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S.Manoff*, B.Dimitrov

$(\overline{L}_n, g)$-SPACES.
ORDINARY AND TENSOR DIFFERENTIALS

*Permanent address: Institute for Nuclear Research and Nuclear Energy, Department of Theoretical Physics, Blvd. Tzarigradsko Chaussee 72, 1784 Sofia, Bulgaria

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1. -Ordinary differential, covariant and Lie differential

1.1. Ordinary differential d as a contravariant vector field. Let us recall some well known facts about the ordinary (common) differential.

The co-ordinate differntials $\{dx^i , i = 1, ..., n , \dim M = n\}$ of the co-ordinates $\{x^i\}$ in a neighborhood U of $x \in M$ are considered as covariant basic vector fields in $T^*(U \subset M)$. They define the s.c. covariant co-ordinate basis at every point $x \in U$. On the other side, the co-ordinate differentials dx^i can be considered as components of a contravariant vector field $d = dx^i.\partial_i$, which is called ordinary differential given in a co-ordinate basis. The reason for the last interpretation is the following:

Let a co-ordinate transformation in M be given of the type

$$\overline{x}^i = x^i + \varepsilon.\xi^i(x^k) = g^i_k.x^k + \varepsilon.\xi^i(x^k) = \overline{x}^i(x^k) = x^{i'}(x^k) \; , \quad | \varepsilon | \ll 1 \; , \tag{1}$$

where ε is an infinitesimal parameter and ξ^i are components of a contravariant vector field ξ in a co-ordinate basis ($\xi = \xi^i.\partial_i$).

The difference between the new co-ordinates $\overline{x}^i$ and the old co-ordinates x^i for $\varepsilon \to 0$ defines the co-ordinate differential $dx^i(x^k)$ at the point $x \in M$ with co-ordinates x^k

$$dx^i(x^k) = \lim_{\varepsilon \to 0} \frac{\overline{x}^i(x^k) - x^i}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}.\varepsilon.\xi^i(x^k) = \xi^i(x^k) \; . \tag{2}$$

The co-ordinate differentials dx^i appear in this case as components of a contravariant vector field ξ, inducing an infinitesimal co-ordinate transformation of the type $\overline{x}^i = x^i + dx^i(x^k)$.

Remark 1. *The possibility of defining co-ordinate differentials as components of a contravariant vector field is considered in a different variant by [1].*

The notion of ordinary differential operator d can be introduced on the basis of the possibility for considering the covariant basic vector fields as components of a contravariant vector field [2].

Definition 1. *Ordinary differential operator. The contravariant vector field d*

$$d = dx^i.\partial_i = e^\alpha.e_\alpha = d^i.\partial_i = d^\alpha.e_\alpha \; , \tag{3}$$

is called ordinary differential operator.

The basic vectors in $T^*(M)$ (dx^i and e^α) appear as components of the ordinary differential d in a co-ordinate or non-co-ordinate basis:

$$d: \; f \Rightarrow df = dx^i.\partial_i f = e^\alpha.e_\alpha f \; , \qquad f \in C^r(M) \; , r \geq 1 \; . \tag{4}$$

The action of the ordinary differential operator on a function f is called *differentiation*. The result df of the action of d on f is called ordinary (or total) differential of the function f.

The properties of the ordinary differential operator d are determined by the properties of the contravariant vector fields and by the peculiarities of its construction:

(a) Linear differential operator, acting on functions over a manifold M

$$d(\alpha.f+\beta.g)=\alpha.df+\beta.dg\ ,\qquad \alpha,\beta\in R\ (\text{or } C)\ ,$$
$$d(f.g)=(df).g+f.(dg)\ ,\qquad f,g\in C^r(M)\ ,\ r\geq 1\ .$$

(b) The action on a function f can be given in co-ordinate or non-co-ordinate basis

$$d:\ f\Rightarrow df=dx^i.\partial_i f=e^\alpha.e_\alpha f\ ,$$
$$f\in C^r(M)\ ,\ r\geq 1\ .$$

Remark 2. *df in this case is interpreted as a function over M with values in $T^*(M)$, i.e. df is a covariant vector field (Pfaffian form, one form)*

$$df=(\partial_i f).dx^i=(e_\alpha f).e^\alpha\ .$$

Here $\partial_i f$ and $e_\alpha f$ are components of the covariant vector field df in a co-ordinate and in non-co-ordinate basis.

(c) df is form-invariant under the changing of the different types of bases

$$df=dx^i.\partial_i f=A_\alpha{}^i.A_i{}^\beta.e^\alpha.e_\beta f=g^\beta_\alpha.e^\alpha.e_\beta f=e^\alpha.e_\alpha f\ ,$$
$$A_\alpha{}^i.A_i{}^\beta=g^\beta_\alpha\ .$$

(d) dx^i is co-ordinate differential of the co-ordinate x^i at point $x\in M$.

The co-ordinate differentials $\{dx^i\}_x$ can be interpreted in two different ways:

1. $\{dx^i\}_x$ are the components of the contravariant vector field d at $x\in M$;
2. $\{dx^i\}_x$ are a co-ordinate basis in the co-tangent space $T^*_x(M)$ at $x\in M$.

Remark 3. *Many authors [see for example [3], [4]] define df as a mapping by means of the condition*

$$df:\xi\rightarrow df(\xi)=\xi f\ ,\ \xi\in T(M)\ ,\quad f\in C^r(M)\ ,\ r\geq 1\ ,\ df\in T^*(M)\ ,$$
$$df(\xi)=\xi^i\partial_i f=(df)_i.dx^i(\xi^j\partial_j)=(df)_i.\xi^j.dx^i(\partial_j)=(df)_i.\xi^j.g^i_j=\xi^i(df)_i\ ,$$
$$(df)_i=\partial_i f\ .$$

In df the components of the contravariant vector field d are interpreted as basic vectors of the covariant vector field df.

From the definition for df it follows that for $f=x^k$

$$dx^k(\partial_i)=\partial_i x^k=g^k_i\ ,$$

which is in accordance with the action of the contraction operator C on the co-ordinate basic vector fields dx^k and ∂_i $[C(dx^k,\partial_i)=C(\partial_i,dx^k)=dx^k(\partial_i)=g^k_i]$, i.e. the contraction operator S is identified with the operator C.

Remark 4. *The definition of df, given by means of the expression $df(\xi)=\xi f$, when $S=C$, can be generalized for $S\neq C$ in the form*

$$\begin{aligned}
&df:\xi\rightarrow df(\xi)=\bar{\xi}f\ ,\qquad \xi\in T(M)\ ,\quad f\in C^r(M)\ ,\\
&\bar{\xi}f=\xi^{\bar{j}}.\partial_j f=\xi^{\bar{\alpha}}.e_\alpha f\ ,\qquad r\geq 1\ ,\quad df\in T^*(M)\ ,\\
&\xi^{\bar{j}}=f^j{}_k.\xi^k\ ,\qquad \xi^{\bar{\alpha}}=f^\alpha{}_\beta.\xi^\beta\ ,\\
&S(dx^i,\partial_j)=S(\partial_j,dx^i)=dx^i(\partial_j)=f^i{}_j\ .
\end{aligned}\tag{5}$$

Remark 5. *The components of the ordinary differential d in a co-ordinate basis are considered as constant functions, i.e. $(dx^i)_{,j}=0$.*

Action of the covariant differential operator on the ordinary differential

The action of the covariant differential operator on the ordinary differential is determined by its action on a contravariant vector field and by the peculiarity of the construction of d.

In a co-ordinate basis

$$\begin{aligned}
&\nabla_{\partial_j}d=d^i{}_{;j}.\partial_i=(dx^i)_{;j}.\partial_i\ ,\\
&(dx^i)_{;j}=\Gamma^i_{kj}.dx^k\ ,
\end{aligned}\tag{6}$$

where $(dx^i)_{;j}$ is the covariant derivative of the covariant differential dx^i, considered as a component of the contravariant vector field d along the contravariant vector field ∂_j. The covariant derivative $\nabla_\xi d$ will have, in a co-ordinate basis, the form

$$\nabla_\xi d=(dx^i)_{;j}.\xi^j.\partial_i=\Gamma^i_{kj}.dx^k.\xi^j.\partial_i\ .\tag{7}$$

Remark 6. *If the covariant differential operator ∇_ξ acts on a co-ordinate basic vector field dx^i (the other interpretation of dx^i), the result of its action is different from that when dx^i is considered as a component of d:*

$$\begin{aligned}
&\nabla_{\partial_j}dx^i=P^i_{kj}.dx^k\ ,\ \nabla_\xi dx^i=P^i_{kj}\xi^j.dx^k\ ,\\
&dx^i\ \text{- basic covariant co-ordinate vector field,}\\
&\nabla_{\partial_j}dx^i=\partial_j dx^i=0\ ,\\
&dx^i\ \text{- component of contravariant vector } d\ .
\end{aligned}\tag{8}$$

The differences in the action of ∇_ξ on dx^i in the cases of different interpretations of dx^i have to be taken into account when co-ordinate differentials are used, if additional conditions for identification of both the results of the action of ∇_ξ on d are not required, i.e. if the condition $P^i_{jk}=0$ is not required. On the other side, these conditions have to be in accordance with the result of the action of the Lie differential operator on dx^i.

In a non-co-ordinate basis

$$\begin{aligned}
&\nabla_{e_\gamma}d=\nabla_{e_\gamma}(e^\alpha.e_\alpha)=e^\alpha{}_{/\gamma}.e_\alpha=d^\alpha{}_{/\gamma}.e_\alpha\ ,\\
&e^\alpha{}_{/\gamma}=e_\gamma(e^\alpha)+\Gamma^\alpha_{\beta\gamma}.e^\beta\ .
\end{aligned}\tag{9}$$

Action of the Lie differential operator on the ordinary differential

The action of the Lie differential operator on the ordinary differential is determined by its action on a contravariant vector field and by the peculiarities of the ordinary differential.

In a co-ordinate basis

$$\pounds_{\partial_j} d = 0 \ , \quad \pounds_\xi d = -\xi^i{}_{,j} . dx^j . \partial_i = (\pounds_\xi d^i) . \partial_i \ , \qquad \pounds_{\partial_j} dx^i = 0 \ , \quad dx^i \text{ - component of the contravariant vector } d \ . \tag{10}$$

On the other side, the Lie derivative of dx^i as a covariant basic vector field is

$$\pounds_{\partial_j} dx^i = (P^i_{kj} + \Gamma^{\bar{i}}_{kj}) . dx^k \ . \tag{11}$$

The compatibility conditions for both interpretations (taking into account the compatibility condition also for the action of the covariant operator) are

$$P^i_{jk} = 0 \ , \ \Gamma^i_{jk} = 0 \ . \tag{12}$$

Therefore, the compatibility of both interpretations of the co-ordinate differential dx^i can be fulfilled only in the following cases:

(a) $P^i_{jk} = 0 \ , \ \Gamma^i_{jk} = 0$ at a given point $x \in M$.

(b) $P^i_{jk} = 0 \ , \ \Gamma^i_{jk} = 0$ on a given trajectory $x(\tau)$ in M, $\tau \in R$.

(c) $P^i_{jk} = 0 \ , \ \Gamma^i_{jk} = 0$ at every point $x \in M$, i.e. when $[R(\xi, u)]v = 0$, $\forall \xi, u, v \in T(M)$, $[R(\xi, u)]p = 0$, $\forall \xi, u \in T(M)$, $\forall p \in T^*(M)$.

Since P and Γ cannot vanish simultaneously in $(\overline{L}_n, g)$-spaces, these conditions can be fulfilled only in (L_n, g)-spaces, where $S = C$ and $P = -\Gamma$.

In the general case of differentiable manifolds with different (not only by sign) contravariant and covariant affine connections and metric the two interpretations of the co-ordinate differential have to be used separately and independently of each other without mixing the contents of the notion of the ordinary differential.

1.2. Covariant differential as a special case of the covariant differential operator. The covariant differential operator along the ordinary differential d defines the notion covariant differential

$$D := \text{covariant differential} \ , \qquad D = \nabla_d = dx^i \nabla_{\partial_i} = e^\alpha \nabla_{e_\alpha} \ . \tag{13}$$

The properties of the covariant differential D are determined by the properties of the covariant differential operator and by the construction of the ordinary differential d:

(a) Action on a function over the manifold M:

$$Df = \nabla_d f = df \ , \qquad f \in C^r(M) \ , \ r \geq 1 \ .$$

(b) Action on a contravariant vector field:

$$Dv = \nabla_d v = v^i{}_{;j} . dx^j . \partial_i = Dv^i . \partial_i = $$
$$= v^\alpha{}_{/\beta} . e^\beta . e_\alpha = Dv^\alpha . e_\alpha \ , \qquad v \in T(M) \ , \qquad Dv \in T(M) \ .$$

$Dv^i = v^i{}_{;j} . dx^j$ is called *covariant differential* of the components of the contravariant vector field v in a co-ordinate basis and $Dv^\alpha = v^\alpha{}_{/\beta} . e^\beta$ is called *covariant differential* of the components of the contravariant vector field v in a non-co-ordinate basis.

