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ON THE PROPERTY OF BEING A BASIS FOR A DENUMERABLE SET OF SOLUTIONS OF A NONLINEAR SCHRÖDINGER-TYPE BOUNDARY-VALUE PROBLEM

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1 Introduction. Definitions. Results

The boundary-value problem we consider is the following

$$u'' = f(u^2)u, \quad u = u(x), \quad x \in (0,1), \tag{1}$$

$$u(0) = u(1) = 0. \tag{2}$$

Here all quantities are real, f is a given sufficiently smooth function and $u(x)$ is an unknown function of the argument $x \in [0,1]$. We consider solutions of the problem (1)-(2) continuous on the segment $[0,1]$ and twice continuously differentiable in the interval $(0,1)$. The function $u(x) \equiv 0$ obviously satisfies the problem (1)-(2) and we look for nontrivial solutions of this problem. Among others, we assume that

$$f(0) \geq 0 \quad \text{and} \quad \lim_{|u|\to\infty} f(u^2) = -\infty. \tag{3}$$

It is known (see, for example, [1]) that the condition (3) provides the existence of a denumerable set $\{u_n\}_{n=0,1,2,...}$ of solutions of the problem (1)-(2) where for each integer $n \geq 0$ the corresponding solution $u_n(x)$ has precisely n roots in the interval $(0,1)$. In the present paper, we are interested in the natural question, which is originated from the similarity in the qualitative behavior of functions from the system $\{u_n\}_{n=0,1,2,...}$ and the functions $e_n(x) = \sqrt{2}\sin\pi(n+1)x$ $(n = 0,1,2,...)$ (the latter functions obviously form bases in standard spaces like $L_2(0,1)$), if the system of functions $\{u_n\}_{n=0,1,2,...}$ is a basis of a functional space containing "arbitrary functions".

The author knows several papers devoted to investigations of the completeness and related properties for systems of solutions (or eigenfunctions) of nonlinear differential equations. In [2] results in this direction are announced (without proofs) for a nonlinear problem which arises from a linear one under small nonlinear perturbations. In our paper [3], the property of being a basis in the space L_2 for the system of eigenfunctions of a nonlinear Sturm-Liouville-type (or Schrödinger-type) eigenvalue problem (considered in a finite interval) is proved. In fact, this paper contains some mistakes which are corrected in the note [4]. In [5] we present an independent and shorter proof of the above-indicated result from [3,4]. Also, in [6] the results of paper [3] are reestablished without proofs. Our approach in [5] is based on a theorem of N.K. Bary [7-9] stating that a system of functions from the space L_2 minimal (or linearly independent) and quadratically close to a Riesz basis is a Riesz basis. In [3] we exploit other methods the main idea of which consists in a reduction of the nonlinear eigenvalue problem under consideration to a linear Sturm-Liouville eigenvalue problem with a potential depending on the spectral parameter.

Let us introduce some notation. Let $L_2(a,b)$, where $a<b$, be the usual Lebesque space consisting of real-valued functions of the argument $x \in (a,b)$, square integrable over the interval (a,b), with the scalar product $(g,h)_{L_2(a,b)} = \int_a^b g(x)h(x)dx$ and the norm $||g||_{L_2(a,b)} = (g,g)^{\frac{1}{2}}_{L_2(a,b)}$. By Δ we denote the closure in the space $L_2(a,b)$ of the operator $-\frac{d^2}{dx^2}$ with the domain $C_0^\infty(a,b)$ which consists of real-valued infinitely differentiable functions of the argument $x\in[a,b]$ becoming zero at the ends of this segment. Then, it is well-known that Δ is a self-adjoint positive operator in the space $L_2(a,b)$ which also has a bounded inverse operator Δ^{-1}. For an arbitrary $s\le 0$, let $H^s(a,b)$ be the completion of the space $L_2(a,b)$ equipped with the scalar product $(g,h)_{H^s(a,b)} = (\Delta^{\frac{s}{2}}g, \Delta^{\frac{s}{2}}h)_{L_2(a,b)}$ and the norm $||g||_{H^s(a,b)} = (g,g)^{\frac{1}{2}}_{H^s(a,b)}$. Then, it is clear that $H^s(a,b)$ is a Hilbert space for each $s\le 0$ and that $H^0(a,b)=L_2(a,b)$. Also, for a Banach space B with a norm $||\cdot||_B$, by $\mathcal{L}(B;B)$ we denote the space of linear bounded operators acting from B into B, with the norm

$$||A||_{\mathcal{L}(B;B)} = \sup_{\substack{x\in B\\ ||x||_B=1}} ||Ax||_B.$$

Before formulating our results, we introduce some definitions for the convenience of readers. Let B be a real Banach space.

