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EIGENFUNCTION EXPANSIONS ASSOCIATED WITH A NONLINEAR SCHRÖDINGER EQUATION

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1 Introduction. Main result

The eigenvalue problem we consider is the following

$$-u'' + f(u^2)u = \lambda u, \quad u = u(x), \quad x \in (0,1), \tag{1}$$

$$u(0) = u(1) = 0, \tag{2}$$

$$\int_0^1 u^2(x)dx = 1. \tag{3}$$

Here all quantities are real and f is a given sufficiently smooth function. If a pair (λ, u), where $\lambda \in (-\infty, \infty)$ and $u = u(x)$ is a twice continuously differentiable function of the argument $x \in [0,1]$, satisfies the problem (1)-(3), then we call λ the eigenvalue and $u(x)$ the corresponding eigenfunction of this problem.

In the present note, we establish an independent and shorter proof of our result from papers [1,2] which states that the eigenfunctions of the problem (1)-(3) form a Riesz basis in the space $L_2(0,1)$. (In fact, paper [1], where a weaker statement on the basis property of the system of eigenfunctions of the problem (1)-(3) is presented, contains some mistakes and gaps in proofs; in the latter one [2] we correct these errors and make some additions to paper [1]. Also, in note [3], results of paper [1] are reestablished without proofs.) We remark that, to our knowledge, there are almost no investigations of basis properties of systems of eigenfunctions of nonlinear problems similar to (1)-(3). We mention only paper [4], where these questions are considered for a nonlinear problem arising from a linear one under small nonlinear perturbations, and our paper [5].

In the present paper, our hypothesis is the following.

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(f) *Let $f(u^2)u$ be a real continuously differentiable function of the argument $u \in (-\infty, \infty)$ and let $f(s)$ be a non-decreasing function of the argument $s \in [0, +\infty)$.*

The hypothesis (f) is valid, for example, for the function $f(s) = |s|^p$ where $p \geq 0$ is a constant.

In our paper [1], the following statement is proved.

Theorem 1. *Under the hypothesis* (f)

(a) *for an arbitrary integer $n \geq 0$ there exists a pair (λ_n, u_n), consisting of an eigenvalue λ_n and a corresponding eigenfunction u_n of the problem* (1)-(3) *and being unique up to the coefficient ± 1 of the function u_n, such that the function $u_n(x)$ has precisely n roots in the interval* $(0,1)$;

(b) $\lambda_0 < \lambda_1 < ... < \lambda_n < ...$ *and* $\lim_{n\to\infty} \lambda_n = +\infty$;

(c) *for any integer $n \geq 0$ the solution $u_n(x)$ of the equation* (1) *taken with $\lambda = \lambda_n$ can be continued onto the whole real line $x \in (-\infty, \infty)$ and the roots of this solution are precisely the points $x_k^n = \frac{k}{n+1}$ where k runs over all integers;*

(d) *there exists $D > 0$ such that $|u_n(x)| \leq D$ for all numbers $n = 0, 1, 2, ...$ and for all $x \in (-\infty, \infty)$;*

(e) *for any integer $n \geq 0$* $u_n\left(x + \frac{1}{n+1}\right) = -u_n(x)$ *for all* $x \in (-\infty, \infty)$;

(f) $u_n\left(\frac{r}{n+1} + x\right) = -u_n\left(\frac{r}{n+1} - x\right)$ *for any non-negative integer n, integer r and $x \in (-\infty, \infty)$.*

Remark. Really, in paper [1] statements (c),(d),(e) and (f) of theorem 1 are presented in a weaker form. However, proofs from paper [1] can be kept without modifications for proving these statements in their present form.

Let $L_2(a,b)$, where $a < b$ are real numbers, be the usual Lebesque space, consisting of real functions of the argument $x \in (a,b)$, with the scalar product $(v,w)_{L_2(a,b)} = \int_a^b v(x)w(x)dx$ and the norm $||v||_{L_2(a,b)} = (v,v)^{\frac{1}{2}}_{L_2(a,b)}$. Let $L_2 = L_2(0,1)$, $(\cdot,\cdot) = (\cdot,\cdot)_{L_2(0,1)}$ and $||\cdot|| = ||\cdot||_{L_2(0,1)}$. We also present the following definitions for the convenience of readers.

