ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E5-98-188

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SYMMETRIES OF THE DUAL METRICS

Submitted to «Physical Review D»

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1998

1 Introduction

The symmetries of space-times were systematically investigated in terms of the motion of pseudo-classical spinning point particles described by the supersymmetric extension of the usual relativistic point particle.Such a supersymmetric theory possesses a supercharge Q generating the supersymmetry transformation between the particle's position x^μ and the particle's spin χ^a, which must be introduced to forbid the negative norm states of spin due to indefinite Lorentz metric η_{ab}.It was a big success of Gibbons et al. [1] to have been able to show that the Killing-Yano tensor [2], which had long been known to relativistic as a rather mysterious structure, can be understood as an object generating " a non-generic" supersymmetry,i.e a supersymmetry appearing only in specific spacetimes [1]. In a geometrical setting , symmetries are connected with isometries associated with Killing vectors, and more generally, Killing tensors on the configuration space of the system.An example is the motion of a point particle in a space with isometries [3], which is a physicist's way of studying the geodesic structure of a manifold.The non-generic symmetries were investigated in the case of Taub-NUT metric [4] and extended Taub-NUT metric [5]. The geometrical interpretation of Killing tensors was investigated in [6]. Recently Holten [7] have presented a theorem concerning the reciprocal relation between two local geometries described by metrics which are Killing tensors with respect to one another. If $K^{\mu\nu}$ are the contravariant components of a Killing tensor with respect to the inverse metric $g^{\mu\nu}$, then $g^{\mu\nu}$ must represent a Killing tensor with respect to the inverse metric defined by $K^{\mu\nu}$. The physical interpretation of the dual metrics was not clarified. In this paper we investigate the geometric duality which exist between $g_{\mu\nu}$ and $K_{\mu\nu}$. The main aim of this paper is to investigate generic and non-generic symmetries of the dual metrics $K_{\mu\nu}$ and to construct the dual spinning space. Spinning space was constructed without introduction of torsion.The introduction of torsion is not crucial here.

The plan of this paper is the following:

In Section 2 we analysed the generic and non-generic symmetries of the metric $K_{\mu\nu}$ which is dual to $g_{\mu\nu}$. In Section 3 the dual spinning space was constructed. In Section 4 we present our conclusions.

2 Symmetries and geometric duality

Let a space with metric $g_{\mu\nu}$ admits a Killing tensor field $K_{\mu\nu}$. A Killing tensor is a symmetric tensor which satisfies the following relation:

$$D_\lambda K_{\mu\nu} + D_\mu K_{\nu\lambda} + D_\nu K_{\lambda\mu} = 0 \tag{1}$$

where D represent the covariant derivative. The equation of motion of a particle on a geodesic is derived from the action

$$S = \int d\tau (\frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu) . \tag{2}$$

The Hamiltonian is constructed in the following form $H = \frac{1}{2} g_{\mu\nu} p^\mu p^\nu$ .The Poisson brackets are

$$\{x_\mu, p^\nu\} = \delta^\nu_\mu . \tag{3}$$

The equation of motion for a phase space function $F(x,p)$ can be computed from the Poisson brackets with the Hamiltonian

$$\dot{F} = \{F, H\} . \tag{4}$$

From the covariant component $K_{\mu\nu}$ of the Killing tensor we can construct a constant of motion K,

$$K = \frac{1}{2} K_{\mu\nu} p^\mu p^\nu . \tag{5}$$

We can easy verified that

$$\{H, K\} = 0 . \tag{6}$$

If we denote D to be the covariant derivative in respect to $g_{\mu\nu}$ and $\hat{D}$ the covariant derivative with respect to $K_{\mu\nu}$ then:

$$\begin{aligned} D_\lambda g_{\mu\nu} &= 0 \\ D_\lambda K_{\mu\nu} + D_\mu K_{\nu\lambda} + D_\nu K_{\lambda\mu} &= 0 \end{aligned} \tag{7}$$

$$\begin{aligned} \hat{D}_\lambda K_{\mu\nu} &= 0 \\ \hat{D}_\lambda g_{\mu\nu} + \hat{D}_\mu g_{\nu\lambda} + \hat{D}_\nu g_{\lambda\mu} &= 0 . \end{aligned} \tag{8}$$

