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SCALE-DEPENDENT FUNCTIONS IN STATISTICAL HYDRODYNAMICS: A FUNCTIONAL ANALYSIS POINT OF VIEW

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1 Introduction

The problem of adequate mathematical description of hydrodynamic turbulence is one of the oldest but not yet solved problems in physics. The transition from laminar fluid flow to highly irregular chaotic regime was first discussed yet by Leonardo da Vinci. Later, when the Navier-Stokes equation (NSE)

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nu \Delta \mathbf{v}, \quad \nabla \cdot \mathbf{v} = 0, \tag{1}$$

which describes an incompressible fluid flow was written down, it was believed that (1) itself may contain all turbulence. It seems quite natural in the light of modern developments in dynamical chaos that even systems with a few degrees of freedom sometimes show unpredictable chaotic behavior.

The turbulence problem is much more complicated. The velocity *field* $\mathbf{v}(\mathbf{x}, t)$ has $3 \times 3D$ *continuum* degrees of freedom – therefor we have a field theory problem. The dynamical object, the velocity field $\mathbf{v}(\mathbf{x}, t)$ itself, is a square-integrable function defined on $\mathbb{R}^3 \times \mathbb{R}^1$ space.

The crucial step in turbulence theory was done by A. N. Kolmogorov in a number of short papers [Kol41b, Kol41c, Kol41a], known as the K41 theory. It was argued that the turbulence, as a chaotic phenomenon, should be described in terms of random functions $\mathbf{v}(\mathbf{x}, t, \cdot)$. (The ideas of statistical description of turbulence have been already discussed by Taylor[Tay35].) *It was suggested to consider the turbulence as a multi-scale phenomenon, formed of velocity fluctuations of all possible scales.* The typical size L of dominating fluctuations of average amplitude U is related to the Reynolds number $Re = \frac{UL}{\nu}$, which determines the transition from laminar to turbulent flow. Following [Fri95] we present the K41 hypothesis in the form

H1 In the limit $Re \to \infty$, all possible symmetries of the NSE, usually broken by the mechanisms reducing the turbulent flow, are restored in a statistical sense at small scales and away from boundaries.

H2 Under the same assumptions as in H1, the turbulent flow is self-similar at small scales, i.e.

$$\delta v(r, \lambda l) \stackrel{law}{=} \lambda^h \delta v(r, l) \tag{2}$$

where $v(r, \lambda l) = v(r + \lambda l) - v(r)$. (The equality-in-law means the coincidence of all statistical momenta.)

H3 Under the same assumptions as in H1 and H2, the turbulent flow has a finite non-vanishing rate ϵ of dissipation of energy per unit of mass.

Using these three assumptions Kolmogorov has derived the two-thirds law

$$\langle(\delta v(l))^2\rangle = C\epsilon^{2/3}l^{2/3}, \tag{3}$$

the basic empirical law of fully developed turbulence.

The scaling behavior (3) of the velocity field v means that we have to deal with functions of typical behavior

$$|v(r+l) - v(r)| \sim l^h \tag{4}$$

with $h \leq 1$, *i.e. with non-differentiable functions*; with nontrivial Hölder exponent $h = 1/3$, if the H3 hypothesis is strictly valid. The Hölder condition $|v(r+l) - v(r)| \leq l^h$ is not very rare in physical problems: it can be found in condensed matter physics, quantum field theory etc. — it becomes very significant only since the resolution (l) itself becomes a physical parameter, and that is why we ought to consider some objects of the form $v(x, l, \cdot)$, which are not yet defined. This problem is actively discussed amongst the turbulence community, but up to the author's knowledge it is beyond functional analysis research, except for the problems directly related to wavelet analysis [Hol94].

In the present paper we investigate the problem of resolution-dependent functions. The research was highly inspired by the fact that there are at least two approaches to resolution-dependent objects. The first approach, based on the decomposition with respect to the group of affine transformations, is known as wavelet analysis. The second one was exported from quantum field theory to turbulence theory, and is based on the consideration of the Fourier transforms of functions cut at some large momentum $k = \Lambda_{cutoff}$.

An attempt to consider the problem as a whole and join these two approaches is presented below.

2 Problem

Self-similarity is a synonym of scale-invariance. To be scale-invariant means to have same properties at different scales. Classical fractals are scale-invariant by construction. The Brownian motion is self-similar: if we look at the trajectory of a Brownian particle at different resolutions of a microscope we will observe more or less the same picture.

