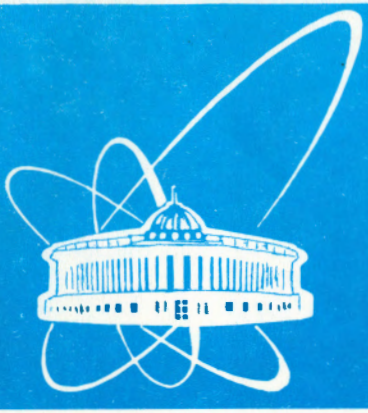

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q-DEFORMED SCHRÖDINGER EQUATION

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q-деформированное уравнение Шредингера

Найдена эрмитова реализация радиуса вектора $\vec{r}$, углового момента $\vec{\Lambda}$ и линейного импульса $\vec{p}$, в которой данные величины ведут себя как векторы относительно алгебры $SU_q(2)$. Эта алгебра порождается L_0 и $L_\pm$. Данная реализация используется для написания q-деформированного уравнения Шредингера, решения которого для случая кулоновского и осцилляторного потенциалов кратко обсуждаются.

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q-Deformed Schrödinger Equation

We found hermitian realization of the position vector $\vec{r}$, angular momentum $\vec{\Lambda}$ and linear momentum $\vec{p}$ behaving like vectors with respect to the $SU_q(2)$ algebra, generated by L_0 and $L_\pm$. They are used to write the q-deformed Schrödinger equation, whose solutions for Coulomb and oscillator potentials are briefly discussed.

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I. INTRODUCTION

The quantum mechanics of a point-like particle is constructed starting with two vectors: the position vector, $\vec{r}$, and the linear momentum, $\vec{p} = -ih\frac{\partial}{\partial \vec{r}}$, having the well known commutation relations. These two vectors are used to build all the other quantities, like the angular momentum, the interacting potential, etc. In general, these new quantities are noncommutative, their commutation relations being determined by the commutation relations satisfied by $\vec{r}$ and $\vec{p}$.

In a q-deformed quantum mechanics the commutation relations between the generators of the $SU_q(2)$ algebra, $\vec{L}$, and the position vector, $\vec{r}$, are well defined and it is natural to take these vectors as the basic quantities from which all the other must be built.

Wishing to build a q-deformed Schrödinger Hamiltonian we searched for a realization of $\vec{p}$ entering the kinetic energy term. First it was necessary to find a realization for $\vec{r}$ and for $\vec{L}$ as self-adjoint quantities obeying the known commutation relations. Then one has to look for a realization of $\vec{p}$ in terms of $\vec{r}$ and of $\vec{L}$. We found that $\vec{p}$ can be written as a sum of two terms which are respectively parallel and perpendicular to $\vec{r}$. The first one is assumed to have the simplest form and is written as $-i\frac{\vec{r}}{r^2}\,(r\frac{\partial}{\partial r}+1)$ while the second one is expressed as a vector product of $\vec{r}$ and of $\vec{L}$.

The paper is organized as follows: Section II contains the general commutation relations involving the q-angular momentum and some quantities having definite transformation properties with respect to the $SU_q(2)$ algebra, like the invariants C, c and the vector $\vec{\Lambda}$. In the third section we give a realization of the position vector, $\vec{r}$, and of the q-angular momentum[1,2,3], $\vec{L}$, in terms of the polar coordinates r, $x_0 = \cos\theta$, φ. The realization of the linear momentum $\vec{p}$ is given in the fourth section. We first build the part perpendicular to $\vec{r}$, denoted $\vec{\partial}$, using the cross product $\vec{r}\times\vec{\Lambda}$ and find that it satisfies[1] some commutation relations similar to those satisfied by $\vec{r}$. The part parallel to $\vec{r}$, supposed to have the simplest form, is just that coming from the ordinary $\frac{\partial}{\partial \vec{r}}$. Section V contains the eigen functions of the q-angular momentum[2,3], written like series in $x_0 = \cos\theta$. The result is a generalization of the hypergeometric functions ${}_2\mathrm{F}_1(a,b,c;\frac{1}{2};x_0^2)$ and ${}_2\mathrm{F}_1(a,b,c;\frac{3}{2};x_0^2)$ which can be related to the q-deformed spherical functions $Y_{lm}(q,x_0,\varphi)$. Some properties and relations satisfied by the eigen functions are also listed. In the last section the q-deformed Schrödinger equation[1,4,5] with scalar potential is given. Its solutions for Coulomb and three dimensional oscillator potentials are briefly discussed.