On the other hand between the connections $\hat{\Gamma}^\mu_{\nu\lambda}$ and $\Gamma^\mu_{\nu\lambda}$ is an interesting relation

$$\hat{\Gamma}^\mu_{\nu\lambda} = \Gamma^\mu_{\nu\lambda} - K^{\mu\delta} D_\delta K_{\nu\lambda} . \tag{9}$$

As is well known for a given metric $g_{\mu\nu}$ the conformal transformation is defined as

$$\begin{aligned} \hat{g}_{\mu\nu} &= e^{2U} g_{\mu\nu} \\ \hat{g}^{\mu\nu} &= e^{-2U} g^{\mu\nu} \qquad , U = U(x) . \end{aligned} \tag{10}$$

From(10)the relation between the corresponding connections is

$$\hat{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + 2\delta^\lambda_{(\mu} U_{\nu)} - g_{\mu\nu} U'^\lambda \tag{11}$$

where $U'^\lambda = \frac{dU}{dx}$.

After calculations we conclude that the dual transformation (9) is not a conformal transformation.

It is easy to check that the metric $K_{\mu\nu} = f_{\mu\lambda} f_{\nu}^{\lambda}$ has an inverse if the dimension of manifold are even.

The next step is to investiagte the symmetries of the dual manifold $K_{\mu\nu}$.

Theorem 1

The metric and the dual metric have the same generic symmetries if

$$D_{\delta} K_{\mu\nu} \hat{\chi}^{\delta} = 0. \tag{12}$$

Proof.

Let $\Gamma^{\mu}_{\nu\lambda}$ and $\hat{\Gamma}^{\mu}_{\nu\lambda}$ be the connections corresponding to $g_{\mu\nu}$ and respectively $K_{\mu\nu}$. We have the following very important relation

$$\hat{\Gamma}^{\mu}_{\nu\lambda} = \Gamma^{\mu}_{\nu\lambda} - K^{\mu\delta} D_{\delta} K_{\nu\lambda}. \tag{13}$$

Taking into account (13) Killing's vectors equations in the dual space are the following form

$$D_{\mu}\hat{\chi}_{\nu} + D_{\nu}\hat{\chi}_{\mu} + 2K^{\delta\sigma} D_{\delta} K_{\mu\nu} \hat{\chi}_{\sigma} = 0 \tag{14}$$

where $\hat{\chi}_{\sigma}$ are Killing vectors in the dual space and D represents the covariant derivative on manifold $g_{\mu\nu}$. Let χ_{σ} be the Killing vectors corresponding to $g_{\mu\nu}$. When we suppose that $\hat{\chi}_{\sigma} = \chi_{\sigma}$ from (12) and (14) we can obtain that

$$D_{\delta} K_{\mu\nu} \hat{\chi}^{\delta} = D_{\delta} K_{\mu\nu} \chi^{\delta} = 0. \tag{15}$$

If we assume that (12) is valid from (14) we can deduce that $\chi_{\sigma} = \hat{\chi}_{\sigma}$. Q.E.D.

An interesting case arises when $K_{\mu\nu}$ satisfies the following relation $D_{\lambda} K_{\mu\nu} = \eta_{\lambda} K_{\mu\nu}$. Then we have the same generic symmetries for given manifold and for dual manifold if

$$\eta_{\delta} \hat{\chi}^{\delta} = \eta_{\delta} \chi^{\delta} = 0. \tag{16}$$

Another very interesting case arises when $D_{\lambda} K_{\mu\nu} = 0$. Then we have the same generic symmetries for the manifold and its dual.

Theorem 2

The metric and the dual metric have the same non-generic symmetries if

$$f_{\nu\delta} K^{\delta\sigma} D_{\sigma} K_{\mu\lambda} + 2 f_{\sigma\lambda} K^{\sigma\delta} D_{\delta} K_{\mu\nu} + f_{\mu\sigma} K^{\sigma\delta} D_{\delta} K_{\nu\lambda} = 0. \tag{17}$$

Proof.