(c) Action on a covariant vector field:

$$Dp = \nabla_d p = p_{i;j} . d^j . dx^i = Dp_i . dx^i = $$
$$= p_{\alpha/\beta} . d^\beta . e^\alpha = Dp_\alpha . e^\alpha \ , \qquad p \in T^*(M) \ , \qquad Dp \in T^*(M) \ .$$

$Dp_i = p_{i;j} . d^j = p_{i;j} . dx^j$ is called *covariant differential* of the components of the covariant vector field p in a co-ordinate basis and $Dp_\alpha = p_{\alpha/\beta} . d^\beta = p_{\alpha/\beta} . e^\beta$ is called *covariant differential* of the components of the covariant vector field p in a non-co-ordinate basis.

(d) Action on a mixed tensor field:

$$DK = \nabla_d K = K^A{}_{B;j} . d^j . \partial_A \otimes dx^B = DK^A{}_B . \partial_A \otimes dx^B =$$
$$= K^A{}_{B/\alpha} . d^\alpha . e_A \otimes e^B = DK^A{}_B . e_A \otimes e^B \ ,$$
$$K \in \otimes^k{}_l(M) \ .$$

$DK^A{}_B = K^A{}_{B;j} . d^j = K^A{}_{B;j} . dx^j$ is called covariant differential of the components of the mixed tensor field K in a co-ordinate basis and $DK^A{}_B = K^A{}_{B/\alpha} . d^\alpha = K^A{}_{B/\alpha} . e^\alpha$ is called the covariant differential of the components of the mixed tensor field K in a non-co-ordinate basis.

Remark 7. *In the definitions of the covariant differential of components of vector and tensor fields dx^i and e^α are considered as components of the ordinary differential d in a co-ordinate and a non-co-ordinate basis. For this type of interpretation of dx^i and e^α we will use the designations d^i and d^α in contrast to the case, when dx^i and e^α are interpreted as covariant basic vector fields. To avoid ambiguity the interpretation of dx^i and e^α have to be explicitly given in every different case.*

(e) Action on the contravariant vector field d:

$$Dd = \nabla_d d = d^i{}_{;j} . d^j . \partial_i = Dd^i . \partial_i = d^\alpha{}_{/\beta} . d^\beta . e_\alpha = Dd^\alpha . e_\alpha \ ,$$
$$Dd^i = d^i{}_{;j} . d^j = \Gamma^i_{kj} . d^k . d^j \ , \qquad Dd^\alpha = d^\alpha{}_{/\beta} . d^\beta = (e_\beta d^\alpha + \Gamma^\alpha_{\gamma\beta} . d^\gamma) . d^\beta \ .$$

Remark 8. *The applications of the covariant differential allow different interpretations, if the covariant differential is considered not as a special case of the covariant differential operator ∇_ξ for $\xi = d$, but as a different-from-∇_d operator, which in its action on tensor fields changes their covariant rank with 1, i.e.*

$$D : K \Rightarrow DK \ , \ K \in \otimes^k{}_l(M) \ , \ DK \in \otimes^k{}_{l+1}(M) \ .$$
$$DK = K^A{}_{B;j} . dx^j \otimes \partial_A \otimes dx^B = K^A{}_{B/\alpha} . e^\alpha \otimes e_A \otimes e^B \ ,$$

(dx^i and e^α are interpreted as basic vector fields).

In the case, where the covariant differential D is considered as covariant differential operator ∇_d, its action does not change the covariant rank, i.e.

$$D = \nabla_d : K \Rightarrow DK = \nabla_d K \ , \qquad K \ , \ DK \in \otimes^k{}_l(M) \ .$$

This is due to the fact that the co-ordinate differentials in d are considered as components of the ordinary differential d and not as covariant basic vector fields.

1.3. Lie differential as a special case of the Lie differential operator. The Lie differential operator along the ordinary differential d defines the notion of the Lie differential

$$\pounds_d := \text{Lie differential} .$$

The properties of the Lie differential $\pounds_d$ are determined by the properties of the Lie differential operator and by the construction of d:

(a) Action on function over manifold M:

$\pounds_d f = df \; , \quad f \in C^r(M) \; , \quad r \geq 1 \; .$

(b) Action on a contravariant vector field:

$\pounds_d v = -\pounds_v d = [d, v] = (\pounds_d v^i).\partial_i = (\pounds_d v^\alpha).e_\alpha \; ,$

$\pounds_d v^i = v^i{}_{;j}.d^j = dv^i \; , \quad v \in T(M) \; ,$

$\pounds_d v^\alpha = (e_\beta v^\alpha).d^\beta - v^\beta.(e_\beta d^\alpha) - C_{\beta\gamma}{}^\alpha.v^\beta.d^\gamma \; .$

$\pounds_d v^i$ is called the *Lie differential* of the components of the contravariant vector field v in a co-ordinate basis.

(c) Action on a covariant vector field:

$\pounds_d p = (\pounds_d p_i).dx^i = (\pounds_d p_\alpha).e^\alpha \; , \quad p \in T^*(M) \; ,$

$\pounds_d p_i = p_{i,j}.d^j + p_j.(P^j_{ik} + \Gamma^{\bar{j}}_{\underline{ik}}).d^k \; ,$

$\pounds_d p_i$ is called the *Lie differential* of the components of the covariant vector field p in a co-ordinate basis

(d) Action on covariant basic vector fields dx^i and e^α:

$\pounds_d dx^i = (P^i_{jk} + \Gamma^{\bar{i}}_{\underline{jk}}).d^k.dx^j \; ,$

$\pounds_d e^\alpha = [e_{\overline{\beta}} d^{\overline{\alpha}} + (P^\alpha_{\beta\gamma} + \Gamma^{\overline{\alpha}}_{\underline{\beta\gamma}} + C_{\underline{\beta\gamma}}{}^{\overline{\alpha}}).d^\gamma].e^\beta \; .$

The introduction of the notion of the ordinary differential d as a contravariant vector field allows another way of introducing notions of the covariant differential and Lie differential. On the other side, the so-defined notion is different from the notion of the ordinary differential df of a function f, which has found many applications in the calculus of (exterior) differential forms.

2. Tensor differentials

2.1. Tensor differential as a mixed tensor field (ordinary tensor differential). In the previous section we have considered the ordinary differential as a contravariant vector field $d = dx^i.\partial_i = e^\alpha.e_\alpha$ acting on functions and tensor fields. Moreover, the co-ordinate differentials dx^i are considered as components of the ordinary differential in a co-ordinate basis. They are not interpreted as covariant co-ordinate basic vector fields but as constant increments dx^i of a function $f \in C^r(M)$:

$$df(x) = \partial_i f(x).dx^i , \qquad dx^i \in R^n , \qquad (dx^i)_{,k} := 0 \; ,$$
$$d = dx^i.\partial_i = e^\alpha.e_\alpha, dx^i , \qquad e^\alpha \in C^r(M) \; .$$

On the other side, one can construct on the analogy of the ordinary differential operator d a new differential operator $\overline{d}$. The only difference between both operators is that in the new operator $\overline{d}$ the co-ordinate differentials are considered as covariant co-ordinate basic vector fields:

Definition 2. *Tensor differential $\overline{d}$. The mixed tensor field*

$$\overline{d} = dx^i \otimes \partial_i = g^j_i.dx^i \otimes \partial_j = e^\alpha \otimes e_\alpha = g^\beta_\alpha.e^\alpha \otimes e_\beta \; , \qquad dx^i, e^\alpha \in T^*(M) \; .$$

is called *(ordinary) tensor differential.*

The tensor differential $\overline{d}$ appears as a mixed tensor field of second rank of type 2 $[\overline{d} \in \otimes_1{}^1(M)]$ in contrast to the Kronecker tensor field $Kr = g^j_i.\partial_j \otimes dx^i = g^\beta_\alpha.e_\alpha \otimes e^\beta$ which is a mixed tensor field of second rank of type 1 $[Kr \in \otimes^1{}_1(M)]$.

The properties of the tensor differential follow from its construction and from the properties of the contravariant and covariant basis vector fields:

(a) Action on a function

$$\overline{d} : f \to \overline{d}f \; , \qquad f \in C^r(M) \; ,$$
$$\overline{d}f = f_{,i}.dx^i \in T^*(M) \; , \qquad dx^i \in T^*(M) \; .$$

The tensor differential $\overline{d}$ has also the property

$$\overline{d}(\overline{d}f) = f_{(,i,j)}.dx^i.dx^j .$$

Proof:

$$\overline{d}(\overline{d}f) = (dx^i \otimes \partial_i)(f_{,j}.dx^j) = f_{,j,i}.dx^i \otimes dx^j = \tfrac{1}{2}.(f_{,i,j} + f_{,j,i}).dx^i \otimes dx^j =$$
$$= \tfrac{1}{2}.(f_{,i,j} + f_{,j,i}).\tfrac{1}{2}.(dx^i \otimes dx^j + dx^j \otimes dx^i) = f_{(,i,j)}.dx^i.dx^j \; ,$$

where

$$f_{(,i,j)} = \tfrac{1}{2}.(f_{,i,j} + f_{,j,i}) \; , \quad dx^i.dx^j = \tfrac{1}{2}.(dx^i \otimes dx^j + dx^j \otimes dx^i) \; ,$$
$$f \in C^r(M) \; , \quad r \geq 2 \; , \qquad f_{,i,j} = f_{,j,i} \; .$$

(b) Action on a tensor field

$$\overline{d} : K \to \overline{d}K \; , \qquad K \in \otimes^k{}_l(M) \; , \qquad \overline{d}K \in {}^A\otimes^k{}_{l+1}(M) \; ,$$

${}^A\otimes^k{}_{l+1}(M)$ is the linear (vector) space of affine tensor fields of rank $(k, l+1)$,

$$K = K^A{}_B.\partial_A \otimes dx^B = K^C{}_D.e_C \otimes e^D \; ,$$
$$\overline{d}K = (dx^i \otimes \partial_i)(K^A{}_B.\partial_A \otimes dx^B) =$$
$$= K^A{}_{B,i}.dx^i \otimes \partial_A \otimes dx^B = \overline{d}K^A{}_B \otimes \partial_A \otimes dx^B \; ,$$
$$\overline{d}K^A{}_B = K^A{}_{B,i}.dx^i \; .$$

$\overline{d}K^A{}_B$ are called tensor differentials of the components $K^A{}_B$ in a co-ordinate basis.

The tensor differential $\overline{d}K$ is a tensor field only with respect to constant (affine) co-ordinate transformations. $\overline{d}K^A{}_B$ transform with respect to the basic vector field dx^i as covariant tensor fields of rank 1 under affine co-ordinate transformations. $\overline{d}K^A{}_B$ appear with respect to the basic vector field dx^i as a set of covariant (affine) tensor fields of rank 1 in contrast to $dK^A{}_B$ appearing as a set of functions over M.

$\overline{d}K$ is a tensor field only under affine (linear) transformations of the co-ordinate x^k, i. e. $\overline{d}K$ is an affine tensor field of contravariant rank k and covariant rank l.

The action of $\overline{d}$ on the components $K^A{}_B$ of a tensor field K is similar in its form to the action of d on $K^A{}_B \in C^r(M)$. The difference between both actions is due to the different meanings of the co-ordinate differentials dx^i ($i = 1, .., n$):

$$\overline{d}K^A{}_B = K^A{}_{B,i}.dx^i \ , \quad dx^i \in T^*(M) \ ,$$
$$dK^A{}_B = K^A{}_{B,i}.dx^i \ , \quad dx^i \in C^r(M) \ , \ r \geq 1 \ .$$

In a non-co-ordinate basis

$$K = K^C{}_D.e_C \otimes e^D \ ,$$
$$\overline{d}K = (e^\alpha \otimes e_\alpha)(K^C{}_D.e_C \otimes e^D) =$$
$$= (e_\alpha K^C{}_D).e^\alpha \otimes e_C \otimes e^D = \overline{d}K^C{}_D \otimes e_C \otimes e^D \ ,$$
$$\overline{d}K^C{}_D = (e_\alpha K^C{}_D).e^\alpha \ .$$

$\overline{d}$ has also the property:

$$\overline{d}(\overline{d}K) = K^A{}_{B(,i,j)}.dx^j.dx^i \otimes \partial_A \otimes dx^B \ , \qquad K^A{}_{B,j,i} = K^A{}_{B,i,j} \ .$$

(c) Linear operator with respect to tensor fields (including functions)

$$\overline{d}(\alpha.K_1 + \beta.K_2) = \alpha.\overline{d}K_1 + \beta.\overline{d}K_2 \ ,$$
$$\alpha, \beta \in R \text{ (or } C), \quad K_i \in \otimes^k{}_l(M) \ , \quad k = 0, 1, ..., \quad l = 0, 1, ... \quad , i = 1, 2.$$

(d) Differential operator with respect to functions over M

$$\overline{d}(f.g) = \overline{d}f.g + f.\overline{d}g \ , \quad f, g \in C^r(M) \ , \quad r \geq 1 \ , \quad \overline{d}f, \overline{d}g \in T^*(M) \ .$$

Proof: $\overline{d}(f.g) = (dx^i \otimes \partial_i)(f.g) = dx^i \otimes \partial_i(f.g) = [\partial_i(f.g)].dx^i =$
$= (\partial_i f.g + f.\partial_i g).dx^i = (\partial_i f.g).dx^i + (f.\partial_i g).dx^i =$
$= (\partial_i f.dx^i).g + f.(\partial_i g.dx^i) = \overline{d}f.g + f.\overline{d}g$, where $(\partial_i f.g).dx^i = (\partial_i f.dx^i).g$, $dx^i.g = g.dx^i$.

