ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ

Дубна

E5-97-198

A.L.Baranovski

CONTROLLING CHAOS IN BREAKABLE DYNAMIC SYSTEMS

Submitted to «Journal of Statistical Physics»

1997

A huge variety of studies in all areas of natural science was brought to the concept "dynamic chaos" . From the view point of an orientation of research in formation of the concept , we mark essential prevalency of the direct problems over inverse ones. If first, one way or another, prove existence of complex non-periodic movements with a continuous spectrum, second, by accepting for initial result of these proofs, ascend from the characteristics of chaos to structures of dynamic systems, producing it. Clearly there are many alternatives of this ascension that indicate an ambiguity of the solution of an inverse problem of chaotic dynamics.

On the one hand, we narrow a spectrum of alternatives by consideration of the inverse problem for breakable dynamic systems in a context of the theory of impulse control by dynamic chaos in this work. On the other hand, we would like to hope that

1) expand understanding of the controlling chaos [1] by consideration of impulse controlling dynamic chaos [2] and interrelation between parametric (parameter is one dimensional map) control of ensemble of trajectories and structures of dynamic systems;

2) develop and use research on controlling chaos both in discrete systems - one-dimensional maps [3,4] and in continuous systems of small dimension - nonlinear and harmonic oscillators with kicked [5,6].

Let us determine the design of dynamic systems as consisting of three stages: choose of appropriate mathematical model of chaotic motions, investigation of main probabilistic characteristics and properly a construction of differential equation.

1. Mathematical model of random process

Constructing a mathematical model, we shall proceed from the following: if to consider x (t) as a mathematical image of a dynamic signal, reflecting character changes in time of a physical condition (for example, current or voltage), it can be hypothetically spread out on the following infinite sum:

$$x(t)=\sum_{i=1}^{\infty}\Theta_{\tau_{i-1},\tau_i}(t)\sum_{j=0}^{n_i}D_{ij}t^{j}, \qquad (1)$$

where $\Theta_{\tau_{i-1},\tau_i}(t)=\begin{cases}1, & t\in[\tau_{i-1},\tau_i)\\ 0, & t\,\overline{\in}\,[\tau_{i-1},\tau_i)\end{cases}$; $\tau_{i-1}<\tau_i$, $\tau_0=0$, $\bigcup_{i=1}^{\infty}[\tau_{i-1},\tau_i)=[0,\infty)$.

The given model characterizes an exponential way of approximation to values of the chaotic process «observable» on elementary intervals of time $[\tau_{i-1},\tau_i)$.

When the internal sum in (1) describes some i-th elementary oscillation, determined by the unique form and amplitude (in a radio engineering such oscillations named by impulses), then the ratio (1) is an infinite sequence of elementary impulses adjoining to each other, i.e.

$$x(t)=\sum_{i=1}^{\infty} A_i g_i\left(\frac{t-\tau_{i-1}}{T_i}\right). \qquad (2)$$

We have denoted a random value of i-th impulse amplitude by A_i, and its random duration by $T_i=\tau_i-\tau_{i-1}$, deterministic form - by $g_i(y)$ at $0<y\leq 1$; $g_i(y)=0, \forall y \overline{\in}(0,1)$. Such a sequence is named an impulse random process [7].

For the important special case, when the duration of impulses is deterministic: $T_i=T$, $\forall i$, uneasy to lead analogy between representation of impulse random process with a deterministic tact interval and description of random process on a method of canonical decomposition [7].

Really, following a terminology [7], if A_i are coefficients of decomposition, $f_i(t)=g_i\left(\frac{t-(i-1)T}{T}\right)$ - coordinate functions,

$$x(t)=\sum_{i=1}^{\infty} A_i f_i(t) \quad - \qquad (3)$$

decomposition of random process. Moreover, if A_i are not correlated, then such a decomposition is said to be canonical.

The inverse problem in a class of breakable dynamic systems of a small order was considered in [5] for canonical representation of x (t), in [6] - for a more general description of x (t) in the form (2).

Clearly, the internal sum in (1) can describe not only a single impulse but a superposition of impulses with unique form and random amplitude. Let us study the general case when x (t) is a sum n of impulse random processes of a kind (3), i.e.

$$x(t)=\sum_{i=1}^{n} z_i(t), \qquad (4)$$

where $z_i(t)=\sum_{k=1}^{\infty} A_k^{(i)} g_i\left(\frac{t-(k-1)T}{T}\right)$ is an infinite sequence of impulses adjoining to each other with a given form g_i and random amplitudes $A_k^{(i)}$ distributed according to $p_{A_i}(x), \forall k$, and fixed duration equal to T; $g_i(t)$:$g_i(0)$ = $g_i(1)=0$, $\forall$ *i*.

