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TRANSPORTS ALONG MAPS IN FIBRE BUNDLES

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1 Introduction

In previous papers (see, e.g., [1, 2]) we have studied the transports along paths in fibre bundles. In them is not always essential the fact that the transports are along paths. This suggests a way of generalizing these investigations which is the subject of the present work.

Sect. 2 gives and discusses the basic definition of transports along maps in fibre bundles. Sect. 3 studies in details the case when the map's domain is a Cartesian product of two sets. Here presented are certain examples too. Sect. 4 is devoted to linear transports along maps in vector bundles. Partial derivations along maps are introduced as well as the general concepts of curvature and torsion. It is pointed out how a number of the already obtained results concerning linear transports along paths can *mutatis mutandis* be transferred in the investigated here general case. Sect. 5 investigated, in analogy with [3], the consistency (compatibility) of transports along maps in fibre bundles with bundle morphisms between them. Sect. 6 closes the paper with a discussion of different problems: An interpretation is given of the obtained in Sect. 3 decomposition of transports along maps whose domain is a Cartesian product of two sets. A scheme is proposed for performing operations with elements of different fibres of a bundle as well as with its sections. It is proved that the Hermitian metrics on a differentiable manifold are in one to one correspondence with the transports along the identity map in an appropriate tensor bundle over it. At the end, some remarks concerning tensor densities are discussed.

2 The basic definition. Special cases and discussion

The fact that γ is a path in definition 2.1 of [1] for a transport along paths in fibre bundles is insignificant from a logical view-point. This observation, as well as other reasons, leads to the following generalization.

Let (E, π, B) be a topological fibre bundle with base B, total space E, projection $\pi : E \to B$, and homeomorphic fibres $\pi^{-1}(x)$, $x \in B$. Let the set N be not empty ($N \neq \emptyset$) and there be given a map $\varkappa : N \to B$. By id_M is denoted the identity map of the set M.

Definition 2.1 *A transport along maps in the fibre bundle (E, π, B) is a map K assigning to any map $\varkappa : N \to B$ a map $K^\varkappa$, transport along $\varkappa$, such that $K^\varkappa : (l, m) \mapsto K^\varkappa_{l\to m}$, where for every $l, m \in N$ the map*

$$K^\varkappa_{l\to m} : \pi^{-1}(\varkappa(l)) \to \pi^{-1}(\varkappa(m)), \tag{2.1}$$

called transport along $\varkappa$ from l to m, satisfies the equalities:

$$K^\varkappa_{m\to n} \circ K^\varkappa_{l\to m} = K^\varkappa_{l\to n}, \quad l, m, n \in N, \tag{2.2}$$

$$K^{\varkappa}_{l\to l} = id_{\pi^{-1}(\varkappa(l))}, \quad l \in N. \tag{2.3}$$

The formal analogy of this definition with definition 2.1 of [1] is evident. In particular, if $\varkappa$ is a path in B, i.e. if N is an $\mathbb{R}$-interval, the former definition reduces to the latter. The two definitions coincide also in the 'flat' case when $N = B$ and $\varkappa = id_B$. In fact, in this case $I^{\gamma}_{s\to t} := K^{id_B}_{\gamma(s)\to\gamma(t)}$ for a path $\gamma : J \to B$, J being an $\mathbb{R}$-interval, $s, t \in J$, defines a transport along paths in (E, π, B) which depends only on the points $\gamma(s)$ and $\gamma(t)$ but not on the path γ itself. On the opposite, if I is a transport along paths having the last property, then $K^{id_B}_{\gamma(s)\to\gamma(t)} := I^{\gamma}_{s\to t}$ is a transport along the identity map of B in (E, π, B). By [4, theorem 6.1] the so defined transports along paths are flat, i.e. their curvature vanishes in the case when they are linear and (E, π, B) is a vector bundle. Due to these facts, we call the transports along the identity map *flat transports*.

The general form of a transport along maps is given by

Theorem 2.1 *Let $\varkappa : N \to B$. The map $K : \varkappa \mapsto K^{\varkappa} : (l, m) \mapsto K^{\varkappa}_{l\to m}$, $l, m \in N$ is a transport along $\varkappa$ if and only if there exist a set Q and a family of one-to-one maps $\{F^{\varkappa}_n : \pi^{-1}(\varkappa(n)) \to Q,\ n \in N\}$ such that*

$$K^{\varkappa}_{l\to m} = (F^{\varkappa}_m)^{-1} \circ (F^{\varkappa}_l), \quad l, m \in N. \tag{2.4}$$

The maps $F^{\varkappa}_n$ are defined up to a left composition with 1:1 map depending only on $\varkappa$, i.e. (2.4) holds for given families of maps $\{F^{\varkappa}_n : \pi^{-1}(\varkappa(n)) \to Q,\ n \in N\}$ and $\{'F^{\varkappa}_n : \pi^{-1}(\varkappa(n)) \to 'Q,\ n \in N\}$ for some sets Q and $'Q$ iff there is 1:1 map $D^{\varkappa} : Q \to 'Q$ such that

$$'F^{\varkappa}_n = D^{\varkappa} \circ F_n, \quad n \in N. \tag{2.5}$$

Proof. This theorem is a trivial corollary of lemmas 3.1 and 3.2 of [1] for $Q_n = \pi^{-1}(\varkappa(n))$, $n \in N$ and $R_{l\to m} = K^{\varkappa}_{l\to m}$, $l, m \in N$. ∎

The formal analogy is evident between transports along maps and the ones along paths. The causes for this are definition 2.1 of this work and [1, definition 2.1], as well as theorem 2.1 of the present work and [1, theorem 3.1]. Due to this almost all results concerning transports along paths are valid *mutatis mutandis* for transports along maps. Exceptions are the results which use explicitly the fact that a path is a map from a real interval to a certain set, viz. in which special properties of the $\mathbb{R}$-intervals, such as ordering, the Abelian structure (the operation addition) an so on, are used. This transferring of results can formally be done by substituting the symbols $\varkappa$ for γ, N for J, K for I, $l, m, n \in N$ for $r, s, t \in J$, and the word map(s) for the word path(s).

For example, definition 2.2, proposition 2.1 and example 2.1 of [1] now read:

Definition 2.2 *The section $\sigma \in \mathrm{Sec}(E, \pi, B)$ undergoes a (K-)transport or is (K-)transported (resp. along $\varkappa : N \to B$) if the equality*

$$\sigma(\varkappa(m)) = K^{\varkappa}_{l\to m}\sigma(\varkappa(l)), \quad l, m \in N \tag{2.6}$$

holds for every (resp. the given) map $\varkappa : N \to B$.

Proposition 2.1 *If (2.6) holds for a fixed $l \in N$, then it is valid for every $l \in N$.*

Example 2.1 *If (E, π, B) has a foliation structure $\{K_\alpha;\ \alpha \in A\}$, then the lifting $\overline{\varkappa}_u : N \to E$ of $\varkappa : N \to B$ through $u \in E$ given by*

$$\overline{\varkappa}_u(l) := \pi^{-1}(\varkappa(l)) \bigcap K_{\alpha(u)},$$

where $K_{\alpha(u)} \ni u$, defines a transport K along maps through

$$K^{\varkappa}_{l\to m}(u) := \overline{\varkappa}_u(m), \quad u \in \pi^{-1}(\varkappa(l)), \quad l, m \in N.$$

On the transports along maps additional restrictions can be imposed, such as (cf. [1, Sect. 2.2 and 2.3]):

- the locality condition:

$$K^{\varkappa|N'}_{l\to m} = K^{\varkappa}_{l\to m}, \quad l, m \in N' \subset N; \tag{2.7}$$

- the 'reparametrization invariance' condition:

$$K^{\varkappa\circ\tau}_{l\to m} = K^{\varkappa}_{\tau(l)\to\tau(m)}, \quad \text{for } 1:1 \text{ map } \tau : N'' \to N, \quad l, m \in N''; \tag{2.8}$$

- the consistency with a bundle binary operation $\beta : x \mapsto \beta_x$, $x \in B$ (e.g. a metric, i.e. a scalar product):

$$\beta_{\varkappa(l)} = \beta_{\varkappa(m)} \circ (K^{\varkappa}_{l\to m} \times K^{\varkappa}_{l\to m}) \tag{2.9}$$

  for $\beta_x : \pi^{-1}(x) \times \pi^{-1}(x) \to M$, $x \in B$, M being a set, e.g. $M = \mathbb{R}, \mathbb{C}$;

- the consistency with the vector structure of a complex (or real) vector bundle:

$$K^{\varkappa}_{l\to m}(\lambda u + \mu v) = \lambda K^{\varkappa}_{l\to m}u + \mu K^{\varkappa}_{l\to m}v, \quad \lambda, \mu \in \mathbb{C} \text{ (or } \mathbb{R}), \quad u, v \in \pi^{-1}(\varkappa(l)). \tag{2.10}$$

The last condition defines the set of *linear transports along maps*.