Definition 1. *A system $\{g_n\}_{n=0,1,2,...} \subset B$ is called complete in the space B if and only if the set of all linear combinations $\sum_{n=0}^{N} a_n g_n$ taken for all integer $N>0$ and for all real coefficients a_n is dense in the space B. A system $\{g_n\}_{n=0,1,2,...} \subset B$ is called incomplete in the space B if and only if it is not complete in this space.*

Definition 2. *A system $\{g_n\}_{n=0,1,2,...} \subset B$ is called linearly independent in the space B if and only if the equality $\sum_{n=0}^{\infty} a_n g_n = 0$, where a_n are real coefficients and the convergence of the infinite sum is understood in the sense of the space B, is possible only in the case $a_n = 0$ for all numbers n.*

Remark 1. Our definition of linearly independent systems in a Banach space is not standard and is taken in view of its convenience for our purposes. Sometimes the linear independence from Definition 2 is called the ω–linear independence.

Definition 3. *A system $\{g_n\}_{n=0,1,2,...} \subset B$ is called a basis of the space B if and only if for an arbitrary point $g\in B$ there exists a unique sequence of real coefficients a_n $(n=0,1,2,...)$ such that $g=\sum_{n=0}^{\infty} a_n g_n$ in the sense of the space B.*

In accordance with papers [7,8] we introduce the following

Definition 4. *A basis $\{g_n\}_{n=0,1,2,...}$ of a Hilbert space H is*

called the Riesz basis of this space iff the series $\sum_{n=0}^{\infty} a_n g_n$ *with real coefficients* a_n *converges in the space* H *if and only if* $\sum_{n=0}^{\infty} a_n^2 < \infty$.
In fact, in [7,8] this definition is given for $H = L_2(a, b)$.

Now, we establish our results. First, we need the following

Theorem 1. (a) *Let* $f(u^2)u$ *be a real-valued continuously differentiable function of the argument* $u \in R$ *and let the function* $f(r)$ *of the argument* $r \in [0, +\infty)$ *be continuous and satisfy the condition* (3). *Then, there exists a denumerable set* $\{u_n\}_{n=0,1,2,\dots}$ *of solutions of the problem* (1)-(2) *such that for any integer* $n \geq 0$ *the solution* $u_n(x)$ *has precisely* n *roots in the interval* $(0, 1)$;

(b) *if in addition to the assumptions from the statement* (a) *the function* $f(r)$ *is nonincreasing on the half-line* $r \geq 0$, *then for any integer* $n \geq 0$ *the solution of the problem* (1)-(2) *possessing precisely* n *roots in the interval* $(0,1)$ *is unique up to the coefficient* ± 1.

Let us consider the function $\xi(s) = \sum_{l=1}^{\infty}(2l+1)^{s-1}$ of the argument $s < 0$. Obviously, since

$$\xi(-1) \leq \int_{\frac{1}{2}}^{\infty} (2x+1)^{-2} dx = \frac{1}{4}$$

(this estimate takes place because $(2x+1)^{-2}$ is a convex function of the argument $x > 0$), the equation

$$2\sqrt{6}\pi^{-1}[1 - 24\pi^{-2}\xi(-1)]^{-\frac{1}{2}}\xi(s) = 1$$

with an unknown s has a unique negative root. We denote this root by s_0.

Our main result is the following.

Theorem 2. *Under the assumptions of Theorem* 1(a) *for any* $s < s_0$ *and for an arbitrary system of solutions* $\{u_n\}_{n=0,1,2,\dots}$ *of the problem* (1)-(2) *given by Theorem* 1(a) *the system of functions* $\{k_n u_n\}_{n=0,1,2,\dots}$, *where* $k_n = ||u_n||^{-1}_{H^s(0,1)}$, *is a Riesz basis of the space* $H^s(0, 1)$.

2 Proof of Theorem 1

Of course, Eq. (1) can be solved by quadratures. However, we believe that the qualitative analysis we use for proving Theorem 1 is simpler. The statement (a) of Theorem 1 is generally well-known (for example, it follows from theorem 1 of paper [1]). Let us prove the statement (b).

Let us consider the following Cauchy problem

$$u'' = f(u^2)u, \quad u = u(x), \quad x \in R, \tag{4}$$

$$u(0) = 0, \quad u'(0) = p \tag{5}$$

where p is a real parameter. Due to our assumptions, for any fixed value of the parameter p the usual local existence, uniqueness and continuous dependence theorems are valid for the problem (4)-(5). Further, one can easily verify that any solution $u(x)$ of the problem (1)-(2) continuous on the segment $[0,1]$ and twice continuously differentiable in the interval $(0,1)$ in view of Eq. (1) has first and second derivatives on the right and on the left, respectively, at the points $x = 0$ and $x = 1$. Therefore, an arbitrary solution of the problem (1)-(2) satisfies the problem (4)-(5) with some value of the parameter p. Also, if $u'(0) = 0$ for a solution $u(x)$ of the problem (1)-(2), then $u(x) \equiv 0$ by the uniqueness theorem. So, $u'(0) \neq 0$ for an arbitrary nontrivial solution of the problem (1)-(2). Hence, since Eq. (1) (or (4)) is invariant with respect to the multiplication of the solution $u(x)$ by -1, we get, up to the coefficient ± 1, all solutions of the problem (1)-(2) considering solutions of the problem (4)-(5) when the parameter p runs over the half-line $(0, +\infty)$ and choosing those solutions which become zero at the point $x = 1$.

