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CONSTRUCTION OF QUASI-JORDAN NORMAL FORM OF RESOLVENT MATRIX IN THE CASE OF UNIFORM PARABOLIC BLOCKS

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Construction of a simple normal form of the resolvent matrix is a principal point of stability investigation for difference initial-boundary value problems. In our previous work [1] we proved the theorem on existence of a quasi-Jordan normal form of the resolvent matrix in the case when all the eigenvalues of the characteristic matrix form a unique parabolic class. In this work we show by concrete example how the quasi-Jordan normal form is constructed in the presence of uniform parabolic blocks. The basis of generalized analytic eigenvectors for the resolvent matrix (of order 12) is constructed from the corresponding basis for the characteristic matrix (of order 4). In the general case an interesting relation for the primitive roots is to be proved. We do this at the end.

We consider a difference system with the characteristic matrix

$$D(e^{i\phi}) = E + G(e^{i\phi} - 2 + e^{-i\phi}) + A_2(e^{2i\phi} - 3e^{i\phi} + 3 - e^{-i\phi}),$$

$$G = \begin{bmatrix} \alpha & -1 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & 2\alpha & -1 \\ 0 & 0 & 0 & 2\alpha \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 & \beta \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ \beta & 0 & 0 & 0 \end{bmatrix}.$$

Put $t = e^{i\phi} - 1$. We have that

$$(D - E)/(-4\sin^2\frac{\phi}{2}) = G + tA_2.$$

The eigenvalues of $G + tA_2$ are solutions of the following equation

$$\det \begin{bmatrix} \alpha - \mu & -1 & 0 & \beta t \\ 0 & \alpha + \beta t - \mu & 0 & 0 \\ 0 & 0 & 2\alpha + \beta t - \mu & -1 \\ \beta t & 0 & 0 & 2\alpha - \mu \end{bmatrix} =$$

$$= (\alpha + \beta t - \mu)(2\alpha + \beta t - \mu)[-\beta^2 t^2 + (\alpha - \mu)(2\alpha - \mu)] = 0.$$

The eigenvalues of $D(e^{i\phi})$ are $\lambda_i = 1 - 4\sin^2\frac{\phi}{2}\,\mu_i, \quad i = 1,2,3,4,$ and

$$\mu_{1,3} = \frac{3\alpha}{2} \pm \sqrt{\frac{\alpha^2}{4} + \beta^2 t^2} = \frac{3\alpha}{2} \pm \frac{\alpha}{2}\sqrt{1 + \frac{4\beta^2 t^2}{\alpha^2}},$$

$$\mu_2 = \alpha + \beta t, \ \mu_4 = 2\alpha + \beta t.$$

If $\alpha > 4\beta$, then $|\mu_i| < 2.5\alpha$, $Re(\mu_i) > 0.5\alpha$ for all $t = e^{i\phi} - 1$, $|t| \le 2$. This gives

$$|\lambda_i|^2 = (1 - 4\sin^2\frac{\phi}{2}\,Re(\mu_i))^2 + (4\sin^2\frac{\phi}{2}\,Im(\mu_i))^2 <$$

$$< 1 - 4\sin^2\frac{\phi}{2}\alpha + 16\sin^4\frac{\phi}{2}(2.5\alpha)^2,$$

which implies $|\lambda_i| < |1 - 10\alpha \sin^2 \frac{\phi}{2}|$. When $10\alpha < 2$, $\phi = 0$ is the unique determining point for the considered matrix $D : |\lambda_i(\phi)| < 1$, for $\phi \neq 0$.

Eigenvectors of the Characteristic Matrix

The right calculating shows that in the vicinity of $\phi = 0$ the matrix D is transformed to the view

$$\hat{D} = E - 4\sin^2\frac{\phi}{2}\begin{bmatrix} \mu_1 & 1 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \\ 0 & 0 & \mu_3 & 1 \\ 0 & 0 & 0 & \mu_4 \end{bmatrix},$$

$$\mu_1 = \alpha - \frac{\beta^2}{\alpha}t^2 + O(t^3), \quad \mu_2 = \alpha + \beta t,$$

$$\mu_3 = 2\alpha + \frac{\beta^2}{\alpha}t^2 + O(t^3), \quad \mu_4 = 2\alpha + \beta t.$$