Examples of results that do not have analogs in our general case are propositions 2.3, 3.3, and 3.4 of [1]. But the all definitions and results of Sect. 4 and Sect. 5 of [1] have analogs in this case. They can be obtained by making the above-pointed substitutions.

3 The composite case

Of special interest are transports along maps whose domain has a structure of a Cartesian product, i.e. maps like $\varkappa : N \to B$ with $N = A \times M$, A and M being not empty sets. In this section, transports $K^{\eta}_{(a,x)\to(b,y)}$ along $\eta : A \times M \to B$ are considered and their general form is found. Here and below $a,b,c \in A$ and $x,y,z \in M$. By $\eta(\cdot,x) : A \to B$ and $\eta(a,\cdot) : M \to B$ are denoted, respectively, the maps $\eta(\cdot,x) : a \mapsto \eta(a,x)$ and $\eta(a,\cdot) : x \mapsto \eta(a,x)$.

Applying (2.2), we get

$$K^{\eta}_{(a,x)\to(b,y)} = K^{\eta}_{(a,y)\to(b,y)} \circ K^{\eta}_{(a,x)\to(a,y)} = K^{\eta}_{(b,x)\to(b,y)} \circ K^{\eta}_{(a,x)\to(b,x)}. \quad (3.1)$$

Using (2.1)–(2.3), we see that ${}^{x}K^{\eta}_{a\to b} := K^{\eta}_{(a,x)\to(b,x)}$ and ${}_{a}K^{\eta}_{x\to y} := K^{\eta}_{(a,x)\to(a,y)}$ satisfy (2.1)–(2.3) with, respectively, $\varkappa = \eta$, $K = {}^{x}K$, $l = a$, $m = b$, and $n = c$ and $\varkappa = \eta$, $K = {}_{a}K$, $l = x$, $m = y$, and $n = z$. Consequently, a transport along η decomposes to a composition of two (commuting) maps satisfying (2.1)–(2.3). Note that if the locality condition (2.7) holds, then these maps are simply the transports along $\eta(\cdot,x)$ and $\eta(a,\cdot)$.

So, applying lemma 3.1 of [1], we find

$$K^{\eta}_{(a,x)\to(b,x)} = ({}^{x}H^{\eta}_{b})^{-1} \circ ({}^{x}H^{\eta}_{a}), \qquad K^{\eta}_{(a,x)\to(a,y)} = ({}^{a}G^{\eta}_{y})^{-1} \circ ({}^{a}G^{\eta}_{x}), \quad (3.2)$$

where ${}^{x}H^{\eta}_{a} : \pi^{-1}(\eta(a,x)) \to Q_H$ and ${}^{a}G^{\eta}_{x} : \pi^{-1}(\eta(a,x)) \to Q_G$ are 1:1 maps on some sets Q_H and Q_G respectively. (The maps ${}^{x}H^{\eta}_{a}$ and ${}^{a}G^{\eta}_{x}$ are defined up to a left composition with 1:1 maps depending on the pairs x and η and a and η respectively - see [1, lemma 3.2].)

The substitution of (3.2) into (3.1) yields

$$\begin{aligned} K^{\eta}_{(a,x)\to(b,y)} &= ({}^{y}H^{\eta}_{b})^{-1} \circ ({}^{y}H^{\eta}_{a}) \circ ({}^{a}G^{\eta}_{y})^{-1} \circ ({}^{a}G^{\eta}_{x}) = \\ &= ({}^{b}G^{\eta}_{y})^{-1} \circ ({}^{b}G^{\eta}_{x}) \circ ({}^{x}H^{\eta}_{b})^{-1} \circ ({}^{x}H^{\eta}_{a}). \end{aligned} \quad (3.3)$$

Separating the terms depending on x and y in the second equality, we see that there exist one-to-one maps $C^{\eta}_{a\to b} : Q_G \to Q_G$ which are independent of x and such that

$$\left({}^{b}G^{\eta}_{x}\right) \circ ({}^{x}H^{\eta}_{b})^{-1} \circ ({}^{x}H^{\eta}_{a}) \circ ({}^{a}G^{\eta}_{x})^{-1} = C^{\eta}_{a\to b}. \quad (3.4)$$

It is trivial to check the equalities $C^{\eta}_{a\to b} = C^{\eta}_{c\to b} \circ C^{\eta}_{a\to c}$ and $C^{\eta}_{a\to a} = id_{Q_G}$. Hence, by [1, lemma 3.1], we have $C^{\eta}_{a\to b} = (C^{\eta}_{b})^{-1} \circ C^{\eta}_{a}$ with certain 1:1 maps $C^{\eta}_{a} : Q_G \to Q_C$ (defined up to a left composition with a map depending only on η [1, lemma 3.2]) on some set Q_C . The substitution of this result into (3.4) and the separation of the terms depending on a and b, shows the existence of 1:1 map $D^{\eta}_{x} : Q_H \to Q_C$ depending on η and x, for which

$$({}^{x}H^{\eta}_{a}) \circ ({}^{a}G^{\eta}_{x})^{-1} \circ (C^{\eta}_{a})^{-1} = (D^{\eta}_{x})^{-1}. \quad (3.5)$$

Hereout

$${}^{x}H^{\eta}_{a} = (D^{\eta}_{x})^{-1} \circ C^{\eta}_{a} \circ ({}^{a}G^{\eta}_{x}) \quad \text{or} \quad {}^{a}G^{\eta}_{x} = (C^{\eta}_{a})^{-1} \circ D^{\eta}_{x} \circ ({}^{x}H^{\eta}_{a}). \quad (3.6)$$

Substituting (3.6) into (3.3), we finally, in accordance with (2.4), get

$$K^{\eta}_{(a,x)\to(b,y)} = \left(F^{\eta}_{(b,y)}\right)^{-1} \circ F^{\eta}_{(a,x)} \quad (3.7)$$

with

$$F^{\eta}_{(a,x)} = C^{\eta}_{a} \circ ({}^{a}G^{\eta}_{x}) = D^{\eta}_{x} \circ ({}^{x}H^{\eta}_{a}) : \pi^{-1}(\eta(a,x)) \to Q_C. \quad (3.8)$$

As we noted above, the maps ${}^{a}G^{\eta}_{x}$, ${}^{x}H^{\eta}_{a}$, and C^{η}_{a} are defined up to the changes

$${}^{a}G^{\eta}_{x} \to {}^{a}P^{\eta}_{G} \circ ({}^{a}G^{\eta}_{x}), \qquad {}^{x}H^{\eta}_{a} \to {}^{x}P^{\eta}_{H} \circ ({}^{x}H^{\eta}_{a}), \quad (3.9)$$

and $C^{\eta}_{a} \to P^{\eta}_{C} \circ C^{\eta}_{a}$, respectively, where ${}^{a}P^{\eta}_{G} : Q_G \to Q_G$, ${}^{x}P^{\eta}_{H} : Q_H \to Q_H$, $P^{\eta}_{G} : Q_C \to Q_C$ are 1:1 mappings. The transformation concerning C^{η}_{a} is valid if $C^{\eta}_{a\to b}$ is defined independently. But this is not our case. Due to (3.4) the changes (3.9) imply $C^{\eta}_{a\to b} \to {}^{b}P^{\eta}_{G} \circ C^{\eta}_{a\to b} \circ ({}^{a}P^{\eta}_{G})^{-1}$. To describe this transformation through C^{η}_{a} we must have

$$C^{\eta}_{a} \to P^{\eta}_{C} \circ C^{\eta}_{a} \circ ({}^{a}P^{\eta}_{G})^{-1}. \quad (3.10)$$

From (3.5) it is easy to verify that the transformations (3.9) and (3.10) imply

$$D^{\eta}_{x} \to P^{\eta}_{C} \circ D^{\eta}_{x} \circ ({}^{x}P^{\eta}_{H})^{-1}. \quad (3.11)$$

At the end, according to (3.8), all this leads to the change

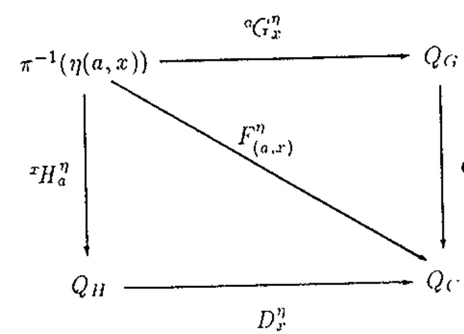
$$F^{\eta}_{(a,x)} \to P^{\eta}_{C} \circ F^{\eta}_{(a,x)}, \quad (3.12)$$

as should be by (2.5).

Together (3.9)–(3.12) form the set of transformations under which our theory is invariant.

Thus we have proved

Proposition 3.1 *The set of maps $\{K^{\eta}_{(a,x)\to(b,y)}\}$ forms a transport along $\eta : A \times M \to B$ iff (3.7) and (3.8) are valid for some 1:1 maps shown on the commutative diagram*



(3.13)

that are defined up to the transformations given by (3.9)-(3.12).