Let us take an arbitrary $p > 0$. One can easily verify that for the solution $u(x)$ of the problem (4)-(5) the following identity takes place

$$\frac{d}{dx}\{[u'(x)]^2 - U(u^2(x))\} = 0$$

where $U(z) = \int_0^z f(t)dt$. Hence, one has

$$p^2 = [u'(x)]^2 - U(u^2(x)) \qquad (6)$$

for all values x for which the solution $u(x)$ exists. Since in view of the condition (3) the function $U(r^2)$ of the argument r is bounded from above and $\lim_{r\to\infty} U(r^2) = -\infty$, the identity (6) implies the boundedness of the functions $u(x)$ and $u'(x)$ in the whole interval of existence of the solution $u(x)$. Also, in view of Eq. (4), the second derivative $u''(x)$ is bounded, too. These facts immediately yield the global solvability of the problem (4)-(5). Indeed, let us suppose that there exists $a > 0$ such that the solution $u(x)$ of the problem (4)-(5) can be continued onto the half-interval $[0, a)$ and cannot be continued on an arbitrary right half-neighborhood of the point $x = a$. Then, we set

$$q = \int_0^a u'(x)dx \quad \text{and} \quad q' = p + \int_0^a u''(x)dx$$

and, considering the Cauchy problem for Eq. (4) with the initial data $u(a) = q$, $u'(a) = q'$, immediately get that our solution $u(x)$ can be continued onto a right half-neighborhood of the point $x = a$, i. e. we arrive at the contradiction. By analogy, the solution $u(x)$ of the problem (4)-(5) can be continued onto the whole half-line $x < 0$.

So, for any $p > 0$ the corresponding solution $u(x)$ of the problem (4)-(5) is global (can be continued onto the whole real line). In what follows, under solutions of the problem (1)-(2) (or (4)-(5)) we mean maximal solutions defined for all $x \in R$ and satisfying boundary conditions (2) (resp., the initial condition (5)) and Eq. (1) (resp., Eq. (4)) for all $x \in R$.

We need some properties of solutions $u_n(x)$ of the problem (1)-(2) given by Theorem 1(a). We establish them with the following

Proposition. *Let the hypotheses of Theorem* 1(a) *be valid and let n be an arbitrary nonnegative integer. Then, for an arbitrary solution $u_n(x)$ of the problem* (1)-(2) *possessing precisely n roots in the interval* $(0,1)$ *the following properties take place:*

1) *the roots of the solution $u_n(x)$ are precisely the points $\frac{k}{n+1}$ where k runs over all integers;*

2) *between any two nearest roots $\frac{k}{n+1}$ and $\frac{k+1}{n+1}$ of the solution $u_n(x)$ this function has the unique point of extremum $\frac{2k+1}{2(n+1)}$, $u_n'(x) \neq 0$ in the interval $\left(\frac{2k-1}{2(n+1)}, \frac{2k+1}{2(n+1)}\right)$ and $u_n\left(\frac{2k+1}{2(n+1)} + x\right) = u_n\left(\frac{2k+1}{2(n+1)} - x\right)$ for any $x \in R$ (here $k = 0, \pm 1, \pm 2, ...$);*

3) $u_n\left(x + \frac{1}{n+1}\right) = -u_n(x)$ *for any $x \in R$.*

Proof of Proposition. Let us fix an arbitrary integer $n \geq 0$ and consider a solution $u_n(x)$ of the problem (1)-(2) possessing precisely n roots in the interval $(0, 1)$. As earlier, without loss of generality we can accept that $u_n'(0) > 0$. Then, the function $u_n(x)$ satisfies the problem (4)-(5) taken with $p = u_n'(0) > 0$. Further, since $p > 0$, there exists a neighborhood of the point $x = 0$ in which the function $u_n(x)$ strictly increases. Then, since $u_n(1) = 0$, there exists a point $d \in (0,1)$ such that $u_n'(d) = 0$ and $u_n'(x) > 0$ for any $x \in [0, d)$. Due to the uniqueness theorem, the autonomy of Eq. (4) and its invariance with respect to the changes of variables $x + a \to a - x$ and $u(x) \to -u(x)$ (where a

is an arbitrary real constant), we get that $u_n(d+x) = u_n(d-x)$, $u_n(2d) = 0$, $u_n(2d+x) = -u_n(x)$ and $u_n(2d+x) = -u(2d-x)$ for any $x \in R$. Hence, $d = \frac{1}{2(n+1)}$, and Proposition is proved.□

