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LINEAR TRANSPORTS ALONG PATHS IN VECTOR BUNDLES

V. Properties of Curvature and Torsion

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1 Introduction

The properties of curvature and torsion tensors of a linear connection and the satisfied by them Bianchi identities are well-known [1, 2]. Looking over the given in [3] definitions and properties of the curvature and torsion of linear transports along paths, one can expect to find out similar results in this more general case too. To their derivation is devoted this paper. Sect. 2 reviews some definitions and results from [4, 3] and also contains new ones needed for our investigation. Sect. 3 proposes a geometrical interpretation of the torsion of a linear transport along paths on the basis of the question of the existence of an 'infinitesimal' parallelogram. Sect. 4 deals with the geometrical meaning of the curvature of a linear transport along paths. It is shown that the curvature governs the main change of a vector after a suitable transportation along a 'small' (infinitesimal) close path. Sect. 5 gives the derivation of the generalizations of the Bianchi identities in the case of linear transports along paths. This is done by using the developed in [5] method for obtaining many-point generalizations of the Jacobi identity. Sect. 6 closes the paper with some concluding remarks, including a criterion for flatness of a linear transport along paths.

2 Some preliminary definitions and results

Below are summarized some needed for this investigation definitions and results for linear transports along paths in vector bundles and their curvature and torsion.

Let (E, π, B) be a real[1] vector bundle with base B, total space E and projection $\pi : E \to B$. The fibres $\pi^{-1}(x)$, $x \in B$ are supposed to be isomorphic real vector spaces. Let $\gamma : J \to B$, with J being a real interval, be an arbitrary path in B. According to [4, definition 2.5] a linear transport (L-transport) in (E, π, B) is a map $L : \gamma \mapsto L^\gamma$, where the L-transport along γ is $L^\gamma : (s,t) \mapsto L^\gamma_{s\to t}$, $s,t \in J$. Here $L^\gamma_{s\to t} : \pi^{-1}(\gamma(s)) \to \pi^{-1}(\gamma(t))$ is the L-transport along γ from s to t. It satisfies the equations:

$$L^\gamma_{s\to t}(\lambda u + \mu v) = \lambda L^\gamma_{s\to t} u + \mu L^\gamma_{s\to t} v,\ \lambda, \mu \in \mathbb{R},\ u, v \in \pi^{-1}(\gamma(s)), \tag{2.1}$$

$$L^\gamma_{t\to r} \circ L^\gamma_{s\to t} = L^\gamma_{s\to r}, \quad r, s, t \in J, \tag{2.2}$$

$$L^\gamma_{s\to s} = id_{\pi^{-1}(\gamma(s))} \tag{2.3}$$

with id_U being the identity map of the set U.

[1] All of the results of this work are valid mutatis mutandis in the complex case too.

Propositions 2.1 and 2.3 of [4] state that the general structure of $L^{\gamma}_{s\to t}$ is

$$L^{\gamma}_{s\to t} = (F^{\gamma}_t)^{-1} \circ F^{\gamma}_s, \ s,t \in J, \tag{2.4}$$

where the map $F^{\gamma}_s : \pi^{-1}(\gamma(s)) \to V$ is a linear isomorphism on a vector space V. The map F^{γ}_s is defined up to a left composition with a linear isomorphism $D^{\gamma} : V \to \underline{V}$, with $\underline{V}$ being a vector space, i.e. up to the change $F^{\gamma}_s \to D^{\gamma} \circ F^{\gamma}_s$.

Let $\{e_i(s)\}$ be a basis in $\pi^{-1}(\gamma(s))$. Here and below the indices $i, j, k, \ldots$ run from 1 to $\dim(\pi^{-1}(x)) = \text{const} =: n$. The matrix of the L-transport L (see [4, p. 5]), $H(t, s; \gamma) = [H^i_j(t, s; \gamma)] = H^{-1}(s, t; \gamma)$, is defined by $L^{\gamma}_{s\to t} e_j(s) = H^i_j(t, s; \gamma) e_i(t)$, where hereafter summation over repeated indices is assumed. The matrix of the coefficients of L [4, p. 13] is $\Gamma_{\gamma}(s) = [\Gamma^i_j(s;\gamma)] = \partial H(s,t;\gamma)/\partial t\,|_{t=s}$. Therefore for a C^2 L-transport, we have

