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RENORMALIZATION GROUP IN MATHEMATICAL PHYSICS AND SOME PROBLEMS OF LASER OPTICS

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1. Introduction

The paper is devoted to the problem of constructing a special class of symmetries for boundary value problems (BVPs) in mathematical physics, namely renormalization group symmetries (RGSs) and of using this symmetry to solve a non-linear problem. By RGS we mean a symmetry that characterizes a solution of BVP and corresponds to transformations involving both dynamical variables and *parameters* entering into the solution via boundary conditions (and equations).

> Definition :
> In the simplest case the renormalization group (RG) can be defined as a continuous one-parameter group of specific transformations of some solution of a physical problem, solution that is fixed by a boundary condition. The RG transformation involves boundary condition parameter(s) and corresponds to some change in the way of imposing this condition on the *same* partial solution.

Symmetries of this type appeared about forty years ago in the context of the RG concept which originally arose [1, 2] in the "depth" of quantum field theory (QTF) and was successively used there [2, 3] for improving an approximate solution to restore a correct structure of solution singularity.

To elucidate the idea of RGS, let us take some solution and consider its one-argument characteristic $\Gamma(\vartheta)$ that can be fixed by a boundary condition $\Gamma(\vartheta_0) = g_0$. Let this characteristic be formally represented as a function of boundary parameters as well: $\Gamma(\vartheta) = \Gamma(\vartheta, \vartheta_0, g_0)$. (This step can be considered as an *embedding* operation). Then, the renormalization group (RG) transformation corresponds to a changeover of the way of parameterization, say from $\{\vartheta_0, g_0\}$ to $\{\vartheta_1, g_1\}$. Suppose now that our characteristic Γ can be written in a form of the two-argument function $G(\vartheta/\vartheta_0, g_0)$ with the property $G(1, g) = g$. Then, the equality $G(\vartheta/\vartheta_0, g_0) = G(\vartheta/\vartheta_1, g_1)$ points to the fact that under change of a boundary condition the form of the function G itself is not modified. Taking into account that $g_1 = G(\vartheta_1/\vartheta_0, g_0)$ and introducing new variables $x = \vartheta/\vartheta_0$, $g_0 = g$ and $a = \vartheta_0/\vartheta_1$ we obtain

$$G(x, g) = G(x/a, G(a, g)). \tag{1.1}$$

This functional equation corresponds to continuous one-parameter point transformations

$$T_a : \{x' = x/a \ , \ g' = G(a, g) \} \ , \quad G(1, g) = g, \tag{1.2}$$

whereas the functional equation (1.1) guarantees the group property $T_a \cdot T_b = T_{ab}$ fulfillment.

The infinitesimal form of the RG transformation can be written down as a differential equation

$$RG = 0\,, \quad \text{with} \;\; R = x\partial_x - \beta(g)\partial_g\,, \quad \beta(g) = \left.\frac{\partial G(a;g)}{\partial a}\right|_{a=1} \tag{1.3}$$

where R is the infinitesimal RGS operator (RGO) with the coordinate $\beta(g)$ defined by the first derivative of G. This equation, reflecting the invariance of G under the RG transformation (i.e., functional self-similarity property) can be treated as a vanishing condition for the coordinate æ of RGO in the canonical form [4]

$$R = æ\partial_G\,, \quad æ \equiv xG_x - \beta(g)G_g = 0\,, \tag{1.4}$$

identically valid on a particular BVP solution $G = G(x,g)$.

Therefore, instead of Eqs. (1.1), (1.2), RG transformation can also be introduced by means of an RGO with canonical coordinates that identically turn to zero on the BVP solution. And vice versa, being given an RGO, one can easily reconstruct the functional equation for the transformation function G with the help of the corresponding Lie equations [5]

$$\frac{dx'}{x'} = -\frac{da}{a}, \quad \frac{dg'}{\beta(g')} = \frac{da}{a}, \quad dG' = 0 \tag{1.5}$$

with the initial conditions $x'|_{a=1} = x,\; g'|_{a=1} = g,\; G'|_{a=1} = G$. Thus, for a given form of the β-function or, in other words, for the given RGS, one can get an explicit solution for $G(x,g)$.