2. Investigation of main probabilistic characteristics of x(t)

2.1. Probability density

Let us assume the amplitudes $A_k^{(i)}$ are independent identically distributed random values with a following probability density

$p_{A_i}(x)=p_{A_i}(-x)$, $\forall x$, $\forall k=1,2,\ldots$. $A_k^{(i)}$ and $A_m^{(j)}$ are mutually independent values $\forall i\neq j, j\in\{1,\ldots,n\}$, $\forall k,m$.

Then the characteristic function of random process (4) is equal to a product of characteristic functions of the separate random processes $z_i(t)$:

$$\Psi(u,t)=M(e^{jx(t)u})=\prod_{i=1}^{n} M(e^{jz_i(t)u})=\prod_{i=1}^{n}\Psi_i(u,t).$$

At known $\Psi(u,t)$ using the inverse theorem, the probability distribution $F(y,t)=P(x(t)\leq y)$ is identically defined. Thus,

$$\int_{-\infty}^{\infty} y\,dF(y,t)=M(x(t))=\sum_{i=1}^{n} M(A_k^{(i)})g_i\left(\left\{\frac{t}{T}\right\}\right)=0 \ ((k-1)T<t<kT,$$

$\frac{t-(k-1)T}{T}=\left\{\frac{t}{T}\right\}$ - fractional part of number $\frac{t}{T}$, $M(x^2(t))=$

$$=M\left(\sum_{i=1}^{n} A_k^{(i)} g_i\left(\left\{\frac{t}{T}\right\}\right)\right)^2=\sum_{i=1}^{n} M\left(A_k^{(i)} g_i\left(\left\{\frac{t}{T}\right\}\right)\right)^2=\sum_{i=1}^{n} M(z_i^2(t))=\sum_{i=1}^{n} b_i^2(t)=$$

$$=B_n^2(t).$$

If assume $B_n^2(t)\to\infty$ for almost all t one can show

$$\lim_{n\to\infty}\frac{1}{B_n^2(t)}\sum_{i=1}^{n}\int_{|y|\geq\varepsilon B_n(t)} y^2 dF_i(y,t)=0, \ \forall\varepsilon>0, \qquad (5)$$

where $F_i(y,t)$ is probability distribution of $z_i(t)$.Moreover $F_i(y,t)=P(z_i(t)<y)$=

$$=F_{A_i}\left(\frac{y}{g_i\left(\left\{\frac{t}{T}\right\}\right)}\right)\Rightarrow F_i(-y,t)=1-F_i(y,t), \quad \int_{|y|\geq\varepsilon B_n(t)} y^2 dF_i(y,t)=0$$

But at $n\to\infty$ $\int_{|y|<\varepsilon B_n(t)} y^2 dF_i(y,t)\to b_i^2(t)$, $\varepsilon B_n(t)\to\infty$.

Hence, (5) is a necessary and sufficient condition of convergence $F(y,t)$ to normal distribution with parameters M (x (t)) and $B_n^2(t)$, according to the Feller theorem [8].

Let us find a power spectrum of random process x (t) at most general assumptions on its random parameters. By definition [7]

$$S_{x(t)}(\omega)=\lim_{T^*\to\infty}\frac{1}{T^*}M\left(\left|F_{T^*}(j\omega)\right|^2\right), \qquad (6)$$

where $F_{T^*}(j\omega)=\int_0^{T^*} x(t)e^{-j\omega t}dt$ is instant spectrum x (t). In (6) after simple identical transformations we receive a general form of power spectrum:

$$S_{x(t)}(\omega)=\sum_{i=1}^{n}S_{z_i(t)}(\omega)+2T\sum_{\ell=1}^{n-1}\sum_{m=\ell+1}^{n}\lim_{L\to\infty}\frac{1}{L}\sum_{k=1}^{L}\sum_{p=1}^{L}M(A_\ell^{(k)}A_m^{(p)})\times$$
$$,\quad \times(\cos(k-p)\omega T(G_\ell^c(\omega T)G_m^c(\omega T)+G_\ell^s(\omega T)G_m^s(\omega T))+ \qquad (7)$$
$$+\sin(k-p)\omega T(G_\ell^c(\omega T)G_m^s(\omega T)-G_\ell^s(\omega T)G^c{}_m(\omega T)))$$

where $G_\ell^c(z)$, $G_\ell^s(z)$ - cosine and sine Fourier transformation of form $g_i(t)$;

$$S_{z_i(t)}(\omega)=2T\sigma_{A_i}^2\left|S_{g_i(t)}(\omega T)\right|^2(1+2\sum_{p=1}^{\infty}R_{A_i}(p)\cos p\omega T) \qquad (8)$$

is power spectrum of process $z_i(t)$ [9], $\sigma_{A_i}^2$ is a variance of $A_k^{(i)}$, $M\grave{A}_i=0$, $\forall i,k$, $R_{A_i}(p)=\frac{1}{\sigma_{A_i}^2}M(A_i^{(k)}A_i^{(k+p)})$ is the coefficient of the amplitudes correlation.

