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KILLING–YANO TENSORS AND NAMBU MECHANICS

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1 Introduction

In this paper we found a very interesting connection between Killing-Yano tensors[1] and Nambu tensors. Nambu mechanics is a generalization of classical Hamiltonian mechanics introduced by Yoichiro Nambu[2].The fundamental principles of a canonical form of Nambu's generalized mechanics , similar to the invariant geometrical form of Hamiltonian mechanics , has been given by Leon Takhtajan [3].In [4] was demonstrated that several Hamiltonian systems possessing dynamical symmetries can be realized in the Nambu formalism of generalized mechanics. Nambu's generalization of mechanics is based upon a higher order ($n \geq 2$) algebraic structure defined on a phase space M.

Let M be a smooth manifold of dimension n with metric g_{ab}.
An r-form field ($1 \leq r \leq n$) $\eta_{a_1 \cdots a_r}$ is said to be a Killing-Yano tensor[1] of valence r iff

$$\nabla_{(a_1} \eta_{a_2) \cdots a_r} = 0 \tag{1}$$

here ∇_a denotes the covariant derivative and the parenthesis denote complete summetryzation over the components indices.According to (1) the $(r-1)$ form field

$$l_{a_1 \cdots a_{r-1}} = \eta_{a_1 \cdots a_{r-1} m} p^m \tag{2}$$

is parallel transported along affine parametrized geodesics with tangent field p^a. A symmetric tensor field $K_{a_1 \cdots a_r}$ is called a Killing tensor of valence r iff

$$\nabla_{(a_1} K_{a_2 \cdots a_r)} = 0 \tag{3}$$

This equation (3) ensures that $K_{m_1 \cdots m_r} p^{m_1} \cdots p^{m_r}$ is a first integral of the geodesic equation.These two generalizations of the Killing vector equation are related . Let $\eta_{a_1 \cdots a_r}$ be a Killing-Yano tensor, then tensor field $K_{ab} = \eta_{a m_2 \cdots m_r} \eta^{m_2 \cdots m_r}{}_b$ is symmetric and proves to be a Killing tensor called the associated Killing tensor. Therefore

$$l_{m_1 \cdots m_{r-1}} l^{m_1 \cdots m_{r-1}} = \eta_{m_1 \cdots m_{r-1} a} \eta^{m_2 \cdots m_r}{}_b p^a p^b \tag{4}$$

is the quadratic first integral generated by K_{ab}.
We point out that a Killing-Yano tensor $f_{\mu\nu}$ with covariant derivative zero generate a Nambu tensor and any Killing-Yano tensor of order higher than two is a Nambu tensor.Because of this very important result all Killing-Yano tensors with covariant derivative zero have the same physical and geometrical signification. We found that all Killing-Yano tensors from flat space a Nambu tensors. In the case of Kerr-Newmann metric and Taub-NUT metric we found that all Killing-Yano tensor of rank two generate a Nambu tensor of rank three.

2 Nambu Mechanics

M is called a Nambu-Poisson manifold [3] if there exists a R-multilinear map

$$\{,\cdots,\}:[C^{(\infty)}]^{\otimes}\rightarrow C^{\infty}(M) \qquad (5)$$

called a Nambu bracket of order n such that $\forall f_1, f_2\cdots, f_{2n-1}$

$$\{f_1,\cdots,f_n\}=(-)^{P(\sigma)}\{f_{\sigma(1)},\cdots,f_{\sigma(n)}\} \qquad (6)$$

$$\{f_1f_2,f_3,\cdots,f_{n+1}\}=f_1\{f_2,f_3,\cdots,f_{n+1}\}+\{f_1,f_3,\cdots,f_{n+1}\}f_2 \qquad (7)$$

and

$$\begin{aligned}&\{\{f_1,\cdots,f_{n-1},f_n\},f_{n+1},\cdots,f_{2n-1}\}+\{f_n\{f_1,\cdots f_{n-1},f_{n+1}\},f_{n+2},\cdots,f_{2n-1}\}\\ +\cdots+\ &\{f_n,\cdots,f_{2n-2}\{f_1,\cdots,f_{n-1},\{f_n,\cdots,f_{2n-1}\}\}\\ =\ &\{f_1,\cdots,f_{n-1},\{f_n,\cdots,f_{2n-1}\}\}\end{aligned} \qquad (8)$$