The corresponding basic vectors are

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ (\beta/\alpha^2)t + O(t^2) \\ (\beta/\alpha)t + O(t^2) \end{bmatrix}, \quad e_2 = \begin{bmatrix} (\beta/\alpha^2)t + O(t^2) \\ -1 \\ (2\beta/\alpha^3)t + O(t^2) \\ (\beta/\alpha^2)t + O(t^2) \end{bmatrix},$$

$$e_3 = \begin{bmatrix} (\beta^2/\alpha)t^2 + O(t^3) \\ 0 \\ 1 \\ \beta t - (\beta^2/\alpha)t^2 + O(t^3) \end{bmatrix}, \quad e_4 = \begin{bmatrix} (-\beta/\alpha)t + O(t^2) \\ 0 \\ 0 \\ -1 \end{bmatrix}.$$

We present here only the principal parts of the expansions of eigenvalues and eigenvectors. On the whole all expansions here have integer powers of ϕ only. They can be found by simple iterations from the corresponding relations defining μ_i and e_i. The structure of $\hat{D}$ and λ_i expansions show [2,3,4], that $D^n(e^{i\phi})$ generates bounded semigroups in the space L_2 and in the uniform metric: $||D^n(e^{i\phi})|| \leq c$. Here c is a positive constant, independent of n.

Eigenvalues and Eigenvectors of the Resolvent Matrix

The resolvent matrix here is a matrix of order 12, depending linearly on spectral parameter z :

$$M(z) = \begin{bmatrix} -A_2^{-1}A_1 & -A_2^{-1}(A_0 - zI) & -A_2^{-1}A_{-1} \\ I & 0 & 0 \\ 0 & I & 0 \end{bmatrix}.$$

Here $A_{-1} = G - A_2$, $A_0 = I - 2G + 3A_2$, $A_1 = G - 3A_2$. The eigenvalues of the resolvent matrix are solutions of the determinant equation, which in the case of $z = 1$ is $\det[G(\kappa - 1)^2 + A_2(\kappa - 1)^3] =$

$$= (\kappa - 1)^8(\beta(\kappa - 1) + \alpha)(\beta(\kappa - 1) + 2\alpha)[2\alpha^2 - \beta^2(\kappa - 1)^2] = 0.$$

Besides the principal eigenvalue $\kappa = 1$ (of multiplicity 8), we get $\theta_3 = 1 - \gamma$, $\theta_4 = 1 - 2\gamma$, $\gamma = \alpha/\beta$; $\theta_{1,2} = 1 \pm \sqrt{2}\gamma$. This implies relations $\theta_1 + \theta_2 = 2$, $\theta_1 - \theta_2 = 2\sqrt{2}\gamma$. Using formulas for eigenvectors of the characteristic matrix, we obtain formulas for $[E_1, E_2, E_3, E_4, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4]$ vectors of the primitive basis [1] , corresponding to the principal eigenvalues $\kappa = 1$ of the resolvent matrix. We denote

$$F_1 = \mathcal{E}_1 - \frac{\beta}{\alpha^2}E_3 + \frac{\beta}{\alpha}E_4, F_2 = \mathcal{E}_2 - \frac{\beta}{\alpha^2}E_1 - \frac{2\beta}{\alpha^3}E_3 + \frac{\beta}{\alpha^2}E_4, F_3 = \mathcal{E}_3 + \beta E_4, F_4 = \mathcal{E}_4 + \frac{\beta}{\alpha}E_1.$$

The basic vectors corresponding to the principal eigenvalues are

$$[E_1, E_2, E_3, E_4, F_1, F_2, F_3, F_4] =$$

$$\left[\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -2 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{array}\right]$$

The eigenvectors corresponding to $\theta_2, \theta_{3,4}$ are

$$\Phi_2 = E_3 - 2\gamma F_3 + 4\gamma^2[0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0]^*,$$

$$\Phi_{3,4} = \theta_{3,4}E_1 - E_4 + (\theta_{3,4} - 1)(\theta_{3,4}F_1 - F_4) + 2\gamma^2[\theta_{3,4}, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0]^*.$$

Sign $*$ transforms row-vector into column-vector.

The basic vectors are expanded in basic vectors $E_i, \mathcal{E}_j$. From these expansions we can find easily basis of vectors $[*, *, *, *, 0, 0, 0, 0, 0, 0, 0, 0]$. Such vectors are in ME_i, $M\mathcal{E}_j$. The following relations

$$\frac{\Phi_3 + \Phi_4}{4\gamma^2} = \frac{1}{2\gamma^2}(E_1 - E_4) + F_1 + [1, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0]^*,$$

$$\frac{\Phi_3 - \Phi_4}{2\gamma^2(\theta_3 - \theta_4)} = \frac{1}{2\gamma^2}(E_1 + F_1) + [1,0,0,0,0,0,0,0,0,0,0,0]^*$$