If random amplitudes are mutually independent, the expression (7) is significantly becomes simpler: $S_{x(t)}(\omega)=\sum_{i=1}^{n}S_{z_i(t)}(\omega)$.

So, the power spectrum is completely defined by Fourier transformations of the forms of impulses gi (t), covariance $M(A_\ell^{(k)}A_m^{(p)})$ between heterogeneous amplitudes $A_\ell^{(k)}$, $A_m^{(p)}$, $\ell\neq m$, ,coefficient of correlation $R_{A_i}(p)$ of homogeneous amplitudes $A_i^{(k)}$,$A_i^{(k+p)}$. Therefore, the purposeful change $S_{x(t)}(\omega)$ by appropriate variation of the characteristics determining power spectrum becomes possible. This question will be considered below when designing dynamic systems inducing the chaotic solutions..

3. Construction of a system of differential equations. A case of simple real roots of the characteristic polynomial

We shall turn to construction on a system of differential equations with the solution x (t) of form (4). We shall previously limit a class of functions determining the form of a impulse by assuming $g_i(z)=e^{\lambda_i z}$ where all λ_i are- various real numbers. Then, denoting $\tilde\lambda_i=\lambda_i/T$, $\tilde A_i^{(k)}=A_i^{(k)}e^{-\lambda_i T(k-1)}$, we rewrite process x (t) in the following form

$$x(t)=\sum_{k=1}^{\infty}x_k(t)\Theta_{(k-1)T,kt}(t),$$
$$x_k(t)=\sum_{i=1}^{n}\tilde A_i^{(k)}e^{\tilde\lambda_i t},\quad (k-1)T\le t<kT. \qquad (9)$$

Function $x_k(t)$ on each piece [(k-1) T, kT] corresponds a homogeneous differential equation with constant coefficients

$$\frac{d^n x}{dt^n}+\sum_{i=0}^{n-1}\gamma_i\frac{d^i x}{dt^i}=0. \qquad (10)$$

Constants γ_i are uniquely expressed through $\tilde\lambda_1,...,\tilde\lambda_n$ under the Vieta formulas:

$$\gamma_{n-1}=-(\tilde\lambda_1+\tilde\lambda_2+...+\tilde\lambda_n)$$
$$\gamma_{n-2}=\tilde\lambda_1\tilde\lambda_2+\tilde\lambda_1\tilde\lambda_3+...+\tilde\lambda_{n-1}\tilde\lambda_n \qquad (11)$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$\gamma_0=(-1)^n\tilde\lambda_1...\tilde\lambda_n,$$

where $\tilde\lambda_i$ are roots of characteristic polynomial

$$D(\tilde\lambda)=\tilde\lambda^n+\sum_{i=0}^{n-1}\gamma_i\tilde\lambda^i. \qquad (12)$$

Obviously, $\tilde\lambda_i$, $i=\overline{1,k}$ are simple real roots of the equation $D(\tilde\lambda)=0$. Let us write (10) in the matrix form:

$$\dot{\bar x}=F\cdot\bar x, \qquad (13)$$

where

$$\bar x=(x(t)\dot x(t)...x^{(n-1)}(t))^T,\quad F=\begin{pmatrix}0&1&0&...&0\\0&0&1&...&0\\...&...&...&...&...\\0&0&0&...&1\\-\gamma_0&-\gamma_1&-\gamma_2&...&-\gamma_{n-1}\end{pmatrix}$$

Let a state of designing system at $(k-1)T\le t<kT$ be characterized by output variable $x_k(t)$ and its derivatives $\dot x_k(t)$, $\ddot x_k(t)$, $x_k^{(n-1)}(t)$. Let us introduce the following notations:

$$\bar x_k(t)=\begin{pmatrix}x_k(t)\\\dot x_k(t)\\...\\x_k^{(n-1)}(t)\end{pmatrix},\ \bar A_k=\begin{pmatrix}A_1^{(k)}\\A_2^{(k)}\\...\\A_n^{(k)}\end{pmatrix},\ \tilde\Lambda=\begin{pmatrix}1&1&...&1\\\tilde\lambda_1&\tilde\lambda_2&...&\tilde\lambda_n\\...&...&...&...\\\tilde\lambda_1^{n-1}&\tilde\lambda_2^{n-1}&...&\tilde\lambda_n^{n-1}\end{pmatrix},$$

$$E(t)=\begin{pmatrix}e^{\tilde\lambda_1 t}&0&0&0\\0&e^{\tilde\lambda_2 t}&0&0\\0&0&...&0\\0&0&0&e^{\tilde\lambda_n t}\end{pmatrix}.$$

