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$(\overline{L}_n, g)$-SPACES.
GENERAL RELATIVITY OVER $\overline{V}_4$-SPACES

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1. Introduction

As we have pointed out in previous papers [1], the recent evolution of the mathematical models for describing the gravitational interaction on the classical level shows a tendency to generalizations [2], [3] using spaces with an affine connection and a metric [the s.c. (L_n, g)-spaces]. It has been shown [4] - [6] that *every differentiable manifold with affine connection and metric can be used as a model for space-time in which the equivalence principle holds.* In (L_n, g)-spaces the connection for cotangent vector fields (as being dual to the tangent vector fields) differs from the connection for the tangent vector fields only by sign. This fact is due to the definition of dual bases of vector spaces over points of a manifold, which is a trivial generalization of the notion of dual bases of algebraic dual vector spaces from the multilinear algebra. The (L_n, g)-spaces can also be generalized using the freedom of the differential-geometric preconditions and especially by means of a generalization of the definition of dual vector spaces. In the last case, if the manifold has two different (not only by sign) connections for tangent and cotangent vector fields [the s.c. $(\overline{L}_n, g)$-spaces] [1], the situation changes and is worth being investigated. On the other side, the gauge theories of gravity in (pseudo) Riemannian spaces with torsion (Einstein-Cartan's spaces) U_4 show some peculiarities related to the existence of the torsion tensor field [7] - [9]. If one uses manifolds with contravariant and covariant affine connections and Riemannian covariant metric related to the covariant affine connection as a Levi-Civita (Christoffel, symmetric) connection [the s.c. $\overline{V}_n$-spaces as a special case of $(\overline{L}_n, g)$-spaces], then a model of the gravitational interaction [in analogy to Einstein's theory of gravitation (ETG)] can be considered. It has been shown that in $\overline{V}_4$-spaces the notion of spherical symmetry can be introduced and spherically symmetrical metrics of the type of the Schwarzschild metric can exist [10]. In contrast to the general relativity and to the gravitational theories in U_4-spaces some new results induced by the existing torsion field for the contravariant affine connection could be expected.

In the present paper the Einstein theory of gravitation is considered over (pseudo) Riemannian spaces $\overline{V}_4$, where the contravariant affine connection Γ is induced by the symmetric covariant connection $P \neq -\Gamma$ and an invariant function φ : $\Gamma^i_{jk} = -P^i_{jk} + \varphi_{,k}.g^i_j$. The invariant function $\varphi = \varphi(x^k)$ is introduced by the action of the contraction operator S on the contravariant and covariant basic vector fields $\{\partial_i\}$ ($\{e_\alpha\}$) and $\{dx^j\}$ ($\{e^\alpha\}$). In contrast to the action of the common (canonical) contraction operator C on ∂_i and dx^j as $C(\partial_i, dx^j) = C(dx^j, \partial_i) = dx^j(\partial_i) = g^j_i$ used in V_4-spaces and in ETG, the action of the new contraction operator S (S instead of C) in $\overline{V}_4$-spaces is determined as $S(\partial_i, dx^j) = S(dx^j, \partial_i) = f^j{}_i = \varphi.g^j_i$, $\varphi \in C^r(M)$, $r \geq 2$. (In the common case, where $S = C$, the relation $\Gamma = -P$ is fulfilled and the change of Γ in the

form $\overline{\Gamma}^i_{jk} = \Gamma^i_{jk} + \varphi_{,k}.g^i_j$ induces automatically a covariant affine connection $\overline{P} = -\overline{\Gamma}$ with components $\overline{P}^i{}_{jk} = -\overline{\Gamma}^i{}_{jk} = P^i_{jk} - \varphi_{,k}.g^i_j$.)

The notion of $\overline{V}_n$-space is introduced. The method of Lagrangians with partial derivative (MLPD) and the method of Lagrangians with covariant derivatives (MLCD) are sketched for the case of $\overline{V}_n$-spaces. The MLCD leads directly to covariant Euler-Lagrange's equations and to the energy-momentum tensors for a Lagrangian theory of tensor fields. The Euler-Lagrange equations as a generalization of Einstein's field equations for a $\overline{V}_4$-space and their corresponding energy-momentum tensors are obtained. The geodesic equation is found by means of the variational principle (in the standard manner). By the use of the representation of the contravariant affine connection in terms of generalized Christoffel symbols and an additional tensor field (generalized contorsion tensor), the geodesic equation is compared with the equation of the corresponding auto-parallel vector field. The Einstein field equations and the geodesic and auto-parallel equations in $\overline{V}_4$-spaces are compared with these in V_4-spaces.

2. (Pseudo) Riemannian spaces with different (not only by sign) contravariant and covariant affine connections ($\overline{V}_n$-spaces)

The (pseudo) Riemannian spaces with different (not only by sign) contravariant and covariant affine connections are considered as a special case of $(\overline{L}_n, g)$-spaces.

Definition 1. *(Pseudo) Riemannian space with contravariant and covariant affine connections ($\overline{V}_n$-space).*

A $(\overline{L}_n, g)$-space with $S(dx^i, \partial_j) = f^i{}_j = e^\varphi.g^i_j$, $\varphi \in C^r(M)$, $r \geq 2$ and a covariant affine connection $\nabla = P$, connected with the covariant metric g [by means of the relation $\nabla_\xi g = 0$ for $\forall \xi \in T(M)$ (or $g_{ij;k} = g_{ij,k} + P^l_{ik}.g_{lj} + P^l_{jk}.g_{il} = 0$)] and defined as a Levi-Civita (symmetric) connection, is called a $\overline{V}_n$-space.

Remark 1. *The definition of $\overline{V}_n$-space could be made for the general case, where $S(dx^i, \partial_j) = f^i{}_j$, $f^i{}_j \in C^r(M)$, $r \geq 2$, and without the explicit form of $f^i{}_j$. Since in the present paper we will consider only the special case, where $f^i{}_j = e^\varphi.g^i_j$, $\varphi \in C^r(M)$, $r \geq 2$, the (pseudo) Riemannian space is indicated here as $\overline{V}_n$-space.*

2.1. General characteristics of a $\overline{V}_n$-space.

On the grounds of its definition and the properties of the $(\overline{L}_n, g)$-spaces, a $\overline{V}_n$-space has the characteristics:

(a) Contraction operator S obeying the relations
$S(dx^i, \partial_j) = f^i{}_j = e^\varphi.g^i_j$, $S \circ \nabla_\xi = \nabla_\xi \circ S$, $S \circ \pounds_\xi = \pounds_\xi \circ S$, $\forall \xi \in T(M)$.

(b) Contravariant affine connection Γ. Γ will have in a co-ordinate basis the components Γ^i_{jk} : $\nabla_{\partial_k}\partial_j = \Gamma^i_{jk}.\partial_i$, $\Gamma^i_{kj} - \Gamma^i_{jk} = T_{jk}{}^i \neq 0$ ($T_{jk}{}^i$ are the components of the contravariant torsion tensor T).

(c) Covariant affine connection P. P will have in a co-ordinate basis the components P^i_{jk} : $\nabla_{\partial_k} dx^i = P^i_{jk}.dx^j$, $P^i_{jk} - P^i_{jk} = U_{jk}{}^i = 0$ ($U_{jk}{}^i$ are the components of the covariant torsion tensor U), ($U = 0$ follows from the definition of a $\overline{V}_n$-space).

(d) Covariant metric tensor g. g has in a co-ordinate basis the components g_{ij} obeying the conditions $g_{ij;k} = g_{ij,k} + P^l_{ik}.g_{lj} + P^l_{jk}.g_{il} = 0$, and $g(\partial_i, \partial_j) := g_{\overline{ij}} = f^k{}_i.f^l{}_j.g_{kl} = e^{2\varphi}.g_{ij}$.

(e) Contravariant metric tensor field $\overline{g}$. $\overline{g}$ fulfils the relations $\overline{g}(dx^i, dx^j) := g^{\overline{i}\overline{j}} = f^i{}_k . f^j{}_l . g^{kl} = e^{2\varphi}.g^{ij}$ and has in a co-ordinate basis the components g^{ij} related to g_{ij} by means of the conditions $g^{\overline{i}\overline{j}}.g_{jk} = f^i{}_l . f^j{}_m . g^{lm}.g_{jk} = e^{2\varphi}.g^{ij}.g_{jk} = g^i_k$. It can be proved that the conditions for $g_{ij} : g_{ij;k} = 0$ induce the conditions for $g^{ij} : g^{ij}{}_{;k} = 0$.

From (a), (c), (d) and (e) there follows that

$$P^i_{jk} = -\frac{1}{2}.g^{\overline{i}\overline{l}}.(g_{lj,k} + g_{kl,j} - g_{jk,l}) . \tag{1}$$

From (a), (b) and (c) the relations follow

$$P^i_{jk} = -\Gamma^i_{jk} + \varphi_{,k}.g^i_j , \quad \Gamma^i_{jk} = -P^i_{jk} + \varphi_{,k}.g^i_j , \tag{2}$$

$$T_{jk}{}^i = \Gamma^i_{kj} - \Gamma^i_{jk} = \varphi_{,k}.g^i_j - \varphi_{,j}.g^i_k , \tag{3}$$

$$R^l{}_{ijk} = -P^l{}_{ijk} ,$$

$$\begin{aligned} R^l{}_{ijk} &= \Gamma^l_{ik,j} - \Gamma^l_{ij,k} + \Gamma^l_{mj}.\Gamma^m_{ik} - \Gamma^l_{mk}.\Gamma^m_{ij} , \\ P^l{}_{ijk} &= P^l_{ik,j} - P^l_{ij,k} + P^l_{mk}.P^m_{ij} - P^l_{mj}.P^m_{ik} . \end{aligned} \tag{4}$$

From the last relations follow that, despite of the difference between the contravariant and covariant affine connections Γ and P, the corresponding contravariant and covariant curvature tensors are different only by sign as in the case of V_n-spaces.

2.2. Properties of the Riemann (curvature) tensor $R^i{}_{jkl}$.

The Riemann (Riemannian) (curvature) tensor $R^i{}_{jkl}$ in a $\overline{V}_n$-space fulfils the relations

(a) $R^i{}_{jkl} = -R^i{}_{jlk}$.