(e) Differential operator with respect to tensor fields with rank > 1 not obeying the Leibniz rule

$$\overline{d}(Q \otimes S) = \overline{d}Q \otimes S + \overline{d}P_{Q \otimes S} \ , \qquad \overline{d}P_{Q \otimes S} = \overline{d}S^C{}_D \otimes Q \otimes \partial_C \otimes dx^D \ .$$

or

$$\overline{d}(Q \otimes S) = \overline{d}Q \otimes S + dx^i \otimes Q \otimes \partial_i S = \overline{d}Q \otimes S + e^\alpha \otimes Q \otimes e_\alpha S \ .$$

Proof:

$$\overline{d}(Q \otimes S) = \overline{d}(Q^A{}_B.\partial_A \otimes dx^B \otimes S^C{}_D.\partial_C \otimes dx^D) =$$
$$= (dx^i \otimes \partial_i)(Q^A{}_B.\partial_A \otimes dx^B \otimes S^C{}_D.\partial_C \otimes dx^D) =$$
$$= (Q^A{}_B.S^C{}_D)_{,i}.dx^i \otimes \partial_A \otimes dx^B \otimes \partial_C \otimes dx^D =$$
$$= (Q^A{}_{B,i}.S^C{}_D + Q^A{}_B.S^C{}_{D,i}).dx^i \otimes \partial_A \otimes dx^B \otimes \partial_C \otimes dx^D =$$
$$= (\overline{d}Q^A{}_B.S^C{}_D + Q^A{}_B.\overline{d}S^C{}_D) \otimes \partial_A \otimes dx^B \otimes \partial_C \otimes dx^D =$$
$$= \overline{d}Q^A{}_B \otimes \partial_A \otimes dx^B \otimes S^C{}_D.\partial_C \otimes dx^D + \overline{d}S^C{}_D \otimes Q^A{}_B.\partial_A \otimes dx^B \otimes \partial_C \otimes dx^D =$$
$$= \overline{d}Q \otimes S + \overline{d}S^C{}_D \otimes Q \otimes \partial_C \otimes dx^D \ , \quad Q \in \otimes^k{}_l(M) \ , \ S \in \otimes^m{}_n(M) \ .$$

The contraction operator S acting on $\overline{d}f$ and on a contravariant vector field ξ leads to the relations:

$$S(\overline{d}f, \xi) = \overline{d}f(\xi) = \overline{\xi}f \ .$$
$$\overline{d}f(\partial_j) = f^i{}_j.\partial_i f = \partial_{\overline{j}} f = f_{,i}.f^i{}_j = f_{,\overline{j}} \ .$$

Proof:

$$\overline{d}f(\xi) = \overline{d}f(\xi^j.\partial_j) = (\partial_i f.dx^i)(\xi^j.\partial_j) = \xi^j.\partial_i f.S(dx^i, \partial_j) = \xi^j.f^i{}_j.\partial_i f =$$
$$= \xi^{\overline{i}}.\partial_i f = \xi^j.\partial_{\overline{j}} f = \overline{\xi}f \ , \qquad \overline{\xi} = f^i{}_j.\xi^j.\partial_i = \xi^{\overline{i}}.\partial_i = \xi^j.\partial_{\overline{j}} \ , \quad \partial_{\overline{j}} = f^i{}_j.\partial_i \ .$$

Special case: $f = x^k$.

$$\overline{d}x^k = \partial_i x^k.dx^i = g^k_i.dx^i = dx^k.$$
$$S(\overline{d}x^k, \xi) = S(dx^k, \xi) = dx^k(\xi) = f^k{}_j.\xi^j = \xi^{\overline{k}}.$$

2.2. Covariant tensor differential. By the use of the covariant differential operator ∇_{∂_i} (instead of the partial differential operator ∂_i) we can construct the covariant tensor differential.

Definition 3. *Covariant tensor differential. The operator*

$$\overline{D} = dx^i \otimes \nabla_{\partial_i} = e^\alpha \otimes \nabla_{e_\alpha}$$

is called *covariant tensor differential.*

The properties of the covariant tensor differential $\overline{D}$ are determined by its construction and the properties of the covariant differential operator along a contravariant basic vector field:

(a) Action on a function

$$\overline{D} : f \to \overline{D}f \ , \quad \overline{D}f = \overline{d}f \ , \quad f \in C^r(M) \ , \quad r \geq 1 \ ,$$
$$\overline{D}f = (dx^i \otimes \nabla_{\partial_i})f = dx^i.\nabla_{\partial_i} f = \partial_i f.dx^i = \overline{d}f \ .$$

$$\overline{D}(\overline{D}f) = f_{;j;i}.dx^i \otimes dx^j \ , \qquad f_{;j} = f_{,j} \ .$$

Proof:

$$\overline{D}(\overline{D}f) = \overline{D}(\overline{d}f) = (dx^i \otimes \nabla_{\partial_i})(f_{,j}.dx^j) = f_{,j,i}.dx^i \otimes dx^j + P^k_{ji}.f_{,k}.dx^i \otimes dx^j =$$
$$= (f_{,j,i} + P^k_{ji}.f_{,k}).dx^i \otimes dx^j = f_{;j;i}.dx^i \otimes dx^j \ , \qquad f_{,j} = f_{;j} \ .$$

In a non-co-ordinate basis

$$\overline{D}(\overline{D}f) = (e_\alpha e_\beta f + P^\gamma_{\beta\alpha}.e_\gamma f).e^\alpha \otimes e^\beta = f_{/\beta/\alpha}.e^\alpha \otimes e^\beta \ , \qquad f_{/\beta} = e_\beta f \ .$$

(b) Action on a tensor field:

$$\overline{D}K = (dx^i \otimes \nabla_{\partial_i})K = dx^i \otimes \nabla_{\partial_i} K = dx^i \otimes K^A{}_{B;i}.\partial_A \otimes dx^B =$$
$$= K^A{}_{B;i}.dx^i \otimes \partial_A \otimes dx^B = \overline{D}K^A{}_B \otimes \partial_A \otimes dx^B \ ,$$
$$K = K^A{}_B.\partial_A \otimes dx^B \in \otimes^k{}_l(M) \ , \qquad \overline{D}K^A{}_B = K^A{}_{B;i}.dx^i \ ,$$
$$\nabla_{\partial_i} K = K^A{}_{B;i}.\partial_A \otimes dx^B \ , \ \nabla_{e_\alpha} K = K^C{}_{D/\alpha}.e_C \otimes e^D \ ; \quad \overline{D}K^C{}_D = K^C{}_{D/\alpha}.e^\alpha \ .$$

$$\overline{D}(\overline{D}K) = K^A{}_{B;i;j}.dx^j \otimes dx^i \otimes \partial_A \otimes dx^B \ .$$

$\overline{D}$ appears as an operator increasing the covariant rank of a tensor field with 1:

$$\overline{D} : K \to \overline{D}K \ , \qquad K \in \otimes^k{}_l(M) \ , \quad \overline{D}K \in \otimes^k{}_{l+1}(M) \ .$$

$\overline{D}K^A{}_B$ are called covariant tensor differentials of the components $K^A{}_B$ of the tensor field K in a co-ordinate basis.

(c) Linear operator with respect to tensor fields:

$$\overline{D}(\alpha.K_1 + \beta.K_2) = \alpha.\overline{D}K_1 + \beta.\overline{D}K_2 \ ,$$
$$K_i \in \otimes^k{}_l(M) \ , \quad i = 1,2 \ , \quad \alpha, \beta \in R \text{ (or } C) .$$

(d) Differential operator with respect to functions:

$$\overline{D}(f.g) = \overline{D}f.g + f.\overline{D}g = \overline{d}f.g + f.\overline{d}g \ , \qquad f, g \in C^r(M) \ .$$

(e) Differential operator not obeying the Leibniz rule with respect to tensor fields with rank > 0:

$$\overline{D}(Q \otimes S) = \overline{D}Q \otimes S + \overline{D}P_{Q \otimes S} \ ,$$
$$\overline{D}P_{Q \otimes S} = \overline{D}S^C{}_D \otimes Q \otimes \partial_C \otimes dx^D = dx^i \otimes Q \otimes \nabla_{\partial_i} S =$$
$$= \overline{D}S^C{}_D \otimes Q \otimes e_C \otimes e^D = e^\alpha \otimes Q \otimes \nabla_{e_\alpha} S \ ,$$
$$Q \in \otimes^k{}_l(M) \ , \quad S \in \otimes^m{}_r(M) \ .$$

Proof:

$$\overline{D}(Q \otimes S) = \overline{D}(Q^A{}_B.S^C{}_D.\partial_A \otimes dx^B \otimes \partial_C \otimes dx^D) =$$
$$= (Q^A{}_B.S^C{}_D)_{;i}.dx^i \otimes \partial_A \otimes dx^B \otimes \partial_C \otimes dx^D =$$
$$= (Q^A{}_{B;i}.S^C{}_D + Q^A{}_B.S^C{}_{D;i}).dx^i \otimes \partial_A \otimes dx^B \otimes \partial_C \otimes dx^D =$$
$$= Q^A{}_{B;i}.dx^i \otimes \partial_A \otimes dx^B \otimes S^C{}_D.\partial_C \otimes dx^D +$$
$$+ S^C{}_{D;i}.dx^i \otimes Q^A{}_B.\partial_A \otimes dx^B \otimes \partial_C \otimes dx^D =$$
$$= \overline{D}Q \otimes S + \overline{D}S^C{}_D \otimes Q \otimes \partial_C \otimes dx^D \ .$$

2.3. Lie tensor differential. By the use of the Lie differential operator $\pounds_{\partial_i}$ (instead of the covariant differential operator ∇_{∂_i}) we can construct the Lie tensor differential.

Definition 4. *Lie tensor differential. The operator*

$$\overline{\pounds} = dx^i \otimes \pounds_{\partial_i} = e^\alpha \otimes \pounds_{e_\alpha}$$

is called *Lie tensor differential.*

The properties of the Lie tensor differential are determined by its construction and the properties of the Lie differential operator along a contravariant basic vector field.

