ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E5-98-187

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SYMMETRY ASPECTS OF MULTIPOLE MOMENTS IN ELECTROMAGNETISM AND MECHANICS OF CONTINUA

Submitted to «Journal of Physics A: Mathematical and General»

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1998

I. INTRODUCTION

In the expansion of charge and currents in electromagnetism, three families of multipole moments arise : the charge, the magnetic and the toroid moments [1]. Among the first members of these multipolar families, the time derivative of the charge dipole $\dot{\boldsymbol{d}}$, charge quadrupole $\dot{Q}_{ij}$ and the magnetic dipole $\dot{\boldsymbol{\mu}}$, correspond to infinitesimal translations, shears and rotations of the points of a continuous distribution of charged matter. For example the charge multipole moments, $Q_{i_1 i_2 \cdots i_n}$, are related to the n-th order inertia moments of a continuous distribution of mass [2]. In view of the correspondence between the electric charge e, which is connected to gauge invariance and the gravitational mass m, which is related to the Poincaré invariance, we make the formal change of the current density $\boldsymbol{j}$ by the momentum vector $\boldsymbol{p}$. In this way we obtain the following associations for these tensors

$$\dot{d}_i \longrightarrow p_i, \tag{1}$$

$$Q_{ij} \longrightarrow x_i x_j - \frac{1}{3}\boldsymbol{r}^2 \delta_{ij}, \tag{2}$$

$$\dot{Q}_{ij} \longrightarrow x_i p_j + x_j p_i - \frac{2}{3}(\boldsymbol{r}\cdot\boldsymbol{p})\delta_{ij}, \tag{3}$$

$$\mu_i \longrightarrow L_i \tag{4}$$

These tensors can be found as generators of many Lie dynamical symmetries like for example the three-dimensional rotation group $SO(3)$, which is generated by the three components of the angular momentum L_i, the group of the rigid rotator $ROT(3)$, generated by the mass quadrupole tensor Q_{ij} and L_i [3], or the linear motion group $SL(3)$ which in its turn is generated by the shear tensor $S_{ij} \equiv \dot{Q}_{ij}$ and L_i [4]. It is then natural to seek the symmetries and the geometrical features of the higher-rank tensors arising in the multipole expansion. An important point that one should mention is that the above tensors are written in the position space. Unfortunately the components of the higher-rank multipoles does not satisfy the closure relations for the Lie symmetry. For our purposes it will turn out to be useful to consider the same tensors in the momentum space too. It is more convenient to write these tensors in a form which will allow the generalization to higher dimensions. For this we introduce a Killing-Yano tensor $f_{\mu\nu}$ on a smooth manifold M of dimension n which satisfies the following equation [5]

$$\mathcal{D}_\lambda f_{\nu\mu} + \mathcal{D}_\nu f_{\lambda\mu} = 0 \tag{5}$$

Here $\mathcal{D}$ denotes the covariant derivative.

Next, we consider a pair of Killing-Yano tensors $(f, \tilde{f})$, where f is a second-order Killing-Yano tensor in the position space and $\tilde{f}$ in the momentum space. In the case of three-dimensional flat space these tensors have the following form :

$$f_{ij} = \varepsilon_{kij} x_k \quad \tilde{f}_{ij} = \varepsilon_{kij} p_k \tag{6}$$

Eq.(6) can be inversed and, thus one may express the position x and momentum p variables in terms of the tensors f and $\tilde{f}$

$$x_i = \frac{1}{2}\varepsilon_{ijk} f_{jk}, \qquad p_i = \frac{1}{2}\varepsilon_{ijk}\tilde{f}_{jk} \tag{7}$$

The Poisson bracket of f and $\tilde{f}$ reads

$$\{f_{ij}, \tilde{f}_{kl}\} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} \tag{8}$$

A scalar product can be defined for these Killing-Yano tensors. For example, the square of f can be written as follows

$$f^2 \equiv f \cdot f \equiv f_{ij} f_{ij} \tag{9}$$

Equation (7) enables us to construct the phase-space in terms of the Nambu tensor ϵ_{ijk} and the Killing-Yano tensors f_{ij} and $\tilde{f}_{ij}$. When a manifold admits a Killing-Yano tensor f_{ij} then we can construct a Killing tensor $K_{ij} = x_i x_j - r^2\delta_{ij}$. This Killing tensor corresponds to a constant of motion $K = K_{ij} p_i p_j$. Because ϵ_{ijk} is a Nambu tensor of rank three it defines a Nambu mechanics with the constants of motion $H = p^2$ and K [6,7]. In [8] an interesting relation between Killing-Yano tensors and Nambu tensors was found. These results can be generalized in the flat space of arbitrary dimension [8]. In this case we have

$$x_i = \frac{1}{n!}\epsilon_{ii_1\cdots i_n} f_{i_1\cdots i_n} \qquad p_i = \frac{1}{n!}\epsilon_{ii_1\cdots i_n}\tilde{f}_{i_1\cdots i_n} \tag{10}$$

Equation (10) enables us to construct the phase space in terms of Nambu tensor $\epsilon_{ii_1\cdots i_n}$ and Killing-Yano tensors $f_{i_1\cdots i_n}$ and $\tilde{f}_{i_1\cdots i_n}$.