To illustrate the RGS application, take the so-called *effective coupling* in QFT. In the simple case of quantum electrodynamics (QED) the effective coupling is just the effective charge of an electron $\bar{e}(r)$, i.e., a function describing a spatial distribution of the electric charge density around the center of an electron; this distribution is related to processes of polarization of quantum vacuum. In conventional use, there is a Fourier transform of $\bar{e}(r)$ squared, the QED effective coupling $\bar{\alpha}(Q)$ that is a function of Q, momentum transfer modulus. In the ultraviolet limit, it can be considered as a function of two variables $x = Q^2/\mu^2$ with μ, some reference value of momentum transfer, and α_μ, the value of effective coupling at $Q = \mu$. This function, $\bar{\alpha}(Q^2/\mu^2, \alpha_\mu)$ being an RG invariant satisfies the group functional equation

$$\bar{\alpha}(x,\alpha) = \bar{\alpha}(\frac{x}{a}, \bar{\alpha}(a,\alpha)), \tag{1.6}$$

that is equivalent to the previous one.

In QFT, generally, the main tool of theoretical calculations is the Feynman perturbation theory that gives a power expansion in coupling constant (squared). In the QED case, this is a small parameter $\alpha_\mu \simeq 1/137$, the fine structure constant.

The first perturbative approximation

$$\bar{\alpha}^{(1)}_{PTh}(x,\alpha) = \alpha + \frac{\alpha^2}{3\pi}\ell + \ldots\,, \quad \ell = \ln x \tag{1.7}$$

does not satisfy this functional equation. However, the application of the algorithm proposed in Ref.[2] and based on the Sophus Lie group theory[5], that is known as *renormalization group method*, after short and elementary manipulation produces the result

$$\bar{\alpha}^{(1)}_{RG}(x,\alpha) = \frac{\alpha}{1 - \dfrac{\alpha}{3\pi}\ln x} \tag{1.8}$$

which, on one hand is an exact partial solution to Eq.(1.6), and, on the other, being expanded into powers of α, corresponds to the perturbative input, Eq.(1.7).

From this example we learn that asymptotic $\sim \ln Q^2$ is not compatible with Eq.(1.6), i.e., with RGS (1.3), whereas $\sim (\ln Q^2)^{-1}$ is compatible with it. Remind, that a basic property of a strong interaction in the microworld, the famous property of "asymptotic freedom"[2], has theoretically been discovered just by the RG method. It is described by an expression for the effective coupling of strong interaction $\bar{\alpha}^{(1)}_s(x,\alpha_s) = 1/(1+\alpha_s\beta_0\ln x); \beta_0 > 0$ which is formally analogous to Eq.(1.8) and in the UV limit tends to zero as $(\ln Q^2)^{-1}$.

In the seventies it was shown [6] that the RG concept is fruitful in some other fields of microscopic physics, more specifically for description of phase transition in large statistical systems like spin lattice, polymers, turbulence, and so on. Here, typically, series in log's powers are summed by the RG method into the exponent, that is into powers of log's argument.

Later on, symmetry underlying the renormgroup invariance was also found in a number of problems of macroscopic physics like, e.g., mechanics, transfer theory, hydrodynamics. A close relation of RGS to the concept

[2]Property of weakening the interaction intensity at small distances, i.e., in the UV limit of large Q^2.

of self-similarity (well known in mathematical physics) has been established and the notion of functional self-similarity (as a synonym of RG that is more adequate in general context of mathematical physics) has been introduced[7].

However, the procedure of finding this symmetry in any particular case is usually based upon atypical manipulations [8, 6, 9] and the constructing of a regular approach to revealing RGS is of principal interest. Such an approach to finding these symmetries has recently been devised[10] for the class of BVPs. In Section II, we briefly describe this approach, and its application to some simple problems of nonlinear optics is discussed in Section III.

2. General description of the approach

An idea of a regular approach to the RGS constructing for a class of mathematical models of physical systems that are described by differential equations (DEs) is based [11] upon the well-known fact that these models can be analyzed by algorithms of modern group analysis. The scheme that describes devising of RGS is then formulated [10] as follows.

Firstly, a specific RG-manifold should be constructed. Secondly, some auxiliary symmetry, i.e. the most general symmetry group admitted by this manifold is then calculated. Thirdly, this symmetry should be restricted on a particular solution to get RGS. Fourthly, the RGS allows one to improve an approximate solution[3].