are true. As a result we obtain the following expansions for the basic vectors:

$$[1,0,0,0,0,0,0,0,0,0,0,0]^* = \frac{\beta^3}{4\sqrt{2}\alpha^3}(\Phi_3 - \Phi_4) + (\frac{\beta^3}{2\alpha^3} - \frac{\beta^2}{2\alpha^2})E_1$$

$$+\frac{\beta^3}{2\alpha^4}E_3 - \frac{\beta^3}{2\alpha^3}E_4 - \frac{\beta^2}{2\alpha^2}\mathcal{E}_1 + \frac{\beta^2}{2\alpha^2}\mathcal{E}_4,$$

$$[0,0,1,0,0,0,0,0,0,0,0,0]^* = \frac{\beta^2}{4\alpha^2}\Phi_2 - \frac{\beta^2}{4\alpha^2}E_3 + \frac{\beta^2}{2\alpha}E_4 + \frac{\beta}{2\alpha}\mathcal{E}_3,$$

$$[0,0,0,1,0,0,0,0,0,0,0,0]^* = -\frac{\beta^3}{4\sqrt{2}\alpha^3}(\Phi_3 - \Phi_4) + \frac{\beta^2}{4\alpha^2}(\Phi_3 + \Phi_4)$$

$$-\frac{\beta^3}{2\alpha^3}E_1 + (\frac{\beta}{\alpha^2} - \frac{\beta^3}{2\alpha^4})E_3 - (\frac{\beta}{\alpha} + \frac{\beta^2}{2\alpha^2} - \frac{\beta^3}{2\alpha^3})E_4 - (1 - \frac{\beta^2}{2\alpha^2})\mathcal{E}_1 - \frac{\beta^2}{2\alpha^2}\mathcal{E}_4.$$

Therefore the resolvent matrix takes the following form in the primitive basis (we mark elements of order $O(z-1)$ by ψ and omit elements of order $o(z-1)$):

$$M(e^{i\psi}) = \begin{bmatrix} \Omega & L_1 & L_2 \\ 0 & G_1 + \psi A & \psi C \\ 0 & \psi D & G_2 + \psi B \end{bmatrix} =$$

$$= \left[\begin{array}{cccc|cccc|cccc}
\omega_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \omega_2 & 0 & 0 & 0 & 0 & 0 & \psi & 0 & 0 & 0 & 0 \\
0 & 0 & \omega_3 & 0 & 0 & \psi & 0 & \psi & 0 & \psi & 0 & \psi \\
0 & 0 & 0 & \omega_4 & 0 & \psi & 0 & \psi & 0 & 0 & 0 & \psi \\
\hline
0 & 0 & 0 & 0 & \kappa_1 & \psi & 1 & \psi & 0 & 0 & 0 & \psi \\
0 & 0 & 0 & 0 & 0 & \kappa_1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \psi & \kappa_2 & \psi & 0 & 0 & 0 & \psi \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \kappa_2 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & \psi & 0 & \psi & \kappa_3 & \psi & 1 & \psi \\
0 & 0 & 0 & 0 & 0 & \psi & 0 & \psi & 0 & \kappa_3 + \psi & 0 & 1 + \psi \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \psi & 0 & \psi & \kappa_4 & 0 \\
0 & 0 & 0 & 0 & 0 & \psi & 0 & \psi & 0 & 0 & 0 & \kappa_4 + \psi
\end{array}\right] \qquad (1)$$

with $a_{32} = -1 + \beta^2/(2\alpha^2)$, $d_{42} = -\beta^2/(2\alpha^2)$ and so on. Note, that blocks A_{21}, B_{21} are nilpotents.

Separation of the Matrix Jordan Boxes

We remove the block D in (1) by performing a number of similarity transformations. First we reduce the blocks $G_1 + \psi A$, $G_2 + \psi B$ to the block-triangular form. These blocks have the following structure

$$G_1 + \psi A = \begin{bmatrix} \kappa_1 I + \psi A_{11} + \psi^{3/2}R^1_{11} & I + \psi A_{12} + \psi^{3/2}R^1_{12} \\ \psi A_{21} + \psi^{3/2}R^1_{21} & \kappa_2 I + \psi A_{22} + \psi^{3/2}R^1_{22} \end{bmatrix},$$

$$G_2 + \psi B = \begin{bmatrix} \kappa_3 I + \psi B_{11} + \psi^{3/2}R^2_{11} & I + \psi B_{12} + \psi^{3/2}R^2_{12} \\ \psi B_{21} + \psi^{3/2}R^2_{21} & \kappa_4 I + \psi B_{22} + \psi^{3/2}R^2_{22} \end{bmatrix}.$$