We suppose that metric $g_{\mu\nu}$ admits a Killing-Yano tensor $f_{\mu\nu}$.A Killing-Yano tensor is an antisymmetric tensor[7,8] which satisfies the following equations

$$D_{\lambda} f_{\mu\nu} + D_{\mu} f_{\nu\lambda} = 0. \tag{18}$$

Because

$$K_{\mu\nu} = f_{\mu\beta} f_{\nu}^{\beta} == f_{\mu a} f_{\nu}^{a} = \hat{e}_{\mu}^{a} \hat{e}_{\nu a} \tag{19}$$

we have that

$$K_{\mu\nu} = f_{\mu}^{a} f_{\nu a} = \hat{e}_{\mu}^{a} \hat{e}_{\nu a} . \tag{20}$$

If we take the dual in (20) we obtain that

$$g_{\mu\nu} = e_{\mu}^{a} e_{\nu a} = \hat{f}_{\mu}^{a} \hat{f}_{\nu a} . \tag{21}$$

Taking into account that

$$f_{\mu\nu} = f_{\mu a} e_{\nu}^{a} \tag{22}$$

from (20) ,(21) and (22) we deduce

$$f_{\mu\nu} = -\hat{f}_{\mu\nu} . \tag{23}$$

From (18) and (23) the metric and its dual must have the same Killing-Yano tensors.

On the other hand from (13) Killing-Yano equations in the dual space have the form

$$\begin{aligned} \hat{D}_{\mu} f_{\nu\lambda} + \hat{D}_{\nu} f_{\mu\lambda} &= D_{\mu} f_{\nu\lambda} + D_{\nu} f_{\mu\lambda} + f_{\nu\delta} K^{\delta\sigma} D_{\sigma} K_{\mu\lambda} + 2 f_{\sigma\lambda} K^{\sigma\delta} D_{\delta} K_{\mu\nu} \\ &+ f_{\mu\sigma} K^{\sigma\delta} D_{\delta} K_{\nu\lambda} = 0 \end{aligned} \tag{24}$$

where D represents the covariant derivative corresponding to $g_{\mu\nu}$. If we suppose that (17) is satisfied then $\hat{f}_{\mu\nu} = -f_{\mu\nu}$ is a Killing-Yano tensor.If we suppose that $\hat{f}_{\mu\nu} = -f_{\mu\nu}$ is a Killing-Yano tensor then from (24) we deduce (17). When this Theorem is satisfies the metric and its dual have the same non-generic symmetries. QED

A solution of Eq.(31) is $f_{\mu\nu} = b_{\mu\nu}$ where $b_{\mu\nu}$ is a constant antisymmetric tensor.

If the manifold admits a Killing tensor $K_{\mu\nu}$ satisfying $D_{\lambda} K_{\mu\nu} = 0$ then dual manifold has the same symmetries like initial manifold.

An interesting example is Kerr-Newmann metric [1].

The Kerr-Newmann geometry describes a charged spinning black hole: in a standard choice of coordinates the metric is given by the following line element:

$$ds^2 = -\frac{\Delta}{\rho^2} \left[dt - a \sin^2(\theta) d\varphi\right]^2 + \frac{\sin(\theta)^2}{\rho^2} \left[(r^2 + a^2) d\varphi - a dt\right]^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \tag{25}$$

Here

$$\Delta = r^2 + a^2 - 2Mr + Q^2, \rho^2 = r^2 + a^2 \cos^2\theta \qquad (26)$$

with Q the background electric charge, and $J = Ma$ the total angular momentum. The expression for ds^2 only describes the fields *outside* the horizon, which is located at

$$r = M + (M^2 - Q^2 - a^2)^{1/2}. \qquad (27)$$

The Killing Yano tensor for the Kerr–Newman is defined by [1]

$$\frac{1}{2} f_{\mu\nu} dx^\mu \wedge dx^\nu = a \cos\theta \, dr \wedge (dt - a \sin^2\theta \, d\phi) + r \sin\theta \, d\theta \wedge [-a \, dt + (r^2 + a^2) \, d\phi]. \qquad (28)$$

The Kerr-Newman metric admits a second-rank Killing tensor field.It can be described in this coordinate system by the quadratic form

$$dK^2 = K_{\mu\nu} dx^\mu dx^\nu = \frac{a^2 \cos^2\theta \Delta}{\rho^2} [dt - a \sin^2\theta d\varphi]^2 + \frac{r^2 \sin^2\theta}{\rho^2} [(r^2 + a^2) d\varphi - a dt]^2 - \frac{\rho^2}{\Delta a^2 \cos^2\theta} dr^2 + \frac{\rho^2}{r^2} d\theta^2. \qquad (29)$$

