98-96

E5-98-96

E.N.Bukina, V.M.Dubovik

THE GAUGE FREEDOMS
OF ENLARGED HELMHOLTZ THEOREM
AND THE NEUMANN–DEBYE POTENTIALS;
THEIR MANIFESTATION IN THE MULTIPOLE
EXPANSION OF CONSERVED CURRENT

1998

1 Introduction

In considering problems of mathematical physics with definite spatial and/or dynamical symmetries, one commonly uses various decompositions of vector *fields over scalar potentials. These decompositions supplement the famous* Helmholtz theorem and reduce its "gauge freedoms".

Let $\exists\, \mathbf{V} : \mathbf{r} \in R^3 \to \mathbf{V}(\mathbf{r}) \in R^3$ with all "good" properties. We may represent the given field $\mathbf{V}$ in different ways

$$\mathbf{V} \to \{V_x, V_y, V_z\} \to \{\varphi, \mathbf{A}\}_{\mathrm{div}\mathbf{A}=0} \to \{\varphi, \psi, \chi\}.$$

In any case, it is necessary to constrain superfluous components, if they take place, by introducing conditions similar to $\mathrm{div}\mathbf{A} = 0$.

The last variant, diffeomorphic scalarization of a vector field, is the most economical and convenient approach to vector boundary-value problems of mathematical physics. But to use this approach, we have to be able to invert decomposition formulas, i.e. to deduce the integral representations of scalar potentials through the original vector field.

Approach. To obtain a vector function from a (pseudo)scalar function set $\varphi(\mathbf{r}), \psi(\mathbf{r})$ and $\chi(\mathbf{r})$, one must act on them by some vector operator $\hat{F}(\mathbf{r}, \nabla)$. Besides $\mathbf{r}$ and ∇ themselves, we may from them construct three simplest operators: $\mathbf{L} := -\mathbf{r} \times \nabla$, $\mathbf{N} := \nabla \times \mathbf{L}$ and $\mathbf{M} := -\mathbf{r} \times \mathbf{L}$. One can see their symplectic nature because of their correspondence to the frames of reference in a phase space: $\mathbf{r}$, $\mathbf{k}$, $\mathbf{r} \times \mathbf{k}$, $\mathbf{r} \times (\mathbf{r} \times \mathbf{k})$ and $\mathbf{k} \times (\mathbf{k} \times \mathbf{r})$. The *trio* of vectors $\mathbf{k}, \mathbf{L}, \mathbf{N}$ and $\mathbf{r}, \mathbf{L}, \mathbf{N}$, being immersed into the spaces $R_\mathbf{r}^3$ and $R_\mathbf{p}^3$, respectively, form orthogonal bases in them, which are important for different applications.

One may verify the following projection and commutation properties of $\hat{F}$:

$$[\mathbf{L}, r^2] = [\mathbf{L}, p^2] = 0$$
$$[L^2, \mathbf{r}] = i[\mathbf{r} \times \mathbf{L}] - i[\mathbf{L} \times \mathbf{r}]$$
$$[L^2, \mathbf{k}] = i[\mathbf{k} \times \mathbf{L}] - i[\mathbf{L} \times \mathbf{k}]$$
$$\mathbf{r} \cdot \mathbf{L} = \nabla \mathbf{L} = \mathbf{r} \cdot \mathbf{N} = \mathbf{L} \cdot \mathbf{N} = \mathbf{L} \cdot \mathbf{M} = \mathbf{M} \cdot \mathbf{L} = 0, \mathrm{curl}\mathbf{N} = -\mathbf{L}\triangle;$$
$$[\mathbf{L}, \triangle] = 0, [\mathbf{N}, \triangle] = 0, [\mathbf{M}, \triangle] = -6\nabla, \text{etc.}$$
$$[r_i, \nabla_k] = -\delta_{ik}, \quad [r_i, L_k] = -\epsilon_{ikj}r_j, \quad [\nabla_i, L_k] = -\epsilon_{ikj}\nabla_j,$$
$$[\nabla_i, N_k] = \nabla_i\nabla_k - \triangle\delta_{ik}, \quad [r_i, M_k] = r_i r_k - r^2\delta_{ik}$$
$$[L_i, M_k] = \epsilon_{ikj}r_j - r^2\epsilon_{ikj}\nabla_j.$$

....