(b) $R_{ijkl} = -R_{jikl}$. This property can be proved by means of the relations

$$\begin{aligned} g(\partial_m, R^i{}_{jkl}.\partial_i) &= -g(\partial_j, R^i{}_{mkl}.\partial_i) + ([\nabla_{\partial_l}, \nabla_{\partial_k}]g)(\partial_j, \partial_m) , \\ g(\partial_m, R^i{}_{jkl}.\partial_i) &= R^i{}_{jkl}.g(\partial_m, \partial_i) = R^i{}_{jkl}.g_{\overline{m}\overline{i}} := R_{\overline{m}jkl} , \\ g(\partial_j, R^i{}_{mkl}.\partial_i) &= R^i{}_{mkl}.g(\partial_j, \partial_i) = R^i{}_{mkl}.g_{\overline{j}\overline{i}} := R_{\overline{j}mkl} , \\ \nabla_{\partial_i} g &= 0 , \; f^i{}_j = e^{\varphi}.g^i_j . \end{aligned} \tag{5}$$

(c) Bianchi identity of the 1. type

$$\begin{aligned} R^l{}_{<ijk>} &\equiv T_{<ij}{}^l{}_{;k>} + T_{<ij}{}^m.T_{mk}{}^l{}_{>} \equiv 0 , \\ R^l{}_{ijk} + R^l{}_{kij} &+ R^l{}_{jki} \equiv 0 . \end{aligned} \tag{6}$$

The proof follows immediately after direct computation of $T_{<ij}{}^l{}_{;k>} + T_{<ij}{}^m.T_{mk}{}^l{}_{>}$ by the use of (3).

From the last identity follows the symmetry of the Ricci tensor $R_{ij} := g^l_k.R^k{}_{ijl}$

$$R_{ij} = R_{ji} . \tag{7}$$

(d) Bianchi identity of the 2. type

$$\begin{aligned} R^l{}_{i<jk;p>} &\equiv -R^l{}_{im<j}.T_{kp}{}^m{}_{>} \equiv -2.R^l{}_{i<jk}.\varphi_{,p>} , \\ R^l{}_{ijk;p} + R^l{}_{ipj;k} + R^l{}_{ikp;j} &\equiv 2(R^l{}_{ikj}.\varphi_{,p} + R^l{}_{ijp}.\varphi_{,k} + R^l{}_{ipk}.\varphi_{,j}) . \end{aligned} \tag{8}$$

By the use of the both types of Bianchi identities and the characteristics of the $\overline{V}_n$-space one can prove the relations:

$$\begin{aligned} g^k_l.(e^{2\varphi}.R^l{}_{ijk})_{;p} &= (e^{2\varphi}.R_{ij})_{;p} + e^{2\varphi}.\varphi_{,p}.R_{ij} , \\ g^k_l.(e^{2\varphi}.R^l{}_{ipj})_{;k} &= (e^{2\varphi}.R^l{}_{ipj})_{;l} , \\ g^k_l.(e^{2\varphi}.R^l{}_{ikp})_{;j} &= -(e^{2\varphi}.R_{ip})_{;j} - e^{2\varphi}.\varphi_{,j}.R_{ip} , \\ g^{\overline{i}\overline{j}}.(e^{2\varphi}.R_{ij})_{;p} &= [g^{\overline{i}\overline{j}}.(e^{2\varphi}.R_{ij})]_{;p} = (e^{2\varphi}.\overline{R})_{;p} , \\ g^{\overline{i}\overline{j}}.(e^{2\varphi}.R^l{}_{ipj})_{;l} &= (e^{2\varphi}.g^{\overline{i}\overline{j}}.R^l{}_{ipj})_{;l} , \\ g^{\overline{i}\overline{j}}.(e^{2\varphi}.R_{ip})_{;j} &= (e^{2\varphi}.g^{\overline{i}\overline{j}}.R_{ip})_{;j} - e^{2\varphi}.\varphi_{.j}.g^{\overline{i}\overline{j}}.R_{ip} , \\ (e^{2\varphi}.g^{\overline{i}\overline{j}}.R^k{}_{ipj})_{;k} - (e^{2\varphi}.g^{\overline{i}\overline{j}}.R_{ip})_{;j} &= -2(e^{2\varphi}.R^{\overline{k}}{}_p)_{;k} , \\ \text{where } R^{\overline{j}}{}_p &= g^{\overline{i}\overline{j}}.R_{ip} , \end{aligned} \tag{9}$$

leading to the identity

$$[e^{2\varphi}.(R^{\overline{j}}{}_i - \frac{1}{2}.\overline{R}.g^j_i)]_{;j} \equiv 0 , \qquad \overline{R} = g^{\overline{k}\overline{l}}.R_{kl} , \tag{10}$$

which can also be written in the form

$$(R^{\overline{j}}{}_i - \frac{1}{2}.\overline{R}.g^j_i)_{;j} = -2.\varphi_{,j}.(R^{\overline{j}}{}_i - \frac{1}{2}.\overline{R}.g^j_i) . \tag{11}$$

2.3. Length of a contravariant or a covariant vector field and the cosine between two vector fields of the same type.

The (pseudo) Riemannian spaces with different (not only by sign) contravariant and covariant affine connections are spaces with metric transport of the vector fields. The scalar product of two contravariant vector fields is independent of the transport of the covariant metric tensor field g

$$\nabla_\xi[g(u, v)] = g(\nabla_\xi u, v) + g(u, \nabla_\xi v) , \quad u, v \in T(M) .$$

The rate of change of the length l_u of a contravariant vector field u under metric transport is determined (see the chapter about the length of a vector field and cosine of the angle between two vector fields) by the relation

$$\xi l_u = \pm\frac{1}{l_u}.g(\nabla_\xi u, u) \text{ for } l_u \neq 0 .$$

In the special case of parallel transport of u along ξ ($\nabla_\xi u = 0$) the length of the vector u does not change

$$\xi l_u = 0 . \tag{12}$$

The length of the vector u does not change under auto-parallel transport $\nabla_u u = 0$

$$u l_u = u^i.\partial_i l_u = 0 . \tag{13}$$

The rate of change of a cosine between two contravariant vector fields depends only on their transport along a contravariant vector field and not on the transport of the metric tensor g. In the spacial case of a parallel transport of two vector fields u and v along a contravariant vector field ξ ($\nabla_\xi u = 0$, $\nabla_\xi v = 0$) the cosine (and respectively the angle) between u and v does not change

$$\xi[\cos(u, v)] = 0 . \tag{14}$$

The same statements are valid for the length of a covariant vector field and for the cosine of the angle between two covariant vector fields.

3. Lagrangian theory of tensor fields over $\overline{V}_n$-spaces

A Lagrangian theory of tensor fields has three essential structures: the *Lagrangian density*, the *Euler-Lagrange equations* and their corresponding *energy-momentum tensors*.

The Lagrangian density can be considered in two different ways as a tensor density of rank 0 with the weight $q = \frac{1}{2}$, depending on tensor fields' components and their first and second covariant derivatives:

(a) as a tensor density $\mathbf{L}$, depending on tensor fields' components and their first (and second) partial derivatives (and the components of a contravariant and a covariant affine connections), i. e.

$$\mathbf{L} = \sqrt{-d_g}.L(g_{ij}, g_{ij,k}, g_{ij,k,l}, V^A{}_B, V^A{}_{B,i}, V^A{}_{B,i,j}) ,$$

where $L(x^k) = L'(x^{k'})$ is a Lagrangian invariant, g_{ij} are the components of the covariant metric tensor field g, $V^A{}_B$ are components of tensor fields V or components Γ^i_{jk} (or P^i_{jk}) of an affine connection Γ (or P), $d_g = \det(g_{ij}) < 0$,

$$V^A{}_{B,i} = \frac{\partial V^A{}_B}{\partial x^i} , V^A{}_{B,i,j} = \frac{\partial^2 V^A{}_B}{\partial x^j \partial x^i} .$$

(b) as a tensor density $\mathbf{L}$, depending on tensor fields' components and their first (and second) covariant derivatives, i. e.

$$\mathbf{L} = \sqrt{-d_g}.L(g_{ij}, V^A{}_B, V^A{}_{B;i}, V^A{}_{B;i;j}) ,$$

where $L(x^k) = L'(x^{k'})$ is a Lagrangian invariant, g_{ij} are the components of the covariant metric tensor field g, $V^A{}_B$ are components of tensor fields $V = V^A{}_B.e_A \otimes e^B = V^A{}_B.\partial_A \otimes dx^B$ with finite rank, $A, B, ...$ are multi-indices: $A = i_1...i_k$, $B = j_1...j_l$, $k, l \in \mathbf{N}$.

The Euler-Lagrange equations can be obtained by means of the functional variation of a Lagrangian density and of these of its field variables considered as *dynamic field variables* (in contrast to the non-varied field variables considered as fixed and *non-dynamic field variables*).

The corresponding energy-momentum tensors can be found by means of the Lie variations (Lie derivatives) of a Lagrangian density and all of its field variables (dynamic and non-dynamic field variables). By means of Lie variations (change of the field variables by draggings-along of the tensor fields and their covariant derivatives) the corresponding energy-momentum tensors can be found.

There are two possible methods for the application of a Lagrangian formalism in finding out the Euler-Lagrange's equations as field equations and the corresponding energy-momentum tensors in a field theory over differentiable manifolds with contravariant and covariant affine connection and metric: *method of Lagrangians with partial derivative* (MLPD) and *method of Lagrangians with covariant derivatives* (MLCD). These two methods corresponds to the two different ways (a) and (b) of considering the dependence of the Lagrangian density on different field variables.