$$\begin{aligned} H^{\pm1}(s+\varepsilon, s;\gamma) &= H^{\mp1}(s, s+\varepsilon;\gamma) = \\ &= \mathbb{1} \mp \varepsilon\Gamma_{\gamma}(s) + \frac{\varepsilon^2}{2}\left(\Gamma_{\gamma}(s)\Gamma_{\gamma}(s) \mp \frac{\partial\Gamma_{\gamma}(s)}{\partial s}\right) + O(\varepsilon^2), \end{aligned} \tag{2.5}$$

with $\mathbb{1}$ being the unit matrix. Here we have used

$$\begin{aligned} \left.\frac{\partial^2 H(s,t;\gamma)}{\partial t^2}\right|_{t=s} &= \Gamma_{\gamma}(s)\Gamma_{\gamma}(s) + \frac{\partial\Gamma_{\gamma}(s)}{\partial s}, \\ \left.\frac{\partial^2 H(t,s;\gamma)}{\partial t^2}\right|_{t=s} &= \Gamma_{\gamma}(s)\Gamma_{\gamma}(s) - \frac{\partial\Gamma_{\gamma}(s)}{\partial s}. \end{aligned} \tag{2.6}$$

These equations follow from the fact that the general form of the matrix H is $H(t, s; \gamma) = F^{-1}(t;\gamma)F(s;\gamma)$ for some nondegenerate matrix function F [4].

Let $\eta : J \times J' \to M$, J, with J' being $\mathbb{R}$-intervals, be a C^2 map on the real differentiable manifold M with a tangent bundle $(T(M), \pi, M)$. Let $\eta(\cdot, t) : s \mapsto \eta(s,t)$ and $\eta(s,\cdot) : t \mapsto \eta(s,t)$, $(s,t) \in J \times J'$. Here by $\eta'(\cdot,t)$ and $\eta''(s,\cdot)$ we denote the tangent to $\eta(\cdot,t)$ and $\eta(s,\cdot)$, respectively, vector fields.

By [3, definition 2.1] the torsion (operator) of a C^1 L-transport L in $(T(M), \pi, M)$ is a map

$$\mathcal{T} : \eta \mapsto \mathcal{T}^{\eta} : J \times J' \to T(M)$$

such that

$$\mathcal{T}^{\eta}(s,t) := \mathcal{D}^{\eta(\cdot,t)}_s \eta''(\cdot,t) - \mathcal{D}^{\eta(s,\cdot)}_t \eta'(s,\cdot) \in T_{\eta(s,t)}(M), \tag{2.7}$$

where $\mathcal{D}^{\gamma}_s$ is the associated with L differentiation along paths [4], defined by

$$\mathcal{D}^{\gamma}_s\sigma := (\mathcal{D}^{\gamma}\sigma)(\gamma(s)) := \left.\left[\frac{\partial}{\partial\varepsilon}\left(L^{\gamma}_{s+\varepsilon\to s}\sigma(s+\varepsilon)\right)\right]\right|_{\varepsilon=0}$$

for a C^1 section σ.

Analogously [3], for $\eta : J \times J' \to B$ the curvature (operator) of an L-transport L in the vector bundle (E, π, B) is a map

$$\mathcal{R} : \eta \mapsto \mathcal{R}^{\eta} : (s,t) \mapsto \mathcal{R}^{\eta}(s,t) : \mathrm{Sec}^2(E,\pi,B) \to \mathrm{Sec}(E,\pi,B)$$

such that

$$\mathcal{R}^{\eta}(s,t) := \mathcal{D}^{\eta(\cdot,t)} \circ \mathcal{D}^{\eta(s,\cdot)} - \mathcal{D}^{\eta(s,\cdot)} \circ \mathcal{D}^{\eta(\cdot,t)}. \tag{2.8}$$