As physical systems are considered, the word self-similarity is more frequently attributed to their dynamics than geometry.

The self-similarity of hydrodynamic velocity field fluctuations $\langle(\delta v(l))^2\rangle \sim l^{2/3}$ is attributed to the behavior of the turbulent velocity field measured at different spatial scales. For the hydrodynamical velocity field it is physically clear that the measurement at scale l_0 necessarily implies averaging of molecular velocities over certain space domain of typical size l_0. This procedure can be generalized to "an averaging of a function up to scale l" [DG96]

$$\phi_l(\mathbf{x}) = l^{-D} \int_{|y|<l} \phi(\mathbf{x} - \mathbf{y}) d^D\mathbf{y}. \tag{5}$$

There are at least two conjectures here:

1. The existence of a "true" (with no scale) field $\phi_l(\mathbf{x}) : l \to 0$.
2. The homogeneity of the measure $d\mu(\mathbf{y}) = d^D\mathbf{y}$.

Physically, it is quite clear that two different fields $\phi_l(\mathbf{x})$ and $\phi_{l'}(\mathbf{x})$ live in different functional spaces if $l \neq l'$. It is meaningless, say, to subtract their values. Therefore, the velocity field of hydrodynamic turbulence is something more than a random vector field defined on $R^D \times R$.

To characterize the turbulent velocity at a certain point $\mathbf{x}$ we ought to know the collection of velocity values $\{\phi_l(\mathbf{x})\}$ at a set of scales labeled by l. The set of scales may be countable

$$l = l_0, kl_0, k^2l_0, k^3l_0, \ldots,$$

say, $k = 1/2$ for period-doubling decomposition, or continuous.

To characterize this set it was proposed in [DG96] to use a collection of unit fields at different scales – a "reference field" $\{\mathcal{R}_l(x)\}$.

The principal question arising here is how to describe the interaction of the fluctuations of different scales. Practically, this problem is often tackled by decomposition of "real" (out-of-scale) field into slow (large-scale) and fast (small-scale) components

$$\phi = V + v, \quad \text{where} \quad \mathrm{M}v = 0.$$

(M means the averaging, or mathematical expectation, here and after.) In this approach the slow component V governs the equation for v and the even-order moments of v contribute to the equation for V.

On the other hand, as we know from both the Kolmogorov theory and renormalization group (RG) approach, there are no absolute scales in hydrodynamics, except for dissipative scale and external scale (the size of the system). So, at least at this middle – the Kolmogorov range – the equations should be scale-covariant. The structure which reveals itself here looks like a fiber bundle over R^D, with leaves labeled by scale. The fluctuations of different scales may be dependent or independent of each other for various physical situations; but at least some similarity should be present.

To construct a basic system on this bundle let us follow the ideas of *multi-resolution analysis* (MRA) [Mal86]. Let us construct a system of functional subspaces $\{V_i : V_i \subset \mathcal{H}\}$, where $\mathcal{H}$ is a space of physical observables. Let the system $\{V_i\}$ be such that

1. $\ldots \subset V_2 \subset V_1 \subset V_0 \subset \ldots$
2. $\cap_\infty V_i = \emptyset, \quad \overline{\cup_\infty V_i} = \mathcal{H}.$
3. Subspaces V_i and V_{i+1} are similar: $f(x) \in V_j \leftrightarrow f(kx) \in V_{j-1}$, if $\{\phi_i(x)\}_{i\in I}$ forms basis in V_j then $\{\phi_i(kx)\}_{i\in I}$ forms the basis in V_{j-1}.

If the sequence $\{V_i\}$ is bounded from above, the maximal subspace is called the highest resolution space. Let it be V_0. Then any function from V_1 can be represented as a linear span of V_0 basic vectors. Therefore, the basis ϕ_0 of the highest resolution space provides a basis for the whole bundle.

It seems attractive to generalize MRA axioms to the case of continuous set of scales. Since the chain of subspaces described above

implies sequential coarse graining of the finest resolution field, some details are being lost in course of this process. The lost details can be stored into the set of orthogonal complements

$$V_0 = V_1 \oplus W_1, \quad V_1 = V_2 \oplus W_2, \ldots \tag{6}$$

So, $\mathcal{H} = \overline{\sum_k W_k}$, $W_k \cap W_j = \emptyset$ if $j \neq k$, and the system $\{W_i\}$ can be considered instead of $\{V_i\}$. The former has the structure of σ-algebra, and thus it is suitable integration.