In the first method (MLPD) a tensor density of rank 0 and weight $q = \frac{1}{2}$ is considered as a Lagrangian density $\mathbf{L}$ depending on tensor fields' components and their first (and second) *partial derivatives* along co-ordinate basic vector fields. By means of the corresponding

functional variation of $\mathbf{L}$ and the field variables (the components of tensor fields with finite rank, their partial derivatives and the components of the affine connections (if they are considered as independent of the metric)

$$\delta\mathbf{L} = \frac{\partial\sqrt{-d_g}}{\partial g_{ij}}.\delta g_{ij}.L + \sqrt{-d_g}[\frac{\partial L}{\partial g_{ij}}.\delta g_{ij} + \frac{\partial L}{\partial g_{ij,k}}.\delta(g_{ij,k}) + \frac{\partial L}{\partial g_{ij,k,l}}.\delta(g_{ij,k,l}) + \frac{\partial L}{\partial V^A{}_B}.\delta V^A{}_B + \frac{\partial L}{\partial V^A{}_{B,i}}.\delta(V^A{}_{B,i}) + \frac{\partial L}{\partial V^A{}_{B,i,j}}.\delta(V^A{}_{B,i,j})] ,$$

$$\frac{\partial L}{\partial V^A{}_{B,i}} = \frac{\partial L}{\partial(V^A{}_{B,i})} , \quad \frac{\partial L}{\partial V^A{}_{B,i,j}} = \frac{\partial L}{\partial(V^A{}_{B,i,j})} .$$

the Euler-Lagrange equations follow after using the Stokes theorem on a common divergency term (separated from $\delta\mathbf{L}$) and imposing boundary conditions on the variations of the dynamic field variables. This is the canonical (classical, conventional) approach for considering a field theory by means of a Lagrangian formalism. One of the main assumptions here is the *commutation of the functional variation with the partial derivatives*, i. e.

$$\delta(V^A{}_{B,i}) = (\delta V^A{}_B)_{,i} , \delta(g_{ij,k}) = (\delta g_{ij})_{,k} ... .$$

In differentiable manifolds without affine connections (or in functional spaces), this assumption is a priory fulfilled on the grounds of the independence of the functional change of the form of a function from the change of the maps (or the co-ordinates) over the manifolds. But in the case of differentiable manifolds with affine connections this assumption leads to relations between the covariant derivatives and the functional variation. In this case as necessary and sufficient conditions (for example) for

$$\delta(V^A{}_{B,i}) = (\delta V^A{}_B)_{,i}$$

appear the conditions

$$\delta(V^A{}_{B;i}) = (\delta V^A{}_B)_{;i} + V^C{}_B.\delta\Gamma^A_{Ci} + V^A{}_D.\delta P^D{}_{Bi} ,$$

where

$$V^A{}_{B;i} = V^A{}_{B,i} + \Gamma^A_{Ci}.V^C{}_B + P^D_{Bi}.V^A{}_D ,$$
$$\Gamma^A_{Ci} = -S_{Ck}{}^{Al}.\Gamma^k_{li} , \; P^D_{Bi} = -S_{Bk}{}^{Dl}.P^k_{li} ,$$
$$S_{Am}{}^{Bi} = -\sum_{k=1}^{l} g^i_{j_k}.g^{i_k}_m.g^{i_1}_{j_1}...g^{i_{k-1}}_{j_{k-1}}.g^{i_{k+1}}_{j_{k+1}}...g^{i_l}_{j_l} ,$$
$$A = j_1...j_l , \; B = i_1...i_l .$$

The quantities (multi-contraction symbols) $S_{Am}{}^{Bi}$ obey the following relations

(a) $S_{Bi}{}^{Aj}.S_{Ak}{}^{Cl} = -g^l_i.S_{Bk}{}^{Cj}$, $\dim M = n$, $l = 1, ..., N$,

(b) $S_{Bi}{}^{Bj} = -N.n^{N-1}.g^j_i$,

(c) $S_{Bi}{}^{Ai} = -N.g^A_B$,

where

$$g^A_B = g^{j_1}_{i_1}...g^{j_{m-1}}_{i_{m-1}}.g^{j_m}_{i_m}.g^{j_{m+1}}_{i_{m+1}}...g^{j_l}_{i_l}$$

is defined as multi-Kronecker symbol of rank l

$$g^A_B = 1 \;\; i_k = j_k \text{ (for all } k \text{ simultaneously)},$$
$$= 0 \;\; i_k \neq j_k , \; k = 1, ..., l .$$

The proof of the above statement follows immediately from the equality

$$\begin{aligned}\delta(V^A{}_{B;i}) - (\delta V^A{}_B)_{;i} = \delta(V^A{}_{B,i}) - (\delta V^A{}_B)_{,i} + \\ + V^C{}_B.\delta\Gamma^A_{Ci} + V^A{}_D.\delta P^D_{Bi} \ .\end{aligned} \qquad (15)$$

A sufficient condition for the application of the method of Lagrangians with partial derivatives is the commutation relation between the variation operator δ and the Lie differential operator along a contravariant basic vector field

$$\delta \circ \pounds_{\partial_i} = \pounds_{\partial_i} \circ \delta \ , \qquad (16)$$

or the commutation relation between the variational operator δ and the Lie differential operator along an arbitrary given contravariant vector field ξ

$$\delta \circ \pounds_\xi - \pounds_\xi \circ \delta = \pounds_{\delta\xi} \ \text{ for } \forall \xi \in T(M) \ . \qquad (17)$$

In the second method (MLCD) a tensor density of rank 0 and weight $q = \frac{1}{2}$ is considered as a Lagrangian density depending on tensor fields' components and their first (and second) *covariant derivatives* along basic vector fields. By means of a functional variation (change of the functional structure of the field functions without changing their tensorial character) of the field variables (the components of tensor fields and their covariant derivatives)

$$\begin{aligned}\delta\mathbf{L} \ = \ & \frac{\partial\sqrt{-d_g}}{\partial g_{ij}}.\delta g_{ij}.L + \sqrt{-d_g}[\frac{\partial L}{\partial g_{ij}}.\delta g_{ij} + \frac{\partial L}{\partial g_{ij;k}}.\delta(g_{ij;k}) + \frac{\partial L}{\partial g_{ij;k;l}}.\delta(g_{ij;k;l}) + \\ & + \frac{\partial L}{\partial V^A{}_B}.\delta V^A{}_B + \frac{\partial L}{\partial V^A{}_{B;i}}.\delta(V^A{}_{B;i}) + \frac{\partial L}{\partial V^A{}_{B;i;j}}.\delta(V^A{}_{B;i;j})] \ ,\end{aligned}$$

$$\frac{\partial L}{\partial V^A{}_{B;i}} = \frac{\partial L}{\partial(V^A{}_{B;i})} \ , \qquad \frac{\partial L}{\partial V^A{}_{B;i;j}} = \frac{\partial L}{\partial(V^A{}_{B;i;j})} \ ,$$

the covariant or canonical Euler-Lagrange equations follow. Since in $\overline{V}_n$-spaces $g_{ij;k} = 0$ and $g_{ij;k;l} = 0$, $\delta\mathbf{L}$ will have the form

$$\begin{aligned}\delta\mathbf{L} \ = \ & \frac{\partial\sqrt{-d_g}}{\partial g_{ij}}.\delta g_{ij}.L + \sqrt{-d_g}[\frac{\partial L}{\partial g_{ij}}.\delta g_{ij} + \\ & + \frac{\partial L}{\partial V^A{}_B}.\delta V^A{}_B + \frac{\partial L}{\partial V^A{}_{B;i}}.\delta(V^A{}_{B;i}) + \frac{\partial L}{\partial V^A{}_{B;i;j}}.\delta(V^A{}_{B;i;j})] \ .\end{aligned}$$

In the case, when additional conditions are imposed on the affine connections, the type of the Euler-Lagrange equations and their corresponding $(\overline{L}_n, g)$-spaces depend on the separated by the variation term and the conditions on the affine connections for transforming this term in a common divergency term necessary for the application of the boundary conditions for the variations after the use of the Stokes theorem. One of the main assumptions here is the *commutation of the functional variation with the covariant derivatives*, i. e.

$$\delta(V^A{}_{B;i}) = (\delta V^A{}_B)_{;i} \ , \qquad \delta(g_{ij;k}) = (\delta g_{ij})_{;k} \ .$$

In this case as necessary and sufficient conditions for

$$\delta(V^A{}_{B;i}) = (\delta V^A{}_B)_{;i}$$

appear the conditions

$$\delta(V^A{}_{B,i}) = (\delta V^A{}_B)_{,i} - V^C{}_B.\delta\Gamma^A_{Ci} - V^A{}_D.\delta P^D{}_{Bi} \ .$$

The proof of the statement follows immediately from (15).

A sufficient condition for the application of the method of Lagrangians with covariant derivatives is the commutation relation between the variation operator δ and the covariant differential operator along a contravariant basic vector field

$$\delta \circ \nabla_{\partial_i} = \nabla_{\partial_i} \circ \delta \ , \qquad (18)$$

or the commutation relation between the variation operator δ and the covariant differential operator along an arbitrary given contravariant vector field ξ

$$\delta \circ \nabla_\xi - \nabla_\xi \circ \delta = \nabla_{\delta\xi} \ \text{ for } \forall \xi \in T(M) \ . \qquad (19)$$

In the case of the MLCD the affine connections appear as non-dynamic fields variables ($\delta\Gamma^i_{jk} = 0$, $\delta P^i_{jk} = 0$) and the variation commutes simultaneously with the partial and the covariant derivatives of the tensor field components. At the same time $\delta R^i{}_{jkl} = 0$ and $\delta P^i{}_{jkl} = 0$.

The use of the MLCD requires the use of covariant (and form-invariant) methods only, related to the applications of the covariant differential operators and the Lie differential operators to tensor fields and their covariant derivatives.

