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SYMMETRIES OF TAUB–NUT DUAL METRICS

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1 Introduction

Recently was demonstrated that for a given manifold $g_{\mu\nu}$ which admits a Killing tensor $K_{\mu\nu}$ we have two types of dual metrics [1]. In [2] a geometric duality between a metric $g^{\mu\nu}$ and its Killing tensor $K^{\mu\nu}$ was disscused. The relation was generalized to spinning spaces, but only at the expense of introducing torsion. The physical interpretation of the dual metrics was not clarified [2]. On the other hand the geometrical interpretation of Killing tensors was investigated in [3]. In [1] geometric duality between $g_{\mu\nu}$ and a Killing tensor $K_{\mu\nu}$. In this case the dual spinning space was constructed without introduction of torsion. An interesting class of metrics with Killing-Yano tensor are Einstein's metrics of D or N types in Petrov's clasiffication.

Taub-NUT geometry is involved in many modern studies in physics. For example the Kaluza-Klein monopole of Gross and Perry [4] and of Sorkin [5] was obtained by embedding the Taub-NUT gravitational instanton into five-dimensional Kaluza-Klein theory. Remarkably, the same object has re-emerged in the study of monopole scattering. In the long distance limit, neglecting radiation, the relative motion of the BPS monopoles is described by the geodesics of this space [6][7]. The dynamics of well-separated monopoles is completely soluble and has a Kepler type symmetry [8, 9, 10, 11]

On the other hand the geodesic motion of pseudo-classical spinning particles in Euclidian Taub-NUT were analised in [12].

Symmetries of extended Taub-NUT metrics recently were investigated in [13][14][15]

The aim of this paper is to investigate generic and non-generic symmetries of the dual metrics. In section one we present briefly some notions about geometric duality. In section two we investigate the symmetries coresponding to Taub–NUT dual metrics.

In Anexa 1 we present the graphics of the curvature in the case of two pairs of dual metrics. In Anexa 2 we write down Christoffel symbols and the scalar curvature for dual metrics. The calculus for all Taub-NUT metrics were provided but due to their huge and complicated expressions we cannot write them out in this paper. Nevertheless we present two quite interesting cases.

2 Geometric Duality

The importance of symmetries in the description of physical systems can hardly be overestimated. In the case of dynamical systems in particular, continuous symmetries determine the structure of the algebra of observables by Noether's theorem, giving rise to constants of motion in classical mechanics and quantum numbers labeling stationary states in quantum theory. In a geometrical setting, symmetries are connected with isometries associated with Killing vectors and, more generally, Killing tensors on the configurations space of the system. An example is the motion of a point particle in a space with isometries [16], which is a physicist's way of studying the geodesic structure of a manifold. Contact with the algebraic approach is made through Lie-derivatives and their commutators. In [16] such studies were extended to spinning space-times described by supersymetric extensions of the geodesic motion, and in [17] it was shown that this can give rise to interesting new types of supersymmetry as well.

Let a space with metric $g_{\mu\nu}$ admits a Killing tensor field $K_{\mu\nu}$. A Killing tensor is a symmetric tensor which satisfies the following relation:

$$D_\lambda K_{\mu\nu} + D_\mu K_{\nu\lambda} + D_\nu K_{\lambda\mu} = 0 \tag{1}$$

where D_μ... denote covariant derivatives. The equation of motion of a particle on a geodesic is derived from the action

$$S = \int d\tau \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \tag{2}$$

The Hamiltonian has the form $H = \frac{1}{2} g_{\mu\nu} p^\mu p^\nu$ The Poisson brackets are

$$\{x_\mu, p^\nu\} = \delta^\nu_\mu \tag{3}$$

The equation of motion for a phase space function $F(x,p)$ can be computed from the Poisson brackets with the Hamiltonian

$$\dot{F} = \{F, H\} \tag{4}$$

From the covariant components $K_{\mu\nu}$ of the Killing tensor one can construct a constant of motion K,

$$K = \frac{1}{2} K_{\mu\nu} p^\mu p^\nu \tag{5}$$

It can be easy verified that

$$\{H, K\} = 0 \tag{6}$$

We have two different ways to investigate the symmetries of the manifold. First, we consider the metric $g^{\mu\nu}$ and the Killing tensor $K^{\mu\nu}$ and second we consider $K^{\mu\nu}$ like a metric and the Killing tensor $g^{\mu\nu}$. In this paper we will use the duality which exist between $g_{\mu\nu}$ and $K_{\mu\nu}$ [1]. Killing's vectors equations in the dual space have the following form [1].

$$D_\mu \hat{\chi}_\nu + D_\nu \hat{\chi}_\mu + 2K^{\delta\sigma} D_\delta K_{\mu\nu} \hat{\chi}_\sigma = 0 \tag{7}$$

where $\hat{\chi}_\sigma$ are Killing vectors in dual spaces.