In terms of the coefficient matrix Γ the components of torsion and curvature are respectively [3]

$$(\mathcal{T}^{\eta}(s,t))^i = \Gamma^i_j(s;\eta(\cdot,t))\,(\eta''(s,t))^j - \Gamma^i_j(t;\eta(s,\cdot))\,(\eta'(s,t))^j, \tag{2.9}$$

$$\begin{aligned} \left[(\mathcal{R}^{\eta}(s,t))^i_j\right] &= \frac{\partial}{\partial s}\Gamma_{\eta(s,\cdot)}(t) - \frac{\partial}{\partial t}\Gamma_{\eta(\cdot,t)}(s) + \\ &+ \Gamma_{\eta(\cdot,t)}(s)\Gamma_{\eta(s,\cdot)}(t) - \Gamma_{\eta(s,\cdot)}(t)\Gamma_{\eta(\cdot,t)}(s). \end{aligned} \tag{2.10}$$

Below we shall need the following definitions:

Definition 2.1 *The torsion vector field (operator) of an L-transport in the tangent to a manifold bundle is a section $T^{\eta} \in \mathrm{Sec}\left((T(M),\pi,M)|_{\eta(J,J')}\right)$ defined by*

$$T^{\eta}(\eta(s,t)) := \mathcal{T}^{\eta}(s,t). \tag{2.11}$$

Defining $(\mathcal{D}^{\gamma}\sigma)(\gamma(s)) := \mathcal{D}^{\gamma}_s\sigma$, from (2.7) we get

$$T^{\eta}(\eta(s,t)) = \left(\mathcal{D}^{\eta(\cdot,t)}\eta''(\cdot,t) - \mathcal{D}^{\eta(s,\cdot)}\eta'(s,\cdot)\right)(\eta(s,t)). \tag{2.12}$$

Definition 2.2 *The curvature vector field (operator) of an L-transport is a C^2 section $R^{\eta} \in \mathrm{Sec}^2\left((E,\pi,B)|_{\eta(J,J')}\right)$ defined by*

$$R^{\eta}(\eta(s,t)) := \mathcal{R}^{\eta}(s,t). \tag{2.13}$$

Definition 2.3 *An L-transport along paths is called flat ($\equiv$curvature free) on a set $U \subseteq B$ if its curvature operator vanishes on U. It is called flat if it is flat on B, i.e. in the case $U = B$.*

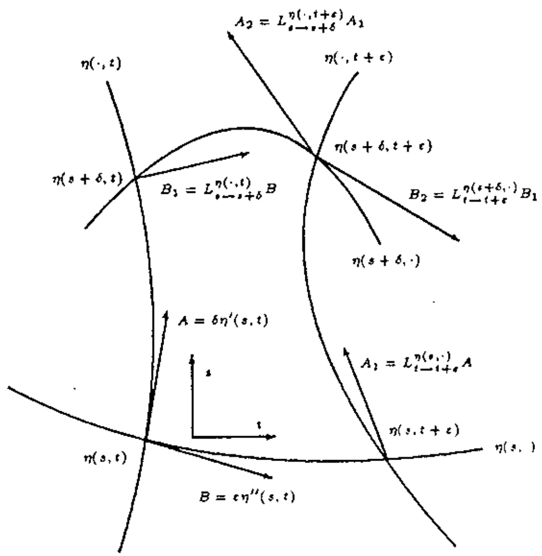

Figure 1: Geometrical interpretation of the torsion

3 Geometrical interpretation of the torsion

Let $\eta : J \times J' \to M$ be a C^1 map into the manifold M, $(s,t) \in J \times J'$, and $\delta, \varepsilon \in \mathbb{R}$ be such that $(s+\delta, t+\varepsilon) \in J \times J'$. Below we consider δ and ε as 'small' (infinitesimal) parameters with respect to which expansions like (2.5) will be used.