4. Einstein's field equations and energy-momentum tensors over $\overline{V}_4$-spaces

4.1. Einstein's field equations. The Lagrangian density for obtaining the Euler-Lagrange equations and their corresponding EMT-s for a material distribution and its gravitational field in $\overline{V}_n$-spaces is given in an analogy with the Lagrangian density for the gravitational field and its sources in V_4-spaces

$$\begin{aligned}\mathbf{L} = \mathbf{L}_g + \mathbf{L}_m &= \sqrt{-d_g}.(L_g + L_m) \ , \\ L_g = \tfrac{1}{\alpha.\kappa_0}.(\overline{R} - \lambda); & \quad \alpha = \pm 2, \ \kappa_0 = \text{const.}, \\ L_m &= L_m(g_{ij}, \ V^A{}_B, \ V^A{}_{B;i}, \ V^A{}_{B;i;j}) \ .\end{aligned} \qquad (20)$$

The variation of the Lagrangian density $\mathbf{L}$ can be written in the form

$$\delta\mathbf{L} = \delta_g\mathbf{L} + \delta_v\mathbf{L}_m \ , \ \delta_g\mathbf{L} = \delta_g\mathbf{L}_g + \delta_g\mathbf{L}_m \ , \ \delta_v\mathbf{L} = \delta_v\mathbf{L}_m \ . \qquad (21)$$

The variation of the Lagrangian density $\mathbf{L}_g$ with respect to the components g_{ij} of the metric g can be found in the form

$$\delta_g\mathbf{L}_g = \frac{\delta_g\mathbf{L}_g}{\delta g_{ij}}.\delta g_{ij} \ , \text{ where } \frac{\delta_g\mathbf{L}_g}{\delta g_{ij}} = -\sqrt{-d_g}.\frac{1}{\alpha.\kappa_0}.[g^{\overline{ik}}.g^{\overline{jl}}.R_{kl} - \frac{1}{2}.(\overline{R} - \lambda).g^{\overline{ij}}] \ , \qquad (22)$$

by means of the relations:

$$\frac{\partial\sqrt{-d_g}}{\partial g_{ij}} = \frac{1}{2}.\sqrt{-d_g}.g^{\overline{ij}} \ , \ \frac{\partial\overline{R}}{\partial g_{ij}} = \frac{\partial}{\partial g_{ij}}(g^{\overline{kl}}.g^r_m.R^m{}_{klr}) = \frac{\partial g^{\overline{kl}}}{\partial g_{ij}}.R_{kl} \ , \tag{23}$$

$$\frac{\partial g^{\overline{kl}}}{\partial g_{ij}} = -\frac{1}{2}(g^{\overline{ki}}.g^{\overline{jl}} + g^{\overline{kj}}.g^{\overline{il}}) \ .$$

The Einstein equations in $\overline{V}_4$-spaces follow in the form

$$R^{\overline{i}}{}_j - \frac{1}{2}(\overline{R} - \lambda).g^i_j = \alpha.\kappa_0.(\frac{1}{\sqrt{-d_g}}.\frac{\delta_g \mathbf{L}_m}{\delta g_{ik}}.g_{jk}) = -\frac{\alpha.\kappa_0}{2}.(-\frac{2}{\sqrt{-d_g}}.\frac{\delta_g \mathbf{L}_m}{\delta g_{ik}}.g_{jk}) \ . \tag{24}$$

After introducing the abbreviations

$$G^{\overline{i}}{}_j = R^{\overline{i}}{}_j - \frac{1}{2}\overline{R}.g^i_j \ , \quad {}_{mgsh}T_j{}^i = -\frac{2}{\sqrt{-d_g}}.\frac{\delta_g \mathbf{L}_m}{\delta g_{ik}}.g_{jk} \ , \tag{25}$$

where $G^{\overline{i}}{}_j$ is the Einstein tensor and ${}_{mgsh}T_j{}^i$ is the symmetric EMT of Hilbert for $\mathbf{L}_m$ in $\overline{V}_4$-spaces, the equations (24) will have the forms

$$G^{\overline{i}}{}_j + \frac{\lambda}{2}.g^i_j = -\frac{\alpha.\kappa_0}{2}.{}_{mgsh}T_j{}^i \ , \quad G^{\overline{i}}{}_j = -\frac{\alpha.\kappa_0}{2}.({}_{mgsh}T_j{}^i + \frac{\lambda}{\alpha.\kappa_0}.g^i_j) \ . \tag{26}$$

Taking into account the identity in the forms

$$[e^{2\varphi}.(R^{\overline{j}}{}_i - \frac{1}{2}.\overline{R}.g^j_i)]_{;j} \equiv 0 \ , \tag{27}$$

$$(R^{\overline{j}}{}_i - \frac{1}{2}.\overline{R}.g^j_i)_{;j} = -2.\varphi_{,j}.(R^{\overline{j}}{}_i - \frac{1}{2}.\overline{R}.g^j_i) \ , \tag{28}$$

the covariant divergency of the Einstein tensor will have the forms

$$G^{\overline{j}}{}_{i;j} \equiv -2.\varphi_{,j}.G^{\overline{j}}{}_i \ , \qquad (e^{2\varphi}.G^{\overline{j}}{}_i)_{;j} \equiv 0 \ . \tag{29}$$

From the first form of the identity follows the covariant divergency of ${}_{mgsh}T_j{}^i$

$$({}_{mgsh}T_i{}^j + \frac{\lambda}{\alpha.\kappa_0}.g^j_i)_{;j} = -2.\varphi_{,j}.({}_{mgsh}T_i{}^j + \frac{\lambda}{\alpha.\kappa_0}.g^j_i) \ , \tag{30}$$

and from the second form of the identity follows the covariant conservation law for ${}_{mgsh}T_j{}^i$

$$[e^{2\varphi}.({}_{mgsh}T_i{}^j + \frac{\lambda}{\alpha.\kappa_0}.g^j_i)]_{;j} = 0 \ . \tag{31}$$

In the case, where $\lambda = 0$, the last expressions will have the simple forms

$$(e^{2\varphi}.{}_{mgsh}T_i{}^j)_{;j} = 0 \ , \ ({}_{mgsh}T_i{}^j)_{;j} = -2.\varphi_{,j}.{}_{mgsh}T_i{}^j \ . \tag{32}$$

The Euler-Lagrange equations for the non-metric fields $V^A{}_B$ can be found in an analogous way as the Einstein equations.

From the relations in the MLCD $\delta\Gamma^i_{jk} = 0$ and $\delta P^i_{jk} = 0$, there follows that $(\delta\varphi)_{,i} = 0$. Therefore, $\delta\varphi$ could be only arbitrary constants: $\delta\varphi =$const. $\in R$ (or C) and the variation of φ allows only translation: $\overline{\varphi} = \varphi + a$, $a = \delta\varphi$.

4.2. Euler-Lagrange's equations for the non-metric tensor fields $V^A{}_B$. After variation of the Lagrangian density $\mathbf{L}$ with respect to the components of the non-metric tensor field $V^A{}_B$ the Euler-Lagrange equations for $V^A{}_{,B}$ can be found. Since $V^A{}_B$ are only elements of $\mathbf{L}_m$, the variation is restricted on $\mathbf{L}_m$. By the use of the standard variation in the MLCD one would find the relations

$$\delta_v \mathbf{L} = \delta_v \mathbf{L}_m = \sqrt{-d_g}.[\frac{\delta_v \mathbf{L}_m}{\delta V^A{}_B}.\delta V^A{}_B + {}_v\mathbf{j}^i{}_{;i} \ , \tag{33}$$

$$\frac{\delta_v \mathbf{L}_m}{\delta V^A{}_B} = \frac{\partial \mathbf{L}_m}{\partial V^A{}_B} - (\frac{\partial \mathbf{L}_m}{\partial V^A{}_{B;i}})_{;i} + (\frac{\partial \mathbf{L}_m}{\partial V^A{}_{B;i;j}})_{;j;i} \ , \tag{34}$$

$${}_v\mathbf{j}^i = [\frac{\partial \mathbf{L}_m}{\partial V^A{}_{B;i}} - (\frac{\partial \mathbf{L}_m}{\partial V^A{}_{B;i;j}} + \frac{\partial \mathbf{L}_m}{\partial V^A{}_{B;j;i}})_{;j}].\delta V^A{}_B + (\frac{\partial \mathbf{L}_m}{\partial V^A{}_{B;j;i}}.\delta V^A{}_B)_{;j} \ . \tag{35}$$

The introduction of the boundary conditions for the variations $\delta V^A{}_B \mid_{(V_n)} = 0$; and their first covariant (or partial) derivatives $(\delta V^A{}_B)_{;i}$ [or $(\delta V^A{}_B)_{,i}$, because of $\delta\Gamma^i_{jk} = 0$ and $\delta P^i_{jk} = 0$]: $(\delta V^A{}_B)_{,i} \mid_{(V_n)} = 0$, on the shell of the volume V_n $(n = 4)$ in which the action $\overline{S} = \int_{V_n} L.d\omega$ is determined, requires the application of the Stokes theorem [11]. This is connected with the transformation of the term ${}_v\mathbf{j}^i{}_{;i}$ in a common divergency term ${}_v\mathbf{j}^i{}_{,i}$. Since

$${}_v\mathbf{j}^i{}_{;i} = {}_v\mathbf{j}^i{}_{,i} + (T_{ki}{}^k + g^l_k.g^k_{l;i}).\mathbf{j}^i \ , \tag{36}$$

as the necessary and sufficient condition for ${}_v\mathbf{j}^i{}_{;i} = {}_v\mathbf{j}^i{}_{,i}$ for *every* $\mathbf{j}^i$ (regardless to its explicit form) appears the condition for $(\overline{L}_n, g)$-spaces

$$(T_{ki}{}^k + g^l_k.g^k_{l;i}).{}_v\mathbf{j}^i = 0 \ . \tag{37}$$

For $\overline{V}_n$-spaces $(n = 4)$ it will have the form

$$\varphi_{,i}.{}_v\mathbf{j}^i = 0 \ , \tag{38}$$

and for $\forall\, \mathbf{j}^i$, it follows that $\varphi_{,i} = 0$, i. e. $\varphi =$const. This case leads to the common V_n-spaces.