It is taken into account that

$$\mathbf{N} = -\mathbf{r}\triangle + \nabla(\mathbf{r}\nabla) + \nabla,$$
$$\mathbf{r} \times \mathbf{N} = -(1 + \mathbf{r}\nabla)\mathbf{L} = -\mathbf{L}(1 + \mathbf{r}\nabla),$$
$$\mathrm{curl}\mathbf{N} = -\mathbf{L}\triangle, \text{etc.}$$

Note that in the space $R_\mathbf{r}^3$, $\mathbf{L}$ and $\mathbf{N}$ form the following algebra of differential operators

$$[L_i, L_j] = \epsilon_{ijk}L_k, \quad [L_i, N_j] = \epsilon_{ijk}N_k, \quad [N_i, N_j] = -\epsilon_{ijk}L_k\triangle.$$

After the rescaling of $\mathbf{N} \to \mathbf{N}/\sqrt{\triangle}$ over a supporting function space, we acquire the $O(3,1)$ Lie algebra representation in terms of the $\hat{F}(\mathbf{r}, \nabla)$ operators in $R_\mathbf{r}^3$. They act in the $O(3) \times O(3)$ function space suitable for arranging the multipole phenomenology in electromagnetic theory (see e.g. [2], [3], [6]). The operators $\mathbf{L}$ and $\mathbf{M}$ obey the same algebra in the space of wave vectors $R_\mathbf{k}^3$ that is Fourier-conjugate to $R_\mathbf{r}^3$. In the preceding paper [7] the emphasis has been *made on the inversion of different formulas for decomposition of vector fields* in the mathematical aspect.

In the physical aspect, two vector-potential formulations of the theory of continuous media with taking into account both magnetic and electric toroid polarizations [8], (see also [9], part II) are published for the first time. Here we consider the mathematical underlying reason of uniqueness of division of the transverse electric distribution density $E_{lm}(\mathbf{k}^2, t)$ into two independent multipole specimen [3]:

$$E_{lm}(\mathbf{k}^2, t) = \dot{Q}_{lm}(0, t) + \mathbf{k}^2 T_{lm}(\mathbf{k}^2, t), \tag{1}$$

where $\dot{Q}_{lm}(0, t)$ are the *time-derivative of the Coulomb (charge) multipole* $l-$moment and $T_{lm}(\mathbf{k}^2, t)$ are the toroid multipole form factor of $l-$order. Note that a secularized relation when neglecting toroid contributions and known as the Siegert theorem may be correct for low-symmetric electromagnetic systems only. The mathematical question of condition responsible for the possibility of identification of multipole moments (the leading ones for each given l) in the transverse and longitudinal parts of current raised in [1] will be tried to clear in this paper. From our representation theorem (see also remark II) it follows that definitions of multipole moments are unique.

Recall that the procedure of multipole decomposition of the current and field densities in electrodynamics in fact corresponds to the description of properties of a system by a set of numerical characteristics which are assigned to a point, "center" of the density distribution of the system considered. In this case, for the poloidal and potential parts of the current an additional connection arises between its longitudinal and transverse components [3] due to degeneracy of boundary conditions of longitudinality and transversality in r-space at the self-similar shrinkage of the definition domain of current to the chosen center. To prove this statement, we could probably use the transfer technique of boundary conditions [4], [5]. It might have lead to the separation of multipole moments called in [3] the toroid ones [1]. But ways of this kind is very difficult in the general framework of the distribution theory. Here we use the simple concrete approach.

The main feature of the multipole expansion procedure is a special choice of basis functions, which in actual practice ensues the rapid convergence of multipole series. This circumstance forces us to weaken requirements of the usual Helmholtz theorem and, respectively, to take account of a gauge freedom extension. In sections 1 and 2, we discuss gauge freedoms in the Helmholtz and Neumann-Debye decomposition. In section 3, we turn to their realizations within the multipole expansion of the electromagnetic current.