II. THE q-ANGULAR MOMENTUM

The $SU_q(2)$ algebra is generated by three operators L_+, L_- and L_0, also named

the q-angular momentum, having the following commutation relations:

$$[\, L_0 \,,\, L_\pm \,] \;=\; \pm\, L_\pm \tag{1}$$

$$[\, L_+ \,,\, L_- \,] \;=\; [2\, L_0] \tag{2}$$

where the quantity in square brackets is defined as

$$[n] \;=\; \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{3}$$

In the following we shall introduce quantities having definite transformation properties with respect to the $SU_q(2)$ algebra. They will be further used to build q scalars and q vectors, like, for instance, the q linear momentum, entering the expression of the hamiltonian operator.

First of all we remind that $SU_q(2)$ algebra has an invariant, C, called the Casimir operator

$$\mathrm{C} \;=\; L_-\, L_+ \;-\; [L_0]\; [L_0 \,+1] \tag{4}$$

whose eigenvalue in the $(2l+1)$ dimensional irreducible representation is:

$$\mathrm{C}_l = [l]\; [l+1]. \tag{5}$$

A vector in this algebra is a set of three quantities v_k, $k = \pm 1, 0$ satisfying the following relations:

$$[\, L_0 \,,\, v_k \,] \;=\; k\, v_k \tag{6}$$

$$\left[\, L_\pm\, v_k \;-\; q^k\, v_k\, L_\pm \,\right]\, q^{L_0} \;=\; \sqrt{[2]}\; v_{k\pm 1} \tag{7}$$

where $v_{\pm 2}$ must be set equal to zero in the right-hand side for $k = \pm 1$.

By comparing the relations (1), (2) with (6), (7), we observe that, unlike the $SU(2)$ algebra, the operators L_k do not represent the components of a vector in the above sense. However, one can use $L_\pm$ and L_0 to define a vector, $\vec{\Lambda}$, in the following manner:

$$\Lambda_{\pm 1} \;=\; \frac{\mp 1}{\sqrt{[2]}}\; q^{-L_0}\; L_\pm \tag{8}$$

$$\Lambda_0 \;=\; \frac{1}{[2]}\, \left(q\, L_+\, L_- \;-\; q^{-1}\, L_-\, L_+\right). \tag{9}$$

It is now an easy matter to show that Λ_k satisfy the relations (6) and (7) as required.

Two vectors $\vec{u}$ and $\vec{v}$ can be used to build a scalar, S, according to the following definition:

$$S \;=\; \vec{u}\,\vec{v} \;=\; -\frac{1}{q}\, u_1\, v_{-1} \;+\; u_0\, v_0 \;-\; q\, u_{-1}\, v_1. \tag{10}$$

By introducing a generalization of the cross product, the vectors can be used also to build a new vector, as it will be further shown.

In the case $\vec{u} = \vec{v} = \vec{\Lambda}$, the scalar product $\vec{\Lambda}^2$ defines a second invariant[6], C′, which is not independent of C. The eigenvalue of C′ is

$$\mathrm{C}'_l \;=\; \frac{[2l]}{[2]}\; \frac{[2l+2]}{[2]}\,. \tag{11}$$

In this paper a third invariant, c, defined as

$$\mathrm{c} \;=\; q^{-2L_0} \;+\; \lambda\; \Lambda_0 \tag{12}$$

with

$$\lambda \;=\; q - \frac{1}{q} \tag{13}$$

will be frequently used in order to write the formulae in a more compact form. Its eigenvalue is:

$$\mathrm{c}_l \;=\; \frac{q^{2l+1} \;+\; q^{-2l-1}}{[2]}. \tag{14}$$

It is worth noticing that, in the limit $q = 1$ the first and second invariants C, C′ both go into the Casimir invariant C=$\vec{L}^2 = l(l+1)$, while the third one, c, becomes equal to unity. The results listed in this section are valid for any realization of the $SU_q(2)$ algebra.