If we take use of the explicit form of $\mathbf{j}^i$, then $\delta_v \mathbf{L}_m$ can be written in the form

$$\begin{aligned} \delta_v \mathbf{L}_m &= \sqrt{-d_g}.(\frac{\delta_v L_m}{\delta V^A{}_B}.\delta V^A{}_B + {}_v j^i{}_{;i}) = \\ &= \sqrt{-d_g}.(\frac{\delta_v L_m}{\delta V^A{}_B} + P_A{}^B).\delta V^A{}_B + (\sqrt{-d_g}.{}_v\overline{j}^i)_{,i} \ . \end{aligned} \tag{39}$$

where

$$P_A{}^B = q_i.[\frac{\partial L_m}{\partial V^A{}_{B;i}} - (\frac{\partial L_m}{\partial V^A{}_{B;i;j}} + \frac{\partial L_m}{\partial V^A{}_{B;j;i}})_{;j}] + (q_i.q_j - q_{i;j}).\frac{\partial L_m}{\partial V^A{}_{B;j;i}} \ ,$$

$$\frac{\delta_v L_m}{\delta V^A{}_B} = \frac{\partial L_m}{\partial V^A{}_B} - (\frac{\partial L_m}{\partial V^A{}_{B;i}})_{;i} + (\frac{\partial L_m}{\partial V^A{}_{B;i;j}})_{;j;i} \ ,$$

$${}_v\overline{j}^i = {}_v j^i + q_j.\frac{\partial L_m}{\partial V^A{}_{B;i;j}}.\delta V^A{}_B \ , \qquad q_i = \varphi_{,i} \ ,$$

$$ {}_{v}j^{i} = [\frac{\partial L_m}{\partial V^A{}_{B;i}} - (\frac{\partial L_m}{\partial V^A{}_{B;i;j}} + \frac{\partial L_m}{\partial V^A{}_{B;j;i}})_{;j}].\delta V^A{}_B + (\frac{\partial L_m}{\partial V^A{}_{B;j;i}}.\delta V^A{}_B)_{;j} . \quad (40) $$

The covariant Euler-Lagrange equations for $V^A{}_B$ follow in the form

$$ \frac{\delta_v L_m}{\delta V^A{}_B} = - P_A{}^B \quad \text{for } \forall \delta V^A{}_B . \quad (41) $$

4.3. Energy-momentum tensors. By means of the Lie variation of the Lagrangian density **L** and the covariant Noether identities valid in $\overline{V}_n$-spaces the energy-momentum tensors (EMT-s) for all tensor fields which components appear in the structure of the Lagrangian density can be found. The covariant Noether identities can be written in the form

$$ \overline{F}_i + \overline{\theta}_i{}^j{}_{;j} \equiv 0 \ , \ \overline{\theta}_i{}^j - {}_sT_i{}^j \equiv \overline{Q}_i{}^j . \quad (42) $$

$\overline{F}_i$ are the components of the volume force density, $\overline{\theta}_i{}^j$ are the components of the generalized canonical EMT (GC-EMT), ${}_sT_i{}^j$ are the components of the symmetric EMT of Belinfante (S-EMT-B) and $\overline{Q}_i{}^j$ are the components of the variational EMT of Euler-Lagrange (V-EMT-EL). In the case of the ETG L_g and L_m lead to different quantities for all (dynamic and non-dynamic) variables. $L_g = \frac{1}{\alpha\kappa_o}(\overline{R} - \lambda)$ is considered as a Lagrangian invariant responsible for the existence of the gravitational field induced by a material distribution described by $L_m = L_m(g_{ij}, V^A{}_B, V^A{}_{B;i}, V^A{}_{B;i;j})$. On the other side, L_m as a function of g_{ij} take part in the determination of the EMT-s for g_{ij}. In this way, every EMT decomposes in terms constructed by means of L_g and L_m. (The first subscript at the left side of a quantity shows the Lagrangian invariant of its composition ($_g$: L_g or $_m$: L_m); the second subscript at the left shows the field variables to which a quantity corresponds ($_g$: to g_{ij}, $_c$: to $R^i{}_{jkl}$, $_v$: to $V^A{}_B$); The third subscript (if any) at the left shows the type of the quantity itself ($_s$: symmetric):

$$ \overline{\theta}_i{}^j = {}_{gg}\overline{\theta}_i{}^j + {}_{gc}\overline{\theta}_i{}^j + {}_{mg}\overline{\theta}_i{}^j + {}_{mv}\overline{\theta}_i{}^j , \quad (43) $$

$$ {}_sT_i{}^j = {}_{ggs}T_i{}^j + {}_{gcs}T_i{}^j + {}_{mgs}T_i{}^j + {}_{mvs}T_i{}^j , \quad (44) $$

$$ \overline{Q}_i{}^j = {}_{gg}\overline{Q}_i{}^j + {}_{gc}\overline{Q}_i{}^j + {}_{mg}\overline{Q}_i{}^j + {}_{mv}\overline{Q}_i{}^j , \quad (45) $$

$$ \overline{F}_i = {}_{gg}\overline{F}_i + {}_{gc}\overline{F}_i + {}_{mg}\overline{F}_i + {}_{mv}\overline{F}_i . \quad (46) $$

Using the decompositions of $\overline{\theta}_i{}^j$, ${}_sT_i{}^j$, $\overline{Q}_i{}^j$ and $\overline{F}_i$, the covariant Noether identities can be written in the form

$$ \begin{aligned} &({}_{gg}\overline{\theta}_i{}^j - {}_{ggs}T_i{}^j) + ({}_{gc}\overline{\theta}_i{}^j - {}_{gcs}T_i{}^j) + \\ &({}_{mg}\overline{\theta}_i{}^j - {}_{mgs}T_i{}^j) + ({}_{mv}\overline{\theta}_i{}^j - {}_{mvs}T_i{}^j) \equiv \\ &\equiv {}_{gg}\overline{Q}_i{}^j + {}_{gc}\overline{Q}_i{}^j + {}_{mg}\overline{Q}_i{}^j + {}_{mv}\overline{Q}_i{}^j , \end{aligned} \quad (47) $$

$$ \begin{aligned} &{}_{gg}\overline{F}_i + {}_{gc}\overline{F}_i + {}_{mg}\overline{F}_i + {}_{mv}\overline{F}_i + \\ &+ ({}_{gg}\overline{\theta}_i{}^j + {}_{gc}\overline{\theta}_i{}^j + {}_{mg}\overline{\theta}_i{}^j + {}_{mv}\overline{\theta}_i{}^j)_{;j} \equiv 0 . \end{aligned} \quad (48) $$

The explicit form of the different energy-momentum tensors connected with the Lagrangian density **L** can be found for the components g_{ij} of the metric tensor, for the components $R^i{}_{jkl}$ of the curvature tensor and for the components $V^A{}_B$ of the non-metric tensor fields V.

4.4. Energy-momentum tensors for the metric field g_{ij} and the curvature $R^i{}_{jkl}$. The energy-momentum tensors for g_{ij} and $R^i{}_{jkl}$ can be determined on the basis of the structure of L_g and L_m. The general structure of the different EMT-s for g_{ij} and $R^i{}_{jkl}$ is the same as for the components $V^A{}_B$ of the non-metric tensor fields V. One has only to substitute $V^A{}_B$ with g_{ij} ($B = ij$, A to be omitted) or $R^i{}_{jkl}$ ($A = i$, $B = jkl$) (s. the next subsection).

$$ {}_{gg}\overline{\theta}_i{}^j = {}_{ggs}T_i{}^j = -\frac{1}{\alpha\kappa_o}(\overline{R} - \lambda).g^j_i \ , \ {}_{gc}\overline{\theta}^j_i = {}_{gcs}T_i{}^j = -\frac{1}{\alpha\kappa_o}(\overline{R} - \lambda).g^j_i , \quad (49) $$

$$ {}_{mg}\overline{\theta}_i{}^j = {}_{mgs}T_i{}^j = - L_m.g^j_i , \quad (50) $$

$$ {}_{gg}\overline{Q}_i{}^j = \frac{2}{\alpha\kappa_o}.R_{\overline{i}}{}^j \ , \ {}_{gc}\overline{Q}_i{}^j = -\frac{2}{\alpha\kappa_o}.R_{\overline{i}}{}^j \ , \ {}_{mg}\overline{Q}^j_i = -2.\frac{\partial L_m}{\partial g_{jk}}.g_{ik} , \quad (51) $$

$$ {}_{gg}\overline{F}_i = {}_{gc}\overline{F}_i = (L_g.g^j_i)_{;j} = \frac{1}{\alpha\kappa_o}.(\overline{R}.g^j_i)_{;j} \ , \ {}_{mg}\overline{F}_i = (L_m.g^j_i)_{;j} , \quad (52) $$

4.5. Energy-momentum tensors for the non-metric tensor fields $V^A{}_B$. 1. Symmetric energy-momentum tensor of Belinfante ${}_{mvs}T_i{}^j$ for $V^A{}_B$

$$ {}_{mvs}T_i{}^j = {}_vT_i{}^j - g^j_i.L_m , \quad (53) $$

$$ {}_vT_i{}^j = {}_vT_i{}^{jk}{}_k = g_{im}.{}_vT^{\overline{m}jk}{}_k \ , \ {}_vT_i{}^{jk}{}_k = g^l_k.{}_vT_i{}^{jk}{}_l , \quad (54) $$

$$ {}_vT_i{}^{jk}{}_l = g_{\overline{i}\overline{m}}.({}_{vs}\overline{V}_r{}^{kj}{}_l.g^{rm} + {}_{vs}\overline{V}_r{}^{km}{}_l.g^{rj} - {}_{vs}\overline{V}_r{}^{jm}{}_l.g^{rk}) , \quad (55) $$

$$ {}_{vs}\overline{V}_r{}^{kj}{}_l = {}_{vs}\overline{V}_r{}^{jk}{}_l = \frac{1}{2}({}_v\overline{V}_r{}^{kj}{}_l + {}_v\overline{V}_r{}^{jk}{}_l) , \quad (56) $$

$$ {}_v\overline{V}_r{}^{kj}{}_l = {}_{vs}\overline{V}_r{}^{kj}{}_l + {}_{va}\overline{V}_r{}^{kj}{}_l , \quad (57) $$

$$ {}_{va}\overline{V}_r{}^{kj}{}_l = \frac{1}{2}({}_v\overline{V}_r{}^{kj}{}_l - {}_v\overline{V}_r{}^{jk}{}_l) , \quad (58) $$