Remark. In fact ${}^{x}H^{\eta}_{a}$ and ${}^{a}G^{\eta}_{x}$ determine the 'restricted' transports $K^{\eta}_{(a,x)\to(b,x)}$ and $K^{\eta}_{(a,x)\to(a,y)}$ through (3.2). In the case when the locality condition (2.7) holds for $\varkappa = \eta$, they are equal, respectively, to the transports $K^{\eta(\cdot,x)}_{a\to b}$ and $K^{\eta(a,\cdot)}_{x\to y}$ along the restricted maps $\eta(\cdot,x)$ and $\eta(a,\cdot)$. Note also that if Q_G, Q_H, and Q_C are regarded as different typical fibres of (E,π,B), then the shown maps represent different ways for mapping a concrete fibre on them. This interpretation is more natural if one puts $Q_G = Q_H = Q_C = Q$, Q being the typical fibre of (E,π,B). This is possible due to the arbitrariness in ${}^{a}G^{\eta}_{x}$, ${}^{x}H^{\eta}_{a}$ and C^{η}_{a}.

Example 3.1 *Now we shall prove that the considered in [5] transport in a family of (vector) bundles $\{\xi_a :\ \xi_a = (E_a,\pi_a,M),\ a\in A\}$ over one and the same base (manifold) M defined by the maps ${}^{a,b}I_{x\to y} : \pi_a^{-1}(x) \to \pi_b^{-1}(y)$, such that ${}^{b,c}I_{y\to z}\circ{}^{a,b}I_{x\to y} = {}^{a,c}I_{x\to z}$ and ${}^{a,a}I_{x\to x} = id_{\pi_a^{-1}(x)}$, is a (flat) transport along the identity map of the base of a suitably chosen fibre bundle.*

A given family $\{(E_a,\pi_a,M),\ a\in A\}$ of fibre bundles over one and the same base is equivalent to some fibre bundle $(E,\pi,A\times M)$ over the composite base $A\times M$. In fact, if $\{(E_a,\pi_a,M),\ a\in A\}$ is given, we construct the fibre bundle $(E,\pi,A\times M)$ by putting

$$E = \bigcup_{a\in A} E_a, \qquad \pi : E \to A\times M, \quad \pi(u) = (a_u, \pi_{a_u}(u)),$$

where $u\in E$ and a_u is the unique $a_u\in A$ for which $E_{a_u}\ni u$. Conversely, if $(E,\pi,A\times M)$ is given, we can construct $\{(E_a,\pi_a,M),\ a\in A\}$ through

$$E_a = \bigcup_{x\in M}\pi^{-1}(a,x), \quad \pi_a : E_a\to M, \quad \pi_a(u) = x_u,$$

where $a\in A$, $u\in E_a$, and x_u is the unique $x_u\in M$ for which $\pi^{-1}(a,x_u)\ni u$.

Now it is trivial to check that the above-defined transports are ${}^{a,b}I_{x\to y} = K^{id_{A\times M}}_{(a,x)\to(b,y)}$, i.e. they are equivalent to the (flat) transport along the identity map $id_{A\times M} : A\times M \to A\times M$ in $(E,\pi,A\times M)$.

Example 3.2 *The described in the previous example general construction can be specified in the case of a flat transport in the fibre bundle of tensors of a fixed rank $k\in\mathbb{N}\cup\{0\}$ over a differentiable manifold M as follows.*

Let $A_r = \{(p,q) :\ p,q\in\mathbb{N}\cup\{0\},\ p+q=r\}$ and $T^p_q|_x(M)$, $(p,q)\in A_r$ be the tensor space of type (p,q) over $x\in M$. The **tensor bundle of type (p,q)** *is $\xi_{(p,q)} := (T^p_q(M),\pi_{(p,q)},M)$ with $T^p_q(M) := \bigcup_{x\in M}T^p_q|_x(M)$ and $\pi_{(p,q)}(u) = x$, for $u\in T^p_q|_x(M)$, x being the unique $x\in M$ for which $T^p_q|_x(M)\ni u$. The* **tensor bundle $(T_r(M),\pi_r,A_r\times M)$ of rank r** *is constructed by the above scheme: $T_r(M) := \bigcup_{(p,q)\in A_r}T^p_q(M) = \bigcup_{p+q=r}\bigcup_{x\in M}T^p_q|_x(M)$ and $\pi_r(u) = ((p,q),x)$ for $u\in T_r(M)$ with $(p,q)\in A_r$ and $x\in M$ being defined by $T^p_q|_x(M)\ni u$. So $\pi_r^{-1}((p,q),x) = \pi^{-1}_{(p,q)}(x) = T^p_q|_x(M)$. Then the desired transport along $id_{A_r\times M}$ is described by the maps ${}^{r}K^{id_{A\times M}}_{((p',q'),x')\to((p'',q''),x'')}$, $p'+q'=p''+q''=r$, $x',x''\in M$ satisfying the relations:*

$${}^{r}K^{id_{A_r\times M}}_{((p',q'),x')\to((p'',q''),x'')} : T^{p'}_{q'}\Big|_{x'}(M) \to T^{p''}_{q''}\Big|_{x''}(M),$$

$${}^{r}K^{id_{A_r\times M}}_{((p'',q''),x'')\to((p''',q'''),x''')}\circ{}^{r}K^{id_{A_r\times M}}_{((p',q'),x')\to((p'',q''),x'')} = {}^{r}K^{id_{A_r\times M}}_{((p',q'),x')\to((p''',q'''),x''')},$$

$${}^{r}K^{id_{A_r\times M}}_{((p',q'),x')\to((p',q'),x')} = id_{T^{p'}_{q'}|_{x'}(M)},$$
$$p'+q' = p''+q'' = p'''+q''' = r, \qquad x',x'',x'''\in M.$$

Example 3.3 *Every transport along $\varkappa'' : M\to M'$ in a bundle (E_0,π_0,M') induces a transport along $\varkappa'\times\varkappa''$ for $\varkappa' : A\to A'$ in any bundle $(E,\pi,A'\times M')$ for which the fibres $\pi^{-1}(a',x')$ are homeomorphic to $\pi_0^{-1}(y')$ for any $a'\in A'$ and $x',y'\in M'$. In fact, by example 3.1, $(E,\pi,A'\times M')$ is equivalent to the family $\{\xi_{a'} = (E_{a'},\pi_{a'},M'),\ \ a'\in A'\}$ with $E_{a'} = \bigcup_{x'\in M'}\pi^{-1}(a',x')$ and $\pi_{a'}(u) = x'$ for $u\in\pi^{-1}(a',x')$. As the fibres of all the introduced fibre bundles are homeomorphic there are fibre morphisms $(h_{a'},id_{M'})$ from $\xi_{a'}$ on ξ_0, i.e. $h_{a'} : E_{a'}\to E_0$ and $\pi_{a'} = \pi_0\circ h_{a'}$, $a'\in A'$. Then it is easy to verify that the maps*

$$K^{\varkappa'\times\varkappa''}_{(a,x)\to(b,y)} := \Big(h_{\varkappa'(b)}\big|_{\pi^{-1}(\varkappa'(b),\varkappa''(y))}\Big)^{-1}\circ$$
$$\circ\Big(h_{\varkappa'(a)}\big|_{\pi^{-1}(\varkappa'(a),\varkappa''(x))}\Big) : \pi^{-1}(\varkappa'(a),\varkappa''(x))\to\pi^{-1}(\varkappa'(b),\varkappa''(y))$$

define a transport along $\varkappa'\times\varkappa''$ in $(E,\pi,A'\times M')$.

Example 3.4 *This example is analogous to example 3.2 and is obtained from it by replacing p and q by integer functions over M.*

Let $f,g : M\to\mathbb{N}\cup\{0\}$, $f+g = r\in\mathbb{N}\cup\{0\}$, and

$$\xi^{r}_{(f,g)} := \left({}^{r}T^{f}_{g}(M),\pi_{f,g},M\right), \quad {}^{r}T^{f}_{g}(M) := \bigcup_{x\in M}T^{f(x)}_{g(x)}\Big|_{x}(M) \text{ and } \pi_{f,g}(u) := x$$

for $u\in T^{f(x)}_{g(x)}\Big|_{x}(M)$. The transports along maps in the so defined fibre bundle preserve the tensor's rank, but, generally, they change the tensor's type.

4 Linear transports along maps

In this section $\xi := (E, \pi, B)$ is supposed to be a complex (or real) *vector* bundle.