The fact that the fields ϕ_l and $\phi_{l'}$ live on different leaves suggests that their Fourier images should be taken separately at their leaves:

$$\phi_l(x) = \int \exp(-\imath k^{(l)} x^{(l)}) \tilde{\phi}_l(k^{(l)}) d\mu_L^{(l)}(k^{(l)}), \tag{7}$$

or some other care should be taken about it in order not to mix fluctuations with the same wave vectors but contributing to different scales. The choice of the left-invariant measure $d\mu_L^{(l)}(k^{(l)})$ is restricted by the fact that velocity components measured at a certain scale are mainly concentrated close to this scale. So, the measure can be expressed as $d\mu_L^{(l)}(k) = dk W(|l^{-1} - ak|)$, where W vanishes at $x \to \pm\infty$, a is a constant.

The decomposition (7) turns out to be a kind of Gabor transformation [Gab46]. The measure $d\mu$ can be explicitly scale-dependent, since the probability spaces $(\Omega_l, \mathcal{U}_l, P_l)$ depend on scale.

At this point we arrive to the difference from standard wavelet approach, where the probability space is completely determined at finest resolution scale. However, if we accept the hypothesis that fluctuations of different sizes can be statistically independent, we have to define the probability spaces separately.

3 Wavelet realization

We start our construction of multi-scale description with the simplistic one-dimensional case, which is however of practical importance since only one component of velocity field is often measured.

Any square-integrable function $f(t) \in L^2(\mathrm{R})$ can be represented as a decomposition with respect to the representations of the affine group

$$t' = at + b, \tag{8}$$

$$f(t) = C_\psi^{-1}\int \frac{1}{\sqrt{a}}\psi\left(\frac{t-b}{a}\right)W_\psi(a,b)\frac{dadb}{a^2}, \qquad (9)$$

which is just another form of the partition of unity with respect to a representation of the affine group in a Hilbert space [GJP85]

$$\hat{1} = C_\psi^{-1}\int_G U^*(g)|\psi\rangle d\mu_L(g)\langle\psi|U(g),$$

which holds if there exists such $\psi \in \mathcal{H}$ that

$$C_\psi := \frac{1}{\|\psi\|_2^2}\int_G |\langle\psi, |U(g)\psi\rangle|^2 d\mu_L(g) < \infty$$

holds; $d\mu_L(g)$ denotes left-invariant measure on the group G. The scalar products

$$W_\psi(g)f := \langle f, U(g)\psi\rangle$$

are known as wavelet coefficients.

For the case of affine transformation group (8) the normalization constant C_ψ can be easily evaluated in the Fourier space:

$$C_\psi = \int_{-\infty}^{\infty}\frac{|\tilde\psi(k)|}{k}dk = 2\int_0^\infty \frac{|\tilde\psi(k)|}{k}dk, \qquad (10)$$

where $\psi(t) = \frac{1}{2\pi}\int \exp(\imath kt)\tilde\psi(k)dk$. For the affine group (8)

$$U(a,b)\psi(x) := \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right); \quad d\mu_L(a,b) = \frac{dadb}{a^2}. \qquad (11)$$

The corresponding wavelet coefficients are

$$W_\psi(a,b) = \int \frac{1}{\sqrt{a}}\overline{\psi\left(\frac{t-b}{a}\right)}f(t)dt. \qquad (12)$$

For practical analytical calculations it is often more efficient to perform wavelet decomposition and reconstruction in Fourier representation, since multiplication should be done then instead of convolution:

$$W_\psi(a,b)f = \frac{1}{2\pi}\int \sqrt{a}\exp(\imath kb)\overline{\tilde\psi(ak)}\tilde f(k)dk. \qquad (13)$$

The decomposition (12) and its inverse (9) are known as wavelet analysis. (See, e.g., [Dau88] for general review.) The scalar product

(12) is readily seen to be the projection of the original "no-scale" function f to the subspace W_a of the MRA system (6).