Consider the following two paths from $\eta(s,t)$ to $\eta(s+\delta,t+\varepsilon)$ (see figure 1): the first, through $\eta(s+\delta,t)$, being a product of $\eta(\cdot,t) : [s, s+\delta] \to M$ and $\eta(s+\delta,\cdot) : [t, t+\varepsilon] \to M$, and the second one, through $\eta(s,t+\varepsilon)$, being a product of $\eta(s,\cdot) : [t,t+\varepsilon] \to M$ and $\eta(\cdot,t+\varepsilon) : [s,s+\delta] \to M$. (Here δ and ε are considered as positive, but this is inessential.)

Up to $O(\delta^2)$ and $O(\varepsilon^2)$ the vectors $A := \delta\eta'(s,t)$ and $B := \varepsilon\eta''(s,t)$ are the displacement vectors [6] (linear elements [1]), respectively, of $\eta(s+\delta,t)$ and $\eta(s,t+\varepsilon)$ with respect to $\eta(s,t)$.

Using (2.5) and keeping only the first order in ε and δ terms in it, we get the following *component* relation:

$$\left(L^{\eta(\cdot,t)}_{s\to s+\delta} B\right)^i - \left(L^{\eta(s,\cdot)}_{t\to t+\varepsilon} A\right)^i = (B-A)^i - \delta\varepsilon\,(\mathcal{T}^{\eta}(s,t))^i + O(\delta\varepsilon^2) + O(\delta^2\varepsilon). \quad (3.1)$$

According to [7, ch. V, sect. 1] this result has the following interpretation. *After the 'L-transportation' of two linear elements A and B along each other we get, up to second order terms, a pentagon with a closure vector $-\delta\varepsilon\mathcal{T}^{\eta}(s,t)$.* This implies the existence of an infinitesimal parallelogram only in the torsion free case.

Using again (2.5) and keeping only first order terms, after some algebra, we find

$$\begin{aligned}
&\left(L^{\eta(s+\delta,\cdot)}_{t\to t+\varepsilon} \circ L^{\eta(\cdot,t)}_{s\to s+\delta} B - L^{\eta(\cdot,t+\varepsilon)}_{s\to s+\delta} \circ L^{\eta(s,\cdot)}_{t\to t+\varepsilon} A\right)^i = \\
&= \left[\left(L^{\eta(s,\cdot)}_{t\to t+\varepsilon} B\right)^i - \left(L^{\eta(\cdot,t)}_{s\to s+\delta} A\right)^i\right] - \delta\varepsilon\,(\mathcal{T}^{\eta}(s,t))^i + \\
&+ O(\delta^3) + O(\delta^2\varepsilon) + O(\delta\varepsilon^2) + O(\varepsilon^3).
\end{aligned} \quad (3.2)$$

Note that if η is a family of L-paths, i.e. $L^{\eta(\cdot,t)}_{s_1\to s_2}\eta'(s_1,t) = \eta'(s_2,t)$ and $L^{\eta(s,\cdot)}_{t_1\to t_2}\eta''(s,t_1) = \eta''(s,t_2)$, for all $s, s_1, s_2 \in J$ and $t, t_1, t_2 \in J'$, the expression in the square brackets in (3.2) is simply $(B-A)^i$.

So, the torsion describes the first order correction to the difference of two (infinitesimal) displacement vectors when they are (L-)transported in the above-described way.

4 Geometrical interpretation of the curvature

Let (E, π, B) be a vector bundle, $\eta : J \times J' \to B$ be a C^1 map, and L be a C^2 L-transport along paths in (E, π, B). Let $(s,t) \in J \times J'$ and $\delta, \varepsilon \in \mathbb{R}$, $\delta, \varepsilon > 0$ (this condition is insignificant for the final result) be such that $(s+\delta, t+\varepsilon) \in J \times J'$.