Clearly, to prove the statement (b) of Theorem 1, it suffices to prove that there are no an integer $n \geq 0$ and two real values p_v, p_w : $0 < p_v < p_w$ of the parameter p such that each of the corresponding solutions $v(x)$ and $w(x)$ of the problem (4)-(5) taken respectively with $p = p_v$ and $p = p_w$ has precisely n roots in the interval $(0,1)$ and becomes zero at the point $x = 1$. Let us suppose that this is not the case, and such numbers n, p_v, p_w exist. By Proposition, $v'(x) > 0$ and $w'(x) > 0$ in the half-interval $\left[0, \frac{1}{2(n+1)}\right)$ and $v'\left(\frac{1}{2(n+1)}\right) = w'\left(\frac{1}{2(n+1)}\right) = 0$. Further, by the identity (6), we have $0 < v'(x_1) < w'(x_2)$ for any $x_1, x_2 \in \left(0, \frac{1}{2(n+1)}\right)$ such that $v(x_1) = w(x_2)$. Hence, $0 < v(x) < w(x)$ and, consequently, $f(v^2(x)) \geq f(w^2(x))$ for any $x \in \left(0, \frac{1}{2(n+1)}\right)$ (because otherwise there exists $d \in \left(0, \frac{1}{2(n+1)}\right)$ such that $0 < v(x) < w(x)$ for $x \in (0,d)$ and $v(d) = w(d)$, therefore $v'(d) \geq w'(d)$). Also, one can easily observe that there exists an interval $(c,d) \subset \left(0, \frac{1}{2(n+1)}\right)$ such that $f(v^2(x)) > f(w^2(x))$ for all $x \in (c,d)$. Indeed, this follows from the continuity of the function $f(r^2)$ for $r > 0$ and the fact that $\frac{d}{dr} f(r^2) < 0$ for some $r \in \left(0, f\left(w^2\left(\frac{1}{2(n+1)}\right)\right)\right)$ (otherwise $f\left(w^2\left(\frac{1}{2(n+1)}\right)\right) = f(0) \geq 0$ in the contradiction with the maximum principle). Let us multiply Eq. (4) written for $u(x) = v(x)$ by $w(x)$, the same equation written for $u(x) = w(x)$ by $v(x)$, subtract these identities from each other and integrate the obtained equality between 0 and $\frac{1}{2(n+1)}$. Then, we get

$$[v'(x)w(x) - v(x)w'(x)]\Big|_0^{\frac{1}{2(n+1)}} =$$

$$= \int_0^{\frac{1}{2(n+1)}} v(x)w(x)[f(v^2(x)) - f(w^2(x))]dx.$$

But here the left-hand side is equal to zero and the right-hand side is positive. Therefore, we get the contradiction. Thus, Theorem 1 is proved.□

3 Proof of Theorem 2

Let $\{u_n\}_{n=0,1,2,\ldots}$ be a system of solutions of the problem (1)-(2) given by Theorem 1(a). As in Section 2, without loss of generality we accept that $u_n'(0) > 0$ for all $n = 0, 1, 2, \ldots$. Further, let $v_n(x) = \frac{u_n(x)}{\|u_n\|_{L_2(0,1)}}$ $(n = 0, 1, 2, \ldots)$. For an arbitrary integer $n \geq 0$ let us consider the function $w_n(x) = v_n\left(\frac{x}{n+1}\right)$. Due to Proposition, w_n is a continuous function positive in the interval $(0,1)$. Therefore,

$$w_n(x) = \sum_{k=0}^{\infty} a_k^n e_k(x)$$

in the space $L_2(0,1)$ where a_k^n are real numbers and $a_0^n = (w_n, e_0)_{L_2(0,1)} > 0$. Hence,

$$v_n(x) = \sum_{k=0}^{\infty} b_k^n e_k(x) \tag{7}$$

where $b_{(k+1)(n+1)-1}^n = a_k^n$ and $b_m^n = 0$ if $m \neq (k+1)(n+1) - 1$ for all integer $k \geq 0$ (here, of course, the Fourier series converges in the sense of the space $L_2\left(0, \frac{1}{n+1}\right)$). Then, since by Proposition $v_n\left(\frac{m}{n+1} + x\right) = -v_n\left(\frac{m}{n+1} - x\right)$ and since the direct verification shows that $e_{(k+1)(n+1)-1}\left(\frac{m}{n+1} + x\right) = -e_{(k+1)(n+1)-1}\left(\frac{m}{n+1} - x\right)$ for any integer m and for any $x \in R$, the equality (7) also holds in the sense of each of the spaces $L_2\left(\frac{m-1}{n+1}, \frac{m}{n+1}\right)$ where $m = \overline{2, n+1}$.

Hence, the equality (7) is valid in the sense of the space $L_2(0,1)$.

Remark 2. We obviously have $b_n^n = a_0^n > 0$ for each number n. Therefore, taking into account the facts that the matrix $(b_k^n)_{n,k=0,1,2,...}$ is upper triangular and that all elements of its principal diagonal are nonzero, one may think that the system of functions $\{v_n\}_{n=0,1,2,...}$ is always complete (for example, in the space $L_2(0,1)$). Here, we demonstrate with the following simple example that it is not so and generally the system of functions $\{v_n\}_{n=0,1,2,...}$ can be incomplete.