If f is a random function defined on a probability space $(\Omega, \mathcal{A}, P)$, the wavelet coefficients

$$W_\psi(a,b,\cdot) = \int \frac{1}{\sqrt{a}}\overline{\psi\left(\frac{t-b}{a}\right)}f(t,\cdot)dt \qquad (14)$$

are also random; the stochastic integration is implied. As is known from the theory of stochastic processes, any random function $\xi(t;\omega), t \in I\!R^1, \omega \in \Omega$ can be represented in a spectral form

$$\xi(t) = \int \phi(t,\lambda)\eta(d\lambda), \qquad (15)$$

where $\phi(t,\lambda)$ is a square-integrable function, $\eta(d\lambda)$ is a stochastic measure

$$\mathrm{M}\eta(d\lambda) = 0, \quad \mathrm{M}|\eta(d\lambda)|^2 = F(d\lambda).$$

A particular form of the spectral representation (15) is the Fourier representation

$$\xi(t) = \int \exp(\imath\lambda t)\eta(d\lambda).$$

In the multi-scale case we can introduce a collection of random processes, each of which belongs to its own leaf of MRA, labeled by resolution parameter a

$$\xi_a(t) = \int \phi_a(t,\lambda)\eta_a(d\lambda).$$

The peculiarity of the stochastic case is that, in contrast to the decomposition of a function with respect to the given basic wavelet $\psi(t)$, the function $\phi_a(t,\lambda)$, which depends on both the properties of random process itself and filtering properties of measuring equipment, is not known exactly. Therefore, we have to construct a decomposition which has a well-defined limit to deterministic case and can be tackled without the exact specification of the form of the basic wavelet ψ.

A straightforward way to do it is to factorize the scaling part of the left-invariant measure from "the purely stochastic part":

$$\xi(t) = \int \phi_a(t,\lambda)\eta_a(d\lambda)\frac{da}{a}$$

or in the spectral form

$$\xi(t) = \frac{1}{2\pi C_\psi}\int e^{\imath\lambda t}\bar{\psi}(\lambda a)\eta_a(d\lambda)\frac{da}{a}, \tag{16}$$

where $\eta_a(d\lambda)$ can be considered as generalized wavelet coefficients, the existence of which does not require the existence of "no-scale" prototype. The left-invariant measure $d\mu(a) = da/(2\pi a)$ on the multiplicative group $x' = ax$ instead of (8) is used, since translations are already incorporated into the exponent; 2π multiplier is introduced for the convenience of Fourier transform.

The representation (16) was constructed only to meet the non-stochastic limit and is not unique. For instance, we can redefine the spectral measure to incorporate both the properties of the signal and those of the measuring apparatus:

$$\xi(t) = \int e^{\imath\lambda t}\hat{\eta}_a(d\lambda)\frac{da}{a}.$$

The specific energy-per-scale density can be easily evaluated then:

$$\begin{aligned}\int E(a)da &= \mathrm{M}\int \xi(t)\bar{\xi}(t)dt \\ &= \mathrm{M}\int \exp(\imath t(\lambda_1-\lambda_2))\eta_{a_1}(d\lambda_1)\bar{\eta}_{a_2}(d\lambda_2)d\mu(a_1)d\mu(a_2)dt \\ &= \mathrm{M}\int \eta_{a_1}(d\lambda)\bar{\eta}_{a_2}(d\lambda)d\mu(a_1)d\mu(a_2) \\ &= \int F(a_1,a_2;d\lambda)d\mu(a_1)d\mu(a_2) = E.\end{aligned} \tag{17}$$

The equation (17) is a stochastic counterpart of a well-known equation for wavelet energy per scale

$$E(a) = C_\psi^{-1}\int \frac{|W(a,b)|^2}{a^2}db = \frac{1}{2\pi C_\psi}\int \frac{|\hat{W}(a,k)|^2}{a}dk. \tag{18}$$

Similar spectral characteristics have been already used for the analysis of turbulent data [Ast96].

If random functions are considered the equation (18) can be rewritten in the RG-like form

$$\frac{\partial E}{\partial \ln a} = \frac{1}{2\pi C_\psi}\mathrm{M}\int |\hat{W}(a,k)|^2 dk, \tag{19}$$

where $\hat{W}(a,k) = \overline{\hat{\psi}(ak)}\hat{f}(k)$ can be understood as the original noisy signal f perceived by the filter ψ, i.e. as generalized wavelet coefficients, not necessarily having "no-scale" prototypes, cf. Eq. (16).

The logarithmic derivative at the l. h. s. of (19) is exactly that of renormalization group equation. However, the r. h. s of this equation was obtained without any cutoff assumptions, it was formally derived from the decomposition of the initial ("infinite resolution") signal filtered by the window function ψ.