---

[1]In fact, their exact title should be the poloidal moments, see [6], [9].

2   The enlarged Helmholtz theorem

We begin our consideration with the Helmholtz Decomposition:

◁ $\forall \mathbf{V}$ with properties of single-valuedness, continuity, boundedness or convergence

$$|\mathbf{V}| < \frac{k}{r^{2+\epsilon}}, \quad \epsilon > 0 \text{ at } r \to \infty$$

in the space may be represented in the form [see e.g. [12]]:

$$\mathbf{V} = \nabla\varphi + \mathrm{curl}\mathbf{A} \quad \text{with} \quad \mathrm{div}\mathbf{A} = 0.$$

Indeed, the theorem requirements are proved to be sufficient in order to reexpress tautologically the given vector field $\mathbf{V}$ through its divergence and vorticity. The explicit realization of the theorem could be attained due to the following operations

$$\mathrm{div}\mathbf{V}^{\|} = \Delta\varphi, \varphi = \Delta^{-1}\mathrm{div}\mathbf{V},$$

$$\text{where} \quad \Delta^{-1} := -\int_{\Omega \subseteq R^3/\{0\}} \frac{d^3 r'}{4\pi|\mathbf{r}-\mathbf{r}'|},$$

$$\mathrm{curl}\mathbf{V}^{\perp} = \mathrm{curl\,curl}\mathbf{A} = -\Delta\mathbf{A}, \quad \mathbf{A} = -\Delta^{-1}\mathrm{curl}\mathbf{V}.$$

So we have

$$\mathbf{V} \equiv \mathbf{V}^{\|} + \mathbf{V}^{\perp} = \nabla\Delta^{-1}\mathrm{div}\mathbf{V} - \mathrm{curl}\Delta^{-1}\mathrm{curl}\mathbf{V}. \quad \triangleright \qquad (2)$$

Form (1) often produces misunderstanding (e.g. [13]) that the representation of $\mathrm{div}\mathbf{V}$ and $\mathrm{curl}\mathbf{V}$ is equivalent to the representation of $\mathbf{V}$ itself. We discuss here to what extent single-valued is this representation. In fact (1) contains the evident "gauge freedom":

$$\varphi = \varphi + \omega, \quad (\Delta\omega = 0), \qquad \mathbf{A} = \mathbf{A} + \nabla w.$$

If we remove the demands of topological triviality Ω or/and the boundedness of functions ω and w, then as an example important for physical application, we may represent the gauge freedoms in the form of special additional functions to $\mathbf{V}$:

$$0 \neq \mathbf{V}_{\mathbf{N}} = \mathrm{curl}\mathbf{L}\begin{pmatrix} r^l \\ 1/r^{l+1} \end{pmatrix} Y_{lm} \equiv \begin{pmatrix} -(l+1)\nabla r^l Y_{lm} \\ l\nabla r^{-l-1} Y_{lm} \end{pmatrix},$$

which have a nonzero finite value all over the space except $r \to \infty$ and $r \to 0$, respectively, as far as

$$\mathrm{div}\mathbf{V}_{\mathbf{N}} = \mathrm{curl}\mathbf{V}_{\mathbf{N}} \equiv 0, \quad \text{in} \quad R^3/S^2_{r\to\infty} \quad \text{and} \quad R^3/\{0\}.$$

Therefore these functions [2] cannot be represented by the usual Helmholtz decomposition, and manifest the gauge freedom of its enlarged formulation.