Its contravariant components define the inverse metric $\tilde{g}^{\mu\nu}$ of the dual geometry.From [7] we know that the dual line element is

$$d\tilde{s}^2 = \frac{\Delta}{\rho^2 a^2 \cos^2\theta} [dt - a \sin^2\theta d\varphi]^2 + \frac{\sin^2\theta}{\rho^2 r^2} [(r^2 + a^2) d\varphi - a dt]^2 - \frac{\rho^2}{\Delta a^2 \cos^2\theta} dr^2 + \frac{\rho^2}{r^2} d\theta^2 \qquad (30)$$

providing an explicit expression for the dual metric.

Kerr–Newman metric admits two Killing vectors $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \varphi}$.Because $K_{\mu\nu}$ is dual to $g_{\mu\nu}$ and $K^{\mu\nu}$ are dual to $g^{\mu\nu}$ is interesting to investigate the generic and non-generic symmetries in this two cases. Using Theorem 1 the corresponding Killing vectors for dual metrics (29) and (30) are found to be the same as for Kerr-Newman metric(25).

We use now Theorem 2 for investigation non-generic symmetrie of dual metrics (29) and (30). The equations (24) have no solutions for $f_{\mu\nu}$ given by (28). This means that the metric $K_{\mu\nu}$ do not admits Killing–Yano tensors.

The Killing–Yano equations corresponding to dual metric from (30) are [2]

$$D_\mu f_{\nu\lambda} + D_\nu f_{\mu\lambda} + f_\nu^\delta D_\delta K_{\mu\lambda} + 2 f_\lambda^\sigma D_\sigma K_{\nu\mu} + f_\mu^\delta D_\delta K_{\nu\lambda} = 0. \qquad (31)$$

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$$\hat{D}_\mu f_{\nu\lambda} + \hat{D}_\nu f_{\mu\lambda} = D_\mu f_{\nu\lambda} + D_\nu f_{\mu\lambda} + f_{\nu\delta} K^{\delta\sigma} D_\sigma K_{\mu\lambda} + 2 f_{\sigma\lambda} K^{\sigma\delta} D_\delta K_{\mu\nu} + f_{\mu\sigma} K^{\sigma\delta} D_\delta K_{\nu\lambda} = 0 \qquad (24)$$

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Its contravariant components define the inverse metric $\tilde{g}^{\mu\nu}$ of the dual geometry.From [7] we know that the dual line element is

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$$D_\mu f_{\nu\lambda} + D_\nu f_{\mu\lambda} + f_\nu^\delta D_\delta K_{\mu\lambda} + 2f_\lambda^\sigma D_\sigma K_{\nu\mu} + f_\mu^\delta D_\delta K_{\nu\lambda} = 0\,. \tag{31}$$

Here D_μ represents the covariant derivative corresponding to $K_{\mu\nu}$. After very tedious calculations we found that the equations (31) have no solutions.

In conclusion the dual metrics (29) and (30) have no non-generic symmetries.

Now we want to investigate the connection between Riemann curvature tensor, Ricci,Ricci scalar of manifold $g_{\mu\nu}$ and the corresponding expressions for the dual metrics $K_{\mu\nu}$. We know that

$$R^\beta_{\nu\rho\sigma} = \Gamma^\beta_{\nu\sigma,\rho} - \Gamma^\beta_{\nu\rho,\sigma} + \Gamma^\alpha_{\nu\sigma}\Gamma^\beta_{\alpha\rho} - \Gamma^\alpha_{\nu\rho}\Gamma^\beta_{\alpha\sigma} \tag{32}$$

then the corresponding dual Riemann curvature tensor $\hat{R}^\beta_{\nu\rho\sigma}$ has the following expression

$$\hat{R}^\beta_{\nu\rho\sigma} = R^\beta_{\nu\rho\sigma} + R'^\beta_{\nu\rho\sigma} \tag{33}$$