Definition 1. *A system of functions* $\{h_n\}_{n=0,1,2,\ldots} \subset L_2(a,b)$ *is called a basis of the space* $L_2(a,b)$ *if and only if for an arbitrary function* $h \in L_2(a,b)$ *there exists a unique sequence of real coefficients* $\{a_n\}_{n=0,1,2,\ldots}$ *such that* $\lim\limits_{N\to\infty} \left|\left| h - \sum\limits_{n=0}^{N} a_n h_n \right|\right|_{L_2(a,b)} = 0$. *We write this equality as follows:* $h = \sum\limits_{n=0}^{\infty} a_n h_n$.

In accordance with papers [6,7] we introduce the following

Definition 2. *We call a basis* $\{h_n\}_{n=0,1,2,\ldots}$ *of the space* $L_2(a,b)$ *the Riesz basis of this space if and only if the following two conditions are satisfied:*

1) *for any* $h = \sum\limits_{n=0}^{\infty} a_n h_n \in L_2(a,b)$, *where* a_n *are real numbers, one has* $\sum\limits_{n=0}^{\infty} a_n^2 < \infty$;

2) *for any sequence* $\{a_n\}_{n=0,1,2,\ldots}$ *of real numbers such that* $\sum\limits_{n=0}^{\infty} a_n^2 < \infty$ *the series* $\sum\limits_{n=0}^{\infty} a_n h_n$ *converges in the space* $L_2(a,b)$.

Our main result is the following.

Theorem 2. *Under the hypothesis* (f) *the system of eigenfunctions* $\{u_n\}_{n=0,1,2,\ldots}$ *of the problem* (1)-(3) *from theorem* 1 *is a Riesz basis of the space* L_2.

2 Proof of theorem 2

In what follows, we accept that $u_n(x) > 0$ for $x \in (0, \frac{1}{n+1})$ (it is possible in view of theorem 1(c) and invariance of the problem (1)-(3) with respect to the multiplication of the unknown function by -1). Our proof of theorem 2 is based on a result obtained by N.K. Bary. To establish this result, we need the following

Definition 3. *A system of functions* $\{h_n\}_{n=0,1,2,\ldots} \subset L_2(a,b)$ *is called minimal in the space* $L_2(a,b)$ *if and only if for any integer* $n \geq 0$ *there exists* $C_n > 0$ *such that* $\left|\left| h_n + \sum\limits_{\substack{k=0\\k\neq n}}^{N} a_k h_k \right|\right|_{L_2(a,b)} \geq C_n$ *for an arbitrary integer* $N > 0$ *and arbitrary real coefficients* a_k.

The result of N.K. Bary announced in paper [6] and proved in paper [7] is the following.

Bary Theorem. *Let a system of functions* $\{g_n\}_{n=0,1,2,\ldots}$ *be a Riesz basis of the space* $L_2(a,b)$, $\{h_n\}_{n=0,1,2,\ldots}$ *be a sequence of functions from the space* $L_2(a,b)$ *and the following two conditions be satisfied:*

A. *the system of functions* $\{h_n\}_{n=0,1,2,\ldots}$ *is minimal in the space* $L_2(a,b)$;

B. $\sum\limits_{n=0}^{\infty} ||h_n - g_n||^2_{L_2(a,b)} < \infty$.

Then, the system of functions $\{h_n\}_{n=0,1,2,\ldots}$ *is a Riesz basis of the space* $L_2(a,b)$.

Let $e_n(x) = \sqrt{2}\sin\pi(n+1)x$ where $n = 0,1,2,\ldots$. Then, $\{e_n\}_{n=0,1,2,\ldots}$ is an orthonormal basis in the space L_2. In particular, it is obviously a Riesz basis in this space. Let us verify the conditions A and B of the Bary Theorem for the systems of functions $\{e_n\}_{n=0,1,2,\ldots}$ and $\{u_n\}_{n=0,1,2,\ldots}$ and the space L_2.

To verify the condition A of the Bary Theorem, let us for each integer $n \geq 0$ expand the function $u_n\left(\frac{\cdot}{n+1}\right)$ in the space L_2 in the Fourier series: $u_n\left(\frac{\cdot}{n+1}\right) = \sum_{k=0}^{\infty} a_k^n e_k(\cdot)$ where a_k^n are real coefficients. Then, one can easily verify that