Remark I. Note that functions $\mathbf{V}_N$ are longitudinal and transverse simultaneously, since they represent the vector solutions of the Laplace equation ($\Delta \equiv \mathrm{grad\,div} - \mathrm{curl\,curl}$). Moreover, in our context it is important to emphsize that they are topologically equivalent to the poloidal (meridional) harmonics on the toruslike surface covering the whole space of R^3 except one deleted axis. ∎

Thus, under this gauge freedom the Helmholtz decomposition takes the following alternative forms

$$\mathbf{V} \equiv \mathbf{V}^{\|} + \mathbf{V}^{\perp} =$$

$$= \nabla\left(\Delta^{-1}\mathrm{div}\mathbf{V} - (l+1)\sum_{lm}[C_{lm}r^l + C'_{lm}r^{-l-1}]Y_{lm}\right) + \mathrm{curl}\Delta^{-1}\mathrm{curl}\mathbf{V} = \quad (3)$$

$$= \nabla\Delta^{-1}\mathrm{div}\mathbf{V} + \mathrm{curl}\left(\Delta^{-1}\mathrm{curl}\mathbf{V} + \mathbf{L}\sum_{lm}[C_{lm}r^l + C'_{lm}r^{-l-1}]Y_{lm}\right).$$

3   The Neumann-Debye decomposition

The well-known mathematical physicist W.M.Elsasser in [15] has already observed that every vector field of the form $\nabla \times \nabla \times \mathbf{r}\chi + \nabla \times \mathbf{r}\psi$, where ψ and χ are any scalars, is solenoidal. In paper [16] it has been shown that if $\mathrm{div}\mathbf{V} = 0$ in R^3, then for every choice of the origin there exist unique scalars ψ and χ such that $\mathbf{V} = \mathbf{L}\psi + \mathbf{N}\chi$ while ψ and χ average to zero on every spherical surface concentric with the origin. The complete theorem of the possibility of decomposition of $\mathbf{V}(\mathbf{r})$ in terms of scalar functions reads as follows:

Representation Theorem. Given a region $\Omega \subseteq R^3\backslash\{0\}$, with a regular boundary and R^3−vector field, $\mathbf{V}: \mathbf{r} \in \Omega \to \mathbf{V}(\mathbf{r}) \in R^3$. Then, there exist three scalar functions $\varphi(\mathbf{r}), \psi(\mathbf{r})$ and $\chi(\mathbf{r})$ on Ω which define this $\mathbf{V}$ [17]. ∎

The most used decomposition, which we call the Neumann-Debye one, has the form

◁ $$\mathbf{V}(\mathbf{r}) := \nabla\varphi(\cdot) + \mathrm{curl}\mathbf{r}\psi(\cdot) + \mathrm{curl\,curl}\mathbf{r}\chi(\cdot) \equiv \nabla\varphi + \mathbf{L}\psi + \mathbf{N}\chi. \quad (4)$$

Here ψ and χ are the so-called Debye potentials and φ is the usual (electric) scalar one. We found fundamental solutions of the inversion problem of (1) in the form [7]

$$\begin{aligned}
\mathrm{div}\mathbf{V} = \Delta\varphi &\to& \varphi &= \Delta^{-1}\mathrm{div}\mathbf{V}, \\
\mathbf{L}\mathbf{V} = L^2\psi &\to& \psi &= -L^{-2}\mathbf{L}\mathbf{V} \equiv L^{-2}\mathbf{r\,curl}\mathbf{V}. \quad (5) \\
\mathbf{r}\mathbf{V} = (\mathbf{r}\nabla)\varphi + L^2\chi &\to& \chi &= L^{-2}(\mathbf{r}\nabla)\Delta^{-1}\mathrm{div}\mathbf{V} - L^{-2}(\mathbf{r}\mathbf{V}).
\end{aligned}$$

---

[2]Note that similar functions may be generated using the common relation

$$\mathrm{curl}(\mathbf{r} \times \nabla)r^\kappa Y_{lm} = -(\kappa+1)\nabla r^\kappa Y_{lm} + (\kappa-l)(\kappa+l+1)\mathbf{r}r^{\kappa-2}Y_{lm}.$$

and taking into account that

$$\kappa = l \quad \text{and} \quad \kappa = -l-1$$

(see [14], Appendix A).