Example. Let H be a separable real Hilbert space with a scalar product $(\cdot,\cdot)_H$ and the corresponding norm $||\cdot||_H$ and let $\{\overline{e}_n\}_{n=0,1,2,...}$ be an orthonormal basis in this space. Let $\{\overline{v}_n\}_{n=0,1,2,...}$ be a sequence of elements of the space H normalized to 1 and such that the expansions (7) take place with $e_n = \overline{e}_n$, $v_n = \overline{v}_n$ $(n = 0,1,2,...)$, $b_n^n = b \in (0,1)$, $b_{n+1}^n = -\sqrt{1-b^2}$ and $b_k^n = 0$ for all other values of n and k (here b is a constant independent of n and k). Let also $B = \frac{\sqrt{1-b^2}}{b}$. We choose $b \in (0,1)$ to satisfy the condition $B \geq 3$ and want to show that the system of functions $\{\overline{v}_n\}_{n=0,1,2,...}$ is incomplete in the space H. For this aim, it suffices to prove that there exists $c > 0$ such that $||\alpha_m||_H \geq c$ for all $m = 1,2,3,...$, all positive integers N_m and all real coefficients c_n^m $(n = 0,1,...,N_m)$, where $\alpha_m = \sum_{n=0}^{N_m} c_n^m \overline{v}_n - \overline{e}_0$. Let us suppose that this is not right and there exist sequences of positive integers N_m and of real numbers c_n^m, where $m = 1,2,3,...$ and $n = \overline{0,N_m}$, such that $||\alpha_m||_H \to 0$ as $m \to \infty$. Setting $\gamma_{m,n} = (\alpha_m, \overline{e}_n)_H$, we easily derive (by multiplication the expressions for α_m by elements $\overline{e}_n$ in H) that

$$c_0^m = b^{-1}(1+\gamma_{m,0}) \quad \text{and} \quad c_n^m = Bc_{n-1}^m + b^{-1}\gamma_{m,n} \quad \text{for} \quad n = \overline{1,N_m}.$$

Also, since $\alpha_m \to 0$ in H as $m \to \infty$, we have that $\sum_n \gamma_{m,n}^2 \to +0$ as $m \to \infty$. Hence, $c_0^m \to b^{-1} > 1$ as $m \to \infty$ and, using the

induction and the facts that $B \geq 3$ and $|\gamma_{m,n}| \leq b$ for all sufficiently large numbers m and for all n, we get that $|c_n^m| \geq 1$ for all sufficiently large numbers m and for all $n = \overline{1,N_m}$. But then $||\alpha_m||_H^2 \geq (c_{N_m}^m)^2(1-b^2) \geq 1-b^2 > 0$ for all sufficiently large numbers m, and we get the contradiction.□

Let $A_n = \max_x u_n(x)$ and $B_n = \max_x v_n(x)$. In view of Proposition $A_n = u_n\left(\frac{1}{2(n+1)}\right)$ and $B_n = v_n\left(\frac{1}{2(n+1)}\right)$.

Lemma 1. *For any $\epsilon > 0$ there exists a number $N_0 > 0$ such that $B_n \leq \sqrt{3}+\epsilon$ for all numbers $n \geq N_0$.*

Proof. By the usual comparison theorem we get that $A_n \to +\infty$ as $n \to \infty$. Let us take an arbitrary $\gamma \in (0,1)$ and let $D_\gamma^n = \{x \in R: \ u_n(x) \in [0, \gamma A_n]\}$. Let us prove that

$$A_n^{-1} \min_{x \in D_\gamma^n} |u_n'(x)| \to +\infty \quad \text{as} \quad n \to \infty. \tag{8}$$

For this aim, let us consider the identity (6) written for $u(x) = u_n(x)$. Then, we get that $u_n'(0) = [-U(A_n^2)]^{\frac{1}{2}}$. Also, in view of the condition (3) $A_n = o([-U(A_n^2)]^{\frac{1}{2}})$ as $n \to \infty$. Further, for any $x \in R$ we have $[u_n'(x)]^2 = [u_n'(0)]^2 + U(u_n^2(x))$. Hence, since in view of the condition (3) $U(u_n^2(x)) \geq U(\gamma^2 A_n^2)$ for all $x \in D_\gamma^n$ and for all sufficiently large numbers n and since by (3) $\lim_{n\to\infty} \frac{-U(A_n^2)+U(\gamma^2 A_n^2)}{A_n^2} = +\infty$, we get (8).

Clearly, $||u_n||_{L_2(0,1)} \leq A_n$ for all numbers n. Also, solutions of Eq. (1) become concave in the domain $u > c$ for a constant $c > 0$. For an arbitrary number $n \geq 0$ let us consider the linear function $l_n(x) = \frac{2(n+1)(A_nB_n-c)}{A_n(1-2p_n(n+1))}(x-p_n)+\frac{c}{A_n}$ where $p_n \in \left(0, \frac{1}{2(n+1)}\right)$ is a point such that $v_n(p_n) = \frac{c}{A_n}$ (we recall that, according to Proposition, the function $v_n(x)$ monotonously increases in the interval $\left(0, \frac{1}{2(n+1)}\right)$). This function coincides with the function $v_n(x)$ at the points $x = p_n$ and $x = \frac{1}{2(n+1)}$ and $l_n(x) \leq v_n(x)$ for all $x \in \left(p_n, \frac{1}{2(n+1)}\right)$ in view of the concavity of the function $v_n(x)$