II. MULTIPOLE AND DYNAMICAL SYMMETRY TENSORS

The next step consists in writing some quadratic forms, like the square of the radius $\boldsymbol{r}^2$ and of the impulse $\boldsymbol{p}^2$

$$\boldsymbol{r}^2 = \frac{1}{2}f^2 \qquad \boldsymbol{p}^2 = \frac{1}{2}\tilde{f}^2, \tag{11}$$

The magnetic dipole tensor is given by

$$\mu_i = L_i = \frac{1}{2}\varepsilon_{klm} f_{ki}\tilde{f}_{lm}, \tag{12}$$

and the dilatation

$$D \equiv \boldsymbol{r}\cdot\boldsymbol{p} = \frac{1}{2}f_{ij}\tilde{f}_{ij} \tag{13}$$

The quadrupole mass-inertia tensor reads

$$Q_{ij} = \frac{1}{4}(f_{im}f_{mj} - \frac{1}{3}\delta_{ij}f^2) \tag{14}$$

The toroid dipole tensor, a quantity related to the poloidal currents on a torus, can be written in the following manner [9]

$$T_i = \frac{1}{10}(x_i D - 2\mathbf{r}^2 p_i) = \frac{1}{40}\varepsilon_{ijk}\{f_{jk}(f \cdot \tilde{f}) - 2\tilde{f}_{jk} f^2\} \tag{15}$$

In the case of purely transversal velocity fields this expression gets a more simplified form [10]

$$T_i = \frac{1}{2} d_i D = \frac{1}{8}\varepsilon_{ijk} f_{jk}(f \cdot \tilde{f}) \tag{16}$$

Next we pass to other tensors, related to dynamical symmetries. Consider first the conformal operator

$$C_i = 2x_i D - \mathbf{r}^2 p_i = \frac{1}{4}\varepsilon_{ijk}\{2f_{jk}(f \cdot \tilde{f}) - \tilde{f}_{jk} f^2\} \tag{17}$$

Together with the angular momentum L_i, C_i is a generator of a symmetry group which obeys commutation relations isomorphic to those of $SO(4)$. This is a subgroup of the group $SO(4,2)$ [11,12] (isomorphic to the conformal group in Minkowski space) which leaves invariant the free Maxwell's equations [13]. The other generators of this larger group are the impulse p_i and the dilatation D which were defined above. Another interesting tensor is the following particular form of the Runge-Lenz vector [12] with components

$$A_i = \frac{1}{2} x_i p^2 - p_i D - \frac{1}{2} x_i \tag{18}$$

This vector, together with the orbital angular momentum L_i, the dilatation D and other 2 vectors and 2 scalars generates the $SO(4,2)$ group which contains as a subgroup the symmetry group of the Hydrogen atom, i.e. $SO(4)$. Thus, from algebraic point of view the properties of the Runge-Lenz vector are similar to those of the conformal one. If we next take the *momentum conjugate* of (18), we then obtain the following tensor in Killing-Yano form

$$\tilde{A}_i = \frac{1}{8}\varepsilon_{ijk}\{(f^2 - 2)\tilde{f}_{jk} - 2f_{jk}(f \cdot \tilde{f})\} \tag{19}$$

This tensor can be viewed as a symmetry generator like A_i, but in the momentum space. Thence we obtain the following formula for the Killing-Yano tensors in terms of the Runge-Lenz vectors and the conformal operator in the space and momentum subspaces.

$$f_{jk} = -\varepsilon_{ijk}(2\tilde{A}_i + C_i) \qquad \tilde{f}_{jk} = -\varepsilon_{ijk}(2A_i + \tilde{C}_i) \tag{20}$$

In this way the toroid dipole tensor (16) can be directly related to $SO(4,2)$ symmetry generators in the full phase-space :

$$T_i = (2\tilde{A}_i + C_i)D \tag{21}$$

When we move to the next rank multipolar tensors we encounter the charge octupole tensor

$$Q_{ijk} = \frac{1}{4}\left\{\varepsilon_{imn}(\delta_{jk}f^2 + 2f_{jl}f_{lk}) - \frac{1}{5}f^2(\varepsilon_{imn}\delta_{jk} + \varepsilon_{jmn}\delta_{ik} + \varepsilon_{kmn}\delta_{ij})\right\}, \tag{22}$$

the magnetic quadrupole tensor

$$\mu_{ij} = \frac{1}{3}(x_i L_j + x_j L_i) = -\frac{1}{3}\left\{f_{ik}f_{kl}\tilde{f}_{lj} + f_{jk}f_{kl}\tilde{f}_{li}\right\}, \tag{23}$$

and the toroid quadrupole tensor

$$T_{ij} = \left(f_{im}f_{mj} - \frac{1}{4}\delta_{ij}f^2\right)(f \cdot \tilde{f}) - \frac{5}{2}f_{im}\tilde{f}_{mj}f^2 \tag{24}$$

Using again (20) we write the last tensor in terms of dynamical symmetry generators, in the position subspace of $\dot{ROT}(3)$ and $SO(4,2)$, for the purely transversal gauge mentioned above

$$T_{ij} = Q_{ij}D = \frac{1}{8}\left(f_{im}f_{mj} - \frac{1}{3}\delta_{ij}f^2\right)(f \cdot \tilde{f}) \tag{25}$$

In this paper we introduced a pair of Killing-Yano tensors which allowed us to make a natural link with the Nambu's mechanics in the case of a flat space. As an application of this construction we rewrite the expressions of the multipole tensors in terms of the three-dimensional position and impulse subspaces of the full phase-space.

Although we were not able to find new symmetries for the higher order multipoles, as was for the first members (d_i, μ_i, Q_{ij}, $\dot{Q}_{ij}$), we showed that in the full phase space it is possible to relate the toroid dipole and quadrupole tensors to $SO(4,2)$ and $ROT(3)$ generators acting in the full phase-space. This pattern is followed also by the toroid and magnetic tensors with higher multipolarity.

Similar multipolar tensors occur in the theory of continuous media [14,15] and can be related to dynamical symmetry generators in the full phase-space using a geometrical representation valid in flat and curved spaces as we showed above.

The authors are very grateful to dr.C.Şochichiu for the interest manifested during the completion of this paper and his critical remarks.

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Received by Publishing Department
on June 24, 1998.