where $R'^\beta_{\nu\rho\sigma}$ is given by

$$\begin{aligned} R'^\beta_{\nu\rho\sigma} &= -(K^{\beta\delta}D_\delta K_{\nu\sigma})_{,\rho} + (K^{\beta\chi}D_\chi K_{\nu\rho})_{,\sigma} - \Gamma^\alpha_{\nu\sigma}K^{\beta\chi}D_\chi K_{\alpha\rho} \\ &- \Gamma^\beta_{\alpha\rho}K^{\alpha\delta}D_\delta K_{\nu\sigma} + \Gamma^\alpha_{\nu\rho}K^{\beta\chi}D_\chi K_{\nu\rho} + \Gamma^\beta_{\alpha\sigma}K^{\alpha\delta}D_\delta K_{\nu\rho} \\ &- K^{\alpha\delta}D_\delta K_{\nu\sigma}K^{\beta\chi}D_\chi K_{\alpha\rho} + K^{\alpha\delta}D_\delta K_{\nu\rho}K^{\beta\chi}D_\chi K_{\alpha\sigma}\,. \end{aligned} \tag{34}$$

The explicit expression for the Ricci tensor is

$$R_{\mu\nu} = \Gamma^\alpha_{\mu\alpha,\nu} - \Gamma^\alpha_{\mu\nu,\alpha} - \Gamma^\alpha_{\mu\nu}\Gamma^\beta_{\alpha\beta} + \Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\alpha}\,. \tag{35}$$

The dual Ricci tensor can be written as

$$\hat{R}_{\mu\nu} = R_{\mu\nu} + R'_{\mu\nu} \tag{36}$$

where

$$\begin{aligned} R'_{\mu\nu} &= -(K^{\alpha\chi}D_\chi K_{\mu\alpha})_{,\nu} + (K^{\alpha\chi}D_\chi K_{\mu\nu})_{,\alpha} - \Gamma^\alpha_{\mu\nu}K^{\beta\delta}D_\delta K_{\alpha\beta} \\ &- \Gamma^\alpha_{\mu\beta}K^{\beta\delta}D_\delta K_{\nu\alpha} - \Gamma^\beta_{\alpha\beta}K^{\alpha\chi}D_\chi K_{\mu\nu} - \Gamma^\beta_{\nu\alpha}K^{\alpha\chi}D_\chi K_{\mu\beta} \\ &- K^{\beta\delta}K^{\alpha\chi}D_\chi K_{\mu\nu}D_\delta K_{\alpha\beta} + K^{\alpha\chi}D_\chi K_{\mu\beta}K^{\beta\delta}D_\delta K_{\nu\alpha}\,. \end{aligned} \tag{37}$$

If the metric $g_{\mu\nu}$ satisfies Einstein's equations in vacuum

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 0 \tag{38}$$

then the dual metric $K_{\mu\nu}$ satisfies the following equation

$$(\hat{R}_{\lambda\sigma} - R'_{\lambda\sigma})(\delta_{\lambda\mu}\delta_{\sigma\nu} - \frac{1}{2}g_{\mu\nu}g^{\lambda\sigma}) = 0 \tag{39}$$

where $\hat{R}_{\mu\nu}$ and $R'_{\mu\nu}$ are given by (36) and (37).

Here D_μ represents the covariant derivative corresponding to $K_{\mu\nu}$. After very tedious calculations we found that the equations (31) have no solutions.

In conclusion the dual metrics (29) and (30) have no non-generic symmetries.

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The explicit expression for the Ricci tensor is

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} . \tag{35}$$

The dual Ricci tensor can be written as

$$\hat{R}_{\mu\nu} = R_{\mu\nu} + R'_{\mu\nu} \tag{36}$$

where

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where $\hat{R}_{\mu\nu}$ and $R'_{\mu\nu}$ are given by (36) and (37).

