_2)$, which can be represented as an infinite upper triangular matrix $T(k) = (t_{n,m}(k))_{n,m=0,1,2,...}$ with $t_{n,n}(k) = \overline{b}^{-1}_{n,n}(k)$ such that $t(T(k) - [B_0(k)]^{-1}) < \frac{1}{4}$ (here $[B_0(k)]^{-1}$ is the diagonal matrix with $([B_0(k)]^{-1})_{n,n} = \overline{b}^{-1}_{n,n}(k)$).*

Proof. By Proposition 1 and Lemmas 1 and 2, for any $\beta \in (0,1)$ there exists $\tilde{k} = \tilde{k}(\beta) > 0$ such that $t(B_1(k)) < \beta$ for all $k \in (0, \tilde{k}]$. Hence, in view of Lemma 4, as it is well-known, for each of those β and an arbitrary $k \in (0, \tilde{k}]$ there exists an operator $T(k)$ inverse to $\overline{B}(k)$ in the space $\mathcal{L}(l_2; l_2)$,

$$T(k) = \left(\Lambda + \sum_{m=1}^{\infty} (-1)^m B_1^m(k) \right) [B_0(k)]^{-1}.$$

Further, by Lemmas 2 and 3

$$t\left(\left(\sum_{m=1}^{\infty} (-1)^m B_1^m(k) \right) [B_0(k)]^{-1} \right) \leq \hat{b}^{-1} \sum_{m=1}^{\infty} \beta^m = \frac{\hat{b}^{-1}\beta}{(1-\beta)}$$

and by analogy

$$t\left(\left(\sum_{m=M_1}^{M_2} (-1)^m B_1^m(k) \right) [B_0(k)]^{-1} \right) \to 0 \quad \text{as} \quad M_1, M_2 \to \infty.$$

Therefore, the operator $T(k)$ can be written as an infinite upper triangular matrix and, taking $\beta \in \left(0, \frac{1}{4}\right)$ so small that $\frac{\hat{b}^{-1}\beta}{(1-\beta)} < \frac{1}{4}$ and choosing $k_1 = \tilde{k}(\beta)$, we get the statement of Lemma 5.□

Lemma 6. *For any $\overline{k} > 0$ there exists $C > 0$ such that in (15),(16)*

$$|\overline{b}_{n,m}(k)| \leq C l^{s+\frac{1}{2}}(n) l^{-s}(m)$$

for any $s = 1, 2, 3$, all $k \in (0, \overline{k}]$ and all numbers n and m. Thus in particular for any $\overline{k} > 0$ there exists $\overline{C} > 0$ such that $|\overline{b}_{0,n}(k)| \leq \overline{C} l^{-3}(n)$ for all $k \in (0, \overline{k}]$ and $n = 0, 1, 2, ...$.

Proof. Take an arbitrary $\overline{k} > 0$. In some places of this proof we use the uniform boundedness of the family of functions $\{u(k,x)\}_{k \in (0,\overline{k}]}$ with respect to $x \in R$, following from Proposition 1, and the estimate (10). At first, let us show the existence of $C_1 > 0$ such that

$$\left| \frac{d^s \overline{u}(k,x)}{dx^s} \right| \leq C_1 k^{-s-\frac{1}{2}}, \quad s = 1, 2, 3, \tag{17}$$

for all $k \in (0, \overline{k}]$ and $x > 0$. We first consider the problem (1),(2). Due to Lemma 2 we have in this case

$$C_2^{-1} k^{\frac{1}{2}} \leq ||u(k,\cdot)||_{L_2(0,k)} \leq C_2 k^{\frac{1}{2}} \tag{18}$$

for a constant $C_2 > 1$ independent of $k \in (0, \overline{k}]$. Multiplying the equation (1), written for $u(k,x)$, by $2u'_x(k,x)$ and integrating from 0 to x, in view of (2) we get the identity

$$-[u'_x(k,x)]^2 + F(u^2(k,x)) - F(p^2) = \lambda(k)u^2(k,x) - \lambda(k)p^2, \quad x \in R,$$

where $F(z) = \int\limits_0^z f(r)dr$, which together with (18) due to (10) implies (17) with $s = 1$. Then, from the equation (1), written for $u(k,x)$, and (18) we derive (17) with $s = 2$. Finally, differentiating the equation (1), written for $u(k,x)$, over x, due to (18) and (17) with $s = 1$ we get (17) with $s = 3$.