Consider the paths on figure 2. The result of an L-transport of a vector from $\eta(s,t)$ to $\eta(s+\delta,t)$ along $\eta(\cdot,t)|_{[s,s+\delta]}$, then from $\eta(s+\delta,t)$ to $\eta(s+\delta,t+\varepsilon)$ along $\eta(s+\delta,\cdot)|_{[t,t+\varepsilon]}$, then from $\eta(s+\delta,t+\varepsilon)$ to $\eta(s,t+\varepsilon)$ along $\eta(\cdot,t+\varepsilon)|_{[s,s+\delta]}$, and, at last, from $\eta(s,t+\varepsilon)$ to $\eta(s,t)$ along $\eta(s,\cdot)|_{[t,t+\varepsilon]}$ is expressed by

Proposition 4.1 *For any C^2 L-transport, we have*

$$\begin{aligned}
&L^{\eta(s,\cdot)}_{t+\varepsilon\to t} \circ L^{\eta(\cdot,t+\varepsilon)}_{s+\delta\to s} \circ L^{\eta(s+\delta,\cdot)}_{t\to t+\varepsilon} \circ L^{\eta(\cdot,t)}_{s\to s+\delta} = \\
&= id_{\pi^{-1}(\eta(s,t))} - \delta\varepsilon\mathcal{R}^{\eta}(s,t) + O(\delta^3) + O(\delta^2\varepsilon) + O(\delta\varepsilon^2) + O(\varepsilon^3).
\end{aligned} \quad (4.1)$$

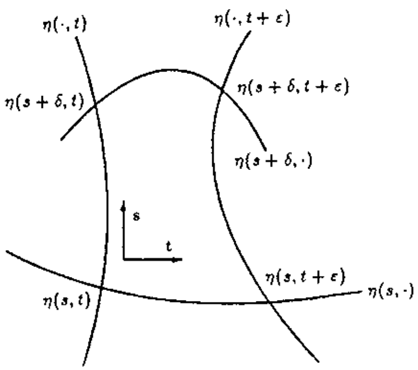

Figure 2: Geometrical interpretation of the curvature

Proof. In a field $\{e_i(s,t),\ (s,t) \in J \times J'\}$ of bases in $\pi^{-1}(\eta(J,J'))$ the matrix of the linear map standing in the l.h.s. of (4.1) is

$$H(t,t+\varepsilon;\eta(s,\cdot))H(s,s+\delta;\eta(\cdot,t+\varepsilon))H(t+\varepsilon,t;\eta(s+\delta,\cdot))H(s+\delta,s;\eta(\cdot,t)).$$

Substituting here (2.5) and using the expressions

$$\Gamma_{\eta(s+\delta,\cdot)}(t) = \Gamma_{\eta(s,\cdot)}(t) + \delta\frac{\partial}{\partial s}\Gamma_{\eta(s,\cdot)}(t) + O(\delta^2),$$

$$\Gamma_{\eta(\cdot,t+\varepsilon)}(s) = \Gamma_{\eta(\cdot,t)}(s) + \varepsilon\frac{\partial}{\partial t}\Gamma_{\eta(\cdot,t)}(s) + O(\varepsilon^2),$$

we find this matrix to be

$$\mathbb{1}+\delta\varepsilon\left(\frac{\partial}{\partial t}\Gamma_{\eta(\cdot,t)}(s) - \frac{\partial}{\partial s}\Gamma_{\eta(s,\cdot)}(t) - \Gamma_{\eta(\cdot,t)}(s)\Gamma_{\eta(s,\cdot)}(t) + \Gamma_{\eta(s,\cdot)}(t)\Gamma_{\eta(\cdot,t)}(s)\right)+ \\ + O(\delta^3) + O(\delta^2\varepsilon) + O(\delta\varepsilon^2) + O(\varepsilon^3).$$

Taking into account (2.10), we get this expression as

$$\mathbb{1} - \delta\varepsilon\left[(\mathcal{R}^\eta(s,t))^i_j\right] + O(\delta^3) + O(\delta^2\varepsilon) + O(\delta\varepsilon^2) + O(\varepsilon^3), \qquad (4.2)$$

which is simply the matrix form of (4.1). ■

Proposition 4.1 shows that *up to third order terms the result of the above-described transportation of a vector* $A \in \pi^{-1}(\eta(s,t))$ *is*

$$A - \delta\varepsilon\,(\mathcal{R}^\eta(s,t))(A). \qquad (4.3)$$

Another corollary of (4.1) is the (equivalent to the definition of the curvature) equality

$$\mathcal{R}^\eta(s,t) = -\lim_{\substack{\delta\to 0\\ \varepsilon\to 0}}\left[\frac{1}{\delta\varepsilon}\left(L^{\eta(s,\cdot)}_{t+\varepsilon\to t}\circ L^{\eta(\cdot,t+\varepsilon)}_{s+\delta\to s}\circ L^{\eta(s+\delta,\cdot)}_{t\to t+\varepsilon}\circ L^{\eta(\cdot,t)}_{s\to s+\delta} - id_{\pi^{-1}(\eta(s,t))}\right)\right]. \qquad (4.4)$$