where [18]

$$\mathbf{L}^{-2} := \int_\sigma \frac{d\omega'}{4\pi} ln(1 - \hat{r} \cdot \hat{r}'). \quad \triangleright$$

Remark II. Eigenfunctions of the square of the operator *angular momentum* iL : $(iL)^2 = -L^2$ are usual spherical functions satisfying the equation

$$L^2 Y_{lm}(\hat{r}) = -l(l+1)Y_{lm}(\hat{r}).$$

The corresponding Green function for this equation can be found with the help of the known Mercer theorem ([18], v.1) which in the given case yields

$$\frac{1}{4\pi} \sum_{l,m} \frac{Y_{lm}^*(\hat{r})Y_{lm}(r')}{-l(l+1)} = -\sum_l \frac{2l+1}{l(l+1)} P_l(\hat{r} \cdot \hat{r}') = 1 - ln2 + ln(1 - \hat{r} \cdot \hat{r}'). \quad \blacksquare$$

Remark III. It is well-known that the gauge freedom of (5) is the following:

$$\varphi \to \varphi + C, \quad \psi \to \psi + \mu(r), \quad \chi \to \chi + \nu(r).$$

Requiring φ to vanish on the boundary, and ψ, χ not to contain spherically symmetric components

$$\int_{S^2} dw\, \psi = \int_{S^2} dw\, \chi = 0;$$

we put these functions in one-to-one correspondence to $\mathbf{V}$. $\blacksquare$

Uniqueness Theorem. If a vector field $\mathbf{V}$ (with the properties determined in the theorem (I)) is defined on every S_r^2 in some range $r_0 < r < r_1$ and in that range $V_r = 0$ while $V_\theta(r, \theta, \varphi)$ and $V_\varphi(r, \theta, \varphi)$ are bounded for each fixed r and are continuously differentiable except possibly at $\theta = 0$ and $\theta = \pi$ and further $div\mathbf{V} = 0$ and $curl\mathbf{V} = 0$, then $\mathbf{V} \equiv 0$.

Our inversion formulas (3) demonstrate that immediately (cp. with [16], p.383, where the condition $\mathbf{LV} = 0$ has been used instead of our $curl\mathbf{V} = 0$). $\blacksquare$

Now we have to reconstruct the representation of χ such that it depends on $curl\mathbf{V}$ and $div\mathbf{V}$ only. Really, the latter quantities have the physical meaning but not the radial component of $\mathbf{V}$. Moreover, we may expect χ not to depend on $div\mathbf{V}$ generally because this potential defines the transverse part of the vector field $\mathbf{V}$. Nevertheless, *because of gauge freedom, the situation is not so simple as it seems to be.*

Indeed, we may substitute the Helmholtz decomposition (the last expression in (3)) into the term with $\mathbf{rV}$ and see that χ takes the form

$$\chi = L^{-2}\mathbf{L}\Delta^{-1}curl\mathbf{V} \sum_{lm} L^{-2}[C_{lm}\nabla r^l + C'_{lm}\nabla r^{-l-1}]Y_{lm}. \quad (6)$$

Further our vector field corresponding to the gauge freedom may be transformed as

$$curl\mathbf{L}L^{-2}(r\nabla)r^l Y_{lm} = (l+1)curlr^l \mathbf{L}L^{-2}Y_{lm} =$$

$$= (l+1)curlr^l \mathbf{L}L^{-2}\frac{L^2}{l(l+1)}Y_{lm} = \frac{1}{l}curl\mathbf{L}r^l Y_{lm} = -\frac{l+1}{l}\nabla r^l Y_{lm}. \quad (7)$$