Then $x_k^{(j)}(t)=\sum_{i=1}^{n}\tilde A_i^{(k)}\tilde\lambda_i^j e^{\tilde\lambda_i t}$, $j=0,1,...,n-1$, or

$$\bar x_k(t)=\tilde\Lambda E(t)\bar A_k,\ (k-1)T<t<kT. \qquad (14)$$

The initial condition of the system is characterized by x(0), $\dot x(0)$, $\ddot x(0),...,x^{(n-1)}(0)$, which define a vector constant $\bar A_1$. Really, the system of linear algebraic equations $\tilde{\Lambda}\overline{A}_1 = \overline{x}_0$ with respect to $\tilde{A}_1^{(1)},...,\tilde{A}_n^{(1)}$ is a nonsingular one, since the determinant of a matrix $\tilde{\Lambda}$, $\det(\tilde{\Lambda})$, being the Vandermonde determinant, does not equal zero, since $\tilde{\lambda}_i$ are various by a condition. From here $\overline{A}_1 = \tilde{\Lambda}^{(-1)}\overline{x}_0$ at $\tilde{\Lambda}^{-1} = \begin{pmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ s_{21} & s_{22} & \cdots & s_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ s_{n1} & s_{n2} & \cdots & s_{nn} \end{pmatrix}$

where $s_{mp} = (M_{mp} / \det\tilde{\Lambda})$, M_{mp} is an appropriate minor Λ.

The motion of a representative point $c_t = (t, \overline{x}(t))$ on a curve $\overline{x}_1(t)$ should come true only up to t=T > 0. After this moment the program motion should be appropriate $\overline{x}_2(t)$

Let us assume the point c_t is instantly thrown by an operator D_t from a state $c_T = (T;\ \overline{x}_1(T))$ to $c_{T+0} = D_T c_T = (T+0;\ \overline{x}_2(T+0))$ in a moment t=T and goes farther on a trajectory $\overline{x}_2(t)$ set by differential equation with the initial conditions $\overline{x}_2(T+0)$. We express action of the operator D_T by a following relation:

$\overline{x}_2(T+0) = \overline{x}_1(T-0) + \overline{\delta}_1$, Where $\overline{\delta}_1 = (00...0\delta_l)^T$. Thus, jump undergoes only (n-1) -th derivative of x (t) on some fixed size δ_1. And so : $\Lambda E(T)\overline{A}_2 = \overline{x}_1(T) + \overline{\delta}_1$. Its solution is a vector $A_2 = \overline{A}_1 + E(-T)\tilde{\Lambda}^{-1}\overline{\delta}_1$ $(E^{-1}(t) = E(-t))$,

which identifies a motion of a representative point on a curve $\overline{x}_2(t) = \tilde{\Lambda}E(t)\overline{A}_2 = \tilde{\Lambda}E(t)\tilde{\Lambda}^{-1}\overline{x}_0 + \tilde{\Lambda}E(t-T)\tilde{\Lambda}^{-1}\overline{\delta}_1$.

If continue a construction of the solution x(t) we have

$$x(t) = \sum_{k=1}^{\infty}\Theta_{(k-1)T,kT}^{(t)}(\tilde{\Lambda}E(t)\tilde{\Lambda}^{-1}\overline{x}_0 + \sum_{\ell=1}^{k-1}\Lambda E(t-\ell T)\tilde{\Lambda}^{-1}\overline{\delta}_\ell). \quad (16)$$

Let us introduce Z-phase space of evolutionary process, point $\overline{x}(t)$ - set of possible states of process. The point $\overline{x}(t)$ at each fixed t can be interpeted as an n-dimensional vector of Euclid space R^n and Z as a set from R^n.

We shall define the topological product $Y=Z\times R$ as extended phase space of process x (t).

Thus we define a law of evolution of process x(t) by:

a) a system of n differential equations $\frac{d\overline{x}}{dt} = F\overline{x},\ \overline{x} \in Z,\ t \in R$;

b) sequence of hyperplanes Γ_t (t=iT) of extended phase space;

c) operator D_t such that $\Gamma_t' = D_t\Gamma_t \in Y$

or by

$$\frac{d\overline{x}}{dt} = F\overline{x} + \sum_{k=1}^{\infty}\delta(t-kT)\overline{\delta}_k .$$

In accordance with [9], the set of relations à) -c) is said to be the system of the differential equations with impulse action. So

$$\begin{aligned} &d\overline{x}/dt = F\overline{x},\ (t,\overline{x}) \in \Gamma_t \\ &\Delta\overline{x} \in \Gamma_t = D_T\overline{x} - \overline{x} \end{aligned} \quad (17)$$

is the analytical representation of the synthesized system.