(a) Action on a function:

$$\overline{\pounds} : f \to \overline{\pounds}f \ , \quad f \in C^r(M) \ , \quad r \geq 1 \ , \quad \overline{\pounds}f = \overline{d}f \in T^*(M) \ ,$$
$$\overline{\pounds}f = (dx^i \otimes \pounds_{\partial_i})f = dx^i.\pounds_{\partial_i} f = dx^i.\partial_i f = f_{,i}.dx^i = \overline{d}f = \overline{D}f \ ,$$
$$\overline{\pounds}f = \overline{D}f = \overline{d}f \ \in T^*(M) \ .$$

$$\overline{\pounds}(\overline{\pounds}f) = [f_{,j,i} + (P^k_{ji} + \Gamma^k_{\underline{ji}}).f_{,k}].dx^i \otimes dx^j = (\pounds_{\partial_i} p_j).dx^i \otimes dx^j \ , \qquad p_j = f_{,j} \ .$$

Proof:

$$\overline{\pounds}(\overline{\pounds}f) = \overline{\pounds}(\overline{d}f) = (dx^i \otimes \pounds_{\partial_i})(f_{,j}.dx^j) = dx^i \otimes \pounds_{\partial_i}(f_{,j}.dx^j) =$$
$$= dx^i \otimes (f_{,j,i}.dx^j + f_{,j}.\pounds_{\partial_i} dx^j) = dx^i \otimes [f_{,j,i}.dx^j + f_{,k}.(P^k_{ji} + \Gamma^k_{\underline{ji}}).dx^j] =$$
$$= [f_{,j,i} + (P^k_{ji} + \Gamma^k_{\underline{ji}}).f_{,k}].dx^i \otimes dx^j = (\pounds_{\partial_i} p_j).dx^i \otimes dx^j \ , \quad p_j = f_{,j} \ .$$

(b) Action on a tensor field $K \in \otimes^k{}_l(M)$:

$$\overline{\pounds} : K \to \overline{\pounds}K \ , \qquad K \in \otimes^k{}_l(M) \ , \quad \overline{\pounds}K \in \otimes^k{}_{l+1}(M) \ ,$$
$$\overline{\pounds}K = \overline{\pounds}K^A{}_B \otimes \partial_A \otimes dx^B \ , \qquad \overline{\pounds}K^A{}_B = (\pounds_{\partial_i} K^A{}_B).dx^i \ .$$

Proof:

$$\overline{\pounds}K = (dx^i \otimes \pounds_{\partial_i})(K^A{}_B.\partial_A \otimes dx^B) = dx^i \otimes \pounds_{\partial_i}(K^A{}_B.\partial_A \otimes dx^B)$$
$$= dx^i \otimes (\pounds_{\partial_i} K^A{}_B).\partial_A \otimes dx^B = (\pounds_{\partial_i} K^A{}_B).dx^i \otimes \partial_A \otimes dx^B =$$
$$= \overline{\pounds}K^A{}_B \otimes \partial_A \otimes dx^B \ .$$

$$\overline{\pounds}(\overline{\pounds}K) = (\pounds_{\partial_j}\pounds_{\partial_i} K^A{}_B).dx^j \otimes dx^i \otimes \partial_A \otimes dx^B \ .$$

$\overline{\pounds}K^A{}_B$ are called Lie tensor differentials of the components $K^A{}_B$ of the tensor field K in a co-ordinate basis.

In a non-co-ordinate basis

$$\overline{\pounds}K = \overline{\pounds}K^C{}_D \otimes e_C \otimes e^D \ , \qquad \overline{\pounds}K^C{}_D = (\pounds_{e_\alpha} K^C{}_D).e^\alpha \ .$$

(c) Linear operator with respect to tensor fields:

$$\overline{\pounds}(\alpha.K_1 + \beta.K_2) = \alpha.\overline{\pounds}K_1 + \beta.\overline{\pounds}K_2 \ ,$$
$$K_i \in \otimes^k{}_l(M) \ , \ i = 1,2 \ , \ \ \alpha, \beta \in R \text{ (or } C) .$$

(d) Differential operator with respect to functions:

$$\overline{\pounds}(f.g) = (\overline{\pounds}f).g + f.(\overline{\pounds}g) \ , \qquad f, g \in C^r(M) \ , \qquad \overline{\pounds}f, \ \overline{\pounds}g \in T^*(M) \ .$$

(e) Differential operator not obeying the Leibniz rule with respect to tensor fields with rank > 0:

$$\overline{\pounds}(Q \otimes S) = \overline{\pounds}Q \otimes S + \overline{\pounds}P_{Q \otimes S} \ ,$$
$$\overline{\pounds}P_{Q \otimes S} = dx^i \otimes Q \otimes \pounds_{\partial_i} S = e^\alpha \otimes Q \otimes \pounds_{e_\alpha} S \ .$$

The proof is analogous to that for $\overline{D}(Q \otimes S)$.

The different types of tensor differentials increase the covariant rank of the tensor fields with 1. It is possible the action of the tensor differentials to be specialized for full symmetric and full anti-symmetric (skew-symmetric) covariant (or contravariant) tensor fields. The additional condition for the action of the tensor differentials on these tensor fields is to map a full symmetric tensor field in a full symmetric tensor field and a full anti-symmetric tensor field in a full anti-symmetric tensor field. Because of the structure of the tensor differentials (they contain a covariant vector basic field and increase the covariant rank with 1) this condition can be fulfilled only for covariant symmetric (or anti-symmetric) tensor fields.

3. Symmetric tensor differentials

The tensor product of two full symmetric covariant (or contravariant) tensor fields is not a symmetric product and the new tensor field is not a full symmetric tensor field. The *symmetric product* of two full symmetric tensor field is defined as [3] (p.89)

$$ {}_s({}_sA \otimes {}_sB) = Sym({}_sA \otimes {}_sB) = {}_sA.{}_sB \ , \quad {}_sA \in {}^s\otimes_k(M) \ , \quad {}_sB \in {}^s\otimes_l(M) \ . $$

Let we now consider the action of the tensor differential on a full symmetric covariant tensor field ${}_sB$.

$$ \overline{d}({}_sB) = B_{(A),i}.dx^i \otimes dx^{(A)} = [e_\alpha B_{(A)}].e^\alpha \otimes e^{(A)} \ , $$

where

$$ dx^{(A)} = dx^{i_1}.....dx^{i_k} \ , \qquad e^{(A)} = e^{\alpha_1}.....e^{\alpha_k} \ , \qquad B_{(A)} = B_{i_1...i_k} \ , \ B_{(A)} = B_{\alpha_1...\alpha_k} \ . $$

If we additionally impose the condition for the full symmetry on the affine tensor field $\overline{d}({}_sB)$, then we have to act with the symmetrisation operator Sym on $\overline{d}({}_sB)$ using the decomposition formula for the Bach brackets for $B_{(i_1...i_k,i)}$

$$ \begin{aligned} Sym[\overline{d}({}_sB)] &= B_{(A,i)}.dx^i.dx^{(A)} = [e_{(\alpha}B_{A)}].e^\alpha.e^{(A)} = \\ &= B_{(i_1...i_k,i)}.dx^i.dx^{i_1}.....dx^{i_k} = [e_{(\alpha}B_{\alpha_1...\alpha_k)}].e^{\hat{\alpha}}.e^{\alpha_1}.....e^{\alpha_k} \ , \end{aligned} $$

where

$$ \begin{aligned} [B_{(A),i}]_{(Ai)} = B_{(A,i)} = B_{(i_1...i_k,i)} = \tfrac{1}{k}.[&B_{(i_1...i_{k-1}i_k),i} + B_{(i_1i_2...i_{k-1}i),i_k} + B_{(i_1i_2...i_{k-2}i_ki),i_{k-1}} + \\ &+ ... + B_{(i_2i_3...i_ki),i_1} \ . \end{aligned} $$

We can now define an operator ${}_s\overline{d}$ by the use of the tensor differential $\overline{d}$ and the symmetrisation operator Sym. It will map a covariant tensor field with rank k in a full symmetric covariant affine tensor field with rank $k+1$.

Definition 5. *Symmetric tensor differential. The operator*

$$ \begin{aligned} &{}_s\overline{d} = Sym \circ \overline{d} : B \to {}_s\overline{d}B = Sym(\overline{d}B) \ , \\ &B \in \otimes_k(M) \ , \qquad {}_s\overline{d}B \in {}^A\otimes_{k+1}(M) \ . \end{aligned} $$

is called symmetric tensor differential.

Remark 9. *Since $\overline{d}$ contains in its construction a covariant basic vector field (dx^i or e^α) the symmetric tensor differential can map only a covariant tensor field in a full symmetric covariant affine tensor field. Contravariant tensor fields cannot be entirely symmetrised by the use of ${}_s\overline{d}$:*

$$ \begin{aligned} {}_s\overline{d}C = Sym(\overline{d}C) = Sym(C^A{}_{,i}.dx^i \otimes \partial_A) = C^{(A)}{}_{,i}.dx^i \otimes \partial_{(A)} \ , \\ C \in \otimes^k(M) \ . \end{aligned} $$

The affine tensor field ${}_s\overline{d}C$ is not a full anti-symmetric contravariant affine tensor field because of the existence of different type of indices (contravariant A and covariant i) in a co-ordinate (or non-co-ordinate) basis.

Remark 10. *We will further consider the action of ${}_s\overline{d}$ only on covariant tensor fields.*

The properties of the symmetric tensor differential ${}_s\overline{d}$ are determined by its construction and by the action of the tensor differential on covariant tensor fields.

(a) Action on a function

$$ {}_s\overline{d}f = \overline{d}f \ , \qquad f \in C^r(M) \ . $$

Proof:

$$ {}_s\overline{d}f = Sym(\overline{d}f) = Sym(\partial_i f.dx^i) = f_{,i}.Sym(dx^i) = f_{,i}.dx^i = \overline{d}f \ , $$
because of $Sym(dx^i) = dx^i$.

${}_s\overline{d}$ has also the property

$$ {}_s\overline{d}({}_s\overline{d}f) = \overline{d}(\overline{d}f) \ . $$

(b) Action on a covariant tensor field

$$ {}_s\overline{d}B =: {}_s\overline{d}({}_sB) \ . $$

Proof:

$$ \begin{aligned} {}_s\overline{d}B &= Sym(\overline{d}B) = Sym(B_{A,i}.dx^i \otimes dx^A) = B_{(A,i)}.dx^{(i} \otimes dx^{A)} = \\ &= B_{(A,i)}.dx^i.dx^{(A)} = B_{(A,i)}.dx^{(A)}.dx^i = {}_s\overline{d}({}_sB) \ , \\ &B \in \otimes_k(M) \ , \quad {}_sB = B_{(A)}.dx^{(A)} \ . \end{aligned} $$

where

$$ {}_s\overline{d}B = B_{(A,i)}.dx^i.dx^{(A)} = [e_{(\alpha}B_{A)}].e^\alpha.e^{(A)} \ , \quad B \in \otimes_k(M) \ . $$

$$ \begin{aligned} {}_s\overline{d}({}_sB) &= Sym(B_{(A),i}.dx^i \otimes dx^{(A)}) = B_{(A,i)}.dx^{(i} \otimes dx^{A)} = B_{(A,i)}.dx^i.dx^{(A)} = \\ &= B_{(A,i)}.dx^{(A)}.dx^i \ . \end{aligned} $$

${}_s\overline{d}$ has also the property

$$ {}_s\overline{d}({}_s\overline{d}B) = B_{(A,i,j)}.dx^j.dx^i.dx^{(A)} \ . $$

(c) Linear operator with respect to covariant tensor fields

$$ \begin{aligned} &{}_s\overline{d}(\alpha.B_1 + \beta.B_2) = \alpha.{}_s\overline{d}B_1 + \beta.{}_s\overline{d}B_2 \ , \\ &B_i \in \otimes_k(M) \ , \quad i = 1,2, \quad \alpha, \beta \in R \ (\text{or } C) \ . \end{aligned} $$

Proof:

$$ \begin{aligned} &{}_s\overline{d}(\alpha.B_1 + \beta.B_2) = (Sym \circ \overline{d})(\alpha.B_1 + \beta.B_2) = Sym[\overline{d}(\alpha.B_1 + \beta.B_2)] = \\ &Sym(\alpha.\overline{d}B_1 + \beta.\overline{d}B_2) = \alpha.Sym(\overline{d}B_1) + \beta.Sym(\overline{d}B_2) = \alpha.{}_s\overline{d}B_1 + \beta.{}_s\overline{d}B_2 \ . \\ &\text{where } Sym(\alpha.B) = \alpha.Sym B = \alpha.{}_sB \ . \end{aligned} $$

(d) Differential operator with respect to covariant tensor fields

$$ {}_s\overline{d}(A \otimes B) = {}_s\overline{d}A.{}_sB + {}_sA.{}_s\overline{d}B \ , \qquad A \in \otimes_k(M) \ , \quad B \in \otimes_l(M) \ . $$

Proof:

$$
\begin{aligned}
{}_{s}\overline{d}(A \otimes B) &= Sym[\overline{d}(A \otimes B)] = Sym[\overline{d}A \otimes B + dx^{i} \otimes A \otimes \partial_{i} B] = \\
&= {}_{s}(\overline{d}A \otimes B) + {}_{s}(dx^{i} \otimes A \otimes \partial_{i} B) = {}_{s}\overline{d}A.{}_{s}B + dx^{i}.{}_{s}A.{}_{s}(\partial_{i} B) = \\
&{}_{s}\overline{d}A.{}_{s}B + {}_{s}A.{}_{s}(dx^{i} \otimes \partial_{i} B) = {}_{s}\overline{d}A.{}_{s}B + {}_{s}A.{}_{s}(\overline{d}B) = \\
&= {}_{s}\overline{d}A.{}_{s}B + {}_{s}A.{}_{s}\overline{d}B \ ,
\end{aligned}
$$

where the relations are fulfilled:

$$
\begin{gathered}
Sym(A \otimes B) = {}_{s}(A \otimes B) = (SymA).(SymB) = {}_{s}A.{}_{s}B \ . \\
Sym(\alpha.B) = \alpha.SymB = \alpha.{}_{s}B \ , \qquad \alpha \in R \text{ (or } C), \\
Sym(\alpha.B_1 + \beta.B_2) = \alpha.SymB_1 + \beta.SymB_2 = \alpha.{}_{s}B_1 + \beta.{}_{s}B_2 \ , \\
B_i \in \otimes_k(M) \ , \quad i = 1, 2 \ , \qquad \alpha, \beta \in R \text{ (or } C), \\
{}_{s}\overline{d}B = Sym(\overline{d}B) = B_{(A,i)}.dx^{(A)}.dx^{i} = Sym(B_{A,i}.dx^{i} \otimes dx^{A}) = \\
= Sym(B_{(A),i}.dx^{i} \otimes dx^{(A)}) = Sym[\overline{d}({}_{s}B)] = {}_{s}\overline{d}({}_{s}B) \ .
\end{gathered}
$$

If A and B are full symmetric covariant tensor field, i. e. $A = {}_{s}A$ and $B = {}_{s}B$, then ${}_{s}\overline{d}$ acts on them as a differential operator obeying the Leibniz rule

$$
\begin{gathered}
{}_{s}\overline{d}({}_{s}A \otimes {}_{s}B) = {}_{s}\overline{d}({}_{s}A).{}_{s}B + {}_{s}A.{}_{s}\overline{d}({}_{s}B) \ , \\
{}_{s}A \in {}^{s}\otimes_k(M) \ , \quad {}_{s}B \in {}^{s}\otimes_l(M) \ .
\end{gathered}
$$

3.1. Covariant symmetric tensor differential. On the analogy of the definition of the symmetric tensor differential we can define the notion of the covariant symmetric tensor differential.