Consider now the case of the problem (1),(3). Multiplying the equation (1), written for $u(k,x)$, by $2u'_x(k,x)$ and integrating between 0 and x, in view of (3) we get

$$-[u'_x(k,x)]^2 + p^2 + F(u^2(k,x)) = \lambda(k)u^2(k,x), \quad x \in R. \qquad (19)$$

Then, due to Lemmas 1 and 2, there exists $C_3 > 1$ such that $C_3^{-1}k \leq |u(k,x)| \leq C_3 k$ and $C_3^{-1}k^{\frac{3}{2}} \leq ||u(k,\cdot)||_{L_2(0,k)} \leq C_3 k^{\frac{3}{2}}$ for all $k \in (0,\overline{k}]$ and $x \in R$ in this case. Hence, multiplying (19) by k^{-3}, due to Lemma 1 and (10) we derive the estimate

$$|\overline{u}'_x(k,x)| \leq C_4 k^{-\frac{3}{2}}$$

with a constant $C_4 > 0$ independent of $k \in (0,\overline{k}]$ and of $x \in R$. By analogy, we derive from the equation (1)

$$|\overline{u}''_{xx}(k,x)| \leq C_8 k^{-\frac{5}{2}}$$

for all $k \in (0,\overline{k}]$ and $x \in R$. These estimates yield (17) with $s = 1, 2$ for the problem (1),(3). The estimate (17) with $s = 3$ in the case of the problem (1),(3) can be obtained by analogy with the use of the differentiation of the equation (1), written for $u(k,x)$, over x. So, the estimate (17) is proved.

Now, we recall that $\overline{b}_{n,m}(k) = 0$ if $l(m) \neq l(n)l(d)$ for all $d = 0,1,2,\ldots$Let numbers n, m and d be such that $l(m) = l(n)l(d)$. Consider for the definiteness the problem (1),(2) and $s = 3$. Using (17) and the integration by parts, we get

$$|\overline{b}_{n,m}(k)| = \left| \int\limits_0^k \overline{u}\left(\frac{k}{l(n)}, x\right) \overline{e}\left(\frac{k}{l(m)}, x\right) dx \right| \leq$$

$$\leq C_{10} k^{\frac{5}{2}} l^{-3}(m) \left| \int\limits_0^k \frac{d^3}{dx^3}\overline{u}\left(\frac{k}{l(n)}, x\right) \sin\left(\frac{\pi l(m) x}{2k}\right) dx \right| \leq C_{11} l^{\frac{7}{2}}(n) l^{-3}(m)$$

with positive constants C_{10} and C_{11} independent of $k > 0$ and of numbers n and m. For other values of s and for the problem (1),(3) proofs are completely analogous. Lemma 6 is proven.□