We suppose that metric $g_{\mu\nu}$ admits a Killing-Yano tensor $f_{\mu\nu}$.A Killing-Yano tensor is an antisymmetric tensor [17] which satisfies the equations

$$D_\lambda f_{\mu\nu} + D_\mu f_{\nu\lambda} = 0 \tag{8}$$

Hence the existence of a Killing Yano tensor of the bosonic manifold is equivalent to the existence of a supersymmetry for the spinning particle with supercharge

$$Q_f = f^\mu_a \Pi_\mu \psi^a - \frac{1}{3} i H_{abc} \psi^a \psi^b \psi^c, \qquad \{Q, Q_f\} = 0. \tag{9}$$

where $H_{\mu\nu\lambda} = f_{\mu\nu;\lambda}$ and semicolon denotes covariant derivative. Then the coresponding Killing–Yano equation in the dual space has the form

$$D_\mu f_{\nu\lambda} + D_\nu f_{\mu\lambda} + f^\delta_\nu D_\delta K_{\mu\lambda} + 2 f^\sigma_\lambda D_\sigma K_{\nu\mu} + f^\delta_\mu D_\delta K_{\nu\lambda} = 0 \tag{10}$$

where D represents the covariant derivative corresponding to $g_{\mu\nu}$.

The metric $g_{\mu\nu}$ and the dual metric $K_{\mu\nu}$ have the same generic symmetries if and only if

$$D_\delta K_{\mu\nu} \hat{\chi}^\delta = 0 \tag{11}$$

3 Generic and non-generic symmetries

The four-dimensional Taub-NUT metric depends on a parameter m which can be positive or negative, depending on the application; for $m > 0$ it represents a nonsingular solution of the self-dual Euclidean equation and as such is interpreted as a gravitational instanton. The standard form of the line element is

$$\begin{aligned} ds^2 &= \left(1 + \frac{2m}{r}\right)(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2) \\ &+ \frac{4m^2}{1 + 2m/r}(d\psi + \cos\theta d\varphi)^2 \end{aligned} \tag{12}$$

The Killing vectors for the metric (12) have the following form:

$$D^{(\alpha)} = R^{(\alpha)\mu} \partial_\mu, \quad \alpha = 1, \cdots, 4 \tag{13}$$

where

$$D^{(1)} = \frac{\partial}{\partial\psi} \tag{14}$$

$$D^{(2)} = -\frac{\partial}{\partial\varphi} \tag{15}$$

$$D^{(3)} = \sin\varphi \frac{\partial}{\partial\theta} + \cos\varphi \cot\theta \frac{\partial}{\partial\varphi} - \frac{\cos\varphi}{\sin\theta} \frac{\partial}{\partial\psi} \tag{16}$$

$$D^{(4)} = -\cos\varphi \frac{\partial}{\partial\theta} + \sin\varphi \cot\theta \frac{\partial}{\partial\varphi} - \frac{\sin\varphi}{\sin\theta} \frac{\partial}{\partial\psi} \tag{17}$$

$D^{(1)}$ which generates the $U^{(1)}$ of λ translations, commutes with other Killing vectors.The remaining three vectors obey an $SU(2)$ algebra with

$$[D^{(2)}, D^{(3)}] = -D^{(4)}, etc... \tag{18}$$

In the Taub-NUT geometry four Killing-Yano tensors are know to exist [18].