Here $A_{11} = A_{12} + A_{21} + A_{22} = \dot{N} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $N^2 = 0$, $B_{11} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$,

$$B_{12} = B_{11},\ B_{21} = N,\ B_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ B_{11}N = B_{12}N = B_{22}N = 0.$$

Calculation shows that

$$\begin{bmatrix} I & 0 \\ X & I \end{bmatrix}(G_1 + \psi A)\begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}$$

takes block-triangular form, if X satisfies the following matrix equation

$$[(\kappa_1 - \kappa_2)I - \psi N - \psi^{3/2}R^1_{22}]X = -\psi N - \psi^{3/2}R^1_{21} - X(\psi N + \psi^{3/2}R^1_{11}) +$$
$$+X(I + \psi N + \psi^{3/2}R^1_{12})X.$$

We solve this equation by the simple iteration method, starting from $X = 0$:

$$[(\kappa_1 - \kappa_2)I - \psi N - \psi^{3/2}R^1_{22}]X^{\nu+1} = -\psi N - \psi^{3/2}R^1_{21} - X^\nu(\psi N + \psi^{3/2}R^1_{11}) +$$
$$+X^\nu(I + \psi N + \psi^{3/2}R^1_{12})X^\nu.$$

Calculation gives

$$X_1 = -\frac{\psi N}{\kappa_1 - \kappa_2} - \frac{\psi^{3/2}}{\kappa_1 - \kappa_2}R^1_{21} + O(\psi^{3/2}),\ X_2 = X_1 + O(\psi^{3/2}).$$

Similarity transformation

$$\begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & X & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & Y & I \end{bmatrix} \qquad (2)$$

with

$$X=-\frac{\psi N}{\kappa_1-\kappa_2}-\frac{\psi^{3/2}R^1_{21}}{\kappa_1-\kappa_2}+O(\psi^{3/2}),$$
$$Y=-\frac{\psi N}{\kappa_3-\kappa_4}-\frac{\psi^{3/2}R^2_{21}}{\kappa_3-\kappa_4}+O(\psi^{3/2})$$

reduces the diagonal blocks $G_1+\psi A$, $G_2+\psi B$ to the views

$$\begin{bmatrix}\kappa_1 I+\psi(\kappa_1-\kappa_2)^{-1}N+O(\psi) & I+\psi A_{12}+O(\psi^{3/2})\\ 0 & \kappa_2 I-\psi(\kappa_1-\kappa_2)^{-1}N+O(\psi)\end{bmatrix},$$
$$\begin{bmatrix}\kappa_3 I+\psi(\kappa_3-\kappa_4)^{-1}N+O(\psi) & I+\psi B_{12}+O(\psi^{3/2})\\ 0 & \kappa_4 I-\psi(\kappa_3-\kappa_4)^{-1}N+O(\psi)\end{bmatrix}$$

respectively. The principal parts of the rest blocks remain unchanged. By analogy we can remove the block ψD. First we transform it to the block-triangular form. Similarity transformation

$$\begin{bmatrix}I&0&0&0&0&0\\0&I&0&0&0&0\\0&0&I&0&0&0\\0&0&0&I&0&0\\0&0&0&0&I&0\\0&0&Z&0&0&I\end{bmatrix}\tag{3}$$

with

$$Z=-\frac{\psi}{\kappa_1-\kappa_4}D_{21}+\frac{\psi^2}{(\kappa_1-\kappa_4)^2(\kappa_3-\kappa_4)}N+O(\psi)$$

reduces the block D to the view

$$\begin{bmatrix}-Z+\psi D_{11} & \psi D_{12}+O(\psi^{3/2})\\ 0 & Z+\psi D_{22}\end{bmatrix}.$$

The principal parts of the rest blocks remain unchanged. But the main diagonal blocks are not block-triangular now: they take the form

$$\begin{bmatrix}\kappa_1 I+\psi(\kappa_1-\kappa_2)^{-1}N+O(\psi) & I+\psi N+O(\psi^{3/2})\\ O(\psi^{3/2}) & \kappa_2 I-\psi(\kappa_1-\kappa_2)^{-1}N+O(\psi)\end{bmatrix},$$
$$\begin{bmatrix}\kappa_3 I+\psi(\kappa_3-\kappa_4)^{-1}N+O(\psi) & I+\psi B_{12}+O(\psi^{3/2})\\ O(\psi^2) & \kappa_4 I-\psi(\kappa_3-\kappa_4)^{-1}N+O(\psi)\end{bmatrix}.$$