Now, for arbitrary $k > k_1$ and integer $M > 0$ consider the matrices $B_M(k) = (\overline{b}_{n,m}(k))_{n,m=\overline{0,M}}$ and $T_M(k) = [B_M(k)]^{-1} = (t^M_{n,m}(k))_{n,m=\overline{0,M}}$. The trivial observations are that the matrices $T_M(k)$ are upper triangular and that for any fixed indices n and m the element $t^M_{n,m}(k)$ does not depend on the number $M \geq M_0 = \max\{n;m\}$ and we rename it simply by $t_{n,m}(k)$. So, for any $k > k_1$ we can construct an infinite matrix $T(k) = (t_{n,m}(k))_{n,m=0,1,2,\ldots}$. Further, one can observe that, completely as in the proof of Lemma 5, for any $\overline{k} > k_1$ there exists a number N_1 independent of $k \in (k_1, \overline{k}]$ such that for the matrix $B^{N_1}(k) = (\overline{b}_{n,m}(k))_{n,m=N_1+1,N_1+2,\ldots}$ there is an upper triangular matrix $T^{N_1}(k) = (t'_{n,m}(k))_{n,m=N_1+1,N_1+2,\ldots}$, where $t'_{n,n}(k) = \overline{b}^{-1}_{n,n}(k)$, for all $k \in (k_1, \overline{k}]$ satisfying the condition $\sum\limits_{n=N_1+1}^{\infty} \sum\limits_{m=n+1}^{\infty} (t'_{n,m}(k))^2 \leq \overline{T}$ with a constant $\overline{T} > 0$ depending only on $\overline{k} > k_1$ and such that $T^{N_1}(k)B^{N_1}(k) = B^{N_1}(k)T^{N_1}(k) = \Lambda$ (since the matrices $T^{N_1}(k)$ and $B^{N_1}(k)$ are upper triangular, expressions for elements of their products contain only finite sums). In addition, one can easily verify that $t'_{n,m}(k) = t_{n,m}(k)$ for all $n, m > N_1$.

Lemma 7. *For any $\overline{k} > k_1$ there exists $C > 0$ such that for any $k \in (0,\overline{k}]$ the following estimates take place:*

$$|t_{n,m}(k)| \leq C l^{\frac{7}{2}}(n) l^{-\frac{3}{2}}(m), \quad n, m = 0, 1, 2, \ldots$$

Thus in particular for any $\overline{k} > 0$ there exists $\overline{C} > 0$ such that $|t_{0,n}(k)| \leq \overline{C} l^{-\frac{3}{2}}(n)$ for all $k \in (0,\overline{k}]$ and all $n = 0, 1, 2, \ldots$.

Proof. For arbitrary nonnegative integer n and m let $N = \max\{n;m\}$ and $B_{m,n}(k)$ be the matrix obtained from $B_N(k)$ by taking away the mth row and the nth column of the latter. Then,

$$t_{n,m}(k) = (-1)^{m+n}\det[B_{m,n}(k)] \times \det[T_N(k)], \quad n, m = \overline{0,N}. \qquad (20)$$

Clearly $\det[B_{m,n}(k)] = 0$ if $m < n$. For $m > n$ we obviously have

$$\det[B_{m,n}(k)] = \prod_{r=0}^{n-1} \overline{b}_{r,r}(k) \times \prod_{r=m+1}^{N} \overline{b}_{r,r}(k) \times \det[B'_{m,n}(k)] \qquad (21)$$

where

$$B'_{m,n}(k) = \begin{pmatrix} \overline{b}_{n,n+1} & \overline{b}_{n,n+2} & \dots\dots & \overline{b}_{n,m-1} & \overline{b}_{n,m} \\ \overline{b}_{n+1,n+1} & \overline{b}_{n+1,n+2} & \dots\dots & \overline{b}_{n+1,m-1} & \overline{b}_{n+1,m} \\ 0 & \overline{b}_{n+2,n+2} & \dots\dots & \overline{b}_{n+2,m-1} & \overline{b}_{n+2,m} \\ \dots & \dots & \dots\dots & \dots & \dots \\ 0 & 0 & \dots\dots & \overline{b}_{m-1,m-1} & \overline{b}_{m-1,m} \end{pmatrix}.$$