5 Bianchi-type identities

The curvature operator (2.8) is simply a commutator of two derivations along paths. As we shall see below, the torsion (2.7) is also a skewsymmetric expression. All this allows one to apply the developed in [5] method for obtaining Jacobi-type identities. This can be done as follows.

Let us take an arbitrary map $\tau^k : J^k \to B$, with $J^k := J \times \cdots \times J$, where J appears k times, $k \in \mathbb{N}$, and B being the base of the vector bundle (E,π,B). Let $s := (s^1,\ldots,s^k) \in J^k$. We define the C^1 paths $\tau_a : J \to B$ by $\tau_a(\sigma) := \tau^k(s)\big|_{s^a=\sigma}$, $\sigma \in J$ and the maps (families of paths) $\tau_{ab} : J \times J \to B$ by $\tau_{ab}(\sigma_1,\sigma_2) := \tau^k(s)\big|_{s^a=\sigma_1,s^b=\sigma_2}$, $\sigma_1,\sigma_2 \in J$, which depend implicitly on s. Hereafter $a,b,c,d = 1,\ldots,k$. We write $\dot\tau_a$ for the tangent to τ_a vector field in the case when $(E,\pi,B) = (T(M),\pi,M)$ for some manifold M.

Proposition 5.1 *The following properties of antisymmetry are valid:*

$$\mathcal{R}^{\tau_{ab}}(s_a,s_b) + \mathcal{R}^{\tau_{ba}}(s_b,s_a) = 0, \qquad (5.1)$$

$$T^{\tau_{ab}} + T^{\tau_{ba}} = 0 \text{ or } \mathcal{T}^{\tau_{ab}}(s_a,s_b) + \mathcal{T}^{\tau_{ba}}(s_b,s_a) = 0. \qquad (5.2)$$

Remark. These equalities are analogues of the usual skewsymmetry of curvature and torsion tensors in the tensor analysis [1].

Proof. The two-point Jacobi-type identity is $\left((A_{ab})_{[a,b]}\right)_{<a,b>} \equiv 0$ (see [5, eq. (5.1)]), where A_{ab} are elements of an Abelian group, $(A_{ab})_{[a,b]} := A_{ab} - A_{ba}$ and $(A_{ab})_{<a,b>} := A_{ab} + A_{ba}$. Substituting here $A_{ab} = \mathcal{D}^{\tau_a}\circ\mathcal{D}^{\tau_b}$ in the case of a vector bundle (E,π,B) and $A_{ab} = \mathcal{D}^{\tau_a}(\dot\tau_b)$ in the case of the tangent to a manifold M bundle $(T(M),\pi,M)$ and using (2.8) and (2.12) (or (2.7)), one gets respectively (5.1) and (5.2). ■

Proposition 5.2 *The following identities are valid:*

$$\{\mathcal{D}^{\tau_a} \circ \mathcal{R}^{\tau_{bc}}(s_b, s_c) - \mathcal{R}^{\tau_{bc}}(s_b, s_c) \circ \mathcal{D}^{\tau_a}\}_{<a,b,c>} \equiv 0 \text{ or} \qquad (5.3)$$
$$\{\mathcal{D}^{\tau_a}(\mathcal{R}^{\tau_{bc}}(s_b, s_c))\}_{<a,b,c>} \equiv 0;$$
$$\{(\mathcal{R}^{\tau_{ab}}(s_a, s_b))(\dot{\tau}_c)\}_{<a,b,c>} \equiv \{\mathcal{D}^{\tau_a}(\mathcal{T}^{\tau_{bc}})\}_{<a,b,c>}. \qquad (5.4)$$

where $< \dots >$ means summation over the cyclic permutations of the corresponding indices.