Definition 6. *Covariant symmetric tensor differential. The operator*

$$
{}_{s}\overline{D} = Sym \circ \overline{D}
$$

is called covariant symmetric tensor differential.

The properties of the covariant symmetric tensor differential ${}_{s}\overline{D}$ are determined by its construction and the properties of the covariant tensor differential:

(a) Action on a function

$$
\begin{gathered}
{}_{s}\overline{D}f = Sym(\overline{D}f) = Sym(\overline{d}f) = \overline{d}f \ , \quad f \in C^{r}(M) \ , r \geq 1 \ , \\
\overline{d}f \in T^{*}(M) \ , \qquad Sym(f) = id(f) \ , \qquad Sym(dx^{i}) = id(dx^{i}) \ .
\end{gathered}
$$

$$
{}_{s}\overline{D}({}_{s}\overline{D}f) = \overline{D}(\overline{d}f) = f_{(,i;j)}.dx^{j}.dx^{i} \ .
$$

(b) Action on a covariant tensor field

$$
\begin{gathered}
{}_{s}\overline{D}B = {}_{s}\overline{D}({}_{s}B) \ , \\
{}_{s}\overline{D}B = B_{(A;i)}.dx^{i}.dx^{(A)} = [e_{(\alpha}B_{A)}].e^{\alpha}.e^{(A)} \ .
\end{gathered}
$$

Proof:

$$
\begin{gathered}
{}_{s}\overline{D}B = Sym(\overline{D}B) = Sym(B_{A;i}.dx^{i} \otimes dx^{A}) = B_{(A;i)}.dx^{(i} \otimes dx^{A)} = \\
= B_{(A;i)}.dx^{i}.dx^{(A)} = B_{(A;i)}.dx^{(A)}.dx^{i} = {}_{s}\overline{D}({}_{s}B) \ , \\
B \in \otimes_k(M) \ , \quad {}_{s}B = B_{(A)}.dx^{(A)} \ ,
\end{gathered}
$$

where

$$
\begin{gathered}
{}_{s}\overline{D}({}_{s}B) = Sym(B_{(A);i}.dx^{i} \otimes dx^{(A)}) = B_{(A;i)}.dx^{(i} \otimes dx^{A)} = B_{(A;i)}.dx^{i}.dx^{(A)} = \\
= B_{(A;i)}.dx^{(A)}.dx^{i} \ .
\end{gathered}
$$

${}_{s}\overline{D}$ has also the property

$$
{}_{s}\overline{D}({}_{s}\overline{D}B) = B_{(A;i;j)}.dx^{j}.dx^{i}.dx^{(B)} = {}_{s}(\overline{D}(\overline{D}B)) \ .
$$

(c) Linear operator with respect to covariant tensor fields:

$$
\begin{gathered}
{}_{s}\overline{D}(\alpha.B_1 + \beta.B_2) = \alpha.Sym(\overline{D}B_1) + \beta.Sym(\overline{D}B_2) = \alpha.{}_{s}\overline{D}B_1 + \beta.{}_{s}\overline{D}B_2 \ , \\
B_i \in \otimes_k(M) \ , \quad i = 1, 2 \ , \qquad \alpha, \beta \in R \text{ (or } C) \ .
\end{gathered}
$$

(d) Differential operator with respect to covariant tensor field:

$$
{}_{s}\overline{D}(A \otimes B) = {}_{s}\overline{D}A.{}_{s}B + {}_{s}A.{}_{s}\overline{D}({}_{s}B) \ .
$$

Proof:

$$
\begin{gathered}
{}_{s}\overline{D}(A \otimes B) = Sym[\overline{D}(A \otimes B)] = Sym[\overline{D}A \otimes B + dx^{i} \otimes A \otimes \nabla_{\partial_i} B] = \\
= Sym(\overline{D}A \otimes B) + Sym(dx^{i} \otimes A \otimes \nabla_{\partial_i} B) = \\
= [Sym(\overline{D}A)].SymB + [Sym(dx^{i})].SymA.Sym(\nabla_{\partial_i} B) = \\
{}_{s}\overline{D}A.{}_{s}B + {}_{s}A.dx^{i}.{}_{s}(\nabla_{\partial_i} B) = {}_{s}\overline{D}A.{}_{s}B + {}_{s}A.{}_{s}(\nabla_{\partial_i} B).dx^{i} = \\
= {}_{s}\overline{D}A.{}_{s}B + {}_{s}A.{}_{s}\overline{D}({}_{s}B) \ ,
\end{gathered}
$$

where

$$
\begin{gathered}
{}_{s}(\nabla_{\partial_i} B).dx^{i} = [\nabla_{\partial_i}({}_{s}B)].dx^{i} = B_{(C);i}.dx^{(C)}.dx^{i} = B_{(C;i)}.dx^{i}.dx^{(C)} = {}_{s}\overline{D}({}_{s}B) \ , \\
{}_{s}\overline{D}A = {}_{s}\overline{D}({}_{s}A) \ , \quad {}_{s}\overline{D}B = {}_{s}\overline{D}({}_{s}B) \ .
\end{gathered}
$$

On the basis of the last relation we obtain that

$$
{}_{s}\overline{D}(A \otimes B) = {}_{s}\overline{D}A.{}_{s}B + {}_{s}A.{}_{s}\overline{D}({}_{s}B) = {}_{s}\overline{D}({}_{s}A).{}_{s}B + {}_{s}A.{}_{s}\overline{D}({}_{s}B) \ .
$$

Therefore, the covariant symmetric tensor differential ${}_{s}\overline{D}$ acts on symmetric covariant tensor fields as a differential operator obeying the Leibniz rule

$$
{}_{s}\overline{D}(A \otimes B) = {}_{s}\overline{D}[{}_{s}(A \otimes B)] = {}_{s}\overline{D}({}_{s}A.{}_{s}B) = {}_{s}\overline{D}({}_{s}A).{}_{s}B + {}_{s}A.{}_{s}\overline{D}({}_{s}B) \ .
$$

3.2. Lie symmetric tensor differential. On the analogy of the definition of the covariant symmetric tensor differential we can define the notion of the Lie symmetric tensor differential.

Definition 7. *Lie symmetric tensor differential. The operator*

$$
{}_{s}\overline{\pounds} = Sym \circ \overline{\pounds}
$$

is called *Lie symmetric tensor differential.*

The properties of the Lie symmetric tensor differential are determined by its construction and the properties of the Lie tensor differential:

(a) Action on a function

$$_s\overline{\pounds}f = Sym(\overline{\pounds}f) = Sym(\overline{d}f) = \overline{d}f \ , \quad f \in C^r(M) \ , r \geq 1 \ ,$$
$$\overline{d}f \in T^*(M) \ , \qquad Sym(f) = id(f) \ , \qquad Sym(dx^i) = id(dx^i) \ .$$

$$_s\overline{\pounds}(_s\overline{\pounds}f) = {_s\overline{\pounds}}(\overline{d}f) = \pounds_{\partial_{(i}}p_{j)}.dx^i.dx^j \ , \qquad p_j = f_{,j} \ , \qquad \pounds_{\partial_{(i}}p_{j)} = \frac{1}{2}.(\pounds_{\partial_i}p_j + \pounds_{\partial_j}p_i) \ .$$

(b) Action on a covariant tensor field

$$_s\overline{\pounds}B = {_s\overline{\pounds}}(_sB) \ .$$

Proof:

$$\begin{aligned} {_s\overline{\pounds}}B &= Sym(\overline{\pounds}B) = Sym[(\pounds_{\partial_i}B_A).dx^i \otimes dx^A] = [\pounds_{\partial_{(i}}B_{A)}].dx^{(i} \otimes dx^{A)} = \\ &= Sym[\pounds_{\partial_i}B_{(A)}.dx^i \otimes dx^{(A)}] = Sym[\overline{\pounds}(_sB)] = \pounds_{\partial_{(i}}B_{A)}.dx^{(A)}.dx^i = {_s\overline{\pounds}}(_sB) \ , \\ & B \in \otimes_k(M) \ , \quad _sB = B_{(A)}.dx^{(A)} \ , \end{aligned}$$

where

$$\begin{aligned} {_s\overline{\pounds}}(_sB) &= Sym(\pounds_{\partial_i}B_{(A)}.dx^i \otimes dx^{(A)}) = \pounds_{\partial_{(i}}B_{A)}.dx^{(i} \otimes dx^{A)} = \pounds_{\partial_{(i}}B_{A)}.dx^i.dx^{(A)} = \\ &= \pounds_{\partial_{(i}}B_{A)}.dx^{(A)}.dx^i \ . \end{aligned}$$

$_s\overline{\pounds}$ has also the properties

$$_s\overline{\pounds}(_s\overline{\pounds}B) = {_s\overline{\pounds}}(_s\overline{\pounds}(_sB)) \ ,$$
$$Sym \circ \overline{\pounds} = Sym \circ \overline{\pounds} \circ Sym \ , \qquad Sym \circ \overline{\pounds} \circ Sym \circ \overline{\pounds} = Sym \circ \overline{\pounds} \circ \overline{\pounds} \ .$$

$$_s\overline{\pounds}(_s\overline{\pounds}B) = {_s\overline{\pounds}}(\overline{\pounds}B) = Sym(\overline{\pounds}(\overline{\pounds}B)) = {_s(\overline{\pounds}(\overline{\pounds}B))} \ ,$$
$$_s\overline{\pounds}(_s\overline{\pounds}B) = \pounds_{\partial_{(j}}\pounds_{\partial_i}B_{A)}.dx^j.dx^i.dx^{(A)} = (\pounds_{\partial_j}\pounds_{\partial_i}B_A)_{(jiA)}.dx^j.dx^i.dx^{(A)} \ .$$

(c) Linear operator with respect to covariant tensor fields

$$_s\overline{\pounds}(\alpha.B_1 + \beta.B_2) = \alpha.Sym(\overline{\pounds}B_1) + \beta.Sym(\overline{\pounds}B_2) = \alpha.{_s\overline{\pounds}}B_1 + \beta.{_s\overline{\pounds}}B_2 \ ,$$
$$B_i \in \otimes_k(M) \ , \quad i = 1, 2 \ , \qquad \alpha, \beta \in R \text{ (or } C) \ .$$

(d) Differential operator with respect to covariant tensor field

$$_s\overline{\pounds}(A \otimes B) = {_s\overline{\pounds}}A.{_sB} + {_sA}.{_s\overline{\pounds}}(_sB) \ .$$

Proof:

$$\begin{aligned} {_s\overline{\pounds}}(A \otimes B) &= Sym[\overline{\pounds}(A \otimes B)] = Sym[\overline{\pounds}A \otimes B + dx^i \otimes A \otimes \pounds_{\partial_i}B] = \\ &= Sym(\overline{\pounds}A \otimes B) + Sym(dx^i \otimes A \otimes \pounds_{\partial_i}B) = \\ &= [Sym(\overline{\pounds}A)].SymB + [Sym(dx^i)].SymA.Sym(\pounds_{\partial_i}B) = \\ & {_s\overline{\pounds}}A.{_sB} + {_sA}.dx^i.{_s(\pounds_{\partial_i}B)} = {_s\overline{\pounds}}A.{_sB} + {_sA}.{_s(\pounds_{\partial_i}B)}.dx^i = \\ &= {_s\overline{\pounds}}A.{_sB} + {_sA}.{_s\overline{\pounds}}(_sB) \ , \end{aligned}$$

where

$$_s(\pounds_{\partial_i}B).dx^i = \pounds_{\partial_i}B_{(C)}.dx^{(C)}.dx^i = \pounds_{\partial_{(i}}B_{C)}.dx^{(C)}.dx^i = {_s\overline{\pounds}}(_sB) \ .$$
$$_s\overline{\pounds}A = {_s\overline{\pounds}}(_sA) \ , \quad {_s\overline{\pounds}}B = {_s\overline{\pounds}}(_sB) \ .$$