Depending on both a mathematical model and boundary conditions, the first step of this procedure can be realized in different ways. In some cases, the desired RG-manifold is obtained by including parameters, entering into a solution via equation(s) and boundary condition, in the list of independent variables. The extension of the space of variables involved in group transformations, e.g., by taking into account the dependence of coordinates of RGO upon differential and/or non-local variables (which leads to the Lie-Bäcklund and non-local transformation groups [4]) can also be used for constructing the RG-manifold. The use of the Ambartsumian's invariant embedding method and of differential constraints sometimes allows reformulations of a boundary condition in a form of additional differential equation(s) and enables one to construct the RG-manifold as a combination of original equations and embedding equations (or differential constrains)

[3]In some lucky cases, to get an exact solution - see below Eqs.(3.12), (3.16).

which are compatible with these equations. At last, of particular interest is the perturbation method of constructing the RG-manifold which is based on the presence of a small parameter.

The second step, the calculating of a most general group $\mathcal{G}$ admitted by the RG-manifold, is a standard procedure in the group analysis and has been described in detail in many texts and monographs – see, for example, [5].

The symmetry group $\mathcal{G}$ thus constructed can not as yet be referred to as a renormgroup. In order to obtain this, the next, third step should be done which consists in restricting $\mathcal{G}$ on a solution of BVP. This procedure utilizes the invariance condition (1.4) and mathematically appears as a "combining" of different coordinates of group generators admitted by the RG-manifold. It should be noticed that in some particular cases the procedure of a group restriction can be partially fulfilled while looking for the admitted group. That means that we can use any additional information about the solution of BVP while making the second step of the scheme.

The final step, i.e., constructing analytical expressions for solutions of BVP on the basis of the RGS obtained, usually presents no specific problems.

3. Application to nonlinear optics

To demonstrate effectiveness of the RGS approach, consider a simple mathematical model, the BVP for a system of geometric optics equations in plane geometry,

$$v_t + vv_x - an_x = 0\,, \quad n_t + vn_x + nv_x = 0\,; \tag{3.1}$$

$$v(0,x) = 0\,, \quad n(0,x) = N(x)\,. \tag{3.2}$$

Here a is a nonlinearity parameter, t and x are coordinates along and transverse to the axis of laser beam propagation, $v(t,x)$ is the derivative of eikonal with respect to x, and $n(t,x)$ is a laser beam intensity. The function N characterizes the beam intensity distribution upon x at the entrance to a medium at $t=0$.

Before proceeding to the discussion of RGSs it is convenient to perform hodograph transformations and to make a change of the variables $\tau = nt$, $\chi = x - vt$ and $w = v/a$, which leads instead of Eq.(3.1) to the linear system

$$\tau_w - n\chi_n = 0\,, \quad \chi_w + a\tau_n = 0\,. \tag{3.3}$$

At first, we look for RGS that is defined by a canonical operator of the Lie-Bäcklund group

$$R = f(w,n,a,\tau_s,\chi_s)\partial_\tau + g(w,n,a,\tau_s,\chi_s)\partial_\chi\,, \tag{3.4}$$

$$\tau_s = \partial^s\tau/\partial n^s\,,\quad \chi_s = \partial^s\chi/\partial n^s\,,$$

considered in the extended space of group variables which includes not only dependent and independent variables τ, χ, w, n but the nonlinearity parameter a and high order derivatives of τ and χ with respect to n as well. In the general case, the coordinates f and g in (3.4) can be presented as a formal power series in a

$$f = \sum_{i=0}^{\infty} a^i f^i\,,\quad g = \sum_{i=0}^{\infty} a^i g^i \tag{3.5}$$

with the coefficients f^i and g^i satisfying the system of recurrence relations

$$f^i = F^i + \int dw\,\{(1-\delta_{i,0})Zf^{i-1} + nYg^i\}\,,$$
$$g^i = G^i + (1-\delta_{i,0})\int dw\,\{Zg^{i-1} - Yf^{i-1}\}\,. \tag{3.6}$$

Here

$$Y = \partial_n + \sum_{s=0}^{\infty}(\tau_{s+1}\partial_{\tau_s} + \chi_{s+1}\partial_{\chi_s})\,,\quad Z = \sum_{s=0}^{\infty}\tau_{s+1}\partial_{\chi_s}\,,$$

and expressions in the curly brackets before integrating should be expressed in terms of $\tilde\tau_s = \tau_s - w(n\chi_{s+1} + s\chi_s)$, n, w and χ_s.