$$u_n(\cdot) = \sum_{m=0}^{\infty} b_m^n e_m(\cdot) \tag{4}$$

in the space L_2 where $b_{(n+1)(k+1)-1}^n = a_k^n$ and $b_m^n = 0$ if $m \neq (n+1)(k+1)-1$ for all $k = 0,1,2,\ldots$. (Indeed, the equality (4) obviously takes place in the space $L_2\left(0, \frac{1}{n+1}\right)$. Then, since in view of theorem 1(f) the function $u_n(x)$ is odd with respect to the points $\frac{r}{n+1}$ where $r = 0, \pm 1, \pm 2, \ldots$ and since the direct verification shows that the functions $e_{(n+1)(k+1)-1}(x)$, where $k = 0,1,2,\ldots$, are odd with respect to these points, too, the equality (4) also holds in the sense of each of the spaces $L_2\left(\frac{r}{n+1}, \frac{r+1}{n+1}\right)$ where $r = \overline{1,n}$. Therefore, it is valid in the sense of the space L_2.) Also, obviously $b_0^n = \ldots = b_{n-1}^n = 0$ for each n and, in view of our acceptation, $b_n^n = a_0^n > 0$.

Lemma 1. *The system of functions $\{u_n\}_{n=0,1,2,\ldots}$ is minimal in the space L_2.*

Proof. Let us suppose that the statement of lemma 1 is invalid. Then, there exist a number n and sequences of numbers $\{N_l\}_{l=1,2,3,\ldots}$ and of real coefficients d_k^l, where $k = \overline{0, N_l}$, such that

$$u_n + \sum_{\substack{k=0 \\ k\neq n}}^{N_l} d_k^l u_k = \alpha_l(x) \quad \text{and} \quad ||\alpha_l|| \to 0 \quad \text{as} \quad l \to \infty.$$

In addition, introducing zero coefficients d_k^l if necessary, we can accept that $N_l > n$ for all l.

First, let $n > 0$. Let P_n be the orthogonal projector in the space L_2 onto the subspace spanned over the functions $\{e_0, \ldots, e_{n-1}\}$

and let L^n be the closure in the same space of the linear span of the functions $\{e_{n+1}, e_{n+2}, \ldots\}$. Then, since $||h|| \geq ||P_n h||$ for any $h \in L_2$ and in view of the expansions (4), we get that

$$P_n\left[u_n + \sum_{\substack{k=0 \\ k\neq n}}^{N_l} d_k^l u_k\right] = \sum_{k=0}^{n-1} d_k^l [P_n u_k] = \beta_l(x)$$

where $||\beta_l|| \to 0$ as $l \to \infty$. Hence, since the vectors $P_n u_0, \ldots, P_n u_{n-1}$ are linearly independent in the space L_2 (because the submatrix $(b_k^m)_{m,k=0,\ldots,n-1}$ in the expansions (4) is non-degenerate), we get that $d_k^l \to 0$ as $l \to \infty$ for all $k = \overline{0, n-1}$. Therefore,

$$u_n + \sum_{k=n+1}^{N_l} d_k^l u_k = \gamma_l(x) \quad \text{where} \quad ||\gamma_l|| \to 0 \quad \text{as} \quad l \to \infty.$$

If $n = 0$, then this property also obviously holds. But in view of the expansions (4) $\sum_{k=n+1}^{N_l} d_k^l u_k \in L^n$, hence

$$\left|\left| u_n + \sum_{k=n+1}^{N_l} d_k^l u_k \right|\right| \geq b_n^n > 0,$$

and we get the contradiction. Thus, lemma 1 is proved.

Lemma 2. $\sum_{n=0}^{\infty} ||u_n - e_n||^2 < \infty$.

Proof. In view of theorem 1(d), we get by the comparison theorem (see [8]) that there exists $D_1 > 0$ such that

$$|\lambda_n - z_n| \leq D_1 \tag{5}$$

for all numbers n where $z_n = (\pi(n+1))^2$ are numbers such that $-e_n''(x) = z_n e_n(x)$. Let us show that there exists $D_2 > 0$ such that

$$|u_n'(0) - e_n'(0)| \leq D_2 \quad \text{and} \quad |u_n'(1) - e_n'(1)| \leq D_2 \tag{6}$$

for all numbers n.