The dynamics on a Nambu -Hamiltonian manifold M(i.e. a phase space) is determined by n-1 so called Nambu-Hamiltonians $H_1,\cdots H_{n-1}\in C^{\infty}(M)$ and is governed by the following equations of motion

$$\frac{df}{dt}=\{f,H_1,\cdots,H_{n-1}\},\forall f\in C^{\infty}(M) \qquad (9)$$

The Nambu bracket is geometrically realized by the Nambu tensor field $\eta\in\wedge^nTM$, a section of the n-fold exterior power $\wedge^nTM$ of a tangent bundle TM, such that

$$\{f_1,\cdots,f_n\}=\eta\{df_1,\cdots,df_n\} \qquad (10)$$

In local coordinates $(x^1,\cdots,x^n)$ it becomes

$$\eta=\eta^{i_1\cdots i_n}(x)\frac{\partial}{\partial x^{i_1}}\wedge\cdots\wedge\frac{\partial}{\partial x^{i_n}} \qquad (11)$$

The fundamental identity (8) is equivalent to the following algebraic and differential constraints on the Nambu tensor $\eta^{i_1}\cdots i_n$

$$N^{i_1i_2\cdots i_nj_1j_2\cdots j_n}+N^{j_1i_2i_3\cdots i_ni_1j_2j_3\cdots j_n}=0 \qquad (12)$$

where

$$\begin{aligned}N^{i_1i_2\cdots i_nj_1j_2\cdots j_n}:\ &=\eta^{i_1i_2\cdots i_n}\eta^{j_1\cdots j_n}+\eta^{j_ni_1i_3\cdots i_n}\eta^{j_1j_2\cdots j_{n-1}i_2}+\cdots\\ +\ &\eta^{j_ni_2i_3\cdots i_{n-1}i_1}\eta^{j_1j_2\cdots j_{n-1}i_n}-\eta^{j_ni_2i_3\cdots i_n}\eta^{j_1j_2\cdots j_{n-1}i_n}\end{aligned} \qquad (13)$$

and one differential [2]

$$\begin{aligned}D^{i_2\cdots i_nj_1\cdots j_n}:\ =\ &\eta^{ki_2\cdots i_n}\partial_k\eta^{j_1j_2\cdots j_n}+\cdots+\eta^{j_ni_2\cdots i_{n-1}k}\partial_k\eta^{j_1j_2\cdots j_{n-1}i_n}\\ -\ &\eta^{j_1j_2\cdots j_{n-1}k}\partial_k\eta^{j_ni_2i_3\cdots i_n}=0\end{aligned} \qquad (14)$$

It has been shown [6] that the algebraic equations (12) and (13) imply that the Nambu tensors are decomposable (as conjectured in [5]) which in particular means that they can be written as determinants of the form

$$\eta^{i_1\cdots i_n}=\epsilon_{\alpha_1\cdots\alpha_n}v^{i_1\alpha_1}\cdots v^{i_n\alpha_n} \qquad (15)$$

3 Nambu and Killing-Yano tensors

In this section we present the connection between Killing-Yano tensors and Nambu tensors.

Theorem 1

If a manifold M of dimension N admits a Killing-Yano tensor $\eta_{\mu\nu}$ with covariant derivative zero then we can construct Killing-Yano tensors of order n

$$\eta^{\mu_1\cdots\mu_r}=\epsilon_{\alpha_1\cdots\alpha_n}\eta^{\mu_1\alpha_1}\cdots\eta^{\mu_r\alpha_n} \qquad (16)$$

with $n=3,\cdots N-1$

Proof.

Let $\eta_{\mu\nu}$ be a Killing-Yano tensor with covariant zero then we have

$$D_\lambda\eta_{\mu\nu}=0, D_\lambda\eta^{\mu\nu}=0 \qquad (17)$$

On the other hand $\eta^{\mu_1\cdots\mu_n}$ from(16) is antisymmetric by construction. Then the corresponding Killing-Yano equations are

$$D_\lambda\eta_{\mu_1\cdots\mu_n}+D_{\mu_1}\eta_{\lambda\cdots\mu_n}=0 \qquad (18)$$

From (16) and (17) we can deduce immediately (18). For $n=N$ the corresponding Killing-Yano tensor is proportional to $\epsilon_{i_1\cdots i_n}$.

QED.

Using the fact that for every Killing-Yano tensors $\eta_{\mu_1\cdots\mu_n}$ we can associate a constant of motion we have the following conserved quantities $H, K^3, K^4,\cdots K^N$, where $K^i=K^i_{ab}p^ap^b$ for $i=3,\cdots N$.

Theorem 2

If $\eta^{i_1\cdots i_n}$ is a Killing-Yano tensor with covariant derivative zero then it is a Nambu tensor

Proof.