As we said above, a *linear transport (or L-transport) along maps* in a vector bundle is one satisfying eq. (2.10). For these transports *mutatis mutandis* valid are almost all definitions and results concerning linear transports along paths in (vector) fibre bundles [2, 6, 7, 8]. This is true for the cases in which the fact that a path is a map from a real interval into some set is not explicitly used. In particular, by replacing the path $\gamma : J \to B$ and the linear transport along paths $L^{\gamma}_{s\to t}$, $s, t \in J$ with, respectively, a map $\varkappa : N \to B$ and a linear transport along maps $L^{\varkappa}_{l\to m}$, $l, m \in N$, one obtains a valid version of sections 2 and 3 of [2], section 3 to proposition 3.3 and section 5 to proposition 5.3 of [6], and sections 1, 2, and 4 of [8]. The other parts of these works, as well as [7], deal more or less with explicit properties of the real interval J, mainly via the differentiation along paths $\mathcal{D}^{\gamma}$ (or $\mathcal{D}^{\gamma}_{s}$) [2]. These exceptional definitions and results can, if possible, be generalized as follows.

Let N be a neighborhood in $\mathbb{R}^k$, $k \in \mathbb{N}$, e.g. one may take $\mathrm{N} = J \times \cdots \times J$ (k-times), J being a real interval. So, any $l \in \mathrm{N}$ has a form $l = (l^1, \ldots, l^k) \in \mathbb{R}^k$. We put $\varepsilon_a := (0, \ldots, 0, \varepsilon, 0, \ldots, 0) \in \mathbb{R}^k$ where $\varepsilon \in \mathbb{R}$ stands in the a-th position, $1 \le a \le k$.

Let $\mathrm{Sec}^p(\xi)$ (resp. $\mathrm{Sec}(\xi)$) be the set of C^p (resp. all) sections over ξ and $L^{\varkappa}_{l\to m}$ be a C^1 (on l) linear transport along $\varkappa : \mathrm{N} \to B$. Now definition 4.1 from [2] is replaced by

Definition 4.1 *The a-th, $1 \le a \le k$ partial derivation along maps generated by L is a map ${}_a\mathcal{D} : \varkappa \mapsto {}_a\mathcal{D}^{\varkappa}$ where the a-th (partial) derivation along $\varkappa$ (generated by L) ${}_a\mathcal{D}^{\varkappa}$ is a map*

$$ {}_a\mathcal{D}^{\varkappa} : \mathrm{Sec}^1\left(\xi|_{\varkappa(\mathrm{N})}\right) \to \mathrm{Sec}\left(\xi|_{\varkappa(\mathrm{N})}\right) \tag{4.1}$$

defined for $\sigma \in \mathrm{Sec}^1\left(\xi|_{\varkappa(\mathrm{N})}\right)$ by

$$ ({}_a\mathcal{D}^{\varkappa}\sigma)(\varkappa(l)) := \lim_{\varepsilon\to 0}\left[\frac{1}{\varepsilon}\left(L^{\varkappa}_{l+\varepsilon^a\to l}\sigma(\varkappa(l+\varepsilon^a)) - \sigma(\varkappa(l))\right)\right]. \tag{4.2}$$

The (partial) a-th derivative of σ along $\varkappa$ with respect to L is ${}_a\mathcal{D}^{\varkappa}\sigma$. Its value at $\varkappa(l)$ is given by the operator

$$ \mathcal{D}^{\varkappa}_{l^a} : \mathrm{Sec}^1\left(\xi|_{\varkappa(\mathrm{N})}\right) \to \pi^{-1}(\varkappa(l)) \tag{4.3}$$

by $\mathcal{D}^{\varkappa}_{l^a}\sigma := ({}_a\mathcal{D}^{\varkappa}\sigma)(\varkappa(l))$.

Evidently, for $k = 1$ this definition reduces to definition 4.1 of [2].

On the basis of the above definition, almost all of the above-mentioned exceptional definitions and results can be modified by replacing in them $\gamma : J \to B$, $L^{\gamma}_{s\to t}$, $\mathcal{D}^{\gamma}$, and $\mathcal{D}^{\gamma}_{s}$, respectively with $\varkappa : \mathrm{N} \to B$, $L^{\varkappa}_{l\to m}$, ${}_a\mathcal{D}^{\varkappa}$, and ${}_a\mathcal{D}^{\varkappa}_{l^a}$. Below we sketch some results in this field.

A corollary of (2.2), (2.3), and (4.2) is

Proposition 4.1 *The operators ${}_a\mathcal{D}^{\varkappa}$ are ($\mathbb{C}$-)linear and*

$$ \mathcal{D}^{\varkappa}_{m^a} \circ L^{\varkappa}_{l\to m} \equiv 0. \tag{4.4}$$

If $\{e_i\}$ is a field of bases on $\varkappa(\mathrm{N})$, i.e. $\{e_i(n)\}$ is a basis in $\pi^{-1}(\varkappa(n))$, $n \in \mathrm{N}$, then in it the L-transport L along maps is described by the matrix $H(m, l; \varkappa) = \left[H^j_i(m, l; \varkappa)\right]^n_{i,j=1}$, $n := \dim_{\mathbb{C}}(\pi^{-1}(\varkappa(l)))$, which is defined by $L^{\varkappa}_{l\to m}e_i(l) =: H^j_i(m, l; \varkappa)e_j(m)$. A simple calculation of the limit in (4.2) verifies

Proposition 4.2 *If $\sigma \in \mathrm{Sec}^1\left(\xi|_{\varkappa(\mathrm{N})}\right)$, then*

$$ \mathcal{D}^{\varkappa}_{l^a}\sigma = \left[\frac{\partial\sigma^i(\varkappa(l))}{\partial l^a} + {}_a\Gamma^i{}_j(l;\varkappa)\sigma^j(\varkappa(l))\right]e_i(l), \tag{4.5}$$

where the components of L are defined by

$$ {}_a\Gamma^i{}_j(l;\varkappa) := \left.\frac{\partial H^i{}_j(l, m; \varkappa)}{\partial m^a}\right|_{m=l} = -\left.\frac{\partial H^i{}_j(m, l; \varkappa)}{\partial m^a}\right|_{m=l}. \tag{4.6}$$

The components of L satisfy

$$ {}_a\mathcal{D}^{\varkappa}e_j = \left({}_a\Gamma_{\varkappa}\right)^i{}_j e_i \tag{4.7}$$

and form k matrices ${}_a\Gamma_{\varkappa}(l) := \left[{}_a\Gamma^i{}_j(l;\varkappa)\right]^n_{i,j=1}$, $a = 1, \ldots, k$ which under the transformation $e_j(l) \mapsto e'_j(l) = A^i_j(l)e_i(l)$ change to

$$ {}_a\Gamma'_{\varkappa}(l) = A^{-1}(l)\left({}_a\Gamma_{\varkappa}(l)\right)A(l) + A^{-1}(l)\frac{\partial}{\partial l^a}A(l) \tag{4.8}$$

with $A(l) := \left[A^i_j(l)\right]$, which is a simple corollary of (4.7). Hence, the difference of the matrices ${}_a\Gamma_{\varkappa}$ of two L-transports along one and the same map behaves like a tensor of type (1,1) under a transformation of the bases.

On the above background one can *mutatis mutandis* reformulate the remaining part of Sect. 4 of [2]. In particular, in this way is established the equivalence of the sets of L-transports along maps $\varkappa : \mathrm{N} \to B$, $\mathrm{N} \subseteq \mathbb{R}^k$ and the one of partial derivations along maps. Sect. 6 and the rest of Sect. 3 and

Sect. 5 of [6] can be modified analogously, only in the last case the tangent vector field $\dot{\gamma}$ to $\gamma : J \to M$ has to be replaced with the set of tangent vectors $\{\dot{\varkappa}_a\}$ to $\varkappa$, $\dot{\varkappa}_a(l) := \left(\frac{\partial \varkappa^i(l)}{\partial l^a}\right)\left(\frac{\partial}{\partial x^i}\big|_{\varkappa(l)}\right)$.

The introduction of torsion and curvature needs more details which will be presented below.

Let M be a differentiable manifold and there be given a C^1 map $\eta : N \times N' \to M$, with N being a neighborhood in $\mathbb{R}^k$ and N' - in $\mathbb{R}^{k'}$, $k, k' \in \mathbb{N}$. Let $\eta(\cdot,m) : l \mapsto \eta(l,m)$, $\eta(l,\cdot) : m \mapsto \eta(l,m)$ for $(l,m) \in N \times N'$. Let $\eta'_a(\cdot,m)$, and $\eta''_b(l,\cdot)$, $l \in N$, $m \in N'$, $a = 1,\dots,k$, $b = 1,\dots,k'$ be the tangent vector fields to $\eta(\cdot,m)$ and $\eta(l,\cdot)$, respectively.