Thus, the solution to the system $\overline{x}(t)$ is a function satisfying to the equation (13) outside of set Γ_t and have breaks in points Γ_t with jumps $\Delta\overline{x} = \overline{x}(t+0) - \overline{x}(t-0) = D_T\overline{x}(t-0) - \overline{x}(t-0)$.

From the construction of system (17) it is clear that $\Delta\overline{x}$ coincides with a vector $\overline{\delta}_k$ in a point t=kT. We require $\delta_k = G(\delta_{k-1})$, where a map G should have an invariant probabilistic measure. Not stopping now on a particular choice G [3,4] let us calculate

$\Delta\overline{x}|_{t=kT} = \overline{\delta}_k = \overline{x}(kT+0) - \overline{x}(kT-0) = \overline{G}(\overline{\delta}_{k-1}) = (00,...0, G(\delta_{k-1}))^T =$
$= \overline{G}(\Delta\overline{x}|_{t=(k-1)T}) = G(\overline{x}((k-1)T+0) - \overline{x}((k-1)T-0))$.

Hence, it is easily to present the action of the operator D_t by a following ratio $\Delta x|_{(t;\overline{x})\in\Gamma_t} = \overline{G}(\overline{x}(t-T)+0) - \overline{x}((k-1)T-0)$, so the synthesized system can be written in the final form :

$$\frac{d\overline{x}}{dt} = F\overline{x}, t \neq iT;\ \Delta\overline{x}|_{t=iT} = \overline{G}(\overline{x}(t-T+0) - \overline{x}(t-T-0) \quad (18)$$

or

$$\frac{d\overline{x}}{dt} = F\overline{x} + \sum_{k=1}^{\infty}\delta(t-kT)\overline{G}(\overline{x}(t-T+0) - \overline{x}(t-T-0)).$$

As the choice of map G is predetermined by required probabilistic properties of x(t), the establishment of connections between probabilistic characteristics of x (t) and δ_k, and on them already reproduction of map G structure is necessary. For this we come back to the initial notations. As $\overline{A}_k = E(-T(k-1))\cdot(A_1^{(k)}, A_2^{(k)},...,A_n^{(k)})^T = E(-T(k-1))\overline{A}^{(k)}$, then the first return function is $\overline{A}^{(k)} = E(T)\overline{A}^{(k-1)} + \tilde{\Lambda}^{-1}\overline{x}_0 \cdot \sum_{\ell=0}^{k-2}E(\ell T)\tilde{\Lambda}^{-1}\overline{\delta}_{k-1-\ell}$. Therefore, according to (16) the solution can be written as follows:

$$x(t) = \sum_{k=1}^{\infty}\Theta_{(k-1)T,kT}(t)(\tilde{\Lambda}E(t)\Lambda^{-1}\overline{x}_0 + \tilde{\Lambda}E(t-(k-1)T)\sum_{\ell=0}^{k-2}E(\ell T)\tilde{\Lambda}^{-1}\overline{\delta}_{k-1-\ell}.$$

In order to make the asymptotic behavior of x (t) be limited, let us assume

$Re\tilde{\lambda}_i < 0$. Really in this case $\tilde{\Lambda}\begin{pmatrix} e^{\tilde{\lambda}_1 t} & & \\ & \dots & \\ & & e^{\tilde{\lambda}_n t}\end{pmatrix}\tilde{\Lambda}^{-1} \xrightarrow[t\to\infty]{} 0$ and the initial condition is forgotten.

Let us consider stationary a mode of oscillations, i.e.

$$\bar{x}(t) = \sum_{k=1}^{\infty}\Theta_{(k-1)T,kT}(t)\tilde{\Lambda}E(t-(k-1)T)\sum_{\ell=0}^{k-2}E(\ell T)\tilde{\Lambda}^{-1}\bar{\delta}_{k-1-\ell}. \quad (19)$$

After matrix transformations in (19),

$$x(t) = \sum_{k=1}^{\infty}\Theta_{(k-1)T,kT}(t)\sum_{\ell=0}^{k-2}\bar{\delta}_{k-1-\ell}\begin{pmatrix} \sum_{p=1}^{n} s_{pn}e^{\tilde{\lambda}_p(t-(k-1)T+\ell T)} \\ \sum_{p=1}^{n} s_{pn}\tilde{\lambda}_p e^{\tilde{\lambda}_p(t-(k-1)T+\ell T)} \\ \dots \\ \sum_{p=1}^{n} s_{pn}\tilde{\lambda}_p^{n-1} e^{\tilde{\lambda}_p(t-(k-1)T+\ell T)} \end{pmatrix}$$

we obtain

or for each component

$$x^{(m)}(t) = \sum_{k=1}^{\infty}\Theta_{(k-1)T,kT}(t)\sum_{\ell=0}^{k-2}\sum_{p=1}^{n}\frac{\lambda_p^m}{T^m}\delta_{k-1-\ell}s_{pn}e^{\lambda_p(\frac{t-(k-1)T}{T})}e^{\lambda_p\ell}$$