In this case in 2-form notation the explicit expressions for the f_i are [18].

$$f_i = 4m(d\psi + \cos\theta d\varphi)\wedge dx_i - \epsilon_{ijk}\left(1+\frac{2m}{r}\right)dx_j\wedge dx_k \tag{19}$$

$$Y = 4m\,(d\psi + \cos\theta d\varphi)\wedge dr + 4r\,(r+m)\left(1+\frac{r}{m}\right)\sin\theta d\theta\wedge d\varphi \tag{20}$$

$$f_if_j + f_jf_i = -2\delta_{ij}, f_if_j - f_jf_i = 2\varepsilon_{ijk}f_k \tag{21}$$

A symmetric Killing tensor with respect to this metric is represented by the quadratic form

$$\begin{aligned} dK^2 &= \left(1+\frac{2m}{r}\right)\left(dr^2 + \frac{r^2}{m^2}(r+m)^2(d\theta^2 + \sin^2\theta d\varphi^2)\right) \\ &+\frac{4m^2}{1+2m/r}(d\psi + \cos\theta d\varphi)^2. \end{aligned} \tag{22}$$

The inverse matrix of the covariant form from (22) gives the dual line element

$$\begin{aligned} d\tilde{s}^2 &= \left(1+\frac{2m}{r}\right)\left(dr^2 + \frac{m^2r^2}{(r+m)^2}(d\theta^2 + \sin^2\theta d\varphi^2)\right) \\ &+\frac{4m^2}{1+2m/r}(d\psi + \cos\theta d\varphi)^2. \end{aligned} \tag{23}$$

If we make the transformation for r to a new variable u using the relation

$$u = re^{\frac{r}{m}} \tag{24}$$

we find that this metric is a particular form for an extended Taub-NUT metric presented in [13, 14]:

$$ds^2 = F(u)(du^2 + u^2(d\theta^2 + \sin^2\theta d\varphi^2)) + G(u)(d\psi + \cos\theta d\varphi^2) \tag{25}$$

where $F(u)$ and $G(u)$ satisfies a very interesting relation

$$G(u)F(u) = 4m^2 f(u) \tag{26}$$

with $f(u)$ having the expression

$$f(re^{\frac{r}{m}}) = \frac{1}{\left(\frac{1}{m}+\frac{1}{r}\right)^2 r^2 e^{\frac{2r}{m}}} \tag{27}$$

The metric has four Killing vectors like a extended Taub-NUT metric [13, 14, 15]. Using the techniques from [1] the properties of the metric were investigated. Because $F(re^{\frac{r}{m}})$ and $G(\frac{r}{m})$ have the following expression

$$F(re^{\frac{r}{m}}) = f(re^{\frac{r}{m}})\left(1+\frac{2m}{r}\right) \tag{28}$$

and

$$G(re^{\frac{r}{m}}) = \frac{4m^2}{1+\frac{2m}{r}} \tag{29}$$

After calculations we have obtained the following results: The dual metric is not a flat metric or conformally flat. It has no Runge-Lenz vectors. The metric do not have Killing-Yano tensors, but admit the same Killing vectors like Taub-NUT. We are ready now to study the symmetries of all dual Taub-NUT metrics

$$\begin{aligned} dK^2_{(i)} &= -\frac{2}{m}\left(1+\frac{2m}{r}\right)\left(1+\frac{m}{r}\right)r_i(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2) + \\ &\frac{8m^2}{r(1+2m/r)}r_i(d\psi + \cos\theta d\varphi)^2 + \frac{2r}{m}\left(1+\frac{2m}{r}\right)^2 dr dr_i \\ &+4\left(1+\frac{2m}{r}\right)(\mathbf{r}\times d\mathbf{r})_i(d\psi + \cos\theta d\varphi) \end{aligned} \tag{30}$$

These conserved quantities define the dual line elements

$$\begin{aligned} d\tilde{s}^2_{(i)} &= \frac{-1}{r_i^2 - (r+2m)^2}\{-\frac{2m^2}{r}\left(1+\frac{2m}{r}\right)r_i(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\varphi^2) \\ &+\frac{8m^3(1+m/r)}{(1+2m/r)}r_i(d\psi + \cos\theta d\varphi)^2 + 2mr\left(1+\frac{2m}{r}\right)^2 dr dr_i \\ &+4m^2\left(1+\frac{2m}{r}\right)(\mathbf{r}\times d\mathbf{r})_i(d\psi + \cos\theta d\varphi)\} \end{aligned} \tag{31}$$

After calculations we found that the dual metrics from (22) and (23) have the same Killing vectors like Taub-NUT metric because relation (11) is identically satisfied. The coresponding metrics for $i = 3$ in (30) and (31) admit two Killing vectors (14) and (15). For $i = 1, 2$ we have only one Killing vector (14) for the coresponding metrics from (30) and (31).