The next step is similarity transformation

$$\begin{bmatrix}I&0&0&0&0&0\\0&I&0&0&0&0\\0&0&I&0&0&0\\0&0&0&I&0&0\\0&0&U&0&I&0\\0&0&0&V&0&I\end{bmatrix}\tag{4}$$

with

$$U=-\frac{\psi D_{21}}{(\kappa_1-\kappa_4)(\kappa_1-\kappa_3)}-\frac{\psi^2 N D_{21}}{(\kappa_1-\kappa_4)(\kappa_1-\kappa_3)^2(\kappa_3-\kappa_4)}+$$
$$+\frac{\psi^2 N}{(\kappa_1-\kappa_4)^2(\kappa_1-\kappa_3)(\kappa_3-\kappa_4)}+O(\sqrt{\psi}),\quad V=\frac{\psi D_{21}}{(\kappa_1-\kappa_4)(\kappa_2-\kappa_4)}-$$
$$-\frac{\psi^2 N D_{21}}{(\kappa_1-\kappa_4)(\kappa_2-\kappa_4)^2(\kappa_3-\kappa_4)}-\frac{\psi^2 N}{(\kappa_1-\kappa_4)^2(\kappa_2-\kappa_4)(\kappa_3-\kappa_4)}+O(\sqrt{\psi}).$$

Remark that U,V are matrices of $O(1)$, but their principal terms are equal, so $U-V=O(\sqrt{\psi})$. We used here simple relations $ND_{21}=N$, $D_{21}N=0$, $\kappa_1-\kappa_3=-(\kappa_2-\kappa_4)$. As a result the blocks D, $G_1+\psi A$ take the form
$$\begin{bmatrix}0 & \sqrt{\psi}R\\ O(\psi^{3/2}) & 0\end{bmatrix},\quad \begin{bmatrix}\kappa_1 I+\psi(\kappa_1-\kappa_2)^{-1}N+O(\psi) & I+\psi cN+O(\psi^{3/2})\\ O(\psi^{3/2}) & \kappa_2 I-\psi(\kappa_1-\kappa_2)^{-1}N+O(\psi)\end{bmatrix}$$
with $c=1-\psi/[(\kappa_1-\kappa_4)(\kappa_2-\kappa_4)]$. The principal parts of the rest blocks remain unchanged.

At last we do similarity trasformation

$$\begin{bmatrix}I&0&0&0&0&0\\0&I&0&0&0&0\\0&0&I&0&0&0\\0&0&0&I&0&0\\0&0&0&W&I&0\\0&0&0&0&0&I\end{bmatrix}\tag{5}$$

with

$$W_1=-\frac{\psi^{1/2}R}{\kappa_2-\kappa_3}-\frac{\psi^{3/2}NR}{(\kappa_2-\kappa_3)^2(\kappa_3-\kappa_4)},$$
$$W=W_1+\frac{\psi W_1 N}{(\kappa_2-\kappa_3)(\kappa_1-\kappa_2)}+\frac{\psi^2 N W_1 N}{(\kappa_1-\kappa_2)(\kappa_2-\kappa_3)^2(\kappa_3-\kappa_4)}+O(\psi^{1/2}).$$

After this the block D is $\begin{bmatrix}O(\psi^{3/2}) & 0\\ O(\psi^{3/2}) & O(\psi^{3/2})\end{bmatrix}$.

Thus we removed terms of order ψ in the block D and in lower, left angles of the main diagonal blocks we have now matrices of order $O(\psi^{3/2})$. In the same way we can remove step by step the terms of orders $O(\psi^{3/2}), O(\psi^2)$ and so on, in order to reduce at the end the resolvent matrix to the block- triangular form:

$$\begin{bmatrix}\Omega & \psi\mathcal{R}_{12} & \psi\mathcal{R}_{13}\\ 0 & \mathcal{G}_1 & \psi\mathcal{R}_{23}\\ 0 & 0 & \mathcal{G}_2\end{bmatrix},\tag{6}$$

where Ω and the pricipal terms of $\mathcal{R}_{23}$ are the same as in original form (1) and

$$\mathcal{G}_1=\begin{bmatrix}\kappa_1 I+\psi(\kappa_1-\kappa_2)^{-1}N+O(\psi) & I+\psi N+O(\psi^{3/2})\\ 0 & \kappa_2 I-\psi(\kappa_1-\kappa_2)^{-1}N+O(\psi)\end{bmatrix},$$

$$\mathcal{G}_2 = \begin{bmatrix} \kappa_3 I + \psi(\kappa_3-\kappa_4)^{-1}N + O(\psi) & I + \psi B_{12} + O(\psi^{3/2}) \\ 0 & \kappa_4 I - \psi(\kappa_3-\kappa_4)^{-1}N + O(\psi) \end{bmatrix}.$$