In this matrix, let us subtract the first column, multiplied by $\overline{b}^{-1}_{n+1,n+1}\overline{b}_{n+1,m}$, from the last column, the second column, multiplied by $\overline{b}^{-1}_{n+2,n+2}\overline{b}_{n+2,m}$, from the last column, and so on. At the last step of this process me subtract from the last column the next to the last column multiplied by $\overline{b}^{-1}_{m-1,m-1}\overline{b}_{m-1,m}$. Then, we get the matrix all elements of the last column of which, except the element in the first row, are equal to zero and the element in the first row is equal to $\overline{b}_{n,m} - \sum_{r=n+1}^{m-1} \overline{b}^{-1}_{r,r}\overline{b}_{r,m}\overline{b}_{n,r}$. Hence,

$$\left|\det[B'_{m,n}(k)]\right| = \left| \prod_{r=n+1}^{m-1} \overline{b}_{r,r} \times \left[\overline{b}_{n,m} - \sum_{r=n+1}^{m-1} \overline{b}^{-1}_{r,r}\overline{b}_{r,m}\overline{b}_{n,r} \right] \right|.$$

Thus, taking also into account Lemmas 2 and 6 with $s = 3$, (20) and (21), we get the existence of $C_1 > 0$ and $C_2 > 0$ such that

$$|t_{n,m}(k)| \le \left| \overline{b}^{-1}_{n,n}\overline{b}^{-1}_{m,m} \left[\overline{b}_{n,m} - \sum_{r=n+1}^{m-1} \overline{b}^{-1}_{r,r}\overline{b}_{r,m}\overline{b}_{n,r} \right] \right| \le$$

$$\le C_1 \left[l^{\frac{7}{2}}(n) l^{-3}(m) + \sum_{r=n+1}^{m-1} \left(l^{\frac{7}{2}}(r) l^{-3}(m) l^{\frac{7}{2}}(n) l^{-3}(r) \right) \right] \le C_2 l^{\frac{7}{2}}(n) l^{-\frac{3}{2}}(m)$$

for all $k \in (0, \overline{k}]$ and numbers n and m, and Lemma 7 is proven.□

Remark 8. Let $\overline{k} > k_1$ and let $N_1 \ge 0$ be the number defined before Lemma 7. According to Lemma 7 there exists $\overline{\overline{T}} > 0$ such that $\sum_{m=0}^{\infty} t^2_{n,m}(k) \le \overline{\overline{T}}$ for all $n = \overline{0, N_1}$. Hence, $\sum_{n=0}^{\infty}\sum_{m=n+1}^{\infty} t^2_{n,m}(k) < \infty$, therefore $T(k) \in \mathcal{L}(l_2; l_2)$ and also, since $T(k)\overline{B}(k) = \overline{B}(k)T(k) = \Lambda$ by construction, $T(k)$ is an inverse operator to $\overline{B}(k)$ in the space l_2. Further, due to Proposition 2 and the arguments after it

$$\overline{e}\left(\frac{k}{l(n)}, \cdot\right) = \sum_{r=0}^{\infty} d_{n,r}(k)\overline{u}\left(\frac{k}{l(r)}, \cdot\right), \quad n = 0, 1, 2, \dots$$

for some real coefficients $d_{n,r}(k)$, such that $d_{n,r}(k) = 0$ if $l(r) \ne l(n)l(d)$ for all $d = 0, 1, 2, \dots$, in the spaces $L_2(0, k)$ and $L_2(a, b)$ with arbitrary $a, b:\ -\infty < a < b < \infty$. Multiplying step by step these equalities by $\overline{e}\left(\frac{k}{l(m)}, \cdot\right)$, $m = 0, 1, 2, \dots$, we easily derive that $d_{n,r}(k) = t_{n,r}(k)$ for all n, r and k; in particular, $t_{0,n}(k) = d_{0,n}(k)$, $n = 0, 1, 2, \dots$.