On the basis of the last relation we obtain that

$$_s\overline{\pounds}(A \otimes B) = {_s\overline{\pounds}}A.{_sB} + {_sA}.{_s\overline{\pounds}}(_sB) = {_s\overline{\pounds}}(_sA).{_sB} + {_sA}.{_s\overline{\pounds}}(_sB) \ .$$

Therefore, the Lie symmetric tensor differential $_s\overline{\pounds}$ acts on symmetric covariant tensor fields as a differential operator obeying the Leibniz rule

$$_s\overline{\pounds}(A \otimes B) = {_s\overline{\pounds}}[_s(A \otimes B)] = {_s\overline{\pounds}}(_sA.{_sB}) = {_s\overline{\pounds}}(_sA).{_sB} + {_sA}.{_s\overline{\pounds}}(_sB) \ .$$

4. Anti-symmetric tensor differentials (external differentials)

The result of the action of the tensor differential $\overline{d}$ on a full anti-symmetric tensor field $_aA \in \Lambda^k(M)$ can be found in the form

$$\overline{d}(_aA) = A_{[B],i}.dx^i \otimes d\widehat{x}^B = A_{[C],\alpha}.e^\alpha \otimes \widehat{e}^B \ , \qquad A_{[C],\alpha} = e_\alpha A_{[C]} \ .$$

If we impose the additional condition that the affine tensor field $\overline{d}(_aA)$ has to be a full anti-symmetric affine tensor field, then we have to act with the anti-symmetrisation operator $Asym$ on $\overline{d}(_aA)$

$$\begin{aligned} Asym(\overline{d}(_aA)) &= Asym(A_{[B],i}).Asym(dx^i \otimes d\widehat{x}^B) = A_{[[B],i]}.dx^i \wedge d\widehat{x}^B = \\ &= Asym(A_{[B],i}).dx^i \wedge d\widehat{x}^B = A_{[B,i]}.dx^i \wedge d\widehat{x}^B \ , \end{aligned}$$

where

$$Asym(A_{[B],i}) = A_{[B,i]} \ , \qquad Asym(dx^i \otimes d\widehat{x}^B) = dx^i \wedge d\widehat{x}^B \ .$$

On the other side, the operator $Asym$ anti-symmetrises the tensor product $A \otimes B$

$$\begin{aligned} Asym(A \otimes B) &= Asym(A_{i_1\ldots i_k}.B_{j_1\ldots j_l}).dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l} = \\ &= A_{[i_1\ldots i_k}.B_{j_1\ldots j_l]}.dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge dx^{j_1} \wedge \ldots \wedge dx^{j_l} = \\ &= A_{[i_1\ldots i_k]}.dx^{i_1} \wedge \ldots \wedge dx^{i_k} \wedge B_{[j_1\ldots j_l]}.dx^{j_1} \wedge \ldots \wedge dx^{j_l} = \\ &= {_aA} \wedge {_aB} = AsymA \wedge AsymB = {_a(A \otimes B)} \ . \end{aligned}$$

From

$$_a(\overline{d}(_aA)) = Asym(\overline{d}(_aA)) = A_{[B,i]}.dx^i \wedge d\widehat{x}^B$$

and by the use of the expression for $Asym(A \otimes B)$ for $\overline{d}A = A_{B,i}.dx^i \otimes dx^B$,

$$Asym(\overline{d}A) = {_a(\overline{d}A)} = Asym(A_{B,i}.dx^i \otimes dx^B) = A_{[B,i]}.dx^i \wedge d\widehat{x}^B \ ,$$

it follows that

$$_a(\overline{d}A) = {_a(\overline{d}(_aA))} \ .$$

We can now define an operator $_a\overline{d}$ constructed of the operator $Asym$ and the operator $\overline{d}$ in the form $_a\overline{d} = Asym \circ \overline{d}$.

Definition 8. *Anti-symmetric tensor differential (external differential). The operator*

$$ {}_a\bar{d} : A \to {}_a\bar{d}A = Asym(\bar{d}A) \, , \qquad A \in \otimes_k(M) \, , \qquad {}_a\bar{d}A \in {}^a\otimes_{k+1}(M) \, . $$

is called *anti-symmetric tensor differential (external differential)*.
${}_a\bar{d}$ maps a covariant tensor field of rank k in a full anti-symmetric covariant affine tensor field of rank $k+1$.

Remark 11. *Since the operator ${}_a\bar{d}$ contains in its structure a covariant basis vector field dx^i or e^α $[{}_a\bar{d} = Asym \circ \bar{d} = Asym \circ (dx^i \otimes \partial_i) = Asym \circ (e^\alpha \otimes e_\alpha)]$, it cannot act on contravariant tensor fields as an anti-symmetrisation operator which maps a contravariant tensor field in a full anti-symmetric affine contravariant tensor field. This is the reason for considering the action of ${}_a\bar{d}$ on covariant tensor fields only.*

The properties of the anti-symmetric tensor differential are determined by its definition and the properties of the tensor differential:

(a) Action on a function

$$ {}_a\bar{d}f = Asym(\bar{d}f) = \bar{d}f \, , \quad {}_a\bar{d}f = \bar{d}f = f_{,i}.dx^i \, , \quad f \in C^r(M) \, , \ {}_a\bar{d}f \in T^*(M) \, , $$
$$ {}_a\bar{d}f(\xi) = \bar{d}f\,(\xi) = f_{,i}.f^i{}_j.\xi^j = \xi^{\bar{i}}.f_{,i} = \bar{\xi}f \, , \quad \bar{\xi} = \xi^{\bar{i}}.\partial_i \, . $$

${}_a\bar{d}$ has also the property

$$ {}_a\bar{d}({}_a\bar{d}f) = 0 . $$

Proof:

$$ {}_a\bar{d}({}_a\bar{d}f) = {}_a\bar{d}(\bar{d}f) = f_{[,i,j]}.dx^j \wedge dx^i = 0 \, , \qquad \text{because of} \quad f_{,i,j} = f_{,j,i} \, . $$

(b) Action on a covariant tensor field $A \in \otimes_k(M)$

$$ {}_a\bar{d}A = {}_a\bar{d}({}_aA) \, . $$

Proof: It follows from the relation ${}_a(\bar{d}A) = {}_a(\bar{d}({}_aA))$ and the definition of ${}_a\bar{d}$.
${}_a\bar{d}$ has the property

Lemma 9. *(Poincaré lemma):*

$$ {}_a\bar{d}({}_a\bar{d}A) = 0 : \qquad {}_a\bar{d} \circ {}_a\bar{d} = 0 \, . $$

Proof:

$$ {}_a\bar{d}({}_a\bar{d}A) = {}_a\bar{d}(\bar{d}A) = Asym[\bar{d}(\bar{d}A)] = Asym[dx^j \otimes \partial_j(A_{B,i} \otimes dx^i \otimes dx^B)] = $$
$$ = Asym[A_{B,i,j}.dx^j \otimes dx^i \otimes dx^B] = A_{[B,i,j]}.dx^j \wedge dx^i \wedge d\hat{x}^B = 0 \, , $$
$$ \text{because of } A_{B,i,j} = A_{B,j,i} : A_{[B,i,j]} = 0 \, . $$

(c) Linear operator with respect to covariant tensor fields

$$ {}_a\bar{d}(\alpha.A_1 + \beta.A_2) = \alpha.{}_a\bar{d}A_1 + \beta.{}_a\bar{d}A_2 \, , $$
$$ A_i \in \otimes_k(M) \, , \quad i = 1, 2 \, , \quad \alpha, \beta \in R \text{ (or } C). $$

Proof:

$$ {}_a\bar{d}(\alpha.A_1 + \beta.A_2) = Asym[\bar{d}(\alpha.A_1 + \beta.A_2)] = Asym[\bar{d}(\alpha.A_1) + \bar{d}(\beta.A_2)] = $$
$$ = Asym[\alpha.\bar{d}A_1] + Asym[\beta.\bar{d}A_2] = \alpha.Asym(\bar{d}A_1) + \beta.Asym(\bar{d}A_2) = $$
$$ = \alpha.{}_a\bar{d}A_1 + \beta.{}_a\bar{d}A_2 \, . $$

The last relation follows also immediately from the linearity of both operators $Asym$ and $\bar{d}$.

(d) Differential operator with respect to covariant tensor fields

$$ {}_a\bar{d}(A \otimes B) = {}_a\bar{d}({}_aA \wedge {}_aB) = {}_a\bar{d}A \wedge {}_aB + (-1)^k.{}_aA \wedge {}_a\bar{d}B \, , $$
$$ A \in \otimes_k(M) \, , \quad B \in \otimes_l(M) \, , \ {}_a\bar{d}A \in {}^a\otimes_{k+1}(M) \, , \ {}_a\bar{d}B \in {}^a\otimes_{l+1}(M) \, . $$

Proof:

$$ {}_a\bar{d}(A \otimes B) = Asym[\bar{d}(A \otimes B)] = Asym[\bar{d}A \otimes B + dx^i \otimes A \otimes \partial_i B] = $$
$$ = Asym(\bar{d}A) \wedge {}_aB + Asym(dx^i \otimes A \otimes \partial_i B) = $$
$$ = {}_a\bar{d}A \wedge {}_aB + Asym(dx^i \otimes A \otimes \partial_i B) \, . $$

For $Asym(dx^i \otimes A \otimes \partial_i B)$ we can find the relations

$$ Asym(dx^i \otimes A \otimes \partial_i B) = Asym(dx^i) \wedge Asym A \wedge Asym(\partial_i B) = $$
$$ = dx^i \wedge {}_aA \wedge [B_{[C],i}.d\hat{x}^C] = (-1)^k.{}_aA \wedge B_{[C,i]}.dx^i \wedge d\hat{x}^C = $$
$$ = (-1)^k.{}_aA \wedge Asym(\bar{d}B) = (-1)^k.{}_aA \wedge {}_a\bar{d}B \, . $$

Therefore,

$$ {}_a\bar{d}(A \otimes B) = {}_a\bar{d}A \wedge {}_aB + Asym(dx^i \otimes A \otimes \partial_i B) = $$
$$ = {}_a\bar{d}A \wedge {}_aB + (-1)^k.{}_aA \wedge {}_a\bar{d}B \, . $$

On the other side, from the relation ${}_a\bar{d}A = {}_a\bar{d}({}_aA)$, it follows that

$$ {}_a\bar{d}(A \otimes B) = {}_a\bar{d}[{}_a(A \otimes B)] = {}_a\bar{d}[Asym(A \otimes B)] = {}_a\bar{d}({}_aA \wedge {}_aB) \, . $$

Putting the last expression in the relation for ${}_a\bar{d}(A \otimes B)$, we obtain

$$ {}_a\bar{d}({}_aA \wedge {}_aB) = {}_a\bar{d}A \wedge {}_aB + (-1)^k.{}_aA \wedge {}_a\bar{d}B \, . $$

By the use of the relations ${}_a\bar{d}A = {}_a\bar{d}({}_aA)$, ${}_a\bar{d}B = {}_a\bar{d}({}_aB)$ we can determine the action of ${}_a\bar{d}$ on the external product ${}_aA \wedge {}_aB$ of two full anti-symmetric tensor fields ${}_aA$ and ${}_aB$

$$ {}_a\bar{d}({}_aA \wedge {}_aB) = [{}_a\bar{d}({}_aA)] \wedge {}_aB + (-1)^k.{}_aA \wedge [{}_a\bar{d}({}_aB)] \, . $$

Therefore, ${}_a\bar{d}$ acts on full anti-symmetric tensor fields as a differential operator obeying the rule for anti-differentiation (i. e. the Leibniz rule with respect to the external product and with a possible change of the sign $[(-1)^k]$ in the second term after differentiation).

4.1. Anti-symmetric covariant tensor differential (covariant external differential). On the analogy of the definition of the anti-symmetric tensor differential we can introduce the notion of anti-symmetric covariant tensor differential. Instead of $\overline{d}$ in ${}_a\overline{d}$ we can put $\overline{D}$ and find an operator of the type ${}_a\overline{D} = Asym \circ \overline{D}$.