If $\eta_{\mu\nu}$ is a Killing-Yano tensor of order 2 then from Theorem 1 we can construct a Killing-Yano tensor of order r

$$\eta^{\mu_1\cdots\mu_n}=\epsilon_{i_1\cdots i_n}\eta^{\mu_1i_1}\cdots\eta^{\mu_ri_r} \qquad (19)$$

Let suppose that $\nabla_k\eta^{i_1\cdots i_n}=0$. Then we have the following relations:

$$\frac{\partial\eta^{i_1\cdots i_n}}{\partial x^k}=-\Gamma^{i_1}_{km}\eta^{m\cdots i_n}-\cdots\Gamma^{i_n}_{km}\eta^{i_1\cdots m}. \qquad (20)$$

The fundamental identity have the following form

$$D^{i_2\cdots i_nj_1\cdots j_n}:=\ \eta^{ki_2\cdots i_n}[\Gamma^{j_1}_{km}\eta^{m\cdots j_n}+\cdots\Gamma^{j_n}_{km}\eta^{j_1\cdots m}]+$$

$$
\begin{aligned}
\cdots + &\quad \eta^{j_n \cdots i_{n-1} k}[\Gamma^{j_1}_{km}\eta^{m \cdots i_n} + \cdots + \Gamma^{i_n}_{km}\eta^{j_1 \cdots m}] \\
- &\quad \eta^{j_1 \cdots j_{n-1} k}[\Gamma^{j_n}_{km}\eta^{m i_2 i_3 \cdots i_n} + \Gamma^{i_n}_{km}\eta^{j_n \cdots m}]
\end{aligned} \qquad (21)
$$

On the other hand (12)is becomes zero when $\eta^{\mu_1 \cdots \mu_r}$ is given by (16).Using this fact in the case when the background has no torsion we found after calculations that (21) is identically zero.QED

When a Killing-Yano tensor is a Nambu tensor it has the same geometrical interpretation.For every Killing-Yano tensor we can associate a constant of motion $K = K_{ab}p^a p^b$,where $K_{ab} = \eta_{am_2 \cdots m_r}\eta^{m_2 \cdots m_r}{}_b$.

Using Theorem 2 we put the algebraic constraint (14) in the following form

$$
\begin{aligned}
D^{i_2 \cdots i_n j_1 \cdots j_n} : \quad = &\quad \eta^{k i_2 \cdots i_n}\nabla\eta^{j_1 j_2 \cdots j_n} + \eta^{j_n k i_3 \cdots i_n}\nabla_k\eta^{j_1 j_2 \cdots j_{n-1} i_2} \\
+ \cdots + &\quad \eta^{j_n i_2 i_3 \cdots i_{n-1} k}\nabla_k\eta^{j_n i_2 i_3 \cdots i_{n-1} k} \\
- &\quad \eta^{j_1 j_2 \cdots j_{n-1} k}\nabla_k\eta^{j_n i_2 i_3 \cdots i_n} = 0
\end{aligned} \qquad (22)
$$

4 Examples

4.1 Flat Space

A very interesting example is the flat space.
Let E^{n+1} be a Euclidean space and $x^\lambda(\lambda = 1 \cdots n+1)$ an orthogonal coordinate system. Killing-Yano equations have this form:

$$\partial_{(\mu} f_{\nu_1)\nu_2 \cdots \nu_n} = 0 \qquad (23)$$

After calculations, from Killing-Yano equations we have the following general solutions

$$f_{\nu_1 \cdots \nu_n} = x^\nu g_{\nu\nu_1\nu_2 \cdots \nu_n} + h_{\nu_1 \cdots \nu_n} \qquad (24)$$

where $g_{\nu\nu_1 \cdots \nu_n}$ and $f_{\nu\nu_1 \cdots \nu_n}$ are constant antisymmetric tensors.In the flat case all the Killing-Yano tensors are Nambu tensors because relations (12) and (14) are satisfied. In this case the associate Killing tensor is

$$
\begin{aligned}
K_{ab} &= f_{a\nu_2 \cdots \nu_n} f^{\nu_2 \cdots \nu_n}{}_b + f_{b\nu_2 \cdots \nu_n} f^{\nu_2 \cdots \nu_n}{}_a \\
&= x^\alpha x^\beta (g_{a\alpha\nu_2 \cdots \nu_n} g^{\nu_2 \cdots \nu_n}{}_{b\beta} + g_{b\alpha\nu_2 \cdots \nu_n} g^{\nu_2 \cdots \nu_n}{}_{a\beta}) \\
&+ x^\alpha (g_{a\alpha\nu_2 \cdots \nu_n} b^{\nu_2 \cdots \nu_n}{}_b + g_{\alpha b\nu_2 \cdots \nu_n} b^{\nu_2 \cdots \nu_n}{}_a + g_{b\alpha\nu_2 \cdots \nu_n} b^{\nu_2 \cdots \nu_n}{}_a + g_{\alpha a\nu_2 \cdots \nu_n} b^{\nu_2 \cdots \nu_n}{}_b) \\
&+ b_{b\nu_2 \cdots \nu_n} b^{\nu_2 \cdots \nu_n}{}_a + b_{a\nu_2 \cdots \nu_n} b^{\nu_2 \cdots \nu_n}{}_b
\end{aligned} \qquad (25)
$$

Because euclidean space has dimension n we have n-2 Killing-Yano tensors with valence $r = 3 \cdots n$.Then we have n-2 constants of motions K^i_{ab} with $i = 3, \cdots n$ associated with Killing-Yano tensors.If we take into account and the Hamiltonian H we have n-1 constants of motion.
This property is very important and make the natural connection with Nambu mechanics.