Lemma 8. *For any $\overline{k} > 0$ the series from the right-hand side of* (9) *converges uniformly with respect to $k \in (0, \overline{k}]$ and $x \in R$. Coefficients $t_{0,n}(k) = d_n(k)$ in* (9) *are continuous functions of the argument $k > 0$.*

Proof easily follows from Lemma 7 and the fact, presented with Proposition 2, that for any $k > 0$ the system of functions $\left\{\overline{u}\left(\frac{k}{l(n)}, \cdot\right)\right\}_{n=0,1,2,\dots}$ is a basis in the space $L_2(0, k)$.□

Lemma 9. *Let $g(\cdot) \in L$. Then, for any point $x \ge 0$ of the continuity of the function $g(\cdot)$ the following equality takes place:*

$$g(x) = \int_a^b \breve{g}(k) k^{-q} e(k, x) dk.$$

Proof is obvious.□

Lemma 10. *For any $\overline{k} > 0$ there exists $C > 0$ such that*

$$||e(l(n)k, \cdot)||_{L_2(0,l(n)k)} ||u(k, \cdot)||^{-1}_{L_2(0,l(n)k)} \le C l^q(n)$$

and

$$||u(l(n)k, \cdot)||_{L_2(0,l(n)k)} ||e(k, \cdot)||^{-1}_{L_2(0,l(n)k)} \le C l^q(n)$$

for all $k \in (0, \overline{k}]$ and all numbers $n = 0, 1, 2, \dots$.

Proof easily follows from Proposition 1 and Lemma 2.□

3 Proof of the Theorem

Let us take an arbitrary continuous function $g(\cdot) \in L$ where $\breve{g}(k) = 0$ if $k \in (0,a) \cup (b,\infty)$ for some $a, b:\ 0 < a < b < \infty$. Due to Lemmas 8 and 9 we have for any point $x > 0$:

$$g(x) = \int_a^b \breve{g}(k) k^{-q} ||e(k,\cdot)||_{L_2(0,k)} \times$$

$$\times \left[\sum_{n=0}^{\infty} t_{0,n}(k) \left\| u\left(\frac{k}{l(n)}, \cdot\right) \right\|_{L_2(0,k)}^{-1} u\left(\frac{k}{l(n)}, x\right) \right] dk.$$

Therefore,

$$g(x) = \sum_{n=0}^{\infty} \int_a^b \breve{g}(k) k^{-q} ||e(k,\cdot)||_{L_2(0,k)} \times$$

$$\times \left\| u\left(\frac{k}{l(n)}, \cdot\right) \right\|_{L_2(0,k)}^{-1} t_{0,n}(k) u\left(\frac{k}{l(n)}, x\right) dk =$$

$$= \sum_{n=0}^{\infty} \int_{l^{-1}(n)a}^{l^{-1}(n)b} l^{1-q}(n) \breve{g}(l(n)k) k^{-q} ||e(l(n)k,\cdot)||_{L_2(0,l(n)k)} \times$$

$$\times ||u(k,\cdot)||_{L_2(0,l(n)k)}^{-1} t_{0,n}(l(n)k) u(k,x) dk. \tag{22}$$

Further, for

$$g_n(k) = l^{1-q}(n) \breve{g}(l(n)k) t_{0,n}(l(n)k) ||e(l(n)k,\cdot)||_{L_2(0,l(n)k)} ||u(k,\cdot)||_{L_2(0,l(n)k)}^{-1}$$

we get by Lemmas 7 and 10 and the finiteness of $\breve{g}(\cdot)$

$$|g_n(k)| \le C l^{-\frac{1}{2}}(n) \tag{23}$$

for a constant $C > 0$ independent of $k \in (0,b]$ and of n. Using (23), we get the existence of $C_1 > 0$ such that

$$\sum_{n=0}^{\infty} |g_n(k)| \le C \sum_{n:\ ak^{-1} \le l(n) \le bk^{-1}} l^{-\frac{1}{2}}(n) \le C_1 k^{-\frac{1}{2}} \tag{24}$$

for all $k \in (0,b]$; also, obviously

$$\sum_{n=0}^{\infty} |g_n(k)| = 0 \tag{25}$$

for $k > b$.

We set

$$\overline{g}(k) = \sum_{n=0}^{\infty} g_n(k).$$

Then, it is clear in view of (24) and (25) that $\overline{g}(\cdot)$ is a continuous function obeying the estimate (4) and that $\overline{g}(k) = 0$ if $k > b$. Hence, in view of (22) for any $x > 0$:

$$g(x) = \int_0^{\infty} \overline{g}(k) k^{-q} u(k,x) dk.$$

Thus, the first part of our Theorem (the existence of the expansion (5)) is proved.