$(m = 0,1,\dots,n-1)$

Taking into account, that the indicator function $\Theta_{(k-1)T,kT}(t)$ is equal zero outside of an interval [(k-1) T, kT), it follows that $\frac{t-(k-1)T}{T} = \left\{\frac{t}{T}\right\}$ and since $\sum_{\ell=0}^{k-2}e^{\lambda_p\ell}\delta_{k-1-\ell} = \sum_{q=1}^{k-1}\delta_q(e^{\lambda_p})^{k-1-q}$ we have

$$x^{(m)}(t) = \sum_{k=1}^{\infty}\Theta_{(k-1)T,kT}(t)\left(\sum_{p=1}^{n}s_{pn}\frac{\lambda_p^m}{T^m}e^{\lambda_p\left\{\frac{t}{T}\right\}}\sum_{q=1}^{k-1}\delta_q e^{\lambda_p(k-q-1)}\right), \quad (20)$$

$m = \overline{0,n-1}$.

From such a representation of the solution it follows that at rather a large t or k in the sum $\sum_{q=1}^{k-1}$ members will stay only at q close to k, since at q << k $e^{\lambda_p(k-q-1)} \to 0$ ($Re\lambda_p < 0$). From here it follows that the chaotic process x (t) is limited if the domain of values δ_q produced by map G is limited.

Process x (t) (m=0) has power spectrum of form (7) where

$$M(A_m^{(k)}A_\ell^{(j)}) = s_{\ell n}s_{mn}\sum_{p=1}^{k-1}\sum_{q=1}^{j-1}e^{\lambda_m(k-1-p)}e^{\lambda_\ell(j-1-q)}M(\delta_p\delta_q)$$

$$R_{A_m}(p) = \frac{s_{mn}^2}{\sigma^2}e^{\lambda_m(p+2(k-1))}\sum_{j=1}^{k-1}\sum_{q=1}^{k-1+p}e^{-\lambda_m^*(1+q)}M(\delta_j\delta_q) \quad (21)$$

(Since $A_m^{(k)} = \sum_{p=1}^{k-1}\delta_p e^{\lambda_m(k-1-p)}s_{mn}$).

Thus, the ratio for power spectrum of the chaotic solution x (t) is analytically established at known correlation function of values δ_k - impacts in the system (18).

Along with the solution of the power spectrum $S_{x(t)}(\omega)$control problem by means of certain change of δ_k correlations is possible.

Let us assume for simplicity that the correlations have exponential decay

$$C_\delta(j) = M(\delta_p\delta_q) = \sigma_\delta^2 R_\delta^{(j)}(1),\ j = |p-q|,\ M(\delta_k) = 0,\ \forall k. \quad (22)$$

It is possible, for example, if δ_k are iterations of piece-wise linear maps given on Markovian partition[3,4]. After substitution $C_\delta(j)$ to (21), covariance of amplitudes $A_m^{(k)}$ and $A_\ell^{(j)}$,coefficient of correlation of $A_m^{(k)}$ as functions of the parameter $R_\delta(1)$ are defined. Hence, power spectrum (7) is function of parameter $R_\delta(1)$. Thus, control by $S_{x(t)}(\omega, R_\delta(1))$ reduces to respective change of unique parameter $R_\delta(1)$. It is possible to put a problem of parametric optimization on a minimum of a functional $\Gamma(R_\delta(1)) = \Gamma(S^*_{x(t)}(\omega) - S_{x(t)}(\omega, R_\delta(1)))$, where Γ is a given functional, $S^*_{x(t)}(\omega)$ is required power spectrum.

Let optimal parameter $R_\delta(1)$ be found. Then on the correlation function $C_\delta(j)$ we design piece-wise linear map G [3,4] and thus solve a problem of reproduction of chaotic process with a given power spectrum. by the system of differential equations (18). The concrete realization of the specified way of spectrum control is carried out in [10] on an example of a breakable dynamic system at n=3 when in (18) t is excluded.

3.1. Case of simple complex and real roots of the characteristic polynomial

Let among roots of the characteristic polynomial $D(\lambda)$ of kind (12) (or among eigenvalues of matrix F) be complex ones. Without loss of generality let us order characteristic numbers as follows: first n_1 numbers are complex, other $(n - n_1)$ are real. Obviously, n_1 is even. By nominating a spectrum of roots λ_1, $i=\overline{1;n}$, $\overline{1;n}$, we thus set n of impulse random processes in the sum (4) such that n_1 processes are with exponential harmonic form of impulses (strictly harmonic form if to put $(\mathrm{Re}\lambda_p = 0, \mathrm{Im}\lambda_p \neq 0)$, $(n - n_1)$ processes with exponential form of impulses $(\mathrm{Re}\lambda_p \neq 0, \mathrm{Im}\lambda_p = 0)$.