We explain shortly, how the terms of order $O(\psi^k)$, $k \geq 3/2$ are removed. Let the resolvent matrix has been transformed to the view with main diagonal blocks

$$\mathcal{G}_1^k = \begin{bmatrix} \kappa_1 I + \psi(\kappa_1-\kappa_2)^{-1}N + O(\psi) & I + \psi N + O(\psi^{3/2}) \\ \psi^k A_{21}^k & \kappa_2 I - \psi(\kappa_1-\kappa_2)^{-1}N + O(\psi) \end{bmatrix},$$

$$\mathcal{G}_2^k = \begin{bmatrix} \kappa_3 I + \psi(\kappa_3-\kappa_4)^{-1}N + O(\psi) & I + \psi B_{12} + O(\psi^{3/2}) \\ \psi^k B_{21}^k & \kappa_4 I - \psi(\kappa_3-\kappa_4)^{-1}N + O(\psi) \end{bmatrix}$$

and let the block $\psi^k D^k$ (D^k is not power of D here) stays on the place of ψD. We perform one after another similarity transformations (1)-(5) with

$$X = -\frac{\psi^k A_{21}^k}{(\kappa_1-\kappa_2)} + \frac{\psi^{k+1}(NA_{21}^k + A_{21}^k N)}{(\kappa_1-\kappa_2)^3} - \frac{2\psi^{k+2}NA_{21}^k N}{(\kappa_1-\kappa_2)^5} + O(\psi^k) = O(\psi^{k-(1/2)}),$$

$$Y = -\frac{\psi^k B_{21}^k}{(\kappa_3-\kappa_4)} + \frac{\psi^{k+1}(NB_{21}^k + B_{21}^k N)}{(\kappa_3-\kappa_4)^3} - \frac{2\psi^{k+2}NB_{21}^k N}{(\kappa_3-\kappa_4)^5} + O(\psi^k) = O(\psi^{k-(1/2)}),$$

$$Z = -\frac{\psi^k D_{21}^k}{(\kappa_1-\kappa_4)} + \frac{\psi^{k+1}ND_{21}^k}{(\kappa_1-\kappa_4)^2(\kappa_3-\kappa_4)} + \frac{\psi^{k+1}D_{21}^k N}{(\kappa_1-\kappa_4)^2(\kappa_1-\kappa_2)}$$
$$-\frac{2\psi^{k+2}ND_{21}^k N}{(\kappa_1-\kappa_4)^3(\kappa_1-\kappa_2)(\kappa_3-\kappa_4)} + O(\psi^k) = O(\psi^{k-(1/2)}),$$

$$U = +\frac{Z}{(\kappa_1-\kappa_3)} + \frac{\psi NZ}{(\kappa_1-\kappa_3)^2(\kappa_3-\kappa_4)} - \frac{\psi ZN}{(\kappa_1-\kappa_3)^2(\kappa_1-\kappa_2)}$$
$$-\frac{2\psi^2 NZN}{(\kappa_1-\kappa_3)^3(\kappa_1-\kappa_2)(\kappa_3-\kappa_4)} + O(\psi^{k-1/2}) = O(\psi^{k-1}),$$

$$V = -\frac{Z}{(\kappa_2-\kappa_4)} + \frac{\psi NZ}{(\kappa_2-\kappa_4)^2(\kappa_3-\kappa_4)} - \frac{\psi ZN}{(\kappa_2-\kappa_4)^2(\kappa_1-\kappa_2)}$$
$$+\frac{2\psi^2 NZN}{(\kappa_2-\kappa_4)^3(\kappa_1-\kappa_2)(\kappa_3-\kappa_4)} + O(\psi^{k-1/2}) = O(\psi^{k-1}),$$

$$W = -\frac{U-V}{(\kappa_2-\kappa_3)} - \frac{\psi N(U-V)}{(\kappa_2-\kappa_3)^2(\kappa_3-\kappa_4)} - \frac{\psi(U-V)N}{(\kappa_2-\kappa_3)^2(\kappa_1-\kappa_2)}$$
$$-\frac{2\psi^2 N(U-V)N}{(\kappa_2-\kappa_3)^3(\kappa_1-\kappa_2)(\kappa_3-\kappa_4)} + O(\psi^{k-1}) = O(\psi^{k-(3/2)}).$$