in the interval $x \in \left(p_n, \frac{1}{2(n+1)}\right)$. Also, $\lim_{n\to\infty} p_n = 0$ by (8). Let us take an arbitrary $\epsilon > 0$. Then, in view of the above arguments and Proposition, we get

$$1 = ||v_n||^2_{L_2(0,1)} = 2(n+1) \int_0^{\frac{1}{2(n+1)}} v_n^2(x)dx \geq$$

$$\geq 2(n+1) \int_{p_n}^{\frac{1}{2(n+1)}} l_n^2(x)dx = \frac{B_n^2}{3} + \delta_n$$

where $\delta_n \to 0$ as $n \to \infty$. Therefore, $0 < B_n \leq \sqrt{3} + \epsilon$ for all sufficiently large numbers n.□

Lemma 2. *For any $\epsilon > 0$ there exists a number $N_0 > 0$ such that for all numbers $n \geq N_0$ and for all positive integers l one has* $|b^n_{(2l+1)(n+1)-1}| \leq \frac{2\sqrt{6}+\epsilon}{\pi(2l+1)}$ and $b^n_{2l(n+1)-1} = 0$ (*we remind that in the expansions* (7) $b^n_m = 0$ *if* $m \geq 0$ *and* $m \neq (k+1)(n+1) - 1$ *for all* $k = 0, 1, 2, \ldots$.)

Proof. We have $b^n_{(2l+1)(n+1)-1} = (v_n, e_{(2l+1)(n+1)-1})_{L_2(0,1)}$. Let us consider the segment $\left[0, \frac{1}{n+1}\right]$ (here $\frac{1}{n+1}$ is the minimal positive root of the function $v_n(x)$). On this segment the function $e_{(2l+1)(n+1)-1}(x)$ has $2l+2$ roots $x_k = \frac{k}{(2l+1)(n+1)}$ where $k = \overline{0, 2l+1}$. Consider the following integrals: $I_k = \int_{x_{k-1}}^{x_k} v_n(x) e_{(2l+1)(n+1)-1}(x)dx$ where $k = \overline{1, 2l+1}$. Then, due to the properties of the function $v_n(x)$ established in Proposition, we have $|I_{l+1}| > |I_l| = |I_{l+2}| > \ldots > |I_1| = |I_{2l+1}|$ and $\mathrm{sign} I_{l+1} = -\mathrm{sign} I_l = -\mathrm{sign} I_{l+2} = \ldots = (-1)^l \mathrm{sign} I_1 = (-1)^l \mathrm{sign} I_{2l+1}$. Hence, due to Proposition $|b^n_{(2l+1)(n+1)-1}| \leq (n+1)|I_{l+1}|$.

Let us take an arbitrary $\epsilon > 0$. Then, for all sufficiently large numbers n we have by Lemma 1

$$|b^n_{(2l+1)(n+1)-1}| \leq$$

$$\leq \left(\sqrt{6} + \frac{\epsilon}{2}\right)(n+1) \int_{\frac{l}{(2l+1)(n+1)}}^{\frac{l+1}{(2l+1)(n+1)}} \sin[\pi(2l+1)(n+1)x]dx = \frac{2\sqrt{6}+\epsilon}{\pi(2l+1)}.$$

Finally, we remark that similar arguments show that for any integer $n \geq 0$ and $l = 1, 2, 3, \ldots$ one has $b^n_{2l(n+1)-1} = 0$ because according to Proposition the function $v_n(x)$ is even with respect to the point $x = \frac{1}{2(n+1)}$ and, for each $l = 1, 2, 3, \ldots$, the function $e_{2l(n+1)-1}(x)$ is odd with respect to this point. □

For an arbitrary $s \leq 0$ and a nonnegative integer n we obviously have $||e_n||^2_{H^s(0,1)} = \pi^{2s}(n+1)^{2s}$. Further, the sequence of real numbers $b^n_n = (v_n, e_n)_{L_2(0,1)}$ where $n = 0, 1, 2, \ldots$ is obviously bounded, hence, by Lemma 2 and since the functions $e_n(x)$, $n = 0, 1, 2, \ldots$, are pairwise orthogonal in the space $H^s(0,1)$ where $s \leq 0$ is arbitrary, for any $s \leq 0$ there exist positive constants C_1 and C_2 such that

$$||v_n||^2_{H^s(0,1)} \leq \pi^{2s}(n+1)^{2s}(b^n_n)^2 + C_1 \sum_{l=1}^{\infty} (2l+1)^{2s-2}(n+1)^{2s} \leq$$

$$\leq C_2(n+1)^{2s}$$

for all sufficiently large numbers n. Therefore, for any $s \leq 0$ there exists $C_3 > 0$ such that

$$||v_n||^2_{H^s(0,1)} \leq C_3(n+1)^{2s} \tag{9}$$

for all $n = 0, 1, 2, \ldots$

Let us estimate the coefficients b^n_n in the expansions (7). In view of Lemma 2, we have for sufficiently small $\epsilon > 0$ and sufficiently large numbers n