Definition 4.2 *The torsion operators of an L-transport along maps in the tangent bundle $(T(M),\pi,M)$ are maps $T_{a,b} : \eta \mapsto T^\eta_{a,b} : N \times N' \to T(M)$ which for $(l,m) \in N \times N'$ are given by*

$$T^\eta_{a,b}(l,m) := \mathcal{D}^{\eta(\cdot,m)}_{l^a}\eta''_b(\cdot,m) - \mathcal{D}^{\eta(l,\cdot)}_{m^b}\eta'_a(l,\cdot) \in T_{\eta(l,m)}(M). \tag{4.9}$$

Similarly, for $\eta : N \times N' \to B$, B being the base of a vector bundle (E,π,B), we have

Definition 4.3 *The curvature operators of an L-transport along maps in (E,π,B) are maps*

$$\mathcal{R}_{a,b} : \eta \mapsto \mathcal{R}^\eta_{a,b} : (l,m) \mapsto \mathcal{R}^\eta_{a,b}(l,m) : \mathrm{Sec}^2(E,\pi,B) \to \mathrm{Sec}(E,\pi,B)$$

defined for $(l,m) \in N \times N'$ by

$$\mathcal{R}^\eta_{a,b}(l,m) := {}_a\mathcal{D}^{\eta(\cdot,m)} \circ {}_b\mathcal{D}^{\eta(l,\cdot)} - {}_b\mathcal{D}^{\eta(l,\cdot)} \circ {}_a\mathcal{D}^{\eta(\cdot,m)}. \tag{4.10}$$

The further treatment of curvature and torsion can be done by the same methods as in [7, 4] (cf. [5, Sect. 8]).

In the composite case there arises a kind of 'restricted' partial derivation along maps generated by L-transports along maps.

Let $N = A \times M$ with M being a neighborhood in $\mathbb{R}^k$ for some $k \in \mathbb{N}$.

In this case instead of definition 4.1, we have

Definition 4.4 *The a-th, $1 \le a \le k$, partial derivative of type β, $\beta \in A$ along the map $\varkappa : A \times M \to B$ generated by an L-transport L along maps in a vector bundle $\xi = (E,\pi,B)$ is a map ${}^\beta_a\mathcal{D} : \varkappa \mapsto {}^\beta_a\mathcal{D}^\varkappa$, where*

$${}^\beta_a\mathcal{D}^\varkappa : \mathrm{Sec}^1\left(\xi|_{\varkappa(A\times M)}\right) \to \mathrm{Sec}\left(\xi|_{\varkappa(A\times M)}\right) \tag{4.11}$$

is the partial derivation along $\varkappa$ (generated by L) which is defined by the equation

$$\left({}^\beta_a\mathcal{D}^\varkappa\sigma\right)(\varkappa(\alpha,x)) := \lim_{\varepsilon\to 0}\left[\frac{1}{\varepsilon}\left(L^\varkappa_{(\alpha,x+\varepsilon^a)\to(\beta,x)}\sigma(\varkappa(\alpha,x+\varepsilon^a)) - L^\varkappa_{(\alpha,x)\to(\beta,x)}\sigma(\varkappa(\alpha,x))\right)\right] \tag{4.12}$$

for $\sigma \in \mathrm{Sec}^1\left(\xi|_{\varkappa(A\times M)}\right)$ and $(\alpha,x) \in A \times M$. The a-th (partial) derivative (of type β) of σ along $\varkappa$ with respect of L is ${}^\beta_a\mathcal{D}^\varkappa\sigma$. Its value at $\varkappa(\alpha,x)$ is given by the operator

$${}^\beta\mathcal{D}^\varkappa_{(\alpha,x^a)} : \mathrm{Sec}^1\left(\xi|_{\varkappa(A\times M)}\right) \to \pi^{-1}(\varkappa(\beta,x)) \tag{4.13}$$

by ${}^\beta\mathcal{D}^\varkappa_{(\alpha,x^a)}\sigma := \left({}^\beta_a\mathcal{D}^\varkappa\sigma\right)\varkappa(\alpha,x)$, with x^a being the a-th component of $x \in M \subseteq \mathbb{R}^k$.

For $A = \emptyset$ this definition reduces to definition 4.1.

Notice that the operator ${}^{a,b}_{\ x}\nabla^I_V$ used in [5, eq. (7.14)] is a special case of ${}^\beta_a\mathcal{D}^\varkappa$, viz. $\left({}^{a,b}_{\ x}\nabla^I_V(\sigma)\right)(a,x) = \sum_i V^i\left({}^b_i\mathcal{D}^{id_M}\sigma\right)(a,x)$ for $i = 1,\dots,\dim M$, $a,b \in A$, $x \in M$, M being a differentiable manifold and V being a vector field on M with local components V^i.

Now the corresponding results from [2, 6, 7, 8] can be modified step by step on the basis of definition 4.4 in the above-described way, where definition 4.1 was used.

5 Consistency with bundle morphisms

The work [3] investigates problems concerning the consistency of transports along paths in fibre bundles and bundle morphisms between them. A critical reading of this paper reveals the insignificance of the fact that the transports in it are along paths; nowhere there is the fact used that the path γ is a map from a real interval J into the base B of some fibre bundle. For this reason all of the work [3] is valid *mutatis mutandis* for arbitrary transports along maps; one has simply to replace the transports along paths, like $I^\gamma_{s\to t}$, $\gamma : J \to B$, $s,t \in J$, with transports along arbitrary maps, like $K^\varkappa_{l\to m}$, $\varkappa : N \to B$, $l,m \in N$. Below are stated *mutatis mutandis* only some definitions and results from [3]. There proofs are omitted as they can easily be obtained from the corresponding ones in [3].

Let there be given two fibre bundles $\xi_h := (E_h,\pi_h,B_h)$, $h = 1,2$ in which defined are, respectively, the transports along maps ${}^1K$ and ${}^2K$. Let (F,f) be a bundle morphism from ξ_1 into ξ_2, i.e. $F : E_1 \to E_2$, $f : B_1 \to B_2$ and $\pi_2 \circ F = f \circ \pi_1$ [9]. Let $F_x := F|_{\pi_1^{-1}(x)}$ for $x \in B_1$ and $\varkappa : N \to B_1$ be an arbitrary map in B_1.

Definition 5.1 *The bundle morphism (F,f) and the pair $({}^1K,{}^2K)$ of transports, or the transports ${}^1K$ and ${}^2K$, along maps are consistent (resp. along the map $\varkappa$) if they commute in a sense that the equality*

$$F_{\varkappa(m)} \circ {}^1K^\varkappa_{l\to m} = {}^2K^{f\circ\varkappa}_{l\to m} \circ F_{\varkappa(l)}, \quad l,m \in N \tag{5.1}$$

is fulfilled for every (resp. the given) map $\varkappa$.

A special case of definition 5.1 is the condition (2.9) for consistency with a bundle binary operation (in particular, a bundle metric), which is obtained from it for: $\xi_1 = (E,\pi,B) \times (E,\pi,B)$, $\xi_2 = (M,\pi_0,m_0)$ with a fixed $m_0 \in M$, $\pi_0 : M \to \{m_0\}$, $F_x = \beta_x$ with $x \in B$, $f : B \times B \to \{m_0\}$, ${}^1K^{\varkappa}_{l\to m} = K^{\varkappa}_{l\to m} \times K^{\varkappa}_{l\to m}$, and ${}^2K^{f\circ\varkappa}_{l\to m} = id_M$.

Let, in accordance with theorem 2.1 (cf. [1, theorem 3.1]), there be chosen sets Q_1 and Q_2 and one-to-one maps ${}^hF^{\varkappa_h}_{l_h} : \pi_h^{-1}(\varkappa_h(l_h)) \to Q_h$, $h = 1,2$, which are associated, respectively, with the maps $\varkappa_h : N_h \to B_h$, $l_h \in N_h$, $h = 1,2$ and are such that (cf. (2.4))

$$ {}^hK^{\varkappa_h}_{l_h\to m_h} = \left({}^hF^{\varkappa_h}_{m_h}\right)^{-1} \circ {}^hF^{\varkappa_h}_{l_h}, \qquad l_h,\ m_h \in N_h, \quad h = 1,2. \tag{5.2} $$

Proposition 5.1 *The bundle morphism (F,f) and the pair $({}^1K,{}^2K)$ of transports along maps, which are given by (5.2) by means of the maps ${}^1F$ and ${}^2F$, are consistent (resp. along a map $\varkappa$) iff there exists a map*

$$ C_0(\varkappa, f\circ\varkappa) : Q_1 \to Q_2, \tag{5.3} $$

such that

$$ F_{\varkappa(l)} = \left({}^2F^{f\circ\varkappa}_l\right)^{-1} \circ C_0(\varkappa, f\circ\varkappa) \circ ({}^1F^{\varkappa}_l), \tag{5.4} $$

or, equivalently, that

$$ F_{\varkappa(l)} = {}^2K^{f\circ\varkappa}_{l_0\to m} \circ C(l_0;\varkappa, f\circ\varkappa) \circ {}^1K^{\varkappa}_{l\to l_0}, \tag{5.5} $$

where $l_0 \in N$ is arbitrary and

$$ C(l_0;\varkappa, f\circ\varkappa) := \left({}^2F^{f\circ\varkappa}_{l_0}\right)^{-1} \circ C_0(\varkappa, f\circ\varkappa) \circ ({}^1F^{\varkappa}_{l_0}) \tag{5.6} $$

for every (resp. the given) map $\varkappa$.