Definition 10. ${}_a\overline{D} = Asym \circ \overline{D}$ *is called anti-symmetric covariant tensor differential (covariant external differential).*

${}_a\overline{D}$ maps a covariant tensor field of rank k in a full anti-symmetric covariant tensor field of rank $k+1$

$$ {}_a\overline{D} : A \to {}_a\overline{D}A = Asym(\overline{D}A) \,, \qquad A \in \otimes_k(M) \,, \quad {}_a\overline{D}A \in \otimes_{k+1}(M) \,. $$

Remark 12. *Since the operator ${}_a\overline{D}$ contains in its structure a covariant basis vector field dx^i or e^α $[{}_a\overline{D} = Asym \circ \overline{D} = Asym \circ (dx^i \otimes \nabla_{\partial_i}) = Asym \circ (e^\alpha \otimes \nabla_{e_\alpha})]$, it cannot act on contravariant tensor fields as an anti-symmetrisation operator which maps a contravariant tensor field in a full anti-symmetric contravariant tensor field. This is the reason for considering the action of ${}_a\overline{D}$ on covariant tensor fields only. This case is analogous with the case of the operator $\overline{d}$.*

The properties of the anti-symmetric covariant tensor differential are determined by its definition and the properties of the covariant tensor differential:

(a) Action on a function

$$ {}_a\overline{D}f = Asym(\overline{D}f) = \overline{D}f = \overline{d}f \,, \quad {}_a\overline{D}f = \overline{d}f = f_{,i}.dx^i \,, \quad f \in C^r(M) \,, \ {}_a\overline{D}f \in T^*(M) \,, $$
$$ {}_a\overline{D}f(\xi) = \overline{d}f\,(\xi) = f_{,i}.f^i{}_{,j}.\xi^j = \overline{\xi}^i.f_{,i} = \overline{\xi}f \,, \quad \overline{\xi} = \xi^{\overline{i}}.\partial_i \,. $$

${}_a\overline{D}$ has also the property

$$ {}_a\overline{D}({}_a\overline{D}f) = f_{[,i;j]}.dx^j \wedge dx^i = $$
$$ = \tfrac{1}{2}.(P^k_{ij} - P^k_{ji}).f_{,k}.dx^j \wedge dx^i = \tfrac{1}{2}.U_{ij}{}^k.f_{,k}.dx^j \wedge dx^i \,, $$
$$ f_{,i,j} = f_{,j,i} \,, \qquad f \in C^r(M) \,, \qquad r \geq 2 \,. $$

(b) Action on a covariant tensor field $A \in \otimes_k(M)$

$$ {}_a\overline{D}A = {}_a\overline{D}({}_aA) \,. $$

Proof:

$$ {}_a\overline{D}A = Asym(\overline{D}A) = Asym(A_{B;i}.dx^i \otimes dx^B) = A_{[B;i]}.dx^i \wedge d\widehat{x}^B \,. $$

On the other side,

$$ {}_a\overline{D}({}_aA) = Asym[\overline{D}({}_aA)] = Asym(A_{[B];i}.dx^i \otimes d\widehat{x}^B) = A_{[[B];i]}.dx^i \wedge d\widehat{x}^B = $$
$$ = A_{[B;i]}.dx^i \wedge d\widehat{x}^B \,, $$

because of the property of the anti-symmetric Bach brackets

$$ A_{[[B];i]} = (A_{[B];i})_{[Bi]} = A_{[B;i]} \,. $$

Therefore,

$$ {}_a\overline{D}A = {}_a\overline{D}({}_aA) \,. $$

${}_a\overline{D}$ has also the property

$$ {}_a\overline{D}({}_a\overline{D}A) = A_{[B;i;j]}.dx^j \wedge dx^i \wedge d\widehat{x}^B \,. $$

(c) Linear operator with respect to covariant tensor fields

$$ {}_a\overline{D}(\alpha.A_1 + \beta.A_2) = \alpha.{}_a\overline{D}A_1 + \beta.{}_a\overline{D}A_2 \,, $$
$$ A_i \in \otimes_k(M) \,, \quad i = 1, 2 \,, \quad \alpha, \beta \in R \text{ (or } C). $$

Proof:

$$ {}_a\overline{D}(\alpha.A_1 + \beta.A_2) = Asym[\overline{D}(\alpha.A_1 + \beta.A_2)] = Asym[\overline{D}(\alpha.A_1) + \overline{D}(\beta.A_2)] = $$
$$ = Asym[\alpha.\overline{D}A_1] + Asym[\beta.\overline{D}A_2] = \alpha.Asym(\overline{D}A_1) + \beta.Asym(\overline{D}A_2) = $$
$$ = \alpha.{}_a\overline{D}A_1 + \beta.{}_a\overline{D}A_2 \,. $$

The last relation follows also immediately from the linearity of both operators $Asym$ and $\overline{D}$.

(d) Differential operator with respect to covariant tensor fields

$$ {}_a\overline{D}(A \otimes B) = {}_a\overline{D}({}_aA \wedge {}_aB) = {}_a\overline{D}A \wedge {}_aB + (-1)^k.{}_aA \wedge {}_a\overline{D}B \,, $$
$$ A \in \otimes_k(M) \,, \quad B \in \otimes_l(M) \,, \ {}_a\overline{D}A \in \otimes_{k+1}(M) \,, \ {}_a\overline{D}B \in \otimes_{l+1}(M) \,. $$

Proof:

$$ {}_a\overline{D}(A \otimes B) = Asym[\overline{D}(A \otimes B)] = Asym[\overline{D}A \otimes B + dx^i \otimes A \otimes \nabla_{\partial_i}B] = $$
$$ = Asym(\overline{D}A) \wedge {}_aB + Asym(dx^i \otimes A \otimes \nabla_{\partial_i}B) = $$
$$ = {}_a\overline{D}A \wedge {}_aB + Asym(dx^i \otimes A \otimes \nabla_{\partial_i}B) \,. $$

For $Asym(dx^i \otimes A \otimes \nabla_{\partial_i}B)$ we can find the relations

$$ Asym(dx^i \otimes A \otimes \nabla_{\partial_i}B) = Asym(dx^i) \wedge Asym A \wedge Asym(\nabla_{\partial_i}B) = $$
$$ = dx^i \wedge {}_aA \wedge [B_{[C];i}.d\widehat{x}^C] = (-1)^k.{}_aA \wedge B_{[C;i]}.dx^i \wedge d\widehat{x}^C = $$
$$ = (-1)^k.{}_aA \wedge Asym(\overline{D}B) = (-1)^k.{}_aA \wedge {}_a\overline{D}B \,. $$

Therefore,

$$ {}_a\overline{D}(A \otimes B) = {}_a\overline{D}A \wedge {}_aB + Asym(dx^i \otimes A \otimes \nabla_{\partial_i}B) = $$
$$ = {}_a\overline{D}A \wedge {}_aB + (-1)^k.{}_aA \wedge {}_a\overline{D}B \,. $$

On the other side, from the relation ${}_a\overline{D}A = {}_a\overline{D}({}_aA)$, it follows that

$$ {}_a\overline{D}(A \otimes B) = {}_a\overline{D}[{}_a(A \otimes B)] = {}_a\overline{D}[Asym(A \otimes B)] = {}_a\overline{D}({}_aA \wedge {}_aB) \,. $$

Putting the last expression in the relation for ${}_a\overline{D}(A \otimes B)$, we obtain

$$ {}_a\overline{D}({}_aA \wedge {}_aB) = {}_a\overline{D}A \wedge {}_aB + (-1)^k.{}_aA \wedge {}_a\overline{D}B \,. $$

By the use of the relations ${}_a\overline{D}A = {}_a\overline{D}({}_aA)$, ${}_a\overline{D}B = {}_a\overline{D}({}_aB)$ we can determine the action of ${}_a\overline{D}$ on the external product ${}_aA \wedge {}_aB$ of two full anti-symmetric tensor fields ${}_aA$ and ${}_aB$

$$ {}_a\overline{D}({}_aA \wedge {}_aB) = [{}_a\overline{D}({}_aA)] \wedge {}_aB + (-1)^k . {}_aA \wedge [{}_a\overline{D}({}_aB)] \; . $$

Therefore, ${}_a\overline{D}$ acts on full anti-symmetric tensor fields as a differential operator obeying the rule for anti-differentiation.

${}_a\overline{D}$ maps a full anti-symmetric covariant tensor field of rank k in a full anti-symmetric covariant tensor field of rank $k+1$.

4.2. Anti-symmetric Lie tensor differential (Lie external differential). On the analogy of the definition of the anti-symmetric covariant tensor differential we can introduce the notion of anti-symmetric Lie tensor differential. Instead of $\overline{D}$ in ${}_a\overline{D}$ we can put $\overline{\pounds}$ and find an operator of the type ${}_a\overline{\pounds} = Asym \circ \overline{\pounds}$.

Definition 11. ${}_a\overline{\pounds} = Asym \circ \overline{\pounds}$ *is called anti-symmetric Lie tensor differential (Lie external differential).*

${}_a\overline{\pounds}$ maps a covariant tensor field of rank k in a full anti-symmetric covariant tensor field of rank $k+1$

$$ {}_a\overline{\pounds} : A \to {}_a\overline{\pounds}A = Asym(\overline{\pounds}A) \; , \qquad A \in \otimes_k(M) \; , \quad {}_a\overline{\pounds}A \in \otimes_{k+1}(M) \; . $$

Remark 13. *Since the operator ${}_a\overline{\pounds}$ contains in its structure a covariant basis vector field dx^i or e^α $[{}_a\overline{\pounds} = Asym \circ \overline{\pounds} = Asym \circ (dx^i \otimes \pounds_{\partial_i}) = Asym \circ (e^\alpha \otimes \pounds_{e_\alpha})]$, it cannot act on contravariant tensor fields as an anti-symmetrisation operator which maps a contravariant tensor field in a full anti-symmetric contravariant tensor field. This is the reason for considering the action of ${}_a\overline{\pounds}$ on covariant tensor fields only. This case is analogous with the case of the operator $\overline{D}$.*

The properties of the anti-symmetric Lie tensor differential are determined by its definition and the properties of the Lie tensor differential:

(a) Action on a function

$$ {}_a\overline{\pounds}f = Asym(\overline{\pounds}f) = \overline{\pounds}f = \overline{d}f \; , \quad {}_a\overline{\pounds}f = \overline{d}f = f_{,i}.dx^i \; , $$
$$ f \in C^r(M) \; , \quad {}_a\overline{\pounds}f \in T^*(M) \; , $$
$$ {}_a\overline{\pounds}f(\xi) = \overline{d}f\,(\xi) = f_{,i}.f^i{}_j.\xi^j = \xi^{\bar{i}}.f_{,i} = \xi f \; , \quad \xi = \xi^{\bar{i}}.\partial_i \; . $$

${}_a\overline{\pounds}$ has also the property

$$ {}_a\overline{\pounds}({}_a\overline{\pounds}f) = \pounds_{\partial_{[j}}p_{i]}.dx^j \wedge dx^i \; , \qquad p_i = f_{,i} \; . $$

(b) Action on a covariant tensor field $A \in \otimes_k(M)$

$$ {}_a\overline{\pounds}A = {}_a\overline{\pounds}({}_aA) \; . $$

Proof:

$$ {}_a\overline{\pounds}A = Asym(\overline{\pounds}A) = Asym(\pounds_{\partial_i}A_B.dx^i \otimes dx^B) = \pounds_{\partial_{[i}}A_{B]}.dx^i \wedge d\widehat{x}^B \; . $$

On the other side,

$$ {}_a\overline{\pounds}({}_aA) = Asym[\overline{\pounds}({}_aA)] = Asym(\pounds_{\partial_i}A_{[B]}.dx^i \otimes d\widehat{x}^B) = \pounds_{\partial_{[i}}A_{[B]]}.dx^i \wedge d\widehat{x}^B = $$
$$ = \pounds_{\partial_{[i}}A_{B]}.dx^i \wedge d\widehat{x}^B \; , $$

because of the property of the anti-symmetric Bach brackets

$$ \pounds_{\partial_{[i}}A_{[B]]} = (\pounds_{\partial_{[i}}A_{[B]]})_{[Bi]} = \pounds_{\partial_{[i}}A_{B]} \; . $$

Therefore,

$$ {}_a\overline{\pounds}A = {}_a\overline{\pounds}({}_aA) \; . $$

${}_a\overline{\pounds}$ has the property

$$ {}_a\overline{\pounds}({}_a\overline{\pounds}A) = \pounds_{\partial_{[j}}\pounds_{\partial_i}A_{B]}.dx^j \wedge dx^i \wedge d\widehat{x}^B \; . $$

(c) Linear operator with respect to covariant tensor fields

$$ {}_a\overline{\pounds}(\alpha.A_1 + \beta.A_2) = \alpha.{}_a\overline{\pounds}A_1 + \beta.{}_a\overline{\pounds}A_2 \; , $$
$$ A_i \in \otimes_k(M) \; , \quad i = 1, 2 \; , \quad \alpha, \beta \in R \text{ (or } C\text{)}. $$

Proof:

$$ {}_a\overline{\pounds}(\alpha.A_1 + \beta.A_2) = Asym[\overline{\pounds}(\alpha.A_1 + \beta.A_2)] = Asym[\overline{\pounds}(\alpha.A_1) + \overline{\pounds}(\beta.A_2)] = $$
$$ = Asym[\alpha.\overline{\pounds}A_1] + Asym[\beta.\overline{\pounds}A_2] = \alpha.Asym(\overline{\pounds}A_1) + \beta.Asym(\overline{\pounds}A_2) = $$
$$ = \alpha.{}_a\overline{\pounds}A_1 + \beta.{}_a\overline{\pounds}A_2 \; . $$

The last relation follows also immediately from the linearity of both operators $Asym$ and $\overline{\pounds}$.