Non-generic symmetries are associated with the existence of Killing Yano tensors For this reason we investigated the non-generic symmetries of dual metrics (22), (23), (30) and (31).

After tedious calculations we have obtained that all dual metrics have no Killing-Yano tensors.Because the curvature is not zero for all dual metrics we have no Runge-Lenz vector. Another important observation is that we cannot construct a dual spinning space for dual Taub-NUT metric because there are no Killing-Yano tensors. Our result differs from that of [2].

Annexa 1

For the metric

$$d\tilde{s}^2 = \left(1 + \frac{2m}{r}\right)\left(dr^2 + \frac{m^2 r^2}{(r+m)^2}(d\theta^2 + \sin^2\theta d\varphi^2)\right) + \frac{4m^2}{1+2m/r}(d\psi + \cos\theta d\varphi)^2.$$

non-vanishing Christoffel components are

$$\Gamma^1_{11} = -\frac{m}{r(r+2m)}, \quad \Gamma^2_{12} = \frac{m^2}{r(r+m)(r+2m)}$$

$$\Gamma^3_{13} = \frac{m^2}{r(r^2+3rm+2m^2)}, \quad \Gamma^4_{13} = \frac{m\cos(\theta)}{r^2+3rm+2m^2}, \quad \Gamma^4_{14} = \frac{m}{r(r+2m)}$$

$$\Gamma^1_{22} = -\frac{rm^4}{(r+m)^3(r+2m)}, \quad \Gamma^3_{23} = -\frac{(r^2-2m^2)\cos(\theta)}{\sin(\theta)(r+2m)^2}$$

$$\Gamma^4_{23} = \frac{3\cos^2(\theta)r^2 - r^2 - 4\sin^2(\theta)rm - 4m^2}{2\sin(\theta)(r+2m)^2}$$

$$\Gamma^3_{24} = -2\frac{(r^2+2rm+m^2)}{\sin(\theta)(r+2m)^2}, \quad \Gamma^4_{24} = 2\frac{\cos(\theta)(r^2+2rm+m^2)}{\sin(\theta)(r^2+4rm+4m^2)}$$

$$\Gamma^1_{33} = -\frac{rm^3(r^2m + 11r^2m\cos^2(\theta) + 4rm^2 + 8rm^2\cos^2(\theta) + 4m^3 + 4r^3\cos^2(\theta))}{r^6 + 9r^5m + 33r^4m^2 + 63r^3m^3 + 66r^2m^4 + 36rm^5 + 8m^6)}$$

$$\Gamma^2_{33} = \frac{r(3r+4m)\cos(\theta)\sin(\theta)}{(r+2m)^2}, \quad \Gamma^1_{34} = -4\frac{rm^3\cos(\theta)}{r^3+6r^2m+12rm^2+8m^3}$$

$$\Gamma^2_{34} = 2\frac{(r+m)^2\sin(\theta)}{(r+2m)^2}, \quad \Gamma^1_{44} = -4\frac{m^3 r}{(r+2m)^3}$$

and the curvature is:

$$R = -2\frac{6m^2+3rm+2r^2}{m(r^3+4r^2m+5rm^2+2m^3)}.$$

For the metric

$$dK^2 = \left(1 + \frac{2m}{r}\right)\left(dr^2 + \frac{r^2}{m^2}(r+m)^2(d\theta^2 + \sin^2\theta d\varphi^2)\right) + \frac{4m^2}{1+2m/r}(d\psi + \cos\theta d\varphi)^2.$$

non-vanishing Christoffel coeficients are

$$\Gamma^1_{11} = -\frac{m}{r(r+2m)}, \quad \Gamma^2_{12} = -\frac{m^2+4mr+2r^2}{r(r^2+3rm+2m^2)}$$

$$\Gamma^3_{13} = \frac{m^2+4rm+2r^2}{r(r^2+3rm+2m^2)}, \quad \Gamma^4_{13} = -\frac{\cos(\theta)(3m+2r)}{r^2+3rm+2m^2}$$