Physically, the existence of the infinite resolution limit is often meaningless. For instance, the behavior of the electromagnetic coupling constant at the Planck scale is just a nonsense. The same happens in hydrodynamics when the scale parameter comes close to the mean free path. However, there is a principal difference between the hydrodynamic turbulence description and the quantum field theory. The scale l (resolution) becomes a physical measurable parameter in hydrodynamics, and thus $\hat{W}(a,k)$ can be considered as Fourier components of the velocity field fluctuations of different typical sizes. The study of the behavior of $\hat{W}(a,k)$ can provide a more consistent picture of what happens at different scales than standard Fourier decomposition.

4 High frequency cutoff

Being utmost scale-invariant at moderate scales, the behavior of turbulent velocity field changes when approaching the smallest and largest scales, between which the hydrodynamical description is valid. The former is the Kolmogorov dissipative scale (η), the latter is the size of the system. The size of the system can often be set to infinity with no harm to physics; whilst the dissipative scale is of physical importance, since the energy dissipation rate $\bar{\epsilon}$ is very constant which determines the turbulence behavior in inertial range.

That is why in RG, as well as in spectral calculations, the cutoff-dependent velocity field is often considered:

$$v_F^{\le}(x) = \frac{1}{(2\pi)^d}\int_{|k|<F}\exp(\imath kx)\tilde{v}(k)dk. \tag{20}$$

The cumulative energy of all harmonics with wave vectors less than or equal to the cutoff value F is one of the main spectral characteristics of developed turbulence:

$$\mathcal{E}(F) = \frac{1}{2}\mathrm{M}\int \overline{v_F^{\le}(x)}v_F^{\le}(x)d^dx = \frac{1}{2}\mathrm{M}\int_{|k|<F}\overline{\tilde{v}(k)}\tilde{v}(k)\frac{d^dk}{(2\pi)^d}. \tag{21}$$

Similarly, we can consider the cumulative energy of all velocity fluctuations with typical size greater than or equal to a given A. For simplicity let us consider a one-component velocity field taken as a function of time

$$\begin{aligned} E(A) &= \tfrac{1}{2}\mathrm{M} \int_{|a| \le A} \overline{v(t)} v(t) dt = \tfrac{1}{C_\psi} \mathrm{M} \int_{a=A}^{\infty} |W_\psi(a,b)v|^2 \tfrac{da db}{a^2} \\ &= C_\psi^{-1} \int_A^\infty \tfrac{|\tilde\psi(y)|^2}{y} dy \cdot \mathrm{M} \int |\tilde v(k)|^2 \tfrac{dk}{2\pi}, \end{aligned} \quad (22)$$

where $\lim_{A\to 0} 2 \int_{a=A}^{\infty} \frac{|\tilde\psi(y)|^2}{y} dy = C_\psi$, and $E = \frac{1}{2}\mathrm{M} \int |\tilde v(k)|^2 \frac{dk}{2\pi}$ is the total energy of all velocity fluctuations.

For non-vanishing A

$$E(A) = F(A) E, \quad \text{where} \quad F(A) = \frac{\int_A^\infty \frac{|\tilde\psi(y)|^2}{y} dy}{\int_0^\infty \frac{|\tilde\psi(y)|^2}{y} dy}. \quad (23)$$

For a better definiteness, let us calculate the filtering function $F(A)$ for a particular family of vanishing momenta wavelets

$$\psi_n(x) = (-1)^n \frac{d^n}{dx^n} \exp(-x^2/2), \quad \tilde\psi_n(k) = \sqrt{2\pi}(-\imath k)^n \exp(-k^2/2), \quad (24)$$

often used for studying hydrodynamical velocity field [MBA91, MBA93].

The normalization constant for this family is

$$C_n = 2\pi \int_{-\infty}^{\infty} k^{2n-1} e^{-k^2} dx = 2\pi\Gamma(n)$$

and so

$$F_n(A) = \frac{\int_{A^2}^\infty y^{n-1} e^{-y} dy}{\Gamma(n)}. \quad (25)$$

The derivative of cumulative energy with respect to the logarithmic measure da/a is

$$\frac{\partial E}{\partial \ln A} = E \frac{\partial F_n(E)}{\partial \ln A} = -\frac{\partial A^2}{\partial \ln A} f_n(A^2)$$

where $f_n(x) = x^{n-1} e^{-x}/\Gamma(n)$. So we arrive at the RG-like equation

$$\frac{\partial E}{\partial \ln A} = -\frac{2A^{2n} \exp(-A^2)}{\Gamma(n)} E. \quad (26)$$

For sufficiently small A the exponential term of the latter equation is close to unity, and thus the behavior is approximately proportional to A^2.