Let us prove the last statement of our Theorem, i. e. the uniqueness of the expansion (5). Suppose the existence of a continuous function $g(\cdot) \in L$ such that for all $x > 0$

$$g(x) = \int_0^{\infty} \overline{g}_i(k) k^{-q} u(k,x) dk, \quad i = 1, 2,$$

where $\overline{g}_1(k) \not\equiv \overline{g}_2(k)$ $(k > 0)$ are two finite continuous functions obeying the estimate (4). Then, for $\overline{g}(k) = \overline{g}_1(k) - \overline{g}_2(k)$ we have

$$\int_0^{\infty} \overline{g}(k) k^{-q} u(k,x) dk \equiv 0, \quad x > 0, \tag{26}$$

and $\overline{g}(k)$ is continuous, obeys the estimate (4), is not equal to zero identically on the half-line $k > 0$, and $\overline{g}(k) = 0$ if $k > b$, for some $b > 0$.

Now, due to (15), (26) and Lemma 6 for any $x > 0$

$$0 = \int_0^b \overline{g}(k) k^{-q} ||u(k,\cdot)||_{L_2(0,k)} \times$$

$$\times \sum_{n=0}^{\infty} \left[\overline{b}_{0,n}(k) \left\| e\left(\frac{k}{l(n)}, \cdot\right) \right\|_{L_2(0,k)}^{-1} e\left(\frac{k}{l(n)}, x\right) \right] dk$$

where the infinite sum in the integrand converges to $\overline{u}(k,x)$ uniformly with respect to $k \in (0,b]$ and $x \in R$. Hence, for any $x > 0$

$$0 = \sum_{n=0}^{\infty} \int_0^{l^{-1}(n)b} l^{1-q}(n)\overline{g}(l(n)k)||u(l(n)k,\cdot)||_{L_2(0,l(n)k)} \times$$

$$\times ||e(k,\cdot)||_{L_2(0,l(n)k)}^{-1} \overline{b}_{0,n}(l(n)k)k^{-q}e(k,x)dk.$$

Let

$$g_n(k) = l^{1-q}(n)\overline{g}(l(n)k)||u(l(n)k,\cdot)||_{L_2(0,l(n)k)}||e(k,\cdot)||_{L_2(0,l(n)k)}^{-1}\overline{b}_{0,n}(l(n)k).$$

In view of Lemmas 6 with $s = 3$ and 10 we have

$$|g_n(k)| \leq Cl^{-\frac{5}{2}}(n)k^{-\frac{1}{2}}$$

for a constant $C > 0$ independent of $k \in (0,b]$ and of $n = 0,1,2,\ldots$ and $g_n(k) = 0$ for $k > b$. Therefore, the function $\overline{\overline{g}}(k) = \sum_{n=0}^{\infty} g_n(k)$ is determined, is continuous on the half-line $k > 0$ and equal to zero for $k > b$, and obeys the estimate (4). Also, as earlier

$$\int_0^b \overline{\overline{g}}(k)k^{-q}e(k,x)dk = \int_{b^{-1}}^{\infty} r^{-2}\overline{\overline{g}}(r^{-1})r^q e(r^{-1},x)dr \equiv 0, \quad x > 0.$$

Hence, since $\hat{g}(r) = r^{-2}\overline{\overline{g}}(r^{-1}) \in L_2(0,\infty)$, we have that $\hat{g}(\cdot)$ is a (renormalized) L_2– Fourier transform of the identical zero, and therefore $\hat{g}(r) \equiv 0$, $r > 0$. But letting $P = \{k > 0 : \overline{\overline{g}}(k) \neq 0\}$ and $\overline{k} = \sup P$ and taking $k_0 \in (0,\overline{k}) \cap P$ so close to $\overline{k}$ that $l(1)k_0 > \overline{k}$, we get $\overline{\overline{g}}(k_0) = g_0(k_0) \neq 0$, i. e. we get the contradiction because $\hat{g}(\cdot)$ is a continuous function. Thus, our Theorem is completely proven.□