Calculation gives that main diagonal blocks are transformed into

$$\begin{bmatrix} \kappa_1 I + \psi(\kappa_1-\kappa_2)^{-1}N + O(\psi) & I + \psi N + O(\psi^{3/2}) \\ O(\psi^{k+(1/2)}) & \kappa_2 I - \psi(\kappa_1-\kappa_2)^{-1}N + O(\psi) \end{bmatrix},$$

$$\begin{bmatrix} \kappa_3 I + \psi(\kappa_3-\kappa_4)^{-1}N + O(\psi) & I + \psi B_{12} + O(\psi^{3/2}) \\ O(\psi^{k+(1/2)}) & \kappa_4 I - \psi(\kappa_3-\kappa_4)^{-1}N + O(\psi) \end{bmatrix}$$

and a matrix of order $O(\psi^{k+(1/2)})$ stays on the place of ψD. The principal terms of the rest blocks of the resolvent matrix remain the same as in original form (1). Thus we proved, that the resolvent matrix reduces to the block-triangular form by nonsingular analytic similarity transformation. We mean here rows in fractional powers of ψ : $\psi = 0$ is branch point.

Remark. We can't in general hope to find holomorphic similarity transformation even for separation blocks, corresponding to the cycles of the eigenvalues: "The sum of the eigenprojections in each cycle is single valued at $\psi = 0$ but need not be holomorphic (it may have a pole)"[5].

The terms (not equel to 1) in blocks lying upper the main diagonal blocks can be removed easily by using upper triangular matrix similarity transformation. First we remove the blocks $\psi\mathcal{R}_{12}, \psi\mathcal{R}_{13}$ in the first row of (6) by performing similarity transformation

$$\begin{bmatrix} I & Q_{12} & Q_{13} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \text{ with } Q_{12} = -\begin{bmatrix} (1-\omega_1)^{-1} & 0 & 0 & 0 \\ 0 & (1-\omega_2)^{-1} & 0 & 0 \\ 0 & 0 & (1-\omega_3)^{-1} & 0 \\ 0 & 0 & 0 & (1-\omega_3)^{-1} \end{bmatrix} \times$$

$$\times \psi\mathcal{R}_{12} \times \begin{bmatrix} 1 & 0 & -(1-\omega_1)^{-1} & 0 \\ 0 & 1 & 0 & -(1-\omega_2)^{-1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + O(\psi^{3/2}),$$

the matrix Q_{13} is obtained from Q_{12} by substituting $\mathcal{R}_{13}$ in place of $\mathcal{R}_{12}$. In the same manner we remove all perturbations in (6). We must move from above to below in rows and from the left to the right in each row. At the end we construct the basis of analytic generalized eigenvectors in which the resolvent matrix takes the simple matrix Jordan form:

$$\begin{bmatrix} \Omega & 0 & 0 \\ 0 & J_1 & 0 \\ 0 & 0 & J_2 \end{bmatrix}.$$

Here $\Omega = diag(\omega_1, \omega_2, \omega_3, \omega_4)$ is the same as in original form (1). The blocks J_1, J_2 are matrix Jordan boxes:

$$J_1 = \begin{bmatrix} \kappa_1 I + \psi(\kappa_1-\kappa_2)^{-1}N + O(\psi) & I \\ 0 & \kappa_2 I - \psi(\kappa_1-\kappa_2)^{-1}N + O(\psi) \end{bmatrix},$$

$$J_2 = \begin{bmatrix} \kappa_3 I + \psi(\kappa_3-\kappa_4)^{-1}N + O(\psi) & I \\ 0 & \kappa_4 I - \psi(\kappa_3-\kappa_4)^{-1}N + O(\psi) \end{bmatrix}.$$

In general case we use result, proved in lemma below. Above the special simple case of $k + l = 2$ was used.

Lemma. The following relation

$$\sum_{j_1=0}^{k}\sum_{j_2=j_1}^{k}\cdots\sum_{j_l=j_{l-1}}^{k} h^{j_1+j_2+\ldots+j_l}=0,\quad h=exp\{-2\pi i/(k+l)\},$$

holds.

The proof is shown by induction in l with constant $m=k+l$.