III. THE POSITION VECTOR $\vec{r}$ AND A REALIZATION OF $L_\pm$ AND L_0

In $R_q(3)$ space the position vector $\vec{r}$ has three noncommutative components r_1, r_{-1} and r_0, satisfying the following relations:

$$r_0\, r_{\pm 1} \;=\; q^{\mp 2}\, r_{\pm 1}\, r_0 \tag{15}$$

$$r_1\, r_{-1} \;=\; r_{-1}\, r_1 \;+\; \lambda\, r_0^2. \tag{16}$$

The quantity r^2 defined as

$$r^2 \;=\; \vec{r}^{\,2} \;=\; -\frac{1}{q}\, r_1\, r_{-1} \;+\; r_0^2 \;-\; q\, r_{-1}\, r_1 \tag{17}$$

commute with all r_i and with all L_i if $\vec{r}$ satisfies the conditions (6) and (7) to be a vector. For $q = 1$ the scalar r is nothing else than the length of the position vector $\vec{r}$. We shall keep this meaning also for $q \neq 1$.

Searching for the concrete realization of $\vec{r}$, $L_\pm$ and of L_0, we begin by expressing L_0 like in $R(3)$ case:

$$L_0 = -i \frac{\partial}{\partial \varphi}. \tag{18}$$

The next step is to write $\vec{r}$ as a product of r and of a unit vector, $\vec{x}$, depending on angles. We put:

$$r_{\pm 1} = r\, x_{\pm 1} \tag{19}$$

$$r_0 = r\, x_0. \tag{20}$$

It remains now to find a realization of $x_{\pm 1}$ in terms of the azimuthal angle φ and of x_0, which is in fact equal to $\cos\theta$, just as in $R(3)$ case. We found:

$$x_1 = -e^{i\varphi} \sqrt{\frac{q}{[2]}} \sqrt{1 - q^2\, x_0^2}\, q^{2N_0} \tag{21}$$

$$x_{-1} = e^{-i\varphi} \sqrt{\frac{1}{[2]q}} \sqrt{1 - q^{-2}\, x_0^2}\, q^{-2N_0} \tag{22}$$

where the dilatation operator N_0 satisfying the relation

$$[\, N_0\, ,\, x_0^n\,] = n\, x_0^n \tag{23}$$

and having the hermiticity property

$$N_0^+ = -N_0 - 1 \tag{24}$$

has been introduced in the expressions (21) and (22) in order to fulfil the commutation relations (15) and (16).

Taking now into account the relations (19-24), and assuming

$$x_0^+ = x_0 \tag{25}$$

we get for $x_\pm$ the normal hermiticity properties:

$$x_1^+ = \frac{-1}{q}\, x_{-1} \tag{26}$$

$$x_{-1}^+ = -q\, x_1. \tag{27}$$

All these arguments allow us to conclude that eqs.(19-23) define the realization of the position vector $\vec{r}$ in $R_q(3)$ space.

The last step is to search for a realization of the $SU_q(2)$ generators. The expressions we propose for L_+ and L_- are:

$$L_+ = \sqrt{[2]}\, e^{i\varphi}\, \tilde{x}_1^{L_0+1}\, \frac{1}{x_0}\, \frac{1 - q^{-2N_0}}{1 - q^{-2}}\, \tilde{x}_1^{-L_0}\, q^{L_0} \tag{28}$$

$$L_- = \sqrt{[2]}\, e^{-i\varphi}\, \tilde{x}_{-1}^{-L_0+1}\, \frac{1}{x_0}\, \frac{1 - q^{2N_0}}{1 - q^2}\, \tilde{x}_{-1}^{L_0}\, q^{L_0} \tag{29}$$

where $\tilde{x}_{\pm 1} = e^{\mp i\varphi}\, x_{\pm 1}$ depends on x_0 only. Looking at the expressions (28) and (29) it becomes clear why the phase factor is removed from $x_{\pm 1}$: expressions like $x_{\pm 1}^{L_0}$ have no meaning, while $\tilde{x}_{\pm 1}^{L_0}$ is well defined.

Considering now the action of the operator L_+ on the non-normalized eigen function of L_0 and of the Casimir operator C

$$\tilde{Y}_{lm}(q, x_0, \phi) = e^{im\varphi}\, \tilde{x}_1^m\, \Theta_{lm}(x_0) \tag{30}$$

we notice that $\tilde{x}_1^{-L_0}$ in (28) removes the factor $\tilde{x}_1^m$ in $\tilde{Y}_{lm}(q, x_0, \phi)$. In this way one prevents q^{-2N_0} from acting on $\tilde{x}_1$ and producing a troublesome result. The operator q^{-2N_0} in L_+ acts then on $\Theta_{lm}(x_0)$ only and $e^{i\varphi}\, \tilde{x}_1^{L_0+1}$ creates the factor x_1^m in right place.