The functions $F^i(n,\chi_s,\tilde\tau_s)$ and $G^i(n,\chi_s,\tilde\tau_s)$ in (3.6) are restricted by the invariance conditions

$$f = 0\,,\quad g = 0 \tag{3.7}$$

that should remain identically valid on the BVP solution. Note that the boundary conditions (3.2) in terms of $(\tau,\ \chi)$ are reformulated as follows:

$$w = 0\,,\quad \tau_s = 0\,,\quad \chi = H(n)\,. \tag{3.8}$$

Provided that F^i and G^i, $i \geq 1$ identically vanish at $\tau = 0$, the boundary conditions are correlated with the form of the functions F^0 and G^0. Of

special interest are such functions F^0 and G^0 for which infinite series (3.5) are truncated for some $i = i_{max} < \infty$, and finite sums are obtained. Just this situation is realized for a laser beam with an intensity distribution at $\tau = 0$ described by $N = \cosh^{-2}(x)$, and it is the BVP that was solved by Rem Khokhlov with his colleagues more than 30 years ago [12]. The boundary function for this case

$$H(n) = \text{Arccosh}(1/\sqrt{n}) \tag{3.9}$$

is compatible with

$$F^0 = 2n(1-n)\tilde\tau_2 - n\tilde\tau_1\,,\quad G^0 = 2n(1-n)\chi_2 + (2-3n)\chi_1\,. \tag{3.10}$$

Substitution of these expressions into (3.6) enables one to calculate f^0, f^1 and g^1, whilst higher-order terms vanish and we obtain RGO that describes the second-order Lie-Bäcklund symmetry of DEs (3.3)

$$R = \left(2n(1-n)\tau_2 - n\tau_1 - 2nw(\chi_1 + n\chi_2) + \frac{a}{2}nw^2\tau_2\right)\partial_\tau$$
$$+ \left(2n(1-n)\chi_2 + (2-3n)\chi_1 + aw(2n\tau_2 + \tau_1) + \frac{a}{2}w^2(n\chi_2 + \chi_1)\right)\partial_\chi\,. \tag{3.11}$$

Now, to find an explicit analytical expression for a BVP solution with the help of the RGS obtained, one should make use of the invariance condition (3.7). Solving the basic equations (3.3) together with (3.7), where f and g are the RGO coordinates just displayed, yields the desired unique solution which for (3.9) has the form (equivalent to the Khokhlov's solution [12])

$$v = -2ant\tanh(x - vt)\,,\quad an^2t^2 = n\cosh^2(x - vt) - 1\,. \tag{3.12}$$

The example with this solution is not unique. As an illustration, we present another exact solution of the BVP that was obtained [13] with the help of RGS described by the operator of the form (3.4) where f and g are given by truncated series. The distribution of the laser beam intensity at the boundary $\tau = 0$ in this case has the form of the "smoothed" step and is analytically expressed by the function $H(n)$ in (3.8) defined via complete first-order elliptic integrals

$$H(n) = \frac{2}{1+\sqrt{n}}\left[K\left(\frac{2n^{1/4}}{1+\sqrt{n}}\right) - \frac{6}{\pi}K\left(\frac{1-\sqrt{n}}{1+\sqrt{n}}\right)\right]. \tag{3.13}$$

The zero-order functions F^0 and G^0 compatible with this expression are given as

$$F^0 = n(1-n)\tilde{\tau}_2 - n\tilde{\tau}_1\,, \quad G^0 = n(1-n)\chi_2 + (1-2n)\chi_1 - \chi/4\,, \quad (3.14)$$

whereas the remaining nonzero terms f^0, f^1 and g^1 in (3.5) follow from (3.6). The resulting RGO

$$R = \left(n(1-n)\tau_2 - n\tau_1 - nw(\frac{5}{4}\chi_1 + n\chi_2) + \frac{a}{4}nw^2\tau_2\right)\partial_\tau + \left(n(1-n)\chi_2\right.$$

$$\left. +(1-2n)\chi_1 - \frac{1}{4}\chi + aw\left(n\tau_2 + \frac{3}{4}\tau_1\right) + \frac{aw^2}{4}(n\chi_2 + \chi_1)\right)\partial_\chi \quad (3.15)$$

is again the second-order Lie-Bäcklund symmetry operator of the basic equations. Solving the combined system of equations (3.3) and (3.7) in view of the expressions for f and g as in (3.15) yields the following BVP solution:

$$t = \frac{1}{n\sqrt{a}}\left[\left(\frac{qk_1}{2}K(k_1) - \frac{2}{k_1}E(k_1) - \frac{6}{\pi k_1}\left(2E(k_2) - k_1^2 K(k_2)\right)\right)\frac{F(\vartheta,k)}{\sqrt{q_2}}\right.$$

$$\left. -k_1\left(K(k_1) - \frac{6}{\pi}K(k_2)\right)\left(\sqrt{q_2}E(\vartheta,k) - \frac{((q_2-\sqrt{n})(\sqrt{n}-q_1))^{1/2}}{n^{1/4}}\right)\right],$$

$$x = vt + \frac{k_1}{n^{1/4}}\left(K(k_1) - \frac{6}{\pi}K(k_2)\right). \quad (3.16)$$

Here

$$q = \frac{1}{\sqrt{n}}\left(1 + n + \frac{v^2}{4a}\right)\; ; \; k = \sqrt{\frac{q_2 - q_1}{q_2}}\,, \; k_1 = \frac{2}{\sqrt{q+2}}\,, \; k_2 = \sqrt{\frac{q-2}{q+2}}\,,$$

$$2q_{1,2} = q \mp \sqrt{q^2 - 4}; \quad \vartheta = \arcsin\left[(q_2/\sqrt{n})(\sqrt{n} - q_1)(q_2 - \sqrt{n})\right]^{1/2}$$

and $K(k)$, $E(k)$ and $F(\vartheta,k)$, $E(\vartheta,k)$ are complete and non-complete elliptic integrals of the first and the second order in the standard notation. The three-dimensional plots for this solution are presented in Fig.1a,b for $a = 1$.

In the case of arbitrary boundary data (3.2) when infinite series (3.5) are not truncated automatically, one can drop higher order terms provided the analysis is restricted to small values of nonlinearity parameter $a \ll 1$. In this case, RGS appear as an approximate symmetry [14] of equations (3.3). As illustration, we present the for BVP with

$$H(n) = (\ln(1/n))^{1/2}\,, \quad (3.17)$$

which describes space evolution (self-focusing) of the Gaussian beam $N(x) = \exp(-x^2)$ with the plane phase front at the boundary $\tau = 0$. To satisfy the initial distribution (3.17), one can choose

$$F^0 = 1 + 2n\chi\chi_1, \quad G^0 = 0\,. \quad (3.18)$$

Here, the "inherited" point group of BVP is constructed with the help of formulas (3.6) and is given by RGO

$$R = -2\chi\partial_w + 2a\tau\partial_n + \left(1 + (a\tau^2/n)\right)\partial_\tau\,. \quad (3.19)$$

As far as RGS (3.19) appears as an approximate symmetry, the use of the invariant conditions (3.7), which in this case are two partial DEs

$$\chi\chi_w - a\tau\chi_n = 0\,, \quad 2\chi\tau_w - 2a\tau\tau_n + 1 + \frac{a\tau^2}{n} = 0\,, \quad (3.20)$$

produces approximate analytical expressions for the solution of BVP

$$x^2 = (ant^2 - \ln n)\left[1 - P(\sqrt{ant^2})\right]^2, \quad v = -\frac{x}{t}\frac{P(\sqrt{ant^2})}{\left[1 - P(\sqrt{ant^2})\right]}\,,$$

$$P(z) = 2ze^{-z^2/2}\int_0^z dt e^{t^2/2}\,. \quad (3.21)$$

Another much more accurate result for the gaussian beam is obtained when the transformation of the nonlinearity parameter a is taken into account. The corresponding RGS is constructed by further modification of the approach described above (the mathematics omitted here can be found in [13]) and is presented by the following RGO of a point symmetry group

$$R = 2\tau\partial_w + 2n\chi\partial_n + 2a\chi\partial_a - \partial_\chi\,. \quad (3.22)$$

For this, RGS invariance conditions (3.7) have the form of two first-order partial DEs

$$a\chi\tau_a + n\chi\tau_n + \tau\tau_w = 0\,, \quad 1 + 2a\chi\chi_a + 2n\chi\chi_n + 2\tau\chi_w = 0\,. \qquad (3.23)$$