3 Dual spinning space

The purpose of this section is to construct the dual spinning space. The non-generic symmetries on the manifolds are related to the existence of Killing-Yano tensors [3]. Let a theory described by the vielbein e^μ_a and having a Killing-Yano tensor $f_{\mu\nu} = f^a_\mu e_{\nu a}$.This theory has a supersymmetry described by the supercharge $Q = e^\mu_a P_\mu \psi^a$,where the momentum P_μ has the form $P_\mu = \dot{x}^\nu g_{\nu\mu}$. The noncovariant Poisson bracket read

$$Q_f = f^\mu_a P_\mu \psi^a + \frac{i}{6}c_{abc}\psi^a\psi^b\psi^c \tag{40}$$

where

$$c_{abc} = e^\mu_a e^\nu_b e^\lambda_c D_\lambda f_{\mu\nu} . \tag{41}$$

By construction Q and Q_f satisfies

$$\{Q, Q_f\}_I = 0 \tag{42}$$

where noncovariant Poisson brackets are

$$\{F,G\}_I = \frac{\partial F}{\partial x^\mu}\frac{\partial G}{P_\mu} - \frac{\partial F}{P_\mu}\frac{\partial G}{\partial x^\mu} + i(-1)^{a_F}\frac{\partial F}{\partial \psi^a}\frac{\partial G}{\partial \psi_a}\frac{\partial F}{\partial \hat{P}_\mu}\frac{\partial G}{\partial x^\mu} + i(-1)^{a_F}\frac{\partial F}{\psi^a}\frac{\partial G}{\partial \psi_a} . \tag{43}$$

The dual theory is described by a vielbein $\hat{e}^\mu_a$ and the Killing-Yano tensor $\hat{f}_{\mu\nu} = -f_{\mu\nu}$. The momentum $\hat{P}^\mu$ is given by

$$\hat{P}_\mu = \dot{x}^\nu K_{\mu\nu} . \tag{44}$$

The noncovariant Poison bracket are in this case

$$\{F,G\}_{II} = \frac{\partial f}{\partial x^\mu}\frac{\partial G}{\partial \hat{P}_\mu} - \frac{\partial F}{\partial \hat{P}_\mu}\frac{\partial G}{\partial x^\mu} + i(-1)^{a_F}\frac{\partial F}{\psi^a}\frac{\partial G}{\partial \psi_a} . \tag{45}$$

The supercharges has the form

$$\hat{Q} = \psi^a \hat{e}^\mu_a \hat{P}_\mu \tag{46}$$

the new supercharge has the form

$$\hat{Q}_f = \psi^a \hat{f}^a_\mu \hat{P}^\mu + \frac{i}{6}\psi^a\psi^b\psi^c\hat{c}_{abc} \tag{47}$$

with

$$\{\hat{Q}, \hat{Q}_f\}_{II} = 0 . \tag{48}$$

Because we have $\hat{f}_{\mu\nu} = -f_{\mu\nu}$ the supercharges $\hat{Q}$ and $\hat{Q}_f$ have the forms

$$\hat{Q} = (f^{-1})^a_\nu \dot{x}^\nu \psi^a \tag{49}$$

3 Dual spinning space

The purpose of this section is to construct the dual spinning space. The non-generic symmetries on the manifolds are related to the existence of Killing-Yano tensors [3]. Let a theory described by the vielbein e_a^μ and having a Killing-Yano tensor $f_{\mu\nu} = f_\mu^a e_{\nu a}$.This theory has a supersymmetry described by the supercharge $Q = e_a^\mu P_\mu \psi^a$,where the momentum P_μ has the form $P_\mu = \dot{x}^\nu g_{\nu\mu}$. The noncovariant Poisson bracket read

$$Q_f = f_a^\mu P_\mu \psi^a + \frac{i}{6} c_{abc} \psi^a \psi^b \psi^c \tag{40}$$

where

$$c_{abc} = e_a^\mu e_b^\nu e_c^\lambda D_\lambda f_{\mu\nu} \, . \tag{41}$$

By construction Q and Q_f satisfies

$$\{Q, Q_f\}_I = 0 \tag{42}$$

where noncovariant Poisson brackets are

$$\{F,G\}_I = \frac{\partial F}{\partial x^\mu}\frac{\partial G}{P_\mu} - \frac{\partial F}{P_\mu}\frac{\partial G}{\partial x^\mu} + i(-1)^{a_F}\frac{\partial F}{\partial \psi^a}\frac{\partial G}{\partial \psi_a}\frac{\partial F}{\partial \hat{P}_\mu}\frac{\partial G}{\partial x^\mu} + i(-1)^{a_F}\frac{\partial F}{\psi^a}\frac{\partial G}{\partial \psi_a} \, . \tag{43}$$