In the well known $R(3)$ theory of angular momentum a different mechanism prevents $\frac{\partial}{\partial\theta}$ in L_+ from acting on x_1^m: the term given by $\frac{\partial}{\partial\theta}\, x_1^m$ is exactly cancelled by $i\mathrm{ctg}\theta \frac{\partial}{\partial\varphi}\, x_1^m$, finally remaining only the derivative $\frac{\partial}{\partial\theta}\Theta_{lm}(x_0)$.

It can be verified that the expressions (18), (28) and (29) satisfy the commutation relations (1) and (2) and hence one can conclude that they are the realization of the $SU_q(2)$ generators in $R_q(3)$ space. It can also be checked that the position vector $\vec{r}$ defined in (19-22) behaves really like a vector in this $SU_q(2)$ algebra, since it satisfies the relations (6) and (7) with $L_\pm$ given by (28) and (29).

IV. THE LINEAR q-MOMENTUM $\vec{p}$

In order to write down an expression for the linear momentum $\vec{p}$, we separate it into a part perpendicular and another parallel to $\vec{x}$. The first one is defined with the aid of the cross product $\vec{x} \times \vec{L}$ and the second one is assumed to have the form $\vec{x}\, \frac{1}{r}\, f\left(r \frac{d}{dr} + 1\right)$, where f is a function which will be defined in the following. The components of the transverse part, denoted ∂_k, write

$$\partial_1 = q^{-1}\, x_1\, \Lambda_0 - q\, x_0\, \Lambda_1 + x_1\, c \tag{31}$$

$$\partial_0 = x_1\, \Lambda_{-1} - \lambda\, x_0\, \Lambda_0 - x_{-1}\, \Lambda_1 + x_0\, c \tag{32}$$

$$\partial_{-1} = -q\, x_{-1}\, \Lambda_0 + q^{-1}\, x_0\, \Lambda_{-1} + x_{-1}\, c \tag{33}$$

where c is the invariant defined in eq.(12) and the terms $x_k\, c$ have been added to the cross product $\vec{x} \times \vec{\Lambda}$ in order to ensure the well defined character with respect to the hermitian conjugation operation

$$\partial_k^+ = -\left(-\frac{1}{q}\right)^k \partial_{-k}. \tag{34}$$

It can be checked that the quantities ∂_k satisfy the following relations:

$$\partial_0\partial_1 \;=\; q^{-2}\,\partial_1\partial_0 \tag{35}$$

$$\partial_0\partial_{-1} \;=\; q^{2}\,\partial_{-1}\partial_0 \tag{36}$$

$$\partial_1\partial_{-1} \;=\; \partial_{-1}\partial_1 \;+\; \lambda\cdot\,\partial_0^2\,. \tag{37}$$

Eq.(35) has been obtained by commuting ∂_0 with ∂_1, and eq.(36) is the hermitian conjugate of the above one, while eq.(37) can be obtained either from eq.(35) or (36) by using eq. (7).

Also, by multiplying equations (31-33) with the corresponding x_k and taking into account the commutation relations (15,16) one gets:

$$\vec{x}\,\vec{\partial} \;=\; -\vec{\partial}\,\vec{x} = c\,. \tag{38}$$

By commuting the invariant c with $\vec{x}$ one finds:

$$\vec{\partial} \;=\; \lambda^{-2}\;[c,\vec{x}]\,. \tag{39}$$

Taking now the matrix elements of the last relation one obtains:

$$\left\langle\, l+1\;m' \mid \vec{\partial} \mid l\;m \right\rangle \;=\; \frac{[2l+2]}{[2\,]}\;\langle\, l+1\;m' \mid \vec{x} \mid l\;m\rangle \tag{40}$$

$$\left\langle\, l-1\;m' \mid \vec{\partial} \mid l\;m \right\rangle \;=\; -\frac{[2l]}{[2\,]}\;\langle\, l-1\;m' \mid \vec{x} \mid l\;m\rangle\,. \tag{41}$$