Solving these DEs in view of (3.3) and of evident additional conditions of the absence of phase front distortions at the beam axis yields the desired solution of BVP

$$x^2 = \left(1 - 2ant^2\right)^2 \ln\frac{1}{n(1 - ant^2)}\,, \quad v = -\frac{2x\,ant}{1 - 2ant^2}\,. \qquad (3.24)$$

The peculiarity of this solution is that using it in the basic DEs (3.3) yields relations that vanish at $x = 0$. This means that the solution (3.24) appears to be exact at the beam axis and leads to the following evolution law for the laser beam intensity $n(t, 0)$ inside the nonlinear medium

$$n(t,0) = \frac{1}{2at^2}\left(1 - \sqrt{1 - 4at^2}\right)\,. \qquad (3.25)$$

The maximum value $n_{max} = 2$ is achieved at $t_{max} = 1/(2\sqrt{a})$. The three-dimensional plots, illustrating the behavior of this solution for $a = 1/10$, are presented on Fig.2a,b.

The comparison of the approximate solutions (3.21) and (3.24) shows good agreement of the numerical values obtained for these solutions in the off-axis region. The main discrepancy (though not large) is observed on the axis and in the near axis region in the vicinity of a singularity of a solution. Namely, for the approximate solution (3.21) the singularity point (with $n_x \to \infty$) is shifted inside the medium, $t_{appr} = 1/\sqrt{ae}$, with respect to its position for the solution (3.24), and the maximum value of the beam intensity $n_{appr} = e$ for the approximate solution (3.21) is greater than that for the solution (3.24). Note that qualitatively the behavior of the BVP solution for the gaussian beam is quite similar to that of Khokhlov's solution. Moreover, the coordinate of the singularity point and the laser beam intensity at this point have the same numerical values in both cases.

4. Conclusion

An original approach to improving the solution to the boundary value problem (BVP) in classical mathematical physics has been outlined and

applied to the laser beam physics. It is based upon the notion of specific continuous symmetry, the *Renormalization Group Symmetry* (RGS), taken from the modern theoretical physics. The RGS is essentially based on continuous transformations involving boundary parameter(s). The procedure of constructing RGS exploits the apparatus of classical and modern group analysis being different from the methods of the RG devising traditionally used in various fields of theoretical physics[15]. We have reviewed it and illustrated for a simple BVP, Eqs.(3.1), related to geometric optics.

On its basis we have discussed the application of our new method to several problems of laser beam propagating in nonlinear medium and found solutions of the three particular BVPs which differ in the form of boundary conditions. Two of them – the well-known Khokhlov's solution (3.12) and the newly obtained solution (3.16), (Fig.1a,b) – are exact solutions corresponding to the "soliton-like" (3.9) and "smoothed" step (3.13) laser beam intensity distribution at $t = 0$. The third, approximate solution of BVP (3.24), (Fig.2a.b) describes evolution of a gaussian beam.

Naturally, solutions of BVPs discussed in this paper do not exhaust possibilities of the RG approach in the field of geometric optics.

Firstly, within the framework of the mathematical model under consideration it can be successfully employed to find approximate or by chance exact solutions of BVP for practically any smooth boundary conditions. In the case of approximate solutions, this approach enables one to calculate successive higher-order corrections to RGSs and, hence, to improve these solutions.

Secondly, this approach is useful even if a more complicated mathematical model is used which takes into account different physical effects such as laser beam diffraction, cylindrical laser beam geometry, various types of media nonlinearity, nonstationary effects, etc. For instance, the approximate RGS for the axial-symmetrical laser beam is described by RGO (3.19) in which one should replace $a \to 2a$, and the corresponding approximate solution of the BVP results from (3.21) in view of the substitution $a \to 2a$, $P \to P/2$. Examples of the BVP solutions of the equations of (3.1)-type for the medium with rational nonlinearity functions were discussed in Ref.[16].