The dual theory is described by a vielbein $\hat{e}_a^\mu$ and the Killing-Yano tensor $\hat{f}_{\mu\nu} = -f_{\mu\nu}$. The momentum $\hat{P}^\mu$ is given by

$$\hat{P}_\mu = \dot{x}^\nu K_{\mu\nu} \, . \tag{44}$$

The noncovariant Poison bracket are in this case

$$\{F,G\}_{II} = \frac{\partial f}{\partial x^\mu}\frac{\partial G}{\partial \hat{P}_\mu} - \frac{\partial F}{\partial \hat{P}_\mu}\frac{\partial G}{\partial x^\mu} + i(-1)^{a_F}\frac{\partial F}{\psi^a}\frac{\partial G}{\partial \psi_a} \, . \tag{45}$$

The supercharges has the form

$$\hat{Q} = \psi^a \hat{e}_a^\mu \hat{P}_\mu \tag{46}$$

the new supercharge has the form

$$\hat{Q}_f = \psi^a \hat{f}_\mu^a \hat{P}^\mu + \frac{i}{6}\psi^a\psi^b\psi^c \hat{c}_{abc} \tag{47}$$

with

$$\{\hat{Q}, \hat{Q}_{\hat{f}}\}_{II} = 0 \, . \tag{48}$$

Because we have $\hat{f}_{\mu\nu} = -f_{\mu\nu}$ the supercharges $\hat{Q}$ and $\hat{Q}_{\hat{f}}$ have the forms

$$\hat{Q} = (f^{-1})_\nu^a \dot{x}^\nu \psi^a \tag{49}$$

and

$$\hat{Q}_{\hat{f}} = (f^{-1})_\mu^b e_b^\mu (f^{-1})_\nu^a \dot{x}^\nu \psi_a + \frac{i}{3} f_a^\mu f_b^\nu f_c^\lambda H_{\mu\nu\lambda} \psi^a \psi^b \psi^c . \tag{50}$$

Here

$$H_{\mu\nu\lambda} = \frac{1}{3}\left(\frac{\partial f_{\mu\nu}}{\partial x^\lambda} + \frac{\partial f_{\nu\lambda}}{\partial x^\mu} + \frac{\partial f_{\lambda\mu}}{\partial x^\nu}\right) . \tag{51}$$

The dual spinning space was constructed naturally without introduction of torsion. In [7] the torsion was considered but the consistency condition (21) were not taking into account.

4 Conclusions

Recently Holten [7] has presented a theorem concerning the reciprocal relation between two local geometries described by metrics which are Killing tensors with respect to one another.Unfortunately Holten found the geometric duality between $g^{\mu\nu}$ and $K^{\mu\nu}$.

In this paper we have investigate the geometric duality between $g_{\mu\nu}$ and a Killing tensor $K_{\mu\nu}$. We found an interesting relation between connections corresponding to $g_{\mu\nu}$ and $K_{\mu\nu}$. Relation (11) is important from geometrical point of view because it can be generalized when $K_{\mu\nu}$ is not a Killing tensor. Symmetries of the dual manifolds $K_{\mu\nu}$ were investigated.The manifold $g_{\mu\nu}$ and its dual metric $K_{\mu\nu}$ have the same Killing-Yano tensors $f_{\mu\nu}$ if and only if (17) is zero.

The dual manifold $K_{\mu\nu}$ has the same Killing vectors as $g_{\mu\mu}$ in a particular case when $D_\lambda K_{\mu\nu} = 0$ or $D_\lambda K_{\mu\nu} = \eta_\lambda K_{\mu\nu}$.

The dual spinning space was constructed without introduction of torsion. Dual Kerr-Newmann metric has the same Killing vectors as Kerr-Newmann metric but has no Killing-Yano tensors. In this case we can not construct a dual spinning space .This result is different from those obtained in [7].

A special case arises when we investigate dual Einstein's metrics.If $g_{\mu\nu}$ belongs to type D and N in the Petrov's classification,then it has Killing-Yano tensors. A special attention will be devoted to Taub-NUT because it has one Killing-Yano tensor and three complex structures [9].

5 Acknowledgments

I would like to thank J.W.van Holten, M.Flato, M.Cahen, S.Bażański and S.Manoff for helpful discussions and encouragements.

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