From parity arguments one can also write:

$$\langle l\;m' \mid \partial_k \mid l\;m\rangle \;=\; 0. \tag{42}$$

By replacing the matrix elements of $\vec{\partial}$ with those of $\vec{x}$ with the aid of eqs. (40) and (41) one can obtain the eigenvalues of $\vec{\partial}^2$:

$$\left\langle\, l\;m \mid \vec{\partial}^2 \mid l\;m \right\rangle \;=\; -\frac{[2l]}{[2]}\;\frac{[2l+2]}{[2]} \;-\; c_l^2. \tag{43}$$

Taking into account all these relations, the realization we found for the linear momentum $\vec{p}$ is:

$$\vec{p} \;=\; \frac{-i}{r}\left(\vec{x}\,(r\frac{\partial}{\partial r} \;+\; 1) - \vec{\partial}\right). \tag{44}$$

Then, in the $(2l+1)$ dimensional representation $\vec{p}^{\,2}$ writes:

$$\vec{p}^{\,2} \;=\; -\frac{1}{r}\,\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r} \;+\; 1\right) \;+\; \frac{1}{r^2}\left(\frac{[2l]}{[2]}\;\frac{[2l+2]}{[2]} \;+\; c_l^2 \;-\; c_l\right) \tag{45}$$

which in the limit $q = 1$ becomes equal to the radial part of the Laplace operator.

V. THE q-ANGULAR MOMENTUM EIGEN FUNCTIONS

The eigen vectors $\Phi_{lm}(q, x_0, \varphi)$ of the $(2l+1)$ dimensional irreducible representation of the q-angular momentum are eigen functions of L_0 and of the Casimir operator C. We begin by writing them as a polynomial in x_0 multiplied by x_1^m:

$$\Phi_{lm}(q, x_0, \varphi) \;\doteq\; x_1^m \sum_{k\geq 0} a_k\; x_0^k \tag{46}$$

where the sum extends over odd and even k-values for odd $(l-m)$ and respectively even $(l-m)$.

The equation:

$$L_+\;L_-\;\Phi_{lm}(q, x_0, \varphi) \;=\; [l+m]\;[l-m+1]\;\Phi_{lm}(q, x_0, \varphi) \tag{47}$$

gives the recursion relation:

$$a_{k+2} \;=\; -q^{-2m}\;\frac{[l-m-k]\;[l+m+k+1]}{[k+1]\;[k+2]}\;a_k. \tag{48}$$

For $(l-m)$ even we obtain:

$$\Phi_{lm}(q, x_0, \varphi) \;=\; x_1^m\;\left\{1 - \frac{[l-m][l+m+1]}{[2]!}\;\left(q^{-m}x_0\right)^2\right.$$

$$\left. +\frac{[l-m][l-m-2][l+m+1][l+m+3]}{[4]!}\left(q^{-m}x_0\right)^4 - \ldots\right\} \tag{49}$$

while for $(l-m)$ odd we get:

$$\Phi_{lm}(q, x_0, \varphi) \;=\; x_1^m\;\left\{\frac{1}{[1]!}\left(q^{-m}x_0\right) - \frac{[l-m-1][l+m+2]}{[3]!}\;\left(q^{-m}x_0\right)^3\right.$$

$$\left. +\frac{[l-m-1][l-m-3][l+m+2][l+m+4]}{[5]!}\left(q^{-m}x_0\right)^5 - \ldots\right\}. \tag{50}$$

In order to express these results in terms of a q-hypergeometric series it is necessary to write all the q-numbers $[n]$ in the form

$$[n] \;=\; \frac{q^n - q^{-n}}{q - q^{-1}} \;=\; [2]\;\frac{(q^2)^{\frac{n}{2}} \;-\; (q^2)^{-\frac{n}{2}}}{q^2 - q^{-2}} \;=\; [2]\;\left[\frac{n}{2}\right]_{q^2}. \tag{51}$$

For $(l-m)$ even we have then:

$$\Phi_{lm}(q,x_0,\varphi) \;=\; x_1^m \;{}_2\mathrm{F}_1\left(\, q^2\,;\, \frac{l+m+1}{2}\,,\, \frac{-l+m}{2}\,;\, \frac{1}{2}\,;\, q^{-m}x_0^2\,\right) \tag{52}$$

while for $(l-m)$ odd we get:

$$\Phi_{lm}(q,x_0,\varphi) \;=\; x_1^m\; q^{-m}\; x_0\; {}_2\mathrm{F}_1\left(\, q^2\,;\, \frac{l+m+2}{2}\,,\, \frac{-l+m+1}{2}\,;\, \frac{3}{2}\,;\, q^{-m}x_0^2\,\right). \tag{53}$$

The index q^2 in ${}_2\mathrm{F}_1$ means that all the q-numbers in the series development of ${}_2\mathrm{F}_1$ must be calculated with q^2 instead of q.