4 Appendix

Here I briefly consider corrections to my paper [2]. In what follows, the numeration of formulas and the notation from that paper is used. First

of all, in the paper [2] we confuse several times with the concepts of the completeness and the linear independence of a denumerable system of functions in $L_2(0,1)$ on the one hand and the property of being a basis for such a system of functions in $L_2(0,1)$ on the other hand. Of course, generally these properties are not equivalent. Further, in [2] there are several errors in the proof of Theorem 1. Arguments of this paper are sufficient for proving the completeness and linear independence in $L_2(0,1)$ of the system of functions (6) with $n \geq N_3$. However, in general this does not imply the completeness in $L_2(0,1)$ of the system $\{u_n(\lambda_n,\cdot)\}_{n=0,1,2,\ldots}$ in the case when this system is linearly independent, as far as in general it is not right that the codimension of the closure L^n of the linear span of the functions

$$\{u_k(\lambda_k,\cdot)\}_{k=n,n+1,n+2,\ldots}, \qquad (\star)$$

where $n \geq N_3$, is equal to n. Here we correct these mistakes for which we first show that the system of functions $(\star)$ is a basis of the subspace L^n if the number $n \geq N_3$ is sufficiently large.

Lemma. *There exist $N \geq N_3$ and constants c, C : $0 < c < C$ such that for any real coefficients $a = (a_N, a_{N+1}, a_{N+2}, \ldots) \in l_2$ the following inequalities take place:*

$$c\left(\sum_{k=N}^{\infty} a_k^2\right)^{\frac{1}{2}} \leq \left\| \sum_{k=N}^{\infty} a_k u_k(\lambda_k,\cdot) \right\|_{L_2(0,1)} \leq C\left(\sum_{k=N}^{\infty} a_k^2\right)^{\frac{1}{2}}.$$

Proof is completely analogous to the proof of Lemma 4 from the paper [2].

Let $n \geq N$ be arbitrary. The above Lemma easily yields that the system of functions $(\star)$ is a basis of the space L^n and that the infinite sum $\sum_{k=n}^{\infty} a_k u_k(\lambda_k,\cdot)$ converges in $L_2(0,1)$ if and only if $a = (a_n, a_{n+1}, a_{n+2}, \ldots) \in l_2$. (The formal proof of these facts is very simple. On this subject, see also [9-11]. Bases $\{u_k\} \subset L_2(0,1)$ for which the infinite sum $\sum_k a_k u_k$ with real coefficients a_k converges in $L_2(0,1)$ if and only if $\{a_k\} \in l_2$ are also called the Riesz bases, see [9-11].)

Let us now prove that for $n \geq N$ the system of functions (6) is a basis in $L_2(0,1)$. By P_n we denote the orthogonal projector in $L_2(0,1)$ on the subspace L^n. Set $u_k^{\perp} = P_n u_k(\lambda_n,\cdot)$ and $v_k = u_k(\lambda_n,\cdot) - u_k^{\perp}$ where

$k = \overline{0, n-1}$. Then, the functions v_k, $k = \overline{0, n-1}$, are orthogonal to the subspace L^n; in addition, using the fact that the system of functions $(\star)$ is a basis of the space L^n, we easily get that these functions are linearly independent (if we assume the linear dependence of these functions, then we immediately get the linear dependence of the system (6)). But then, since the system of functions (6) is complete in $L_2(0,1)$, the system of functions $v_0, ..., v_{n-1}, u_n(\lambda_n, \cdot), u_{n+1}(\lambda_{n+1}, \cdot), ...$ is complete in $L_2(0,1)$, too, therefore the functions $v_0, ..., v_{n-1}$ form a basis of the orthocomplement $L^n_\perp$ of the subspace L^n (in particular, the codimension of the subspace L^n of the space $L_2(0,1)$ is equal to n). In addition, since the system of functions $(\star)$ is a basis in L^n, the functions $u_k^\perp$ are representable in the form $u_k^\perp = \sum\limits_{m=n}^{\infty} b_m^k u_m(\lambda_m, \cdot)$ where $k = \overline{0, n-1}$ and the series converge in $L_2(0,1)$. Take an arbitrary $g \in L_2(0,1)$. Then, we get