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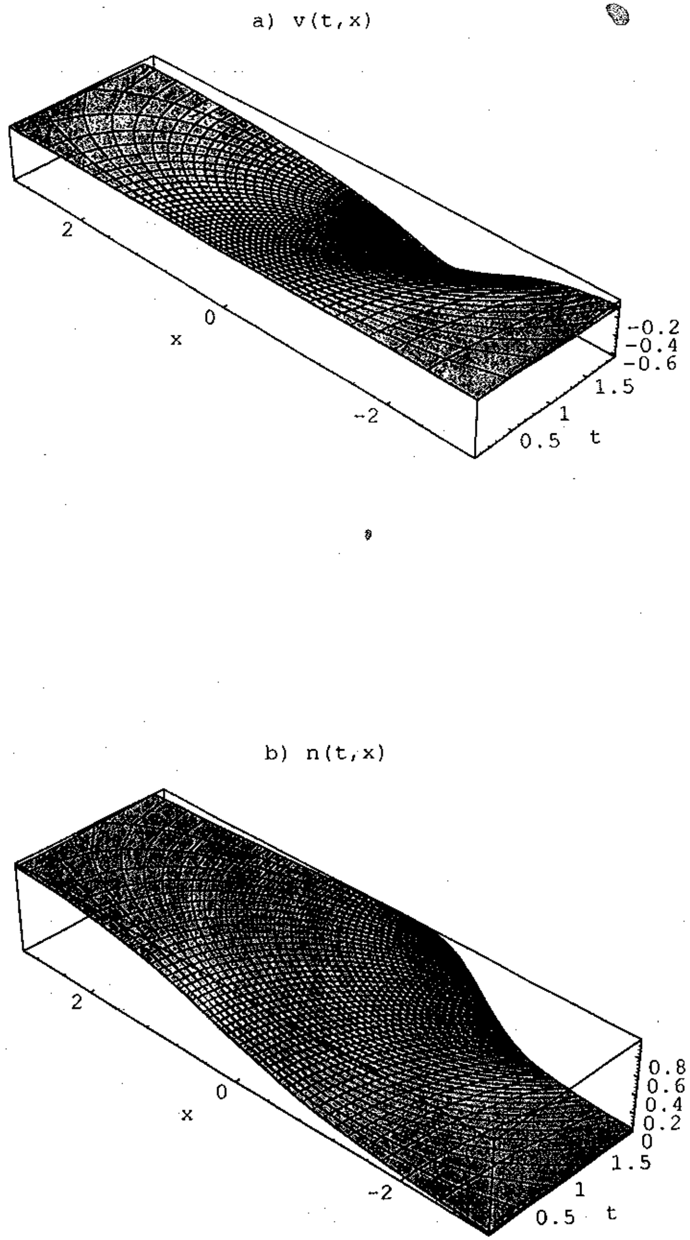

Figure 1. The three-dimensional plots of the functions $v(t, x)$ (a) and $n(t, x)$ (b) for the new exact analytical solution Eq.(3.16) for $a = 1$.