Really the chaotic solution of system (18) on an interval $(k\text{-}1)T \le t < kT$ looks like:

$$x_k(t) = e^{a_1\left\{\frac{t}{T}\right\}}\left(B_1^{(k)}\cos b_1\left\{\frac{t}{T}\right\} + B_2^{(k)}\sin b_1\left\{\frac{t}{T}\right\}\right) + \ldots + e^{a_{n1/2}\left\{\frac{t}{T}\right\}} \times \times\left(B_{n_1-1}^{(k)}\cos b_{n1/2}\left\{\frac{t}{T}\right\} + B_{n_1}^{(k)}\sin b_{n1/2}\left\{\frac{t}{T}\right\}\right) + \sum_{i=n_1+1}^{n} B_i^{(k)} e^{\tilde{\lambda}_i t}, \qquad (23)$$

Where $\lambda_{2p\text{-}1} = a_p + jb_p, \lambda^{*}_{2p-1} = a_p - jb_p$, $p = \overline{1,k}$; $a_p < 0$; $B_i^{(k)}$ - real numbers,

$$B_m^{(k)} = \begin{cases} 2\,\mathrm{Re}\left(\tilde{A}_m^{(k)} e^{\lambda_m (k-1)}\right), & \text{if } m \text{ is odd } (m < n_1) \\ 2\,I_m\left(\tilde{A}_m^{(k)} e^{\lambda_m (k-1)}\right), & \text{if } m \text{ is even } (m \le n_1) \\ \tilde{A}_m^{(k)} e^{\lambda_m (k-1)}, & m = \overline{n_1 + 1, n} \end{cases}$$

The power spectrum of general solution x(t) will include covariance $M\left(B_n^{(k)} B_\ell^{(j)}\right)$ in accordance of (7).

Clearly the way of control by a power spectrum is similar to above considered case of the real roots of the characteristic polynomial.

4. Case of multiple eigenvalues of a matrix F

Let $g_i(t)$ - the form of impulses of the chaotic process $z_i(t)$ be any continuous on (0,1) function. On the basis of the above analysis for individual kinds of functions $g_i(t)$ is necessary to define the multiplicity of characteristic numbers. Really let us assume the multiplicity of λ_i equal k_i such that $1 < k_i < n$, $\sum_{i=1}^{m} k_i = n$, where n is dimension of phase space of linear equations system (10).

Then the sum of quasipolynomial of the following kind $\sum_{i=1}^{m} e^{\lambda_i t} \sum_{j=0}^{k_i - 1} d_{ij} t^j$, $\lambda_i \neq \lambda_j$ satisfies the equation (10).

Thus each characteristic number corresponds the following quasipolynomial

$$\varphi_i(t) = e^{\lambda_i t} \sum_{j=0}^{k_i - 1} d_{ij} t^j, \qquad (24)$$

So by a certain choice of constant coefficients d_{ij} and multiplicity k_i it is possible to come close to the form of impulse $g_i(t)$

Let us write an analytical representation of the general solution of system (18) in the following form: $x(t) = \sum_{k=1}^{\infty} \Theta_{(k-1)T,kT}(t) \sum_{i=1}^{m} \varphi_i(t)$. Comparing it with (4), we shall obtain a ratio

$$\varphi_i(t) = e^{\lambda_i t} \sum_{j=0}^{k_i - 1} d_{ij}(k) t^j = A_i^{(k)} g_i\left(\frac{t - (k-1)T}{T}\right), \quad (k-1)T < t < kT, \qquad (25)$$

which is an equation for $d_{ij}(k)$.

Let us denote $h_i(z) = g_i(z) e^{\tilde{\gamma}_i z}$, $\tilde{\gamma}_i = -\lambda_i T$, $z = \dfrac{t - (k-1)T}{T}$.

Then according to (25)

$$\sum_{j=0}^{k_i - 1} d_{ij}(k) t^j = h_i(z) A_i^{(k)} e^{\tilde{\gamma}_i (k-1)}. \qquad (26)$$

Let us decompose the function $h_i(z)$ to a Teylor series keeping first k_i terms:

$$h_i(z) \approx \sum_{p=0}^{k_i - 1} \frac{h_i^{(p)}(0)}{p!} z^p = \tilde{h}_i(z)$$

We define multiplicity of i-th eigenvalue from a condition:

$$\rho_p(\tilde{h}_i(z), h_i(z)) \le \varepsilon, \qquad (27)$$

where ε- is a given small positive value, ρ_p-is a distance function in L_p..

Let k_i be found satisfying (27). Then the truncated function $\tilde{h}_i(z)$ is enough to substitute in (26) in order to calculate coefficients $d_{ij}(k)$.