Conclusion

In the present paper we give a mathematical framework for the analysis of functions which depend on scale. Usually, the scale-dependent functions express the value of a certain physical quantity measured at a point x by averaging over a box of size l centered at x. Such functions are often used in hydrodynamics, geophysics, signal analysis. One of the most known ways of treating resolution-dependent functions is to identify the size of the box (l) with the inverse wave number of the Fourier transform $k^{-1} = l$. The higher wave numbers are then cut off. Sometimes this procedure leads to confusion (many problems of field theory approach to hydrodynamic turbulence originate from this confusion).

In our approach, using the ideas of wavelet analysis, we keep wave vectors (k) and scales (a) separately. We derive a renormalization group like equations which can be used to study the energy distribution between different scales. The renormalization group like equations of the form

$$\frac{\partial E}{\partial \ln l} = \beta(l) E, \quad (27)$$

Cf.Eqs.(23,26), obtained by means of wavelet decomposition of the stochastic process, provide a possibility to study the energetic budget of experimentally measured stochastic signals with an accuracy, higher than that of Fourier methods. The reason is rather obvious. If something is known about the smoothing function ψ of the measuring equipment, then wavelet decomposition can provide a more detailed estimation of energy-per-scale budget, as compared to just cutting the high frequencies; and *vice versa* the experimentally measured dependence of the form (27) may give additional information not only on the signal in question, but also on the smoothing function of the equipment. The results can be applied in signal analysis, in the investigation of turbulence velocity experimental data and in further theoretical research.

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A Self-similarity in a bounded domain of scales and Scale-relativity

In appendix, we would like to mention some results of other authors, which are not used in present work, but are closely related to the subject in question, and may be used in further investigations of this subject.

A direct approach to scale-dependent functions, which was proposed by Nottale [Not93], is based on the assumption that physics is scale-dependent, but scale-covariant. The former means to give up the differentiability and consider functions $v(x,l)$ with $\lim_{l\to 0}\frac{\partial v(x,l)}{\partial x}$ not necessarily existing; the latter imposes generalized scale-covariant equation

$$\frac{\partial v(x,l)}{\partial \ln l} = \beta(v(x,l)), \qquad (28)$$

which states that the scale behavior of a scale-dependent function is completely determined by the value of this function at a given scale.

The benefit of the scale relativity[Not93] approach is a possibility to account for the processes which admit self-similarity only at a limited domain of scales

$$v(\lambda l) = \lambda^h v(l); \eta \ll l \ll \Lambda.$$

For a homogeneous scale-dependent function (mono-fractal) $v = v_0(\lambda/l)^\delta$, where δ is independent of scale l. To get the additive rather than multiplicative form for two sequential scale transformations, it is convenient to express them in the logarithmic form :

$$\ln\frac{v(l')}{v_0} = \ln\frac{v(l)}{v_0} + \boldsymbol{V}\delta(l), \qquad (29)$$

where $\delta(l') = \delta(l) = \delta, \boldsymbol{V} = \ln(l/l')$. In real hydrodynamic turbulence the multi-fractal behavior is observed: $\delta = \delta(l)$. In the inertial range

$\eta \ll l \ll \Lambda$ the exponent δ is practically a constant, but closer to the limiting scales η and Λ the dependence of scale becomes significant.

The logarithmic form (29) suggests a direct generalization to the multi-fractal case $\delta = \delta(l)$, which is like the generalization of Galilean transform to Lorentz transform:

$$\begin{aligned} X' &= \Gamma(\boldsymbol{V})[X - \boldsymbol{V}T], \\ T' &= \Gamma(\boldsymbol{V})[A(\boldsymbol{V})X + B(\boldsymbol{V})T], \end{aligned} \qquad (30)$$

where $T = \ln(l/l_0), X(T) = \ln \mathrm{M}(v_l/v_0)$, see [DG96] for details.

The composition of two scale transformations of this type behaves like a composition of two Lorentz boosts: when approaching the unpassable limit Λ, instead of "Galilean" scale transform (29), the scale-dependent transform with

$$\delta(l) = \delta_0(1 - \ln^2(\Lambda/l)/\ln^2(\Lambda/\eta))^{-1/2} \qquad (31)$$

is applied.

The transformation group (30) can be used to construct a wavelet decomposition at a limited domain of scales. This, however, is the subject of the next paper (in preparation).

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