So, we found fundamental solutions of the inversion problem of (5) in the form [7]

$$\begin{aligned} div\mathbf{V} = \Delta\varphi \quad &\to \quad \varphi = \Delta^{-1}div\mathbf{V}, \\ \mathbf{LV} = L^2\psi \quad &\to \quad \psi = -L^{-2}\mathbf{LV} \equiv L^{-2}\mathbf{r}\,curl\mathbf{V}, \\ \mathbf{rV} = (r\nabla)\varphi + L^2\chi \quad &\to \quad \chi = L^{-2}(r\nabla)\Delta^{-1}div\mathbf{V} - L^{-2}(\mathbf{rV}), \\ &\qquad = L^{-2}\mathbf{L}\Delta^{-1}curl\mathbf{V} \\ &\qquad + \sum_{lm} L^{-2}[C_{lm}\nabla r^l + C'_{lm}\nabla r^{-l-1}]Y_{lm} \end{aligned} \quad (8)$$

with taking into account the gauge freedom in (3). $\blacksquare$

4   The multipole expansion of the electromagnetic current

Now we compare our abstract exercises and the procedure of multipole expansion of electromagnetic current on the basis of the vector Helmholtz equation solutions constructed through the Neumann-Debye decomposition (8). The multipole representation of current $\mathbf{J}(r)$ by (5) may be obtained by the standard expansion of three scalar densities into series:

$$\begin{aligned} \phi &\sim \sum_{lmk} j_l(kr)Y_{lm}\dot{Q}_{lm}(k^2, t); \\ \psi &\sim \sum_{lmk} j_l(kr)Y_{lm}M_{lm}(k^2, t); \\ \chi &\sim \sum_{lmk} j_l(kr)Y_{lm}E_{lm}(k^2, t). \end{aligned} \quad (9)$$

Therefore the multipole representations of the transverse part of current $(div\,\mathbf{J} = 0)$ are determined by the magnetic form factors $M_{lm}(k^2, t)$, and the transverse electric contributions $E_{lm}(k^2, t)$, and the scalar part of current $(curl\,\mathbf{J} = 0)$ are expressed due to the 4−current conservation law $(div\,\mathbf{J} = -\dot{\rho})$ through the Coulomb (charge) multipole moments $Q_{lm}(0\,t)$ and their mean $2n$−power radii

$$Q_{lm}(k^2, t) = Q_{lm}(0, t) + \sum_{n=1}^\infty \frac{k^{2n}}{n!}Q_{lm}^{(2n)}(0, t).$$

As a result, $\mathbf{J}$ may be represented as [3]

$$\begin{aligned} \mathbf{J}(\mathbf{r}, t) &= (2\pi)^{-3}\sum_{l,m}\int_0^\infty dk(-ik)^l \frac{\sqrt{4\pi(2l+1)}}{l(2l+1)!!}\{-ik\,\mathbf{L}f_l(kr)Y_{lm}(\hat{r})M_{lm}(k^2, t) \\ &-ik curl\mathbf{L}f_l(kr)Y_{lm}(\hat{r})E_{lm}(k^2, t) + l\,\nabla f_l(kr)Y_{lm}(\hat{r})\dot{Q}_{lm}(k^2, t)\}. \quad (10) \end{aligned}$$

We repeat here the procedure used firstly in [3] for the ascertainment of the exact structure of transverse electric contributions $E_{lm}(k^2, t)$. To this end, it one should rewrite explicitly our basis functions in terms of the vector harmonics

$$\text{curl} \mathbf{L} f_l(kr) Y_{lm}(\hat{\mathbf{r}}) = (2l+1)^{-1/2} \{ f_{l-1}(kr)\sqrt{l+1} Y_{ll-1m}(\hat{\mathbf{r}}) +$$
$$+ f_{l+1}(kr)\sqrt{l} Y_{ll+1m}(\hat{\mathbf{r}}) \};$$

$$\nabla f_l(kr) Y_{lm}(\hat{\mathbf{r}}) = (2l+1)^{-1/2} \{ f_{l-1}(kr)\sqrt{l} Y_{ll-1m}(\hat{\mathbf{r}}) -$$
$$- f_{l+1}(kr)\sqrt{l+1} Y_{ll+1m}(\hat{\mathbf{r}}) \},$$

where $Y_{ll-1m}(\hat{\mathbf{r}})$ is a harmonic polynomial function defined as

$$r^{l-1} Y_{ll-1m}(\hat{\mathbf{r}}) = \frac{1}{\sqrt{l(2l+1)}} \nabla r^l Y_{lm}.$$