At first, let us multiply the equation (1) written for $u(x) = u_n(x)$ by $u_n(x)$ and integrate the obtained identity over the segment $[0,1]$. Then, due to theorem 1(d), we get

$$\lambda_n - C_1 \leq \int_0^1 [u_n'(x)]^2 dx \leq \lambda_n + C_1$$

for a constant $C_1 > 0$ independent of n. Further, let us multiply the equation (1) written for $u(x) = u_n(x)$ by $2(1+x)u_n'(x)$ and integrate the obtained identity between 0 and 1. We get after transformations with the use of the previous estimate

$$\left|2[u_n'(1)]^2 - [u_n'(0)]^2 - 2\lambda_n\right| \leq$$

$$\leq C_1 + 2\left|\int_0^1 (1+x)f(u_n^2(x))u_n(x)u_n'(x)dx\right| \leq$$

$$\leq C_1 + \int_0^1 |F(u_n^2(x))|dx$$

where $F(s) = \int_0^s f(t)dt$. Hence, since due to theorem 1(e) $[u_n'(0)]^2 = [u_n'(1)]^2$, theorem 1(d) implies that

$$|[u_n'(1)]^2 - 2\lambda_n| = |[u_n'(0)]^2 - 2\lambda_n| \leq C_2$$

for a constant $C_2 > 0$ independent of n. We also remark that, due to the acceptation that $u_n(x) > 0$ for $x \in \left(0, \frac{1}{n+1}\right)$, we have $u_n'(0) > 0$ because otherwise (if $u_n'(0) = 0$) it must be $u_n(x) \equiv 0$ by the uniqueness theorem. Now, since sign $u_n'(0)$ = sign $e_n'(0)$ and, as it follows from theorem 1(e), sign $u_n'(1)$ = sign $e_n'(1)$, because $[e_n'(0)]^2 = [e_n'(1)]^2 = 2z_n$ and in view of the inequality (5), we get (6).

Let $w_n(x) = u_n(x) - e_n(x)$. Then, due to theorem 1(d) and the inequality (5), we have

$$-w_n''(x) + W_n(x) = z_n w_n(x), \quad x \in (0,1), \tag{7}$$

$$w_n(0) = w_n(1) = 0 \tag{8}$$

where $\{W_n(x)\}_{n=0,1,2,\dots}$ is a uniformly bounded sequence of continuous functions. Multiplying the equation (7) by $2(1+x)w_n'(x)$ and integrating the obtained identity from 0 to 1 with the use of the integration by parts, due to (6) and (8) we get

$$D_3 - \int_0^1 [w_n'(x)]^2 dx - 2\int_0^1 (x+1)W_n(x)w_n'(x)dx \geq z_n \int_0^1 [w_n(x)]^2 dx$$

where $D_3 > 0$ is a constant independent of n. Therefore, applying the inequality $2ab \leq a^2 + b^2$, we obtain that there exists $D_4 > 0$ such that $z_n \int_0^1 w_n^2(x)dx \leq D_4$ for all numbers n. Hence, $\|w_n\| \leq D_4^{\frac{1}{2}}\pi^{-1}(n+1)^{-1}$, and lemma 2 is proved.

Now theorem 2 follows from proved lemmas 1 and 2 and the Bary Theorem.

References

[1] P.E. Zhidkov. Completeness of systems of eigenfunctions for the Sturm-Liouville operator with potential depending on the spectral parameter and for one non-linear problem// *Matematicheskiĭ Sbornik.* 1997. V. **188**, No 7. P. 123-138.

[2] P.E. Zhidkov. Corrections to the paper "Completeness of systems of eigenfunctions for the Sturm-Liouville operator with potential depending on the spectral parameter and for one non-linear problem" (*Matematicheskiĭ Sbornik.* 1997. V. **188**, No 7. P. 123-138) // Submitted to "*Matematicheskiĭ Sbornik*".

[3] P.E. Zhidkov. On the completeness of the system of normalized eigenfunctions of a nonlinear Schrödinger-type operator on a segment// *Int. J. Modern Phys.* **A**. 1997. V. 12, No 1. P. 295-298.

[4] A.P. Makhmudov. Completeness of eigenelements for some non-linear operator equations// *Doklady Akad. Nauk SSSR.* 1982. V. **263**, No 1. P. 23-27.

[5] P.E. Zhidkov. On the property of being a Riesz basis for the system of eigenfunctions of a nonlinear Sturm-Liouville-type problem// Submitted to "*Matematicheskiĭ Sbornik*".

[6] N.K. Bary. On bases in Hilbert space// *Doklady Akad. Nauk SSSR.* 1946. V. **54**, No 5. P. 383-386.

[7] N.K. Bary. Biorthogonal systems and bases in Hilbert space// *Moskov. Gos. Univ. Učenye Zapiski.* 1951. Vip. **148**. *Matematika* **4**. P. 69-107.

[8] B.M. Levitan, I.S. Sargsyan. *Sturm-Liouville and Dirac operators.* 1988, Moscow, Nauka.

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