$$b^n_n \geq \left\{1 - \sum_{l=1}^{\infty} \frac{24+\epsilon}{\pi^2(2l+1)^2}\right\}^{\frac{1}{2}} = [1 - (24+\epsilon)\pi^{-2}\xi(-1)]^{\frac{1}{2}} \tag{10}$$

where $\xi(-1) \leq \frac{1}{4}$ as in Section 1. The above arguments including the estimate (10) imply, in particular, the existence of constants $0 < b_1 < b_2$ such that

$$b_1 \leq b_n^n \leq b_2, \qquad n = 0, 1, 2, \ldots \tag{11}$$

Due to the estimate (11), we have for any $s \leq 0$

$$||v_n||^2_{H^s(0,1)} \geq \pi^{2s}(n+1)^{2s}(b_n^n)^2 \geq b_1^2\pi^{2s}(n+1)^{2s}, \qquad n = 0, 1, 2, \ldots \tag{12}$$

We denote $v_n^s = (b_n^n)^{-1}(n+1)^{-s}\pi^{-s}v_n$. Then, in view of the expansions (7) and by Lemma 2

$$v_n^s = \sum_{k=0}^{\infty} b_k^{n,s} e_k^s, \quad n = 0, 1, 2, \ldots, \tag{13}$$

in the spaces $L_2(0,1)$ and $H^t(0,1)$ $(t \leq 0)$ where

$$e_n^s = \pi^{-s}(n+1)^{-s}e_n \ (n = 0, 1, 2, \ldots),$$

$$b^{n,s}_{(2l+1)(n+1)-1} = (b_n^n)^{-1}(2l+1)^s b^n_{(2l+1)(n+1)-1},$$

$$b^{n,s}_m = 0 \ \text{ if } \ m \neq (2l+1)(n+1) - 1 \ \text{ for all}$$

$$l = 0, 1, 2, \ldots \ \text{ and } \ b_n^{n,s} = 1. \tag{14}$$

In view of the estimates (9) and (12), for any $s \leq 0$ there exist positive constants C' and C'' such that

$$C' \leq ||v_n^s||_{H^s(0,1)} \leq C'', \quad n = 0, 1, 2, \ldots \tag{15}$$

Due to Lemma 2, (10) and (14), for an arbitrary $s \leq 0$ and a sufficiently small $\epsilon > 0$ there exists a number $N_1 = N_1(s, \epsilon) > 0$ such that for all $n \geq N_1$

$$|b^{n,s}_{(2l+1)(n+1)-1}| \leq$$

$$\leq [(1 - 24\pi^{-2}\xi(-1))^{-\frac{1}{2}} + \epsilon]2\sqrt{6}\pi^{-1}(2l+1)^{s-1}, \quad l = 1, 2, 3, \ldots$$

Also, if $s < s_0$, then in view of the definition of the number s_0 there exist $\epsilon_0 = \epsilon_0(s) > 0$ and $\delta = \delta(s) > 0$ such that

$$\sum_{l=1}^{\infty}[(1 - 24\pi^{-2}\xi(-1))^{-\frac{1}{2}} + \epsilon_0]2\sqrt{6}\pi^{-1}(2l+1)^{s-1} =$$

$$= [(1 - 24\pi^{-2}\xi(-1))^{-\frac{1}{2}} + \epsilon_0]2\sqrt{6}\pi^{-1}\xi(s) \leq 1 - \delta.$$

Therefore, if $s < s_0$, then

$$\sum_{l=1}^{\infty} \sup_{n \geq N_1(s,\epsilon_0)} |b^{n,s}_{(2l+1)(n+1)-1}| \leq 1 - \delta. \tag{16}$$

Let us fix an arbitrary $s < s_0$ and let L be the closure in the space $H^s(0,1)$ of the linear span of the functions $\{e_n^s\}_{n \geq N_1(s,\epsilon_0)}$ equipped with the topology of the space $H^s(0,1)$. Let us consider the following linear operators acting in the space L:

Λ is the identity operator;
G_l is the operator which for any integer $n \geq N_1(s, \epsilon_0)$ transforms the function e_n^s in the function $b^{n,s}_{(2l+1)(n+1)-1}e^s_{(2l+1)(n+1)-1}$ $(l = 1, 2, 3, \ldots)$;
$G = \sum_{l=1}^{\infty} G_l$;
$A = \Lambda + G$
(we mean that the operators G_l and G are extended onto all finite linear combinations of functions e_n^s, $n \geq N_1(s, \epsilon_0)$, too). Clearly, $||G_l||_{\mathcal{L}(L;L)} \leq \sup_{n \geq N_1(s,\epsilon_0)} |b^{n,s}_{(2l+1)(n+1)-1}|$. Therefore, due to the estimate (16)

$$||G||_{\mathcal{L}(L;L)} \leq \sum_{l=1}^{\infty} ||G_l||_{\mathcal{L}(L;L)} \leq 1 - \delta. \tag{17}$$

The inequality (17) shows that A is a bounded linear operator from L into L. We denote $N = N_1(s, \epsilon_0)$ and remark that the operator A transforms any e_n^s, where $n \geq N$, into v_n^s. Also, due to (17) this operator possesses a bounded inverse one $A^{-1} = \Lambda + \sum_{r=1}^{\infty}(-1)^r G^r$ (for the proof of this fact, see, for example, [10]).