So, the solution of system (18) on interval of $(k\text{-}1)T \le t << kT$ between δ-*impacts* will be written as:

$$x_k^{(j)}(t) = \frac{d^j}{dt^j}\left(\sum_{i=1}^{m} e^{\lambda_i t} \sum_{p=o}^{k_i - 1} d_{i,p}(k) t^p\right), j = \overline{0, k-1} \qquad (28)$$

From the construction of system (28) it is clear that the constant coefficients $d_{i,p}(k)$ on intervals $((k\text{-}1)T, kT)$ are functions of k-values $\delta_1,\ldots,\delta_k$. Let us denote them by $f_{i,p}(\delta_1,\ldots\delta_k)$ for simplicity.

To define a power spectrum of the x (t) by using general expression (7) it is necessary previously to express amplitudes $A_i^{(k)} = (d_{i,j}(k) / v_{i,j}(k)) = \dfrac{f_{i,j}(\delta_1,\ldots\delta_k)}{v_{i,j}(k)}$ and calculate $M(A_m^{(k)} A_\ell^{(j)})$, $R_{A_m}(p)$, and then to substitute in (7). Thus, the deterministic system of the differential equations with impulse action (18) has chaotic solution as realization of random process x (t) of general kind (4). In addition the elements of a matrix F should be calculated by Vieta formulas (11) on nominated eigenvalues $\lambda_i < 0$ with multiplicities $k_i, i=\overline{1,m}$. We emphasize that the power spectrum calculated on unique realization (in definition (6) a symbol Ì of average on ensemble can be lowered) of chaotic solution of the system (13) coincides with general representation of power spectrum (7) for a process x (t) of kind (1) if the sequence of impacts is ergodic. Therefore it is necessary to choose the ergodic map G. This problem is solved by methods considered in [3,4].

Conclusion

Probabilistic characteristics (probability density and power spectrum) of sum of impulse random processes are analytically investigated. Synthesis of the system of the differential equations with impulse action producing chaotic process with exponential, exponential and harmonic, any continuous on (0,1) forms of impulses is carried out.

The procedure of the power spectrum control on a base of control by one-dimensional map inserted into the system is offered. The given map defines values of impacts arising when the phase trajectories cross given hyperplanes.

References:

1. Ott E., Grebogi C., Yorke J.A. [1990] "Controlling chaos" Phys. Rev. Lett. 64, 1196-1199
2. Baranovski A.L. The synthesis of deterministic dynamic systems forming stochastic oscillations with given probabilistic characteristics. Ph.D. thesis. Tomsk State University, 1989, 250 p.
3. Baranovski A.L. The construction of one dimensional dynamic systems under given probabilistic characteristics of their stochastic motions/ Tomsk Polytechnic Institute. Tomsk, 1988. 13 p. Preprint in VINITI 148-B89
4. A.Baranovski and D.Daems Design of one-dimensional chaotic maps with prescribed statistical properties. International Journal of Bifurcation and Chaos Vol.5, No.6 (1995) 1585-1598
5. Rybin Yu.K., Baranovski A.L. The synthesis of dynamic system with stochastic auto-oscillations// Journal of Communications Technology and Electronics, 1988, V.333, 8, 1643-1651
6. Baranovski A.L. Dynamic models of random processes// In: Methods and algorithms of technology processes automation.(ed. V.Tarasenko), Tomsk, University Press, 1995, p. 219-234
7. Levin B.R. Theoretical bases of statistical radio engineering, Vol. 1, Moscow: Sov. Radio Press, 1974. 552 p.
8. Tarasenko F.P. Nonparametric statistics. Tomsk University Press, 1976. 294 p.
9. Samoylenko A.M., Perestyuk N.A. Differential equations with impulse action. Kiev: Naukova dumka Press, 1987. 287 p.
10. Baranovski A.L. Design of self-oscillating system with controlled chaotic behavior. Preprint JINR, E5-97-197, Dubna, 1997.

Received by Publishing Department
on June 20, 1997.

Барановский А.Л. E5-97-198
Управляемый хаос в разрывных динамических системах

Сформулирована обратная задача для разрывных динамических систем. Решение сводится к двум этапам: построение математической модели хаотических процессов и синтез системы дифференциальных уравнений с импульсным воздействием. Предлагается эффективный способ управления вероятностными характеристиками хаоса на основе обусловленного изменения хаотического отображения, встроенного в систему.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна, 1997

Baranovski A.L. E5-97-198
Controlling Chaos in Breakable Dynamic Systems

The inverse problem of chaotic dynamics for breakable dynamic systems is formulated. The solution is reduced to two stages: chaotic processes mathematical modelling and design of the differential equations system with impulse action. Effective method of probabilistic characteristics chaos control is suggested. It based on stipulated change of chaotic map inserted into the system.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna, 1997