(d) Differential operator with respect to covariant tensor fields

$$ {}_a\overline{\pounds}(A \otimes B) = {}_a\overline{\pounds}({}_aA \wedge {}_aB) = {}_a\overline{\pounds}A \wedge {}_aB + (-1)^k.{}_aA \wedge {}_a\overline{\pounds}B \; , $$
$$ A \in \otimes_k(M) \; , \quad B \in \otimes_l(M) \; , \quad {}_a\overline{\pounds}A \in \otimes_{k+1}(M) \; , \quad {}_a\overline{\pounds}B \in \otimes_{l+1}(M) \; . $$

Proof:

$$ {}_a\overline{\pounds}(A \otimes B) = Asym[\overline{\pounds}(A \otimes B)] = Asym[\overline{\pounds}A \otimes B + dx^i \otimes A \otimes \pounds_{\partial_i}B] = $$
$$ = Asym(\overline{\pounds}A) \wedge {}_aB + Asym(dx^i \otimes A \otimes \pounds_{\partial_i}B) = $$
$$ = {}_a\overline{\pounds}A \wedge {}_aB + Asym(dx^i \otimes A \otimes \pounds_{\partial_i}B) \; . $$

For $Asym(dx^i \otimes A \otimes \pounds_{\partial_i}B)$ we can find the relations

$$ Asym(dx^i \otimes A \otimes \pounds_{\partial_i}B) = Asym(dx^i) \wedge Asym A \wedge Asym(\pounds_{\partial_i}B) = $$
$$ = dx^i \wedge {}_aA \wedge [\pounds_{\partial_i}B_{[C]}.d\widehat{x}^C] = (-1)^k.{}_aA \wedge \pounds_{\partial_{[i}}B_{C]}.dx^i \wedge d\widehat{x}^C = $$
$$ = (-1)^k.{}_aA \wedge Asym(\overline{\pounds}B) = (-1)^k.{}_aA \wedge {}_a\overline{\pounds}B \; , \text{ where} $$
$$ dx^i \wedge Asym(\pounds_{\partial_i}B) = Asym(dx^i \otimes \pounds_{\partial_i}B) = Asym(\overline{\pounds}B) = {}_a\overline{\pounds}B \; . $$

Therefore,

$$ {}_a\overline{\pounds}(A \otimes B) = {}_a\overline{\pounds}A \wedge {}_aB + Asym(dx^i \otimes A \otimes \pounds_{\partial_i}B) = $$
$$ = {}_a\overline{\pounds}A \wedge {}_aB + (-1)^k.{}_aA \wedge {}_a\overline{\pounds}B \; . $$

On the other side, from the relation ${}_a\overline{\mathfrak{L}}A = {}_a\overline{\mathfrak{L}}({}_aA)$, it follows that

$${}_a\overline{\mathfrak{L}}(A \otimes B) = {}_a\overline{\mathfrak{L}}[{}_a(A \otimes B)] = {}_a\overline{\mathfrak{L}}[Asym(A \otimes B)] = {}_a\overline{\mathfrak{L}}({}_aA \wedge {}_aB) \ .$$

Putting the last expression in the relation for ${}_a\overline{\mathfrak{L}}(A \otimes B)$, we obtain

$${}_a\overline{\mathfrak{L}}({}_aA \wedge {}_aB) = {}_a\overline{\mathfrak{L}}A \wedge {}_aB + (-1)^k . {}_aA \wedge {}_a\overline{\mathfrak{L}}B \ .$$

By the use of the relations ${}_a\overline{\mathfrak{L}}A = {}_a\overline{\mathfrak{L}}({}_aA)$, ${}_a\overline{\mathfrak{L}}B = {}_a\overline{\mathfrak{L}}({}_aB)$ we can determine the action of ${}_a\overline{\mathfrak{L}}$ on the external product ${}_aA \wedge {}_aB$ of two full anti-symmetric tensor fields ${}_aA$ and ${}_aB$

$${}_a\overline{\mathfrak{L}}({}_aA \wedge {}_aB) = [{}_a\overline{\mathfrak{L}}({}_aA)] \wedge {}_aB + (-1)^k . {}_aA \wedge [{}_a\overline{\mathfrak{L}}({}_aB)] \ .$$

Therefore, ${}_a\overline{\mathfrak{L}}$ acts on full anti-symmetric tensor fields as a differential operator obeying the rule for anti-differentiation.

${}_a\overline{\mathfrak{L}}$ maps a full anti-symmetric covariant tensor field of rank k in a full anti-symmetric covariant tensor field of rank $k+1$.

There is a relation between the action of the anti-symmetric Lie covariant differential ${}_a\overline{\mathfrak{L}}$ and the tensor differential $\overline{d}$. From

$${}_a\overline{\mathfrak{L}}A = Asym(\overline{\mathfrak{L}}A) = Asym[dx^j \otimes \mathfrak{L}_{\partial_j}(A)] = Asym[(\mathfrak{L}_{\partial_j} A_B).dx^j \otimes dx^B]$$

and the explicit forms of $\mathfrak{L}_{\partial_j} A_B$ and $(\mathfrak{L}_{\partial_j} A_B).dx^j$

$$(\mathfrak{L}_{\partial_j} A_B).dx^j = A_{B,j}.dx^j + (P^C_{Bj} + \widetilde{\Gamma}^C_{Bj}).A_C.dx^j = \overline{d}A_B + P_{Bj}.dx^j \ ,$$

where

$$P_{Bj} = (P^C_{Bj} + \widetilde{\Gamma}^C_{Bj}).A_C \ , \quad \overline{d}A_B = A_{B,j}.dx^j \ , \quad \overline{d}A = \overline{d}A_B \otimes dx^B \ ,$$

we obtain

$$\overline{\mathfrak{L}}A = \overline{d}A + P \ , \qquad P = P_{Bj}.dx^j \otimes dx^B \ .$$

Then,

$${}_a\overline{\mathfrak{L}}A = Asym(\overline{\mathfrak{L}}A) = Asym(\overline{d}A + P) = {}_a\overline{d}A + {}_aP \ ,$$
$${}_aP = P_{[Bj]}.dx^j \wedge d\widetilde{x}^B \ , \qquad {}_a\overline{d}A = A_{[B,j]}.dx^j \wedge d\widetilde{x}^B \ .$$

Special case: $S = C : f^i{}_j = g^i_j : P^i_{jk} + \Gamma^i_{jk} = 0$,

$$P = 0 \ , \qquad \overline{\mathfrak{L}}A = \overline{d}A \ , \qquad {}_a\overline{\mathfrak{L}}A = {}_a\overline{d}A \ .$$

The Lie derivative of a full anti-symmetric covariant tensor field along a contravariant vector field ξ can be found on the basis of the Lie derivative of a covariant tensor field $W \in \otimes_k(M)$

$$\mathfrak{L}_\xi W = (\mathfrak{L}_\xi W_A).dx^A = (\mathfrak{L}_\xi W_B).e^B \ ,$$
$$\mathfrak{L}_\xi W_A = W_{A,k}.\xi^k - S_{A\overline{k}}{}^{Bl}.W_B.\xi^k{}_{,l} + (P^B_{Al} + \widetilde{\Gamma}^B_{Al}).W_B.\xi^l \ .$$

The Lie derivative of ${}_a\overline{d}f$ can be found after direct computation in the form

$$\mathfrak{L}_\xi({}_a\overline{d}f) = \{f_{,i,j}.\xi^j + f_{,j}.[\xi^{\overline{j}}{}_{,\underline{i}} + (P^j_{ik} + \Gamma^{\overline{j}}_{\underline{i}k}).\xi^k]\}.dx^i \ .$$

Special case: $S = C : f^i{}_j = g^i_j : P^i_{jk} + \Gamma^{\overline{i}}_{\underline{j}k} = P^i_{jk} + \Gamma^i_{jk} = 0$. [5] (p.171):

$$\mathfrak{L}_\xi({}_a\overline{d}f) = (f_{,i,j}.\xi^j + f_{,j}.\xi^j{}_{,i}).dx^i = ((f_{,j,i}.\xi^j + f_{,j}.\xi^j{}_{,i}).dx^i = (f_{,j}.\xi^j)_{,i}.dx^i =$$
$$= \overline{d}(f_{,j}.\xi^j) = {}_a\overline{d}(f_{,j}.\xi^j) = {}_a\overline{d}(\xi f) \ .$$

$$\mathfrak{L}_\xi({}_a\overline{d}f) = {}_a\overline{d}(\xi f) = {}_a\overline{d}(\mathfrak{L}_\xi f) \ , \quad \text{because of } \mathfrak{L}_\xi f = \xi f \ .$$

The Lie derivative of a covariant vector field p along a contravariant vector field ξ can be written in the form

$$\mathfrak{L}_\xi p = (\mathfrak{L}_\xi p_i).dx^i = [p_{i,k}.\xi^k + p_j.\xi^{\overline{j}}{}_{\underline{i}} + p_j.(P^j_{ik} + \Gamma^{\overline{j}}_{\underline{i}k}).\xi^k].dx^i \ .$$

In the special case, when $S = C$, $\mathfrak{L}_\xi p$ can be expressed by the use of S and ${}_a\overline{d}$.
Special case: $S = C : f^i{}_j = g^i_j : P^i_{jk} + \Gamma^{\overline{i}}_{\underline{j}k} = P^i_{jk} + \Gamma^i_{jk} = 0$:

$$\mathfrak{L}_\xi p = 2.S(\xi, {}_a\overline{d}p) + {}_a\overline{d}[S(p, \xi)] = i_\xi({}_a\overline{d}p) + {}_a\overline{d}(i_\xi p) =$$
$$= (i_\xi \circ {}_a\overline{d} + {}_a\overline{d} \circ i_\xi)p \ .$$

Proof:

$$\mathfrak{L}_\xi p = (p_{i,k}.\xi^k + p_k.\xi^k{}_{,i}).dx^i = p_{i,k}.\xi^k.dx^i + p_k.\xi^k{}_{,i}.dx^i =$$
$$= p_{i,k}.\xi^k.dx^i + (p_k.\xi^k)_{,i}.dx^i - p_{k,i}.\xi^k.dx^i =$$
$$= (p_{i,k} - p_{k,i}).\xi^k.dx^i + [S(p, \xi)]_{,i}.dx^i =$$
$$= 2.p_{[i,k]}.\xi^k.dx^i + \overline{d}[S(p, \xi)] = 2.S(\xi, {}_a\overline{d}p) + {}_a\overline{d}[S(p, \xi)] \ .$$
$$p_{k,i}.\xi^k.dx^i = p_{k,i}.dx^i.\xi^k = \overline{d}p_k.\xi^k = (\overline{d}p)_k.\xi^k \ .$$
$$[S(p, \xi)]_{,i}.dx^i = \overline{d}[S(p, \xi)] = {}_a\overline{d}[S(p, \xi)] \ .$$

On the other side,

$${}_a\overline{d}p = p_{[k,i]}.dx^i \wedge dx^k = p_{[i,k]}.dx^k \wedge dx^i \ ,$$
$$S(\xi, {}_a\overline{d}p) = p_{[i,k]}.\xi^i.dx^k = \tfrac{1}{2}.i_\xi({}_a\overline{d}p) \ , \qquad S(\xi, p) = S(p, \xi) = i_\xi p \ .$$

Therefore,

$$\mathfrak{L}_\xi p = 2.S(\xi, {}_a\overline{d}p) + {}_a\overline{d}[S(p, \xi)] = i_\xi({}_a\overline{d}p) + {}_a\overline{d}(i_\xi p) \ .$$

Remark 14. *In $(\overline{L}_n, g)$-spaces [in contrast to (L_n, g)-spaces] relations of the type $\mathfrak{L}_\xi \circ {}_a\overline{d} = {}_a\overline{d} \circ \mathfrak{L}_\xi$, $\mathfrak{L}_\xi \circ i_\xi = i_\xi \circ \mathfrak{L}_\xi$ are not fulfilled.*

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