The following Lemmas 3 and 4 are in fact proved in [8,9]. However, since proofs of these statements are short and simple, we present them for the convenience of readers.

Lemma 3. *For our $s < s_0$ the system of functions $\{v_n^s\}_{n=0,1,2,...}$ is linearly independent in the space $H^s(0,1)$.*

Proof. Let us suppose that the statement of this lemma is invalid and there exist real coefficients a_n not all equal to zero and such that

$$\sum_{n=0}^{\infty} a_n v_n^s = 0 \tag{18}$$

in the space $H^s(0,1)$. Let $l \geq 0$ be a number such that $a_0 = ... = a_{l-1} = 0$ and $a_l \neq 0$. Formulas (13) and (14) show that $(v_n^s, e_m^s)_{H^s(0,1)} = 0$ for arbitrary numbers $m < n$. Using this fact and multiplying the equality (18) by e_l^s in the space $H^s(0,1)$, we get $a_l b_l^{l,s} = 0$, i. e. the contradiction.□

Lemma 4. *For our $s < s_0$ $\{v_n^s\}_{n \geq N}$ is a Riesz basis of the space L.*

Proof. Let us take an arbitrary $v \in L$ and let $u = A^{-1}v \in L$. Then, $u = \sum_{n=N}^{\infty} c_n e_n^s$ in the space L for some real coefficients c_n. We have $v = Au = \sum_{n=N}^{\infty} c_n A e_n^s = \sum_{n=N}^{\infty} c_n v_n^s$ where the infinite sums converge again in the space L. Therefore, in view of the linear independence of the system of functions $\{v_n^s\}_{n \geq N}$ in the space L given by Lemma 3, this system of functions is a basis of the space L. Also, obviously, the infinite sum $\sum_{n=N}^{\infty} c_n e_n^s$ with real coefficients

c_n converges in the space L if and only if $\sum_{n=N}^{\infty} c_n^2 < \infty$. Hence, first, if $\sum_{n=N}^{\infty} c_n^2 < \infty$, then the infinite sum $\sum_{n=N}^{\infty} c_n v_n^s$ converges in the space $H^s(0,1)$ and, second, if the latter series converges in the indicated space, then $A^{-1}\sum_{n=N}^{\infty} c_n v_n^s = \sum_{n=N}^{\infty} c_n A^{-1} v_n^s = \sum_{n=N}^{\infty} c_n e_n^s \in L$, i. e. $\sum_{n=N}^{\infty} c_n^2 < \infty$. Thus, Lemma 4 is proved.□

Lemma 5. *For our $s < s_0$ the system of functions $\{v_n^s\}_{n=0,1,2,...}$ is a Riesz basis of the space $H^s(0,1)$.*

Proof. Let again $N = N_1(s, \epsilon_0)$ and P be the orthogonal projector in the space $H^s(0,1)$ onto the subspace L_N spanned over the functions $e_0^s, ..., e_{N-1}^s$. By (13) and (14)

$$Pv_n^s = v_n^s - \sum_{k=N}^{\infty} b_k^{n,s} v_k^s, \quad n = \overline{0, N-1} \tag{19}$$

in the space $H^s(0,1)$. Since $\dim L_N = N$ and since in view of the formulas (13) and (14) the functions $Pv_0^s, ..., Pv_{N-1}^s$ are linearly independent in the space $H^s(0,1)$, these functions form a basis of the space L_N. Therefore, in view of Lemma 4, the system of functions $\{Pv_n^s\}_{n=\overline{0,N-1}} \cup \{v_n^s\}_{n \geq N}$ is a basis of the space $H^s(0,1)$.

Let us take an arbitrary $u \in H^s(0,1)$. Then, due to the above arguments, there exists a unique sequence $\{c_n\}_{n=0,1,2,...}$ of real numbers such that

$$u = \sum_{n=0}^{N-1} c_n P v_n^s + \sum_{n=N}^{\infty} c_n v_n^s \tag{20}$$

in the space $H^s(0,1)$. Substituting the expansions (19) into (20), we get

$$u = \sum_{n=0}^{N-1} c_n v_n^s + \sum_{n=N}^{\infty} \left(c_n - \sum_{m=0}^{N-1} c_m b_n^{m,s} \right) v_n^s$$

in the space $H^s(0,1)$. Hence, in view of Lemma 3, the system of functions $\{v_n^s\}_{n=0,1,2,...}$ is a basis of the space $H^s(0,1)$. Finally, the fact that this system of functions is a Riesz basis of the above space follows from Lemma 4. Thus, Lemma 5 is proved.□

Now Theorem 2 follows from Lemma 5 and the inequalities (15).□

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