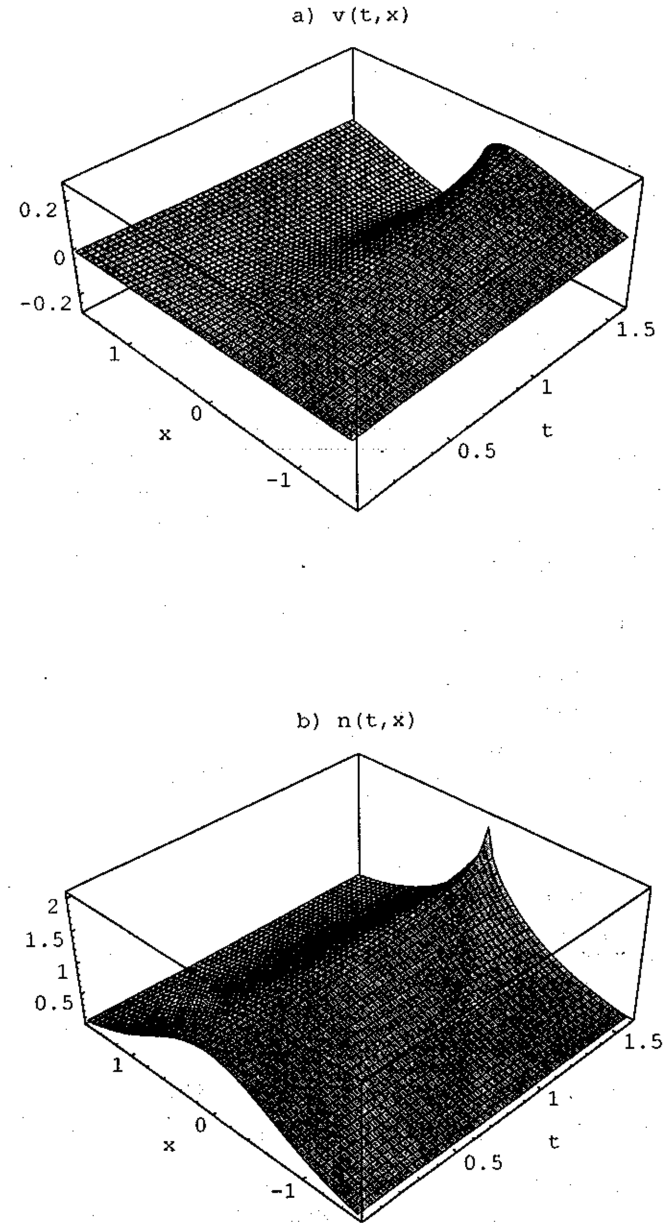

Figure 2. The three-dimensional plots of the functions $v(t, x)$ (a) and $n(t, x)$ (b) for the new approximate analytical solution Eq.(3.24) for the gaussian beam at $a = 1/10$.

References

[1] E.E.C. Stueckelberg and A. Petermann, *Helv. Phys. Acta* **22**, 499-520 (1953) ; M. Gell-Mann and F. Low, *Phys. Rev.* **95**, 1300-1312 (1954).

[2] N.N. Bogoliubov and D.V. Shirkov, *Dokl. Akad. Nauk. SSSR* **103**, 203-206 (1955); *ibid.*, **103**, 391-394 (1955).

[3] N.N. Bogoliubov and D.V. Shirkov, *Nuovo Cim.* **3**, 845-863 (1956).

[4] N.H.Ibragimov *Transformation groups applied to mathematical physics* (Riedel, Dordrecht, 1985).

[5] L.V. Ovsyannikov, *Group analysis of differential equations* (Academic Press, N.-Y., 1982) ; *CRC Handbook of Lie Group Analysis of Differential Equations*, Ed.N.H.Ibragimov, CRC Press, Boca Raton, Florida, USA. Vol.1: Symmetries, Exact Solutions and Conservation Laws, 1994; Vol.2: Applications in Engineering and Physical Sciences, 1995; Vol.3: New Trends in Theoretical Developments and Computational Methods, 1996.

[6] K. Wilson and J. Kogut, "The Renormalization Group and the ε expansion", *Phys. Rep.* **C 12** (1974) pp. 76 - 199 ; P.G. de Gennes, *Scaling Concepts in Polymer Physics* (Ithaca, 1979); J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, (Clarendon Press, Oxford, 1978; 1989);

[7] D.V. Shirkov, *Sov.Phys.Dokl.* **27**, 197-200 (1982); *Theor.& Math. Phys.* **60(2)**, 778-782 (1984).

[8] N. Bogoliubov and D. Shirkov, *Introduction to the Theory of Quantized Fields*, 3rd Russian ed. (Wiley-Interscience, N.Y., 1980).

[9] D.V. Shirkov, *Intern.J.Mod. Physics* **A3**, 1321-1342 (1988).

[10] V.F. Kovalev, V.V. Pustovalov, D.V. Shirkov, "Group analysis and renormgroup symmetries", JINR preprint E5-96-209, Dubna (1996).

[11] D.V. Shirkov, "Several topics on Renormgroup Theory"; V.F. Kovalev, S.V. Krivenko, V.V. Pustovalov, "The Renormalization group method based on group analysis", in *"Renormalization group-91", (Proc. of Second Intern. Conf., Sept. 1991, Dubna, USSR*, Ed. D.V.Shirkov & V.B.Priezzhev, WS, Singapore, 1992, pp 1-10; 300-314.

[12] S.A. Akhmanov, R.V. Khokhlov and A.P. Sukhorukov, *Sov. Phys. JETP* **23**, No 6, 1025-1033 (1966).

[13] V.F. Kovalev, "Renormgroup symmetries in problems of nonlinear geometrical optics", JINR Preprint P5-96-477, Dubna (1996).

[14] V.A. Baikov, R.K. Gazizov, N.H. Ibragimov, *J.Sov.Math.*, **55(1)**, 1450 (1991).

[15] D.V. Shirkov, *Russian Math. Surveys* **49:5**, 155-176 (1994); see also the Second English printing of this paper - Joint Institute for Nuclear Research(JINR) preprint E2-96-15, Dubna, 1996 - with corrected misprints.

[16] Kovalev V.F., *Journal of Nonlinear Mathematical Physics*, **3** (3-4), 351 (1996).

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