As is obvious, in the wave-length approximation, the leading contributions in the latter expressions are delivered by the vector harmonic functions $f_{l-1} Y_{ll-1m}$ from which it follows that

$$\text{curl} \mathbf{L} f_l(kr) Y_{lm}(\hat{\mathbf{r}}) \approx_{k\to 0} \sqrt{(l+1)/l} \nabla f_l(kr) Y_{lm}(\hat{\mathbf{r}}) =$$
$$= \frac{4\pi(ikr)^{l-1}}{(2l+1)!!} \sqrt{\frac{l+1}{l}} \nabla r^l Y_{lm}(\hat{\mathbf{r}}) + O[(kr)^{l+1}]. \tag{11}$$

It is the relation that permits us to identify the leading term in $E_{lm}(k^2, t)$ with time-derivatives of $Q_{lm}(0, t)$. However all functtually independent contributions in $E_{lm}(k^2, t)$ give the so-called toroid moments and their $2n$−power radii

$$T_{lm}^{(2n)}(0, t) = -\frac{\sqrt{\pi l}}{2l+1} \int r^{l+2n+1} \left[Y_{ll-1m}^*(\mathbf{r}) + \frac{2\sqrt{l/(l+1)}}{2l+3} Y_{ll+1m}^*(\mathbf{r}) \right] \mathbf{J}(r, t) d^3 r.$$

As the toroid moments have a distinct geometrical meaning (diverse details and representations are given in [3], [4], [9], [10] and see also [11]), the rejection of $T_{lm}(t)$ is generally invalid like it was done in the Siegert theorem $E_{lm}(k^2, t) \to_{k\to 0} \dot{Q}_{lm}(0, t)$. Neglect of $T_{lm}(t)$ in comparison with $Q_{lm}(t)$ is analogous to the neglect of a higher multipole moment (contributions of highest symmetries of a given system) in comparison with the lower ones, which is of course permissible only when the lower moments (low symmetries) of this system do exist. So, the strict theorem determining the electric part structure has the form (1) $E_{lm}(\mathbf{k}, t) = \dot{Q}_{lm}(0, t) + \mathbf{k}^2 T_{lm}(\mathbf{k}, t)$, and its validity and uniqueness rely on the gauge freedom which has been obtained for the enlarged Helmholtz theorem and transferred to Debye potentials (compare (8) and (11)).

Moreover, by using the exact relation (1), it is not so hard to find an expression for the complete parametrization of the current in terms of generalized functions [3] [9]

$$\mathbf{J}(\mathbf{r}, t) = \sum_{l=1}^{\infty} \sum_{m=-l}^{m=l} \sum_{n=0}^{\infty} \frac{(2l+1)!!}{2^n n!! (2l+2n+1)!!} \sqrt{\frac{4\pi}{2l+1}} \{ M_{lm}^{(2n)}(t)(\mathbf{r} \times \nabla) \Delta^n \delta_{lm}(\mathbf{r})$$
$$\tag{12}$$

$$+ [\dot{Q}_{lm}(t)\delta_{n,0}\Delta^{-1} - T_{lm}^{(2n)}(t)] \text{curl}(\mathbf{r} \times \nabla) \Delta^n \delta_{lm}(\mathbf{r}) - l\, \dot{Q}_{lm}^{(2n)}(t) \nabla \Delta^{n-1} \delta_{lm}(\mathbf{r}) \}.$$

Remark IV. However, the expansion via of spherical harmonics emerges rapidly convergent. We remind (see e.g. [2], p.806) that if the field decreases for large distances very slowly, slower than r^{-2}, the divergence and curl of the vector field considered are assigned arbitrary independent values. Conversely, if we know that $\mathbf{J}$ vanishes identically outside some source radius R, $\nabla \mathbf{J}$ and $\nabla \times \mathbf{J}$ are no longer independent of each other. As far as the expansion via spherical harmonics is rapidly convergent, it is realized in the latter representation immediately.