We found that $\Phi_{lm}(q,x_0,\varphi)$ satisfies the following simple relations:

$$x_1\,\frac{1}{x_0}\,\frac{1-q^{-2N_0}}{1-q^{-2}}\;\Phi_{lm}(q,x_0,\varphi) \;=\; -[l-m]\;[l+m+1]\;\Phi_{l\;m+1}(q,x_0,\varphi), \tag{54}$$

for $(l-m)$ even, and

$$x_1\,\frac{1}{x_0}\,\frac{1-q^{-2N_0}}{1-q^{-2}}\;\Phi_{lm}(q,x_0,\varphi) \;=\; \Phi_{l\;m+1}(q,x_0,\varphi), \tag{55}$$

for $(l-m)$ odd.

The normalized eigen functions are:

$$Y_{lm}(q,x_0,\varphi) = (-1)^{\frac{l-m}{2}}\sqrt{\frac{[2l+1]}{4\pi}}\left(\frac{[l-m-1]!!}{[l-m]!!}\,\frac{[l+m-1]!!}{[l+m]!!}\right)^{1/2}[2]^{\frac{m}{2}}\Phi_{l\;m}(q,x_0,\varphi), \tag{56}$$

for even $(l-m)$, and

$$Y_{lm}(q,x_0,\varphi) = (-1)^{\frac{l-m-1}{2}}\sqrt{\frac{[2l+1]}{4\pi}}\left(\frac{[l-m]!!}{[l-m-1]!!}\,\frac{[l+m]!!}{[l+m-1]!!}\right)^{1/2}[2]^{\frac{m}{2}}\Phi_{l\;m}(q,x_0,\varphi) \tag{57}$$

for odd $(l-m)$ and the orthogonality relation is written as:

$$\int Y^{+}_{l'm'}(q,x_0,\varphi)\;Y_{lm}(q,x_0,\varphi)\;d\varphi\;d[x_0] \;=\; \delta_{ll'}\;\delta_{mm'} \tag{58}$$

where the integral over φ is a normal one, while the integrals over $d[x_0]$ are taken as:

$$\int_0^1 x_0^n\;d[x_0] \;=\; \frac{1}{[n+1]}. \tag{59}$$

For the negative interval $(-1,0)$ from parity arguments we take:

$$\int_{-1}^0 x_0^n\;d[x_0] \;=\; (-1)^n\;\frac{1}{[n+1]}. \tag{60}$$

The relation (59) is in fact the result of a discrete integration of $f(x_0)=x_0^n$, performed by dividing the integration interval $(0,1)$ with an infinite set of points $x_k=q^k$ for $q<1$ and writing

$$\int_0^1 f(x_0)\;d[x_0] \;=\; \sum_{k=0}^{\infty} f(x_{2k+1})\;(x_{2k}\;-\;x_{2k+2})\;. \tag{61}$$

Looking now for the properties of Y_{lm}, we found that, just as in the $R(3)$ case, the product $x_k\;Y_{lm}$ can be expressed in terms of $Y_{l\pm1\;m+k}$ as follows:

$$x_1\;Y_{lm}(q,x_0,\varphi) \;=\; q^{l-m}\sqrt{\frac{[l+m+1][l+m+2]}{[2][2l+1][2l+3]}}\;Y_{l+1\;m+1}(q,x_0,\varphi)$$

$$-\;q^{-l-m-1}\sqrt{\frac{[l-m][l-m-1]}{[2][2l+1][2l-1]}}\;Y_{l-1\;m+1}(q,x_0,\varphi) \tag{62}$$

$$x_0\;Y_{lm} \;=\; q^{-m}\cdot\sqrt{\frac{[l-m+1][l+m+1]}{[2l+1][2l+3]}}\;Y_{l+1\;m}(q,x_0,\varphi)$$

$$-\;q^{-m}\sqrt{\frac{[l-m][l+m]}{[2l+1][2l-1]}}\;Y_{l-1\;m}(q,x_0,\varphi) \tag{63}$$

$$x_{-1}\;Y_{lm}(q,x_0,\varphi) \;=\; q^{l-m}\sqrt{\frac{[l-m+1][l-m+2]}{[2][2l+1][2l+3]}}\;Y_{l+1\;m-1}(q,x_0,\varphi)$$

$$-\;q^{l-m+1}\sqrt{\frac{[l+m][l+m-1]}{[2][2l+1][2l-1]}}\;Y_{l-1\;m-1}(q,x_0,\varphi). \tag{64}$$