Let there be given two fibre bundles $\xi_h = (E_h,\pi_h,B_h)$, $h = 1,2$. We define the *fibre bundle $\xi_0 = (E_0,\pi_0,B_1)$ of bundle morphisms* from ξ_1 onto ξ_2 in the following way:

$$ E_0 := \{(F_{b_1}, f) \colon F_{b_1} \colon \pi_1^{-1}(b_1) \to \pi_2^{-1}(f(b_1)),\ b_1 \in B_1,\ f : B_1 \to B_2\}, \tag{5.7} $$

$$ \pi_0((F_{b_1}, f)) := b_1, \quad (F_{b_1}, f) \in E_0, \quad b_1 \in B_1. \tag{5.8} $$

It is clear that every section $(F,f) \in \mathrm{Sec}(\xi_0)$ is a bundle morphism from ξ_1 into ξ_2 and vice versa, every bundle morphism from ξ_1 onto ξ_2 is a section of ξ_0. (Thus a bundle structure in the set $\mathrm{Morf}(\xi_1,\xi_2)$ of bundle morphisms from ξ_1 on ξ_2 is introduced.)

If in ξ_0 a transport K along the maps in B_1 is given, then, according to definition 2.2 (see eq. (2.6)), the bundle morphism $(F,f) \in \mathrm{Sec}(\xi_0)$ is (K-)transported along $\varkappa : N \to B_1$ if

$$ (F_{\varkappa(m)}, f) = K^{\varkappa}_{l\to m}(F_{\varkappa(l)}, f), \qquad l,m \in N. \tag{5.9} $$

Given in ξ_1 and ξ_2 the respective transports ${}^1K$ and ${}^2K$ along the maps in B_1 and B_2 respectively, they generate in ξ_0 a 'natural' transport ${}^0K$ along the maps in B_1. The action of this transport along $\varkappa : N \to B_1$ on $(F_{\varkappa(l)}, f) \in \pi_0^{-1}(\varkappa(l))$ for a fixed $l \in N$ and arbitrary $m \in N$ is defined by

$$ {}^0K^{\varkappa}_{l\to m}(F_{\varkappa(l)}, f) := \left({}^2K^{f\circ\varkappa}_{l\to m} \circ F_{\varkappa(l)} \circ {}^1K^{\varkappa}_{m\to l}, f\right) \in \pi_0^{-1}(\varkappa(m)). \tag{5.10} $$

Lemma 5.1 *If $(F,f) \in \mathrm{Sec}(\xi_0)$, then (5.1) is equivalent to*

$$ (F_{\varkappa(m)}, f) = {}^0K^{\varkappa}_{l\to m}(F_{\varkappa(l)}, f), \qquad l,m \in M. \tag{5.11} $$

Proposition 5.2 *The bundle morphism (F,f) and the pair $({}^1K,{}^2K)$ of transports along maps are consistent (resp. along the map $\varkappa$) if and only if (F,f) is transported along every (resp. the given) map $\varkappa$ with the help of the defined from $({}^1K,{}^2K)$ in ξ_0 transport along maps ${}^0K$.*

6 Concluding discussion

(1) The substitution of (3.8) into (3.7) gives

$$ K^{\eta}_{(a,x)\to(b,y)} = \left({}^bG^{\eta}_y\right)^{-1} \circ C^{\eta}_{a\to b} \circ ({}^aG^{\eta}_x) = ({}^yH^{\eta}_b)^{-1} \circ D^{\eta}_{x\to y} \circ ({}^xH^{\eta}_a) \tag{6.1} $$

where (cf. (3.4))

$$ C^{\eta}_{a\to b} := (C^{\eta}_b)^{-1} \circ C^{\eta}_a : Q_G \to Q_G, \quad D^{\eta}_{x\to y} := (D^{\eta}_y)^{-1} \circ D^{\eta}_x : Q_H \to Q_H \tag{6.2} $$

are 'transport like' maps. From them transports along the identity map in corresponding fibre bundles can be constructed. For instance, for $D^{\eta}_{x\to y}$ this can be done as follows. Consider the fibre bundle $(M \times Q_H, \pi_1, M)$ with $\pi_1(x,q) := x$, $(x,q) \in M \times Q_H$. Hence $\pi_1^{-1}(x) = \{x\} \times Q_H$. Defining $P_x : \{x\} \times Q_H \to Q_H$ by $P_x(x,q) := q$, we see that $\overline{D}^{\eta}_{x\to y} := P_y^{-1} \circ D^{\eta}_{x\to y} \circ P_x$ is a transport along id_M in $(M \times Q_H, \pi_1, M)$. It depends on η as on a parameter. Consequently, we can write

$$ K^{\eta}_{(a,x)\to(b,y)} = (P_y^{-1} \circ {}^yH^{\eta}_b)^{-1} \circ \overline{D}^{\eta}_{x\to y} \circ (P_x^{-1} \circ {}^xH^{\eta}_a). \tag{6.3} $$

This decomposition is important when $P_x^{-1} \circ {}^xH^{\eta}_a$ is independent of $x \in M$. Such a situation is realized in the special case when $B = A' \times M'$ and $\eta = \eta' \times \eta''$ with $\eta' : A \to A'$ and $\eta'' : M \to M'$, i.e. for (some) transports along $\eta' \times \eta''$ in $(E,\pi,A' \times M')$. According to example 3.1 the fibre bundle $(E,\pi,A' \times M')$ is equivalent to the family $\{\xi_{a'} : \xi_{a'} = (E_{a'},\pi_{a'},M'),\quad E_{a'} = \bigcup_{x'\in M'} \pi^{-1}(a',x'),\quad \pi_{a'}(u) = x'$, for $u \in \pi^{-1}(a',x'),\quad a' \in A'\}$. Let $\xi_0 = (E_0,\pi_0,M')$ be any fibre bundle for which there exist bundle morphisms $(h_{a'}, id_{M'})$ from $\xi_{a'}$ into ξ_0. i.e. $h_{a'} : E_{a'} \to E_0$ and $\pi_{a'} = \pi_0 \circ h_{a'}$, $a \in A'$. (The existence of ξ_0 and $h_{a'}$ is a consequence from the fact that the fibres of all the defined bundles are homeomorphic; e.g. one may put $\xi_0 = \xi_{b'}$ for some fixed $b' \in A'$.)

If ${}^0K^{\varkappa''}_{x\to y}$ is any transport along $\varkappa''$ in ξ_0, then a simple calculation shows that

$$K^{\varkappa'\times\varkappa''}_{(a,x)\to(b,y)} := h^{-1}_{\varkappa'(b)} \circ {}^0K^{\varkappa''}_{x\to y} \circ h_{\varkappa'(a)} \tag{6.4}$$

defines a transport along $\varkappa' \times \varkappa''$ in $(E, \pi, A' \times M')$. The opposite statement is, generally, not valid, i.e. not for every transport along $\varkappa' \times \varkappa''$ in the fibre bundle $(E, \pi, A' \times M')$ there exists a decomposition like (6.4).

(2) In vector bundles, such as the tensor bundles over a differentiable manifold, sometimes the problem arises of comparing or performing some operations with vectors from different fibres, or speaking more freely, with vectors (defined) at different points. A way for approaching such problems is the following one.

Let the fibre bundle (E, π, B) be endowed with (maybe linear) transport K^0 along the identity map of B and, e.g., a binary operation β, $\beta : x \mapsto \beta_x : \pi^{-1}(x) \times \pi^{-1}(x) \to Q_x$ for some sets Q_x, $x \in B$. The problem is to extend the operation β on sets like $\pi^{-1}(y) \times \pi^{-1}(z)$, $y, z \in B$. A possible solution is to replace β with $\{\overline{\beta}_x\}$ for some maps:

$$\overline{\beta}_x : (y, z) \mapsto \overline{\beta}^{y,z}_x : \pi^{-1}(y) \times \pi^{-1}(z) \to Q_x.$$

where

$$\overline{\beta}^{y,z}_x := \beta_x \circ (K^0_{y\to x} \times K^0_{z\to x}) . \tag{6.5}$$

For instance, if $Q_x = \pi^{-1}(x)$ and (E, π, B) is a vector bundle one can define in this way the linear combination of vectors from different fibres by the equality

$$(\lambda u + \mu v)_x := \lambda K^0_{y\to x} u + \mu K^0_{z\to x} v, \qquad \lambda, \mu \in \mathbb{C},\ u \in \pi^{-1}(y),\ v \in \pi^{-1}(z).$$

It depends on x as on a parameter. If K^0 is linear, then $(\lambda u + \mu v)_{x''} = K^0_{x'\to x''}(\lambda u + \mu v)_{x'}$, $x', x'' \in B$.

Also in this way can be introduced different kinds of integrations of the sections of (E, π, B) if on Q_x corresponding measures are defined. Viz. if $d\mu_x$ is a measure on Q_x and $\sigma \in \mathrm{Sec}(E, \pi, B)$, the integral of σ over a set $V \subseteq E$ is defined as $\int_V (K^0_{y\to x}\sigma(y))\, d\mu_x$. This procedure is especially useful in tensor bundles in which there are different possibilities depending on the understanding of the product of the integrand with the measure. e.g. it can be a tensor product that may be combined with some contraction(s) too.