Thus, we strictly demonstrated that the gauge freedom in division of the electromagnetic current into the transverse and longitudinal parts leads to the fact that the multipole contributions to the transverse part of current $E_{lm}(k^2, t)$ are represented in the form (1) and its leading terms may be identified with $\dot{Q}_{lm}(0\,t)$ from the longitudinal part of current for all l. Note that, since the coefficient C_l does not depend on the wave number k, we can use hereafter the Lorentz gauge condition in the calculation of the vector potential.

5 Conclusion

The representation of χ in the Neumann-Debye scalarization already assumes that the prohibition of the electric type of radiation imposes some conditions both on curlJ and divJ. But we could not reveal the ones due to their non-division in the scalarization mentioned. Exploitation of the enlarged Helmholtz theorem for this operation has inserted the extended gauge freedom (let us recall that the Neumann-Debye representation the gauge freedom reduce to functions of the scalar argument $|\mathbf{r}|$ only). It is just this freedom, consideration of which made it possible to identify the coefficients of leading order of the expansion of transverse and longitudinal electric parts of the current!

The form of the expression (1) shows the possibility of compensation of the electric type radiation if the toroid and charge moments are switched on as "anti-phase" ones [3] (see also [19]).

References

[1] French I.B. and Shimamoto Y. *Phys. Rev.*, v.91, 1959, p.898.

---

[3] The contribution of $l = 0$, $n = 0$ to the last term is forbidden by total charge conservation, whereas other terms contain no contribution of $l = 0$, formally, owing to $(\mathbf{r} \times \nabla)\delta(\mathbf{r}) \equiv 0$.

[2]   Blatt J.M. and Weisskopf V.F. *Theoretical Nuclear Physics*, Wiley, New York, 1952.

[3]   Dubovik V.M. and Cheshkov A.A. *Fiz. Elem. Chastits At. Yadra*, v.5,(2), 1974, p.318.

[4]   Vladimirov V.S. *PMM* , v.19, (3), 1955, p.315-324.

[5]   Fedoriuk M.V. *JVM and MP* , v.12, (2), 1972, p.274-287.

[6]   Dubovik V.M. and Tosunyan L.A. *Fiz. Elem. Chastits At. Yadra*, v.14, 1983, p.504.

[7]   Dubovik V.M. and Magar E.N. J. Moscow Phys. Soc., v.3, 1993, p.245-254.

[8]   Dubovik V.M., Tosunyan L.A. and Tugushev V.V. *Sov.Phys. JETP*, v.63, 1986, (2) p.344.

[9]   Dubovik V.M., Tugushev V.V. *Phys.Rep.*, v.187, N 4, 1990, p.145-202.

[10]   Dubovik V.M., Shabanov S.V. Special issue *"Essays on the formal aspects of electromagnetic theory"*. Ed. A.Lakhtakia, WS, Singapore, 1992.

[11]   V. M. Dubovik, M. A. Martsenyuk, B. Saha *Generalized Equations of Electrodynamics of Continuous Media* In S. Jeffers et al. (eds), The present status of the Quantum Theory of Light, (Kluwer Acad. Publ.) 141 - 150 (1997).

[12]   Lee T.D. *Mathematical Methods of Physics*, Columbia Univ., New York, 1963.

[13]   Bosko B. *Phys. Lett.* , v.B40, 1972, p.171.

[14]   Afanasiev G.N. *Fiz. Elem. Chastits At. Yadra*, v.24, 1993, (2), p.512-593.

[15]   Elsasser.*Phys. Rev.*, v.69, 1946, p.106.

[16]   Backus G. *Ann. Phys.*, v.4, 1958, p.372-447.

[17]   Müller E.E. *J. Math. Phys.*, v.28, N 11, 1987, p.2786-2790.

[18]   Courant R. and Hilbert D. *Methods of Mathematical Physics* . New York: Interscience, v.1, 1953.

[19]   Afanasiev G.N. and Dubovik V.M. (in press *J. Particles and Nuclei*), 1998.