$$\Gamma^4_{14} = \frac{m}{r(r+2m)}, \quad \Gamma^1_{22} = -\frac{r(r+m)(m^2+4rm+2r^2)}{(r+2m)m^2}$$

$$\Gamma^3_{23} = \frac{\cos(\theta)(r^4+6r^3m+13r^2m^2+12rm^3+2m^4)}{\sin(\theta)(r^4+6r^3m+13r^2m^2+12rm^3+4m^4)}$$

$$\Gamma^4_{23} = -\frac{(\cos^2(\theta)+1)(r^4+6mr^3+13r^2m^2+12rm^3)+4m^4}{2\sin(\theta)(r^4+6r^3m+13r^2m^2+12rm^3+4m^4)}$$

$$\Gamma^3_{24} = -2\frac{m^4}{\sin(\theta)(r^4+6r^3m+13r^2m^2+12rm^3+4m^4)}$$

$$\Gamma^4_{24} = 2\frac{m^4\cos(\theta)}{\sin(\theta)(r^4+6r^3m+13r^2m^2+12rm^3+4m^4)}$$

$$\Gamma^1_{33} = -\frac{r(2r^5\sin^2(\theta) + 14r^4m\sin^2(\theta) + 37r^3m^2\sin^2(\theta) + 45r^2m^3\sin^2(\theta) + 24rm^4\sin^2(\theta) + 4m^5)}{m^2(r^3+6r^2m+12rm^2+8m^3)}$$

$$\Gamma^2_{33} = -\frac{r\sin(\theta)\cos(\theta)(r^3+6r^2m+13rm^2+12m^3)}{\sin(\theta)(r^4+6r^3m+13r^2m^2+12rm^3+4m^4)}$$

$$\Gamma^1_{34} = -4\frac{m^3 r\cos(\theta)}{r^3+6r^2m+12rm^2+8m^3}$$

$$\Gamma^2_{34} = 2\frac{m^4\sin(\theta)}{r^4+6r^3m+13r^2m^2+12rm^3+4m^4}, \quad \Gamma^1_{44} = -4\frac{m^3 r}{(r+2m)^3}$$

and the curvature

$$R = 2\frac{6m^3+21rm^2+22r^2m+8r^3}{2m^5+9rm^4+16r^2m^3+14m^2r^3+6r^4m+r^5}$$

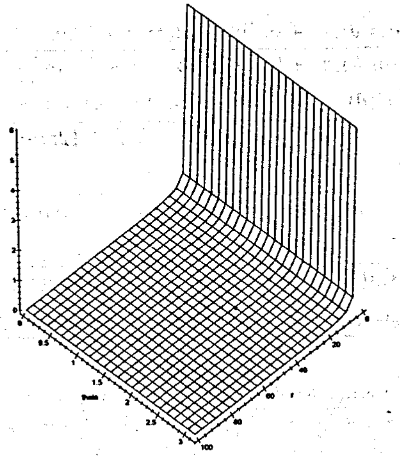

Figure 1: Ricciscalar plot for metric (22)

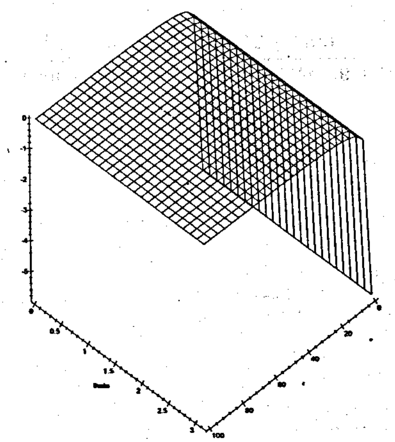

Figure 2: Ricciscalar plot for metric (23) (dual to (22)

4 Conclusions

In this paper were investigated the generic and non-generic symmetries of dual Taub-NUT metric. The scalar curvature of Taub-NUT metric is zero ,but the coresponding dual metrics have non vanishing curvatures. The dual Taub-NUT metrics have no Killing-Yano tensors. These properties tell us that we have no Runge-Lenz vector for dual Taub-NUT metrics. The dual metrics $K^{\mu\nu}$ and $K_{\mu\nu}$ have different topological properties. On the other hand the symmetries of the dual Taub-NUT metrics depend drastically on their particular form.

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