$$ {}_v\overline{V}_r{}^{kj}{}_l = {}_v\overline{Q}_{\overline{r}}{}^{k\underline{j}}{}_l - {}_v\overline{P}_r{}^{kj}{}_l , \quad (59) $$

$$ \begin{aligned} {}_v\overline{P}_r{}^{kj}{}_l = \ & S_{Cr}{}^{Aj}.[\frac{\partial L}{\partial V^A{}_{B;k}}.V^C{}_B + \\ & +(\frac{\partial L}{\partial V^A{}_{B;k;m}} + \frac{\partial L}{\partial V^A{}_{B;m;k}}).V^C{}_{B;m} - (\frac{\partial L}{\partial V^A{}_{B;k;m}}.V^C{}_B)_{;m}]_{;l} + \\ & +(\frac{\partial L}{\partial V^A{}_{B;j;k}}.V^A{}_{B;r})_{;l} , \end{aligned} \quad (60) $$

$$ \begin{aligned} {}_v\overline{Q}_{\overline{r}}{}^{k\underline{j}}{}_l = \ & S_{B\overline{r}}{}^{D\underline{j}}.[\frac{\partial L}{\partial V^A{}_{B;k}}.V^A{}_D + (\frac{\partial L}{\partial V^A{}_{B;k;m}} + \frac{\partial L}{\partial V^A{}_{B;m;k}}).V^A{}_{D;m} - \\ & -(\frac{\partial L}{\partial V^A{}_{D;k;m}}.V^A{}_D)_{;m}]_{;l} , \end{aligned} \quad (61) $$

$$S_{Cm}{}^{Aj} = -\sum_{k=1}^{l} g^{j}_{j_k}.g^{i_k}_{m}.g^{i_1}_{j_1}\dots g^{i_{k-1}}_{j_{k-1}}.g^{i_{k+1}}_{j_{k+1}}\dots g^{i_l}_{j_l} \ , \qquad (62)$$

$$V^A{}_{B;i} = e_i V^A{}_B + \Gamma^A_{Ci}.V^C{}_B + P^D_{Bi}.V^A{}_D = V^A{}_{B,i} + \Gamma^A_{Ci}.V^C{}_B + P^D_{Bi}.V^A{}_D \ , \qquad (63)$$

$$\Gamma^A_{Ci} = -S_{Cm}{}^{Aj}.\Gamma^m_{ji} \ , \qquad P^A_{Ci} = -S_{Cm}{}^{Aj}.P^m_{ji} \ , \qquad (64)$$

2. Generalized canonical energy-momentum tensor ${}_{mv}\overline{\theta}_i{}^j$ for $V^A{}_B$. ${}_{mv}\overline{t}_i{}^j$ is the canonical energy-momentum tensor for $V^A{}_B$.

$${}_{mv}\overline{\theta}_i{}^j = {}_{mv}\overline{t}_i{}^j - {}_vK_i{}^j - {}_v\overline{W}_i{}^{jk}{}_k \ , \qquad (65)$$

$${}_{mv}\overline{t}_i{}^j = [\frac{\partial L}{\partial V^A{}_{B;j}} - (\frac{\partial L}{\partial V^A{}_{B;k;j}} + \frac{\partial L}{\partial V^A{}_{B;j;k}})_{;k}].V^A{}_{B;i} + (\frac{\partial L}{\partial V^A{}_{B;k;j}}.V^A{}_{B;i})_{;k} - g^j_i.L_m \ , \qquad (66)$$

$${}_vK_i{}^j = (S_{Cm}{}^{An}.V^C{}_B - S_{B\overline{m}}{}^{D\overline{n}}.V^A{}_D).\frac{\partial L}{\partial V^A{}_{B;k;j}}.R^m{}_{nik} + \frac{\partial L}{\partial V^A{}_{B;k;j}}.V^A{}_{B;l}.T_{ik}{}^l \ , \qquad (67)$$

$$T_{ij}{}^k = -T_{ji}{}^k = \Gamma^k_{ji} - \Gamma^k_{ij} - C_{ij}{}^k \text{ (in a non-co-ordinate basis)}, \quad T_{ij}{}^k = \Gamma^k_{ji} - \Gamma^k_{ij} \text{ (in a co-ordinate basis)}, \qquad (68)$$

$${}_v\overline{W}_i{}^{jk}{}_k = g^l_k.{}_v\overline{W}_i{}^{jk}{}_l = g^l_k.g_{\overline{im}}.{}_v\overline{W}^{mjk}{}_l \ , \qquad (69)$$

$${}_v\overline{W}^{mjk}{}_l = {}_{vs}\overline{V}_n{}^{jm}{}_l.g^{nk} - {}_{vs}\overline{V}_n{}^{km}{}_l.g^{nj} - {}_{va}\overline{V}_n{}^{jk}{}_l.g^{nm} = -{}_v\overline{W}^{mkj}{}_l \ . \qquad (70)$$

3. Variational energy-momentum tensor of Euler-Lagrange ${}_{mv}\overline{Q}_i{}^j$ for $V^A{}_B$

$${}_{mv}\overline{Q}_i{}^j = (S_{B\overline{i}}{}^{D\underline{j}}.V^A{}_D - S_{Ci}{}^{Aj}.V^C{}_B).\frac{\delta_v L}{\delta V^A{}_B} \ . \qquad (71)$$

4. Volume force density ${}_{mv}\overline{F}_i$ for $V^A{}_B$

$${}_{mv}\overline{F}_i = \frac{\delta_v L}{\delta V^A{}_B}.V^A{}_{B;i} + {}_vW_i \ , \qquad (72)$$

$${}_vW_i = {}_vS_i - {}_vS_k{}^j.T_{ij}{}^k + g^j_{i;j}.L_m \ , \qquad (73)$$

$${}_vS_i = \overline{W}_i{}^{jk}{}_{k;j} + \overline{V}_m{}^{kj}.R^m{}_{jik} - \overline{Q}_m{}^{kj}.(g^m_j{}_{;k;i} - g^m_j{}_{;i;k} + g^m_{j;n}.T_{ik}{}^n) \ , \qquad (74)$$

$${}_v\overline{Q}_r{}^{kj} = S_{Br}{}^{Dj}.[\frac{\partial L}{\partial V^A{}_{B;k}}.V^A{}_D + (\frac{\partial L}{\partial V^A{}_{B;k;m}} + \frac{\partial L}{\partial V^A{}_{B;m;k}}).V^A{}_{D;m} - (\frac{\partial L}{\partial V^A{}_{D;k;m}}.V^A{}_D)_{;m}] \ , \qquad (75)$$

$${}_vS_k{}^j = [\frac{\partial L_m}{\partial V^A{}_{B;j}} - (\frac{\partial L_m}{\partial V^A{}_{B;j;l}})_{;l}].V^A{}_{B;k} + \frac{\partial L_m}{\partial V^A{}_{B;l;j}}.V^A{}_{B;l;k} \ . \qquad (76)$$

5. Geodesic and auto-parallel equations in $\overline{V}_n$-spaces

The geodesic and auto-parallel equations for the general case of $(\overline{L}_n, g)$-spaces are considered in [12]. In this section the main results will be specialized for the case of $\overline{V}_n$-spaces.

By means of the line-element ds the length s of a curve in a manifold M can be found between point p_1 and point p_2 in M with the corresponding co-ordinates x^k_1 and x^k_2 $(k = 1, \dots, n)$

$$s = \int_{p_1}^{p_2} ds + s_0 \ , \qquad s_0 = const. \qquad (77)$$

The extremum of the length s is to be determined using the standard variational method with variation of the co-ordinates. From the variation of $ds^2 = g_{\overline{ij}}dx^i dx^j$, where $dx^i = d^i$, i.e. dx^i are considered as components of the ordinary differential.

$$\delta(ds^2) = 2ds.\delta(ds) = (g_{\overline{ij}})_{,k}.\delta x^k.dx^i dx^j + 2g_{\overline{ij}}.dx^i.\delta(dx^j) \ , \qquad (78)$$

$$2\delta(ds) = (g_{\overline{ij}})_{,k}.\delta x^k.\frac{dx^i}{ds}.dx^j + 2g_{\overline{ij}}.\frac{dx^i}{ds}.\delta(dx^j) \ ,$$

from where (under the commutation condition $\delta(dx^i) = d(\delta x^i)$) the variation of s follows [13] in the form

$$\delta\int_{p_1}^{p_2} ds = \int_{p_1}^{p_2} \delta(ds) = g_{\overline{ij}}.\frac{dx^i}{ds}.\delta x^j \,|^{p_2}_{p_1} + \frac{1}{2}\int_{p_1}^{p_2} [(g_{\overline{ij}})_{,k}.\frac{dx^i}{ds}.\frac{dx^j}{ds} - 2(g_{\overline{ik}})_{,j}.\frac{dx^j}{ds}.\frac{dx^i}{ds} - 2g_{\overline{ik}}.\frac{d^2x^i}{ds^2}].\delta x^k.ds = 0 \ .$$

Under the boundary conditions $\delta x^i \,|_{p_1} = 0$ and $\delta x^i \,|_{p_2} = 0$ and for arbitrary δx^k the equation follows

$$g_{\overline{ik}}.\frac{d^2x^i}{ds^2} - \frac{1}{2}[(g_{\overline{ij}})_{,k} - (g_{\overline{ik}})_{,j} - (g_{\overline{jk}})_{,i}].\frac{dx^i}{ds}.\frac{dx^j}{ds} = 0 \ . \qquad (79)$$

After multiplication with g^{kl} and summation over k the last equation will have the form

$$\frac{d^2x^i}{ds^2} + \frac{1}{2}.g^{ik}.[(g_{\overline{jk}})_{,l} + (g_{\overline{lk}})_{,j} - (g_{\overline{jl}})_{,k}].\frac{dx^j}{ds}.\frac{dx^l}{ds} = 0 \ . \qquad (80)$$