4.2 Taub-NUT metric

In the Taub-NUT geometry four Killing-Yano tensors are know to be exist [8]. In this case in 2-form notation the explicit expression for the f_i are [8].

$$f_i = 4m(d\psi + \cos\theta d\varphi) \wedge dx_i - \epsilon_{ijk}\left(1 + \frac{2m}{r}\right) dx_j \wedge dx_k \qquad (26)$$

$$Y = 4m\,(d\psi + \cos\theta d\varphi) \wedge dr + 4r\,(r+m)\left(1 + \frac{r}{m}\right)\sin\theta d\theta \wedge d\varphi \qquad (27)$$

The symmetries of extended Taub-NUT metric was investigated in [9]. Let consider $i = 1$ in (26)

From **Theorem 1** and **Theorem 2** we found that the independent components of Nambu's tensor $\eta^{i_1 i_2 i_3}$ are η^{124}, η^{134} and η^{234}

$$
\begin{aligned}
\eta^{124} &= 4\frac{\sin\varphi}{rm\sin\theta(r+2m)^2}, \eta^{134} = -4\frac{\cos\varphi}{rm\sin^2\theta(r+2m)^2} \\
\eta^{234} &= -4\frac{\sin\varphi}{r^2 m(r+2m)^2\sin\theta}
\end{aligned} \qquad (28)
$$

When $i = 2$ we obtaining the following results.

$$
\begin{aligned}
\eta^{124} &= -4\frac{\cos\varphi}{rm\sin\theta(r+2m)^2}, \eta^{134} = 4\frac{\sin\varphi}{rm\sin^2\theta(r+2m)^2} \\
\eta^{234} &= -\frac{4\sin\varphi}{r^2 m(r+2m)^2\sin\theta}
\end{aligned} \qquad (29)
$$

For i=3 the independent components for Nambu tensor of order 3 are η^{134} and η^{234}

$$\eta^{134} = -\frac{4\cos\theta}{rm\sin^2\theta(r+2m)^2)}, \eta^{234} = -4\frac{\cos\theta}{r^2 m\sin^2\theta(r+2m)^2} \qquad (30)$$

Killing-Yano tensor Y from (27) has not covariant derivative zero but it generate a Nambu tensor of order 3. In this case we have the following nonzero components for the Nambu tensor

$$\eta^{234} = \frac{2(r^4 + 4r^3 m + 6r^2 m^2 + 4rm^3 + m^4)}{m^3 r^2[-\sin^2\theta(r^4 + 16m^4) + 8rm\cos^2\theta(r^2 + 3rm + 4m^2)]} \qquad (31)$$

4.3 Kerr-Newmann metric

The Kerr-Newmann geometry describes a charged spinning black hole; in a standard choice of coordinates the metric is given by the following line element:

$$ds^2 = -\frac{\Delta}{\rho^2}\left[dt - a\sin^2(\theta)d\varphi\right]^2 + \frac{\sin(\theta)^2}{\rho^2}\left[(r^2 + a^2)d\varphi - adt\right]^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \qquad (32)$$

Here

$$\Delta = r^2 + a^2 - 2Mr + Q^2, \rho^2 = r^2 + a^2\cos^2(\theta) \quad (33)$$

with Q the background electric charge, and $J = Ma$ the total angular momentum.

Killing-Yano tensor for the Kerr-Newmann metric is defined by

$$\frac{1}{2} f_{\mu\nu} dx^\mu \wedge dx^\nu = a\cos\theta\, dr \wedge (dt - a\sin^2\theta\, d\phi) + r\sin\theta\, d\theta \wedge [-a\, dt + (r^2 + a^2)\, d\phi] \quad (34)$$

This Killing-Yano tensor have not covariant derivative zero but again generate a Nambu tensor of order 3.

It has the following nenule components

$$\eta^{134} = \frac{\cos^6\theta \sin\theta a^7 r^3}{sin^2\theta (r^2 + a^2)^2}, \eta^{234} = \frac{r^6 \cos^3\theta a^3}{\sin^2\theta (r^2 + a^2)^2} \quad (35)$$

A very interesting problem is quantization of Nambu mechanics when Nambu tensor is Killing-Yano tensor. There exist different approaches toward a quantization of Nambu mechanics, such a deformation quantization in the spirit of [8] and Feynman path integral approach, based on the action principle for Nambu mechanics [3].

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