In addition, we have three relations which express the noncommutativity of x_k with Y_{lm} and represent a generalization of the equations ():

$$x_0\;Y_{lm}(q,x_0,\varphi) \;=\; q^{-2m}\;Y_{lm}(q,x_0,\varphi)\;x_0 \tag{65}$$

$$x_1\;Y_{lm}(q,x_0,\varphi) \;=\; Y_{lm}(q,x_0,\varphi)\;x_1$$

$$+\;\frac{\lambda}{\sqrt{[2]}}\;q^{-m-1}\;\sqrt{[l-m][l+m+1]}\;Y_{l\;m+1}(q,x_0,\varphi)\;x_0 \tag{66}$$

$$x_{-1}\, Y_{lm}(q,x_0,\varphi) \;=\; Y_{lm}(q,x_0,\varphi)\, x_{-1}$$
$$-\frac{\lambda}{\sqrt{[2]}}\, q^{-m+1}\, \sqrt{[l+m][l-m+1]}\; Y_{l\,m-1}(q,x_0,\varphi)\, x_0. \qquad (67)$$

The last two equations have been obtained from the first one with the aid of L_+ and L_- which rise and lower the index m of Y_{lm}.

VI. q-DEFORMED SCHRÖDINGER EQUATION

Taking into account all the above results, we assume that the Hamiltonian entering the q- deformed Schrodinger equation is:

$$\mathcal{H} \;=\; \frac{1}{2}\,\vec{p}^{\,2} \;+\; V(r) \qquad (68)$$

where the operator $\vec{p}$ has been defined in the fourth section. The eigen functions of this Hamiltonian are:

$$\Psi(r,x_0,\varphi) \;=\; r^L\; u_L(r)\; Y_{lm}(q,x_0,\varphi) \qquad (69)$$

where L is the solution of the following equation:

$$L(L+1) \;=\; \frac{[2l]}{[2]}\frac{[2l+2]}{[2]} \;+\; c_l^2 \;-\; c_l \qquad (70)$$

obtained from the condition that $u_L(r)$ is finite in the limit $r \to 0$.

This Schrödinger equation has simple solutions for Coulomb potential $V(r) = -\frac{1}{r}$ and for the oscillator potential $V(r) = \frac{1}{2}\, r^2$. The eigen values of the two Hamiltonians are:

$$E_{nl} \;=\; -\frac{1}{2(n+L+1)^2} \qquad (71)$$

for the Coulomb potential and respectively:

$$E_{nl} \;=\; (2n+L+\frac{3}{2}) \qquad (72)$$

for the oscillator potential, n being the radial quantum number and L the solution of the equation (70), usually not an integer. We notice that the spectrum is degenerate with respect to the magnetic quantum number m. The solution of the wave equation which does not depend on θ and φ gives for the mean value of x_0^2 the value $R^2/[3]$ instead of $R^2/3$ obtained in the case of spherical symmetry. It results then that the quadrupole momentum as well as all the 2^{2n} poles are different from zero, although the

wave function does not depend on θ and φ. This shows clearly that the Hamiltonian loosed the spherical symmetry. One can mention however that it gained another, namely the symmetry under the $SU_q(2)$ algebra.

Finally, we remark that there are three sources producing the differences between the case of q-deformed Schrödinger equation and the case of spherical symmetry: The first one is that the q-harmonic wave function $Y_{lm}(q,x_0,\varphi)$ differs from the spherical one $Y_{lm}(\theta,\varphi)$. The second reason is that the coefficient of the centrifugal potential in the radial Schrödinger equation is $L(L+1)$, with L given by eq.(70), not $l(l+1)$, as in the sperical case. The third source is that in the q-deformed case the integral over x_0 is performed according to the relations (59-60).

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