Remark that $h^{k+l}=1$. For $l=1$ our relation is true:

$$\sum_{j_1=0}^{k} h^{j_1}=\frac{1-h^{k+1}}{1-h}=0,\quad h^{k+1}=h^{k+l}=1.$$

Let our relation is true for $l\le n$. We show, that it's true then for $l=n+1$:

$$\sum_{j_1=0}^{k}\sum_{j_2=j_1}^{k}\cdots\sum_{j_{n+1}=j_n}^{k} h^{j_1+j_2+\ldots+j_{n+1}}=0,\quad h=exp\{-2\pi i/m\},k=m-n-1.$$

After producing the first summation we have

$$\sum_{j_1=0}^{k}\sum_{j_2=j_1}^{k}\cdots\sum_{j_n=j_{n-1}}^{k} h^{j_1+j_2+\ldots+j_n}\frac{h^{j_n}-h^{k+1}}{1-h}.$$

The last multiplier is zero, when $j_n=k+1$. So we can replace in original sum

$$\sum_{j_1=0}^{k}\sum_{j_2=j_1}^{k}\cdots\sum_{j_{n+1}=j_n}^{k}\quad by\quad \sum_{j_1=0}^{k}\sum_{j_2=j_1}^{k}\cdots\sum_{j_n=j_{n-1}}^{k+1}\sum_{j_{n+1}=j_n}^{k}.$$

And we have the same after every next summation: we get function equal to 0 in $j_s=k+1$. So all upper limits, except the last, can be replace by $k+1$. Now consider the result of the first summation:

$$\sum_{j_1=0}^{k+1}\sum_{j_2=j_1}^{k+1}\cdots\sum_{j_n=j_{n-1}}^{k+1} h^{j_1+j_2+\ldots+j_n}\frac{h^{j_n}-h^{k+1}}{1-h}.$$

The second part

$$\frac{-h^{k+1}}{1-h}\sum_{j_1=0}^{k+1}\sum_{j_2=j_1}^{k+1}\cdots\sum_{j_n=j_{n-1}}^{k+1} h^{j_1+j_2+\ldots+j_n}=0$$

by induction assumption. Now we must prove, that the first part

$$\frac{1}{1-h}\sum_{j_1=0}^{k+1}\sum_{j_2=j_1}^{k+1}\cdots\sum_{j_n=j_{n-1}}^{k+1} h^{j_1+j_2+\ldots+j_{n-1}+2j_n}$$

is equel to 0. After producing the first summation we get

$$\frac{1}{1-h}\sum_{j_1=0}^{k+1}\sum_{j_2=j_1}^{k+1}\cdots\sum_{j_{n-1}=j_{n-2}}^{k+1} h^{j_1+j_2+\ldots+j_{n-1}}\frac{h^{2j_{n-1}}-h^{2(k+2)}}{1-h^2}.$$

By analogy with above we get, that all upper limits except the last in original sum can be replace by $k+2$ and the second part is equel to 0 by induction assumption again. The first part here is

$$\frac{1}{(1-h)(1-h^2)}\sum_{j_1=0}^{k+2}\sum_{j_2=j_1}^{k+2}\cdots\sum_{j_{n-1}=j_{n-2}}^{k+2} h^{j_1+j_2+\ldots+j_{n-2}+3j_{n-1}}.$$

At the end of n–th step we have

$$\sum_{j_1=0}^{k+n}\frac{h^{(n+1)j_1}}{(1-h)(1-h^2)\ldots(1-h^n)}=\frac{(1-h^{(k+n+1)(n+1)})}{(1-h)(1-h^2)\ldots(1-h^{n+1})}=0,$$

because of $n+1=l$ by assumption and $h^{k+l}=1$ by definition of h.

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Построение квазижордановой нормальной формы резольвентной матрицы в случае параболических классов однородной структуры

В нашей предшествующей работе [1] доказана теорема существования квазижордановой нормальной формы резольвентной матрицы в случае, когда собственные значения характеристической матрицы образуют единственный параболический блок. В предлагаемой работе на конкретном примере показано, как строится квазижорданова нормальная форма при наличии нескольких параболических классов однородной структуры. В рассматриваемом случае квазижорданова нормальная форма резольвентной матрицы имеет блочно-диагональную структуру. Каждый параболический блок имеет на диагонали матричный жорданов ящик. В заключение доказывается соотношение для первообразных корней, которое используется при построении нормальной формы.

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Construction of Quasi-Jordan Normal Form of Resolvent Matrix in the Case of Uniform Parabolic Blocks

In our previous work [1] we proved the theorem on existence of a quasi-Jordan normal form of the resolvent matrix in the case when all the eigenvalues of the characteristic matrix form a unique parabolic class. In the proposed work we show by concrete example how the quasi-Jordan normal form is constructed in the presence of uniform parabolic blocks. In this case the quasi-Jordan normal form of the resolvent matrix has block-diagonal structure. Each parabolic block has a matrix Jordan box on the diagonal. An interesting relation for the primitive roots is to be proved in the process of construction. We do this in conclusion.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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