The situation is important when (E, π, B) is endowed with a transport along maps of a given kind, i.e. along $\varkappa \in \mathcal{K}$, where $\mathcal{K}$ is a certain set of

maps onto B. A typical example of such a set is the set of all paths on B, i.e. $\{\gamma :\ \gamma : J \to B,\ J \subseteq \mathbb{R}\}$.

Let for some $x \in B$ there be a neighborhood $U \ni x$ in B with the property that for any $y \in U$ there are a unique map $\varkappa_y : N_y \to B$, $\varkappa_y \in \mathcal{K}$ and a set $M_y \subsetneq N_y$ such that $\varkappa_y|_{M_y} : M_y \to U$, $\varkappa_y|_{M_y}(m_x) = x$, and $\varkappa_y|_{M_y}(m_y) = y$ for some $m_x, m_y \in M_y$. A well-known example of this kind is the case of geodesic paths (curves) on a differentiable manifold endowed with an affine connection [10, 11].

In such a neighborhood U one can repeat the above discussion (of the flat case) with the only change that $K^0_{y\to x}$ has to be replaced with $K^{\varkappa_y}_{m_y\to m_x}$.

The use of transports along the maps $\varkappa_y$ has the disadvantage that the result depends on $\varkappa_y$, but as they are unique in the above sense this is insignificant. If the fibre bundle admits some 'natural' family of such maps, as the above-pointed case of geodesic curves, the question of this uniqueness does not arise at all. If the set of maps with the considered property does not exists or is not unique, then the pointed procedure does not exists or is not unique and, consequently, one gets nothing or not a 'reasonable' result, respectively.

(3) The class of Hermitian (resp. real) metrics on a complex (resp. real) differentiable manifold M turns out to be in one-to-one correspondence with the class of flat linear transports in the tensor bundle of rank 1 over it (see example 3.2). Below is presented the proof of this statement.

The tensor bundle of rank 1 over M is

$$\left(T^1_0(M) \bigcup T^0_1(M), \pi_1, \{(1,0),(0,1)\} \times M\right)$$

with $\pi_1(u) = ((p,q),x)$ for $u \in T^p_q|_x(M)$, $p + q = 1$. According to proposition 3.1 there are the following four kinds of transports along $id_{\{(1,0),(0,1)\}\times M}$:

$$L_{x\to y} := L^{id_{\{(1,0),(0,1)\}\times M}}_{((1,0),x)\to((1,0),y)} = F^{-1}_y \circ F_x = G^{-1}_y \circ G_x, \tag{6.6}$$

$$L^*_{x\to y} := L^{id_{\{(1,0),(0,1)\}\times M}}_{((0,1),x)\to((0,1),y)} = {}^*F^{-1}_y \circ {}^*F_x = {}^*G^{-1}_y \circ {}^*G_x, \tag{6.7}$$

$$L^{1,0}_{x\to y} := L^{id_{\{(1,0),(0,1)\}\times M}}_{((1,0),x)\to((0,1),y)} = {}^*F^{-1}_y \circ F_x = {}^*G^{-1}_y \circ {}^*C^{-1} \circ C \circ G_x, \tag{6.8}$$

$$L^{0,1}_{x\to y} := L^{id_{\{(1,0),(0,1)\}\times M}}_{((0,1),x)\to((1,0),y)} = F^{-1}_y \circ {}^*F_x = G^{-1}_y \circ C \circ {}^*C \circ {}^*G_x. \tag{6.9}$$

Here, for brevity, we have put: $F_x := F^{\cdots}_{((1,0),x)} = C \circ G_x$, ${}^*F_x := F^{\cdots}_{((0,1),x)} = {}^*C \circ {}^*G_x$, $G_x := {}^{(1,0)}G^{\cdots}_x$, ${}^*G_x := {}^{(0,1)}G^{\cdots}_x$, $C := C^{\cdots}_{(1,0)}$, ${}^*C := C^{\cdots}_{(0,1)}$, where the dots $(\ldots)$ stand for $id_{\{(1,0),(0,1)\}\times M}$. These maps act between vector spaces and are linear because of the linearity of the considered transports. Let in all vector spaces, as well as in the fibres $T_x(M) := T^1_0|_x(M)$ and $T^*_x(M) := T^0_1|_x(M)$, $x \in M$, some bases be fixed in which the matrix of a map $X \in \{F_x, {}^*F_x, G_x, {}^*G_x, C, {}^*C, L^{1,0}_{x\to x}\}$ will be written as $[X]$.

A (fibre) Hermitian (resp. real) metric on M is $g : x \mapsto g_x$, $x \in M$. Here $g_x : T_x(M) \times T_x(M) \to \mathbb{C}$ (resp. $\mathbb{R}$) are $1\frac{1}{2}$ linear (resp. bilinear), nondegenerate, and Hermitian (resp. symmetric) maps [12]. Let in the above bases the matrix of g_x be $G(x)$; we have $\det G(x) \neq 0$ and $G^{\dagger}(x) = G(x)$ (resp. $G^{\top}(x) = G(x)$ as in the real case $G^{\dagger} = G^{\top}$), where $\dagger$ (resp. $\top$) means Hermitian conjugation (resp. transposition), i.e. transposition plus complex conjugation (denoted by a bar). Because of this there is a unitary (resp. orthogonal) matrix $D(x)$, i.e. $D^{\dagger} = D^{-1}$ (resp. $D^{\top} = D^{-1}$), such that $G(x) = D^{\dagger}(x) G_{p,q} D(x)$ with $G_{p,q} := \mathrm{diag}(\underbrace{+1,\dots,+1}_{\text{p-times}}, \underbrace{-1,\dots,-1}_{\text{q-times}})$ for some unique $p, q \in \mathbb{N} \bigcup \{0\}$, $p + q = \dim M$ [13].

Now the idea is to interpret the maps (6.8) for $y = x$ as metrics.

In fact, if for $u, v \in T_x(M)$ we put $g_x(u, v) := (g_x(u, \cdot))\overline{v}$ with

$$g_x(u, \cdot) := L^{1,0}_{x \to x} u \in T^*_x(M), \tag{6.10}$$

we find $G(x) = [{}^*F_x]^{-1}[F_x]$. This matrix will be Hermitian if, e.g., we choose $[{}^*F_x]^{-1} = [F_x]^{\dagger}$, i.e. ${}^*F_x^{-1} = F_x^{\dagger}$, which leads to $(L_{x \to x})^{\dagger} = L^*_{x \to x}$. In particular, we can choose ${}^*C^{-1} = C^{\dagger}$ and ${}^*G_x^{-1} = G_x^{\dagger}$. For this selection the maps (6.10) form a Hermitian metric on M.

Conversely, let there be given an arbitrary Hermitian metric g with $G(x) = D^{\dagger}(x) G_{p,q} D(x)$, $D^{\dagger} = D$. Take some constant unitary matrix A ($A^{\dagger} = A^{-1}$) and any C for which $[C]^{\dagger}[C] = A^{\dagger} G_{p,q} A$. Let us define $[{}^*C] := ([C]^{\dagger})^{-1}$. Putting $[G_x] = A^{\dagger} D(x)$, from $D^{\dagger} = D$ we get $[G_x]^{\dagger} = [G_x]^{-1}$. At last, define in a fixed basis $[{}^*G_x] = [G_x]$ (${}^*G_x^{-1} = G_x^{\dagger}$). Thus we have constructed a transport along $id_{\{(1,0),(0,1)\} \times M}$ with $[F_x] = [C] A^{\dagger} D(x)$ and $[{}^*F_x] = ([C]^{\dagger})^{-1} A^{\dagger} D(x)$. In particular, we have $[L^{1,0}_{x \to y}] = D^{\dagger}(y) G_{p,q} D(x)$, so that $[L^{1,0}_{x \to x}] = G(x)$.

In this way we have proved

Theorem 6.1 *The class of Hermitian metrics on a differentiable manifold is in one-to-one correspondence with the class of transports along the identity map in the tensor bundle of rank 1 over that manifold which have decompositions like (6.6)-(6.9) in which ${}^*C^{-1}$ and ${}^*G_x^{-1}$ (and, hence, ${}^*F^{-1}$) are Hermitian conjugate to C and G_x (and F) respectively.*

(4) The problems concerning linear transports along maps in the tensor bundles over a differentiable manifold can be investigated in the same way as in [6, Sect.3] in which the text before proposition 3.3 remains true *mutatis mutandis* in the considered in the present paper general case.

(5) At the end we want to pay attention to tensor densities. Usually [14], a tensor density (field) is defined as a quantity which is locally

represented by a set of numbers (resp. functions) with a suitable transformation law. Our equivalent view is that the tensor densities (density fields) are tensors (resp. tensor fields) that appropriately depend on one fixed basis in the corresponding tensor space and which are referred to modified (with respect to the tensors) bases.