If the abbreviation

$$\overline{\{^{\,i}_{jl}\}} = \frac{1}{2}.g^{ik}.[(g_{\overline{jk}})_{,l} + (g_{\overline{lk}})_{,j} - (g_{\overline{jl}})_{,k}] \ , \qquad (81)$$

is introduced, where $\overline{\{^{\,i}_{jl}\}}$ are called *generalized Christoffel symbols of second kind in* $\overline{V}_n$-space, then the equation takes the form

$$\frac{d^2x^i}{ds^2} + \overline{\{^{\,i}_{jl}\}}.\frac{dx^j}{ds}.\frac{dx^l}{ds} = 0 \qquad (82)$$

and is called *equation of geodesic (geodesic equation)* in a $\overline{V}_n$-space. If we use the abbreviation

$$G^i = \overline{\{^{\,i}_{jk}\}}.\frac{dx^j}{ds}.\frac{dx^k}{ds} \ , \qquad (83)$$

the geodesic equation will take the form

$$\frac{d^2x^i}{ds^2} + G^i = 0 \; . \tag{84}$$

On the other side, the equation of auto-parallel transport of the contravariant vector field $u = d/ds = (dx^i/ds)\partial_i = u^i\partial_i$ along the curve with parameter (length) s is equation of the type $\nabla_u u = 0$ and in a co-ordinate basis will have the form

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{kj}.\frac{dx^k}{ds}.\frac{dx^j}{ds} = 0 \; . \tag{85}$$

or the form

$$\frac{d^2x^i}{ds^2} + \Gamma^i = 0 \; , \tag{86}$$

where

$$\Gamma^i = \Gamma^i_{kj}.\frac{dx^k}{ds}.\frac{dx^j}{ds} \; . \tag{87}$$

The components Γ^i_{kj} of the contravariant affine connection Γ can be represented by means of the generalized Christoffel symbols of second kind for Γ and other additional terms in the form

$$\Gamma^i_{kj} = \overline{\{^{\,i}_{kj}\}} - \overline{S}_{kj}{}^i \; , \tag{88}$$

where

$$\overline{S}_{kj}{}^i = \frac{1}{2}.g^{il}.(g_{\overline{mk}}.T_{jl}{}^m + g_{\overline{mj}}.T_{kl}{}^m + g_{\overline{ml}}.T_{kj}{}^m) \tag{89}$$

are the components of the generalized contorsion tensor $\overline{S}$. By the use of the explicit form of $T_{jk}{}^i : T_{jk}{}^i = g^i_k.\varphi_{,j} - g^i_j.\varphi_{,k}$, they take the form

$$\overline{S}_{ij}{}^k = g^k_j.\varphi_{,i} - g^{kl}.g_{\overline{ij}}.\varphi_{,l} \; . \tag{90}$$

By means of the representation of Γ^i_{kj} the equation of auto-parallel transport can be written in the form

$$\frac{d^2x^i}{ds^2} + \overline{\{^{\,i}_{kj}\}}.\frac{dx^k}{ds}.\frac{dx^j}{ds} - \overline{S}_{kj}{}^i.\frac{dx^k}{ds}.\frac{dx^j}{ds} = 0 \; , \tag{91}$$

or in the form

$$\frac{d^2x^i}{ds^2} + G^i = \overline{T}^i \; , \tag{92}$$

where

$$\overline{T}^i = \overline{S}_{kj}{}^i.\frac{dx^k}{ds}.\frac{dx^j}{ds} \; , \quad \Gamma^i = G^i - \overline{T}^i \; . \tag{93}$$

The difference between the geodesic and auto-parallel equation is obvious. $\overline{T} = \overline{T}^i.\partial_i$ is a contravariant vector field. Since only the symmetric part of $\overline{S}_{kj}{}^i$ has to be taken into account in the term $\overline{T}^i$, $\overline{T}^i$ will have the explicit form

$$\overline{T}^i = \varphi_{,l}.(g^i_k.g^l_j - g^{il}.g_{\overline{jk}}).\frac{dx^j}{ds}.\frac{dx^k}{ds} \; , \tag{94}$$

and the auto-parallel equation will have on its right side a term describing a force induced by the gradient of the scalar field φ.

Therefore, the action of the contraction operator S induces an additional force, related to the contravariant torsion tensor T and different from zero for the parallel transport of contravariant vector fields.

In the special case, when the contravariant connection Γ is chosen to be zero on the curve $x^k(s)$, i.e. $\Gamma^i_{jk}(x^l(s)) = 0$, $(\Gamma^i(x^l(s)) = 0)$, then the geodesic equation will have the form

$$\frac{d^2x^i}{ds^2} = -\overline{T}^i \; , \tag{95}$$

and the auto-parallel equation will take the form

$$\frac{d^2x^i}{ds^2} = 0 \; . \tag{96}$$

The last equation can be interpreted as an equation for the trajectory of a free moving particle in contrast to the geodesic equation.

If the relation between the contravariant affine connection Γ and the covariant affine connection P is used in the form $\Gamma^i_{jk} + P^i_{jk} = g^i_{j;k}$, then the geodesic equation can be written in the form

$$\frac{d^2x^i}{ds^2} - P^i_{jk}.\frac{dx^j}{ds}.\frac{dx^k}{ds} + g^i_{j;k}.\frac{dx^j}{ds}.\frac{dx^k}{ds} + \overline{T}^i = 0 \; . \tag{97}$$

For the auto-parallel equation it follows

$$\frac{d^2x^i}{ds^2} - P^i_{jk}.\frac{dx^j}{ds}.\frac{dx^k}{ds} + g^i_{j;k}.\frac{dx^j}{ds}.\frac{dx^k}{ds} = 0 \; . \tag{98}$$

Now, if one chooses in analogy to the case of $\Gamma(x^k(s)) = 0$ the vanishing of P on the curve $x^k(s)$, i.e. $P^i_{jk}(x^l(s)) = 0$, then the geodesic equation and the auto-parallel equation will have a different form from that in the case of $\Gamma(s) = 0$, which will depend on the explicit form of $g^i_{j;k} = \varphi_{,k}.g^i_j$

$$\frac{d^2x^i}{ds^2} + g^i_{j;k}.\frac{dx^j}{ds}\frac{dx^k}{ds} + \overline{T}^i = 0 \text{ (geodesic equation)}, \tag{99}$$

$$\frac{d^2x^i}{ds^2} + g^i_{j;k}.\frac{dx^j}{ds}.\frac{dx^k}{ds} = 0 \text{ (auto-parallel equation)}. \tag{100}$$

When $P^i_{jk}(x^l(s)) = 0$ (and therefore $P^i_{\overline{jk}} = 0$), then $g^i_{j;k} = f_l{}^i.f^l{}_{j,k}$, and the geodesic equation and the auto-parallel equation can be written in the form

$$\frac{d^2x^i}{ds^2} + \frac{d\varphi}{ds}.\frac{dx^i}{ds} + \overline{T}^i = 0 \text{ (geodesic equation)}, \tag{101}$$

$$\frac{d^2x^i}{ds^2} + \frac{d\varphi}{ds}.\frac{dx^i}{ds} = 0 \text{ (auto-parallel equation)}. \tag{102}$$

The structure of the generalized Christoffel symbols of second kind $\overline{\{^{\,i}_{jk}\}}$ for the case of $\overline{V}_n$-spaces shows that the functions $f^i{}_j(x^k) = e^{\varphi(x^k)}.g^i_j$ have to be taken into account in the consideration and applications of the geodesic equation. The same is also valid for the auto-parallel equation. It is obvious that in the case where $P^i_{jk}(x^l(s)) = 0$ the auto-parallel equation can not be interpreted as an equation of the trajectory of a free moving particle (if the affine parameter s is not changed) in contrast to the case, when $\Gamma^i_{jk}(x^l(s)) = 0$. Therefore, in $\overline{V}_n$-spaces [as in the general case of $(\overline{L}_n, g)$-spaces] *the equivalence principle will be valid only for the case of vanishing the contravariant affine connection in the auto-parallel equation and it will be not valid in all other cases without changing the parameter s of the considered trajectory.*

6. Einstein's and geodesic equations in $\overline{V}_4$- and V_4-spaces

6.1. Einstein's equations. The relations between the Einstein equations in $\overline{V}_n$- and these in V_n-spaces ($n = 4$) can be found on the ground of the relations between the main structure of both types of spaces. In V_n-spaces the contravaraint affine connection $\widetilde{\Gamma}$ is connected to the covariant affine connection $\widetilde{P}$ on the basis of the action of the contraction operator $C : C(dx^i, \partial_j) = dx^i(\partial_j) = g^i_j$ and considered as a symmetric (Levi-Civita, Christoffel) connection

$$\widetilde{\Gamma}^i_{jk} = -\widetilde{P}^i_{jk} = \{^i_{jk}\} = \{^i_{kj}\} = \frac{1}{2} . g^{im} . (g_{mk,j} + g_{jm,k} - g_{jk,m}) \ , \qquad (103)$$

where $\{^i_{jk}\}$ are the Christoffel symbols of second kind in V_n-spaces. From (1) and (2) there follows that

$$\begin{aligned} P^i_{jk} &= e^{2\varphi} . \widetilde{P}^i_{jk} = -e^{2\varphi} . \widetilde{\Gamma}^i_{jk} = -e^{2\varphi} . \{^i_{jk}\} \ , \\ \Gamma^i_{jk} &= e^{2\varphi} . \widetilde{\Gamma}^i_{jk} + \varphi_{,k} . g^i_j = e^{2\varphi} . \{^i_{jk}\} + \varphi_{,k} . g^i_j \ . \end{aligned} \qquad (104)$$

The Riemann (curvature) tensor $\widetilde{R}^l{}_{ijk}$ in V_n-spaces is determined by means of $\{^i_{jk}\}$ in the form

$$\begin{aligned} \widetilde{R}^l{}_{ijk} &= \widetilde{\Gamma}^l_{ik,j} - \widetilde{\Gamma}^l_{ij,k} + \widetilde{\Gamma}^l_{mj} . \widetilde{\Gamma}^m_{ik} - \widetilde{\Gamma}^l_{mk} . \widetilde{\Gamma}^m_{ij} = \\ &= \{^l_{ik}\}_{,j} - \{^l_{ij}\}_{,k} + \{^l_{mj}\} . \{^m_{ik}\} - \{^l_{mk}\} . \{^m_{ij}\} = -\widetilde{P}^l{}_{ijk} \ . \end{aligned} \qquad (105)$$