$$g = \sum_{k=0}^{n-1} a_k v_k + \sum_{k=n}^{\infty} a_k u_k(\lambda_k, \cdot) = \sum_{k=0}^{\infty} a_k u_k(\lambda_{l(k)}, \cdot) - \sum_{k=0}^{n-1} a_k \sum_{m=n}^{\infty} b_m^k u_m(\lambda_m, \cdot) =$$

$$= \sum_{k=0}^{\infty} d_k u_k(\lambda_{l(k)}, \cdot),$$

where $l(k) = n$ if $k < n$ and $l(k) = k$ for $k \geq n$, $d_k = a_k$ for $k = \overline{0, n-1}$, $d_k = a_k - \sum\limits_{m=0}^{n-1} a_m b_k^m$ for $k \geq n$ and the convergence of all series is understood in the sense of the space $L_2(0,1)$. Thus, in view of the linear independence of the system of functions (6) with $n \geq N$, we have proved that this system is a basis of the space $L_2(0,1)$ if $n \geq N$.

In complete analogy, assuming the linear independence of the system of functions $\{u_n(\lambda_n, \cdot)\}_{n=0,1,2,...}$ and using the facts that for $n \geq N$ the system of functions $(\star)$ is a basis of the subspace L^n and that the codimension of this subspace is equal to n, one can show that the system of functions $\{u_n(\lambda_n, \cdot)\}_{n=0,1,2,...}$ is a basis in $L_2(0,1)$. We also remark that, since as it is proved earlier, if $n \geq N$, then the infinite sum $\sum\limits_{k=n}^{\infty} a_k u_k(\lambda_k, \cdot)$ converges in $L_2(0,1)$ when and only when $\sum\limits_{k=n}^{\infty} a_k^2 < \infty$, the series $\sum\limits_{n=0}^{\infty} a_n u_n(\lambda_n, \cdot)$ converges in $L_2(0,1)$ when and only when $\sum\limits_{n=0}^{\infty} a_n^2 < \infty$, i. e. in this case the system of functions $\{u_n(\lambda_n, \cdot)\}_{n=0,1,2,...}$ is a Riesz basis in $L_2(0,1)$.

If the system of functions $\{u_n(\lambda_n, \cdot)\}_{n=0,1,2,...}$ is linearly dependent in $L_2(0,1)$, then using similar arguments one can prove that it is incomplete in $L_2(0,1)$ (for this aim one should use the facts that, first, for $n \geq N$ the system of functions $(\star)$ is a basis in L^n and that the codimension of this subspace is equal to n and, second, that from the linear dependence of the system of functions under consideration we get the linear dependence of the functions $v_0, ..., v_{n-1}$ introduced earlier).

In my opinion, the above-described arguments make complete the proof of Theorem 1 and the proofs of all other results from my paper [2]. In addition, in view of the above-written, Theorem 1 and the statement (b) of Theorem 2 can be formulated as follows.

Theorem 1. *Let the assumption* (V) *be valid. Then, for a standard system of functions* $\{u_n(\lambda_n, \cdot)\}_{n=0,1,2,...}$ *to be complete in the space* $L_2(0,1)$ *it is sufficient and necessary for this system of functions to be linearly independent in* $L_2(0,1)$. *Further, if the indicated system of functions is linearly independent, then it is a Riesz basis in* $L_2(0,1)$.

(b) *the system of eigenfunctions* $\{u_n(x)\}_{n=0,1,2,...}$ *of the nonlinear problem* (3)-(5) *is a Riesz basis in* $L_2(0,1)$.

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