Let M be a differentiable manifold and a basis $\{{}_0E^B_A(x)\}$ be fixed in $T^p_q|_x(M)$, and $\{{}_1E^B_A(x)\}$ be an arbitrary basis in it. Here A and B stand for the corresponding multiindeces (e.g. $A = (\alpha_1, \dots, \alpha_p)$, $B = (\beta_1, \dots, \beta_p)$, $\alpha_1, \dots, \beta_p = 1, \dots, \dim M$). We define a tensor density (field) of type (p, q) and weight $w \in \mathbb{R}$ (with respect to $\{{}_0E^B_A(x)\}$) as a tensor (field) ${}^w_0T(x) \in T^p_q|_x(M)$ whose local components ${}^w_0T^A_B(x)$ are referred to bases like $\{|{}^1_0E(x)|^w {}_1E^B_A(x)\}$, where $|{}^1_0E(x)|$ is the Jacobian between the above bases, i.e. $|{}^1_0E(x)| := \det({}^1_0E(x))$, ${}^1_0E(x) := \left[\partial x^i_1 / \partial x^j_0\right]$. So, we have

$${}^w_0T(x) = {}^w_0T^A_B(x) |{}^1_0E(x)|^w {}_1E^B_A. \tag{6.11}$$

It is easy to verify that the components of the so defined tensor densities have the accepted transformation law [14, ch. II, Sect. 8]. Consequently, the both definitions are equivalent.

We shall mention only two features of the tensor-density case.

(i) There exists a class of transports along maps like $\eta : \mathbb{R} \times N \to M$, i.e. $K^{\eta}_{(v,l) \to (w,m)}$, $v, w \in \mathbb{R}$, $l, m \in N$, which map tensor densities of weight v at one point into such of weight w at another point. For these transports the results of Sect. 3 are valid, in particular, for $N = M$ and $\eta = id_{\mathbb{R} \times M}$ we have the case considered in example 3.1 (with $A = \mathbb{R}$).

(ii) Of course, one can differentiate a tensor-density field as tensor field using (6.11), but this operation does not lead directly to what one expects. In fact, applying (4.5), one finds

$$\mathcal{D}^{\varkappa}_{l^a}({}^w_0T) = \left[\frac{\partial}{\partial l^a}({}^w_0T^C_D(\varkappa(l))) + {}_a\Gamma^{C\ A}_{D\ B}(l;\varkappa)({}^w_0T^B_A(\varkappa(l)))\right] |{}^1_0E(\varkappa(l))|^w \times$$
$$\times\ {}_1E^D_C(\varkappa(l)) + {}^w_0T(\varkappa(l)) \left(|{}^1_0E(\varkappa(l))|^{-w} \frac{\partial}{\partial l^a} |{}^1_0E(\varkappa(l))|^w \right). \tag{6.12}$$

The term in parentheses in the last term is equal to $w \frac{\partial}{\partial l^a} \ln |{}^1_0E(\varkappa(l))|$ which, due to (4.8), can be written as

$$-w\left({}^1_a\Gamma^{\alpha\cdot}_{\cdot\alpha}(l;\varkappa) - {}^0_a\Gamma^{\alpha\cdot}_{\cdot\alpha}(l;\varkappa)\right) = +w\left({}^1_a\Gamma^{\cdot\alpha}_{\alpha\cdot}(l;\varkappa) - {}^0_a\Gamma^{\cdot\alpha}_{\alpha\cdot}(l;\varkappa).\right),$$

where the points in the gamma's stand instead of the absent indices.

Here and above ${}^1_a\Gamma^{C\ A}_{D\ B}(l;\varkappa)$ are the components of $\mathcal{D}$ in the basis $\{{}_1E^B_A\}$, i.e.

$$\mathcal{D}^{\varkappa}_{l^a}(E^A_B) = {}^1_a\Gamma^{C\ A}_{D\ B}(l;\varkappa)\, {}_1E^D_C(\varkappa(l)).$$

and ${}^{0}_{a}\Gamma^{\cdots}_{\cdots} := {}^{0}_{a}\Gamma^{\cdots}_{\cdots}|_{{}_1E^A_B = {}_0E^A_B}$. So, if we put $P^-_a(l;\varkappa) := {}^{1}_{a}\Gamma^{\alpha}_{\cdot\alpha}(l;\varkappa)$, $P^+_a(l;\varkappa) := {}^{1}_{a}\Gamma^{\cdot\alpha}_{\alpha\cdot}(l;\varkappa)$, and ${}^0P^{\pm}_a(l;\varkappa) := P^{\pm}_a(l;\varkappa)|_{{}_1E^A_B = {}_0E^A_B}$, we get

$$\mathcal{D}^{\varkappa}_{l^a}({}^w_0\mathcal{T}) \pm w\,({}^0P^{\pm}_a(l;\varkappa))\,({}^w_0\mathcal{T}(\varkappa(l))) = \Big[\frac{\partial}{\partial l^a}\,({}^w_0\mathcal{T}^C_D(\varkappa(l))) + {}^{1}_{a}\Gamma^{C\ \,A}_{\ D\ B}(l;\varkappa)\times$$
$$\times\ \ ({}^w_0\mathcal{T}^B_A(\varkappa(l))) \pm wP^{\pm}_a(l;\varkappa)\,({}^w_0\mathcal{T}^C_D(\varkappa(l)))\Big]\,|{}^1_0E(\varkappa(l))|^w\,{}_1E^D_C(\varkappa(l)). \quad (6.13)$$

Thus the operator (4.3) when applied on tensor density fields produces, of course, tensor fields which, generally, are not tensor density fields as by (6.12) their components with respect to the corresponding bases depend on them in a way different from that of tensor densities. On the contrary, the right-hand-side of (6.13) is a tensor density field whose components, following [14, ch. V, Sect. 1], should be identified with those of the *a-th partial (plus or minus) derivation along the map* $\varkappa$ of the initial tensor density field; the components of this derivation being defined by the r.h.s. of (6.13).

It can be proved that when $\varkappa$ is a path and the transport along it is a parallel transport assigned to a covariant differentiation (linear connection) (see [2, p. 19]), the components of the r.h.s. of (6.13) coincide with the covariant differentiation along the tangent to the path vector field of the initial tensor density field (see [14, ch. V, Sect. 1]).

In the special case when $\Gamma^{\alpha}_{\cdot\alpha} = \Gamma^{\cdot\alpha}_{\alpha\cdot}$, we have $P^- = -P^+$, i.e. the defined derivation is unique.

The appropriate approach to the derivation of tensor density fields is based on transports of tensor densities mentioned in (i) and the general theory of Sect. 4, but this will be done elsewhere.

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References

[1] Iliev B. Z. Transports along paths in fibre bundles. General theory. JINR Communication E5-93-299, Dubna, 1993.

[2] Iliev B. Z. Linear transports along paths in vector bundles. I. General theory. JINR Communication E5-93-239, Dubna, 1993.

[3] Iliev B. Z. Transports along paths in fibre bundles. III. Consistency with bundle morphisms. JINR Communication E5-94-41, Dubna, 1994.

[4] Iliev B. Z. Linear transports along paths in vector bundles. V. Properties of curvature and torsion. JINR Communication E5-97-1, Dubna, 1997.

[5] Iliev B. Z. Some generalizations of the Jacobi identity with applications to the curvature- and torsion-depending hamiltonians of particle systems. In J. Lawrynowicz, editor, *Hurwitz-type structures and applications to surface physics*, number II in Deformations of Mathematical Structures, pages 161–188. Kluwer Academic Publishers Group, Dordrecht-Boston-London, 1993. (Papers from the Seminars on Deformations 1988–1992).

[6] Iliev B. Z. Linear transports along paths in vector bundles. II. Some applications. JINR Communication E5-93-260, Dubna, 1993.

[7] Iliev B. Z. Linear transports along paths in vector bundles. III. Curvature and torsion. JINR Communication E5-93-261, Dubna, 1993.

[8] Iliev B. Z. Linear transports along paths in vector bundles. IV. Consistency with bundle metrics. JINR Communication E5-94-17, Dubna, 1994.

[9] Husemoller D. *Fibre bundles*. McGrow-Hill Book C., New York-St. Louis-San Francisco-Toronto-London-Sydney, 1966.

[10] Kobayashi S. and Nomizu K. *Foundations of Differential Geometry*, volume I. Interscience Publishers, New York-London, 1963.

[11] Postnikov M. M. *Introduction to the Morse theory*. Nauka, Moscow, 1971. (In Russian).

[12] Greub W., Halperin S., and Vanstone R. *Connections, Curvature, and Cohomology*, volume 1. Academic Press, New York and London, 1972.

[13] Bellman R. *Introduction to matrix analysis*. MCGRAW-HILL book comp., New York-Toronto-London, 1960.

[14] Schouten J. A. *Tensor analysis for physicists*. Clarendon Press, Oxford, 1951.

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