The Riemann (curvature) tensor $R^l{}_{ijk}$ in $\overline{V}_n$-spaces can be expressed by means of the Riemann (curvature) tensor in V_n-spaces through the relation

$$R^l{}_{ijk} = e^{2\varphi} . (\widetilde{R}^l{}_{ijk} + Q^l{}_{ijk}) \ , \qquad Q^l{}_{ijk} = e^{-2\varphi} . R^l{}_{ijk} - \widetilde{R}^l{}_{ijk} \ , \qquad (106)$$

where

$$Q^l{}_{ijk} = (e^{2\varphi} - 1) . (\{^l_{mj}\} . \{^m_{ik}\} - \{^l_{mk}\} . \{^m_{ij}\}) + 2 . \varphi_{,m} . (g^m_j . \{^l_{ik}\} - g^m_k . \{^l_{ij}\}) \ . \qquad (107)$$

$Q^l{}_{ijk}$ are components of a tensor field. They can vanish at a point or on a trajectory in V_n, where the affine connection $\widetilde{\Gamma}(x^k(s))$ is chosen to be zero. In this case $R^l{}_{ijk} = e^{2\varphi} . \widetilde{R}^l{}_{ijk}$. On the other side, $\overline{V}_n$ and V_n could not simultaneously be flat spaces ($R^l{}_{ijk} = 0$, $\widetilde{R}^l{}_{ijk} = 0$) if $Q^l{}_{ijk} \neq 0$.

If we introduce the abbreviations for the case of V_n-spaces:

$$\widetilde{R}^i{}_j = g^{ik} . \widetilde{R}_{kj} = g^{ik} . g^l_m . \widetilde{R}^m{}_{kjl} \ , \qquad \widetilde{R} = g^{ij} . g^k_l . \widetilde{R}^l{}_{ijk} \ , \qquad (108)$$

$$Q^i{}_j = g^{ik} . Q_{kj} = g^{ik} . g^l_m . Q^m{}_{kjl} \ , \qquad Q = g^{ij} . g^k_l . Q^l{}_{ijk} \ . \qquad (109)$$

$$\widetilde{G}^i{}_j = \widetilde{R}^i{}_j - \frac{1}{2} . \widetilde{R} . g^i_j \ \text{(Einstein's tensor in } V_n\text{-spaces)}, \qquad (110)$$

then the Einstein tensor $G^i{}_j$ in $\overline{V}_n$-spaces can be expressed by means of the Einstein tensor in V_n-spaces

$$G^i{}_j = e^{4\varphi} . (\widetilde{G}^i{}_j + {}_qG^i{}_j) \ , \qquad {}_qG^i{}_j = Q^i{}_j - \frac{1}{2} . g^i_j . Q \ . \qquad (111)$$

Therefore, the Einstein equations in $\overline{V}_n$-spaces ($n = 4$) can be written in terms of the Einstein equations in V_n-spaces as

$$\widetilde{G}^i{}_j + \frac{\lambda}{2} . e^{-4\varphi} . g^i_j = -\frac{\alpha . \kappa_0}{2} . e^{-4\varphi} . {}_{mgsh}T_j{}^i - {}_qG^i{}_j \ . \qquad (112)$$

A comparison with the Einstein equations in V_n-spaces ($n = 4$)

$$\widetilde{G}^i{}_j + \frac{\lambda}{2} . g^i_j = -\frac{\alpha . \kappa_0}{2} . {}_{mgsh}T_j{}^i \qquad (113)$$

shows that the consideration of the last equations in $\overline{V}_4$-spaces induces an additional term ${}_qG^i{}_j$ on the right side of the equations and an additional factor [depending on the scalar field $\varphi(x^k)$] to the fundamental constants κ_0 and λ depending on the scalar field $\varphi(x^k)$. If we now introduce the functions

$$\widetilde{\lambda}(x^k) = \lambda . e^{-4\varphi(x^k)} \ , \qquad \widetilde{\kappa}_0(x^k) = \kappa_0 . e^{-4\varphi(x^k)} \ , \qquad (114)$$

then (112) will take the form

$$\widetilde{G}^i{}_j + \frac{\widetilde{\lambda}}{2} . g^i_j = -\frac{\alpha . \widetilde{\kappa}_0}{2} . {}_{mgsh}T_j{}^i - {}_qG^i{}_j \qquad (115)$$

and we will have Einstein's equations in V_n-spaces ($n = 4$) with cosmological and gravitational functions $\widetilde{\lambda}$ and $\widetilde{\kappa}_0$ instead of the cosmological and gravitational constants λ and κ_0 and with an additional gravitational source induced by an invariant function [the scalar field $\varphi(x^k)$].

6.2. Geodesic equation. The geodesic and auto-parallel equations are identical in a co-ordinate basis in V_4-spaces. The geodesic equation can be written in the form

$$\frac{d^2x^i}{d\overline{s}^2} + \widetilde{\Gamma}^i_{jk} . \frac{dx^j}{d\overline{s}} . \frac{dx^k}{d\overline{s}} = \frac{d^2x^i}{d\overline{s}^2} + \{^i_{jk}\} . \frac{dx^j}{d\overline{s}} . \frac{dx^k}{d\overline{s}} = 0 \ . \qquad (116)$$

In $\overline{V}_n$-spaces the square $ds^2 = g_{\overline{ij}} . dx^i dx^j$ of the line element ds is conformal to $d\overline{s}^2 = g_{ij} . dx^i dx^j$ in V_n-spaces

$$ds^2 = e^{2\varphi} . d\overline{s}^2 . \qquad (117)$$

Both types of equations (geodesic and auto-parallel) in $\overline{V}_n$-spaces are different

$$\frac{d^2x^i}{ds^2} + \overline{\{^i_{jk}\}} . \frac{dx^j}{ds} . \frac{dx^k}{ds} = 0 \ \text{(geodesic equation)}, \qquad (118)$$

$$\frac{d^2x^i}{ds^2} + \overline{\{^{\,i}_{kj}\}}.\frac{dx^k}{ds}.\frac{dx^j}{ds} - \overline{S}_{kj}{}^{i}.\frac{dx^k}{ds}.\frac{dx^j}{ds} = 0 \, , \tag{119}$$

$$\text{or } \frac{d^2x^i}{ds^2} + \Gamma^i_{jk}.\frac{dx^j}{ds}.\frac{dx^k}{ds} = 0 \text{ (auto-paralell equation).} \tag{120}$$

By means of the relation between the affine connections in $\overline{V}_n$- and V_n-spaces the geodesic and the autoparallel equations in $\overline{V}_n$-spaces can be written in terms of the affine connection in V_n-spaces and the scalar field φ [s. (82) ÷ (94)]

$$\frac{d^2x^i}{d\widetilde{s}^2} + e^{2\varphi}.\widetilde{\Gamma}^{\,i}_{jk}.\frac{dx^j}{d\widetilde{s}}.\frac{dx^k}{d\widetilde{s}} = e.\varphi_{,m}.e^{2\varphi}.g^{im} - 2\frac{d\varphi}{d\widetilde{s}}.\frac{dx^i}{d\widetilde{s}} \tag{121}$$

(geodesic equation),

$$e = g_{\overline{jk}}.\frac{dx^j}{ds}.\frac{dx^k}{ds} = g(u,u) = e^{2\varphi}.g_{jk}.\frac{dx^j}{ds}.\frac{dx^k}{ds} = g_{jk}.\frac{dx^j}{d\widetilde{s}}.\frac{dx^k}{d\widetilde{s}} = \widetilde{e} \, ,$$

$$\frac{d^2x^i}{d\widetilde{s}^2} + e^{2\varphi}.\widetilde{\Gamma}^{\,i}_{jk}.\frac{dx^j}{d\widetilde{s}}.\frac{dx^k}{d\widetilde{s}} = -\frac{d\varphi}{d\widetilde{s}}.\frac{dx^i}{d\widetilde{s}} \tag{122}$$

(auto-parallel equation).

The geodesic and auto-parallel equations differ from the geodesic equation in V_n-spaces. The additional terms are induced by the scalar field φ. If we chose $\widetilde{\Gamma}^{\,i}_{jk}\,(x^k(s)) = 0$, then both equations described the motion of a particle under different forces caused by the scalar field φ

$$\frac{d^2x^i}{d\widetilde{s}^2} = e.\varphi_{,m}.e^{2\varphi}.g^{im} - 2\frac{d\varphi}{d\widetilde{s}}.\frac{dx^i}{d\widetilde{s}} \text{ (geodesic equation),} \tag{123}$$

$$\frac{d^2x^i}{d\widetilde{s}^2} = -\frac{d\varphi}{d\widetilde{s}}.\frac{dx^i}{d\widetilde{s}} \text{ (auto-parallel equation),} \tag{124}$$

and are not equations of a free moving particle in a V_n-space.

7. Conclusions

In the present paper the main structures of Einstein's theory of gravitation are considered over (pseudo) Riemannian spaces with different (not only by sign) contravariant and covariant affine connections. The covariant affine connection is determined as the common symmetric (Levi-Civita) connection and the metric is the common Riemannian metric. The contravariant affine connection induced by the covariant affine connection and the action of the contraction operator S appears in a co-ordinate basis as a non-symmetric affine connection with torsion tensor induced by the existing scalar field (invariant function) φ. This scalar field changes the whole structure of the Einstein field equations although in their form they are analogous to these in the common (pseudo) Riemannian spaces with one affine connection and Riemannian metric. The difference between Einstein's equations in the spaces with one and the spaces with two affine connections is shown in explicit form by expressing both types of equations in a (pseudo) Riemannian space with one affine connection. In such a space the geodesic and auto-parallel equations from a space with two connections do not appear as equations of free moving particles. Additional terms induced by the scalar field cause the existence of forces due to the torsion tensor field of the contravariant affine connection in the space with two connections. The presented model for describing the gravitational interaction appears as a model lying between the models in Einstein-Cartan spaces and these in Riemannian spaces. Further considerations are necessary for finding out the range of vitality of the considered in this chapter model.

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