Remark. These identities are analogous, respectively, of the second and first Bianchi identities in tensor analysis [7, 2]. This is clear from the fact that due to the antisymmetries (5.1) and (5.2) the cyclization over the indices a, b and c, i.e. the operation $< \dots >$, in (5.3) and (5.4) may be replaced with antisymmetrization over the indices a, b and c. (E.g. if $A_{abc} = -A_{acb}$ and $(A_{abc})_{[a,b,c]} := (A_{abc} + A_{bca} + A_{cab})_{[b,c]}$, then $2(A_{abc})_{<abc>} = (A_{abc})_{[abc]}$.)

Proof. The (3-point) generalized Jacobi identity (see [5, eq. (5.2)]) is $\left((A_{abc})_{[a,[b,c]]}\right)_{<a,b,c>} \equiv 0$, with A_{abc} being elements of an Abelian group, $(A_{abc})_{[a,[b,c]]} := (A_{abc} - A_{bca})_{[b,c]}$ and $(A_{abc})_{<a,b,c>} := A_{abc} + A_{bca} + A_{cab}$.

We put $A_{abc} = \mathcal{D}^{\tau_a} \circ \mathcal{D}^{\tau_b} \circ \mathcal{D}^{\tau_c}$ in the vector bundle case and $A_{abc} = (\mathcal{D}^{\tau_a} \circ \mathcal{D}^{\tau_b})(\dot{\tau}_c)$ in the tangent bundle case. In this way, after some simple algebra (see (2.8), (2.7) and (2.1)-(2.3)), we get respectively (5.3) and (5.4). ∎

The 4-point generalized Jacobi-type identity

$$\left\{(A_{abcd})_{[a,[b,[c,d]]]} + (A_{adcb})_{[a,[d,[c,b]]]}\right\}_{<a,b,c,d>} \equiv 0$$

with $(A_{abcd})_{[a,[b,[c,d]]]} := (A_{abcd} - A_{bcda})_{[b,[c,d]]}$ and $(A_{abcd})_{<a,b,c,d>} := A_{abcd} + A_{bcda} + A_{cdab} + A_{dabc}$ also produces an interesting identity in our case. In fact, putting $A_{abcd} = \mathcal{D}^{\tau_a} \circ \mathcal{D}^{\tau_b} \circ \mathcal{D}^{\tau_c} \circ \mathcal{D}^{\tau_d}$ in the vector bundle case, one can easily prove after some simple calculations

Proposition 5.3 *The identity*

$$\{\mathcal{R}^{\tau_{ab}}(s_a, s_b)(R^{\tau_{cd}})\}_{<a,b,c,d>} \equiv 0. \qquad (5.5)$$

where $R^{\tau_{cd}}$ is the curvature vector field on $\tau^k(J, \dots, J)$ is valid.

Remark. This result generalizes eq. (6.5) of [5] in the classical tensor case.

The last result also follows from the evident chain identity

$$\begin{aligned}
0 &\equiv \{\mathcal{R}^{\tau_{ab}}(s_a, s_b) \circ \mathcal{R}^{\tau_{cd}}(s_c, s_d) - \mathcal{R}^{\tau_{ab}}(s_a, s_b) \circ \mathcal{R}^{\tau_{cd}}(s_c, s_d)\}_{<a,b,c,d>} \equiv \\
&\equiv \{\mathcal{R}^{\tau_{ab}}(s_a, s_b) \circ \mathcal{R}^{\tau_{cd}}(s_c, s_d) - \mathcal{R}^{\tau_{cd}}(s_c, s_d) \circ \mathcal{R}^{\tau_{ab}}(s_a, s_b)\}_{<a,b,c,d>} \equiv \\
&\equiv \{(\mathcal{R}^{\tau_{ab}}(s_a, s_b)(R^{\tau_{cd}}))(\tau_{cd}(s_c, s_d))\}_{<a,b,c,d>} \equiv \\
&\equiv \left(\{\mathcal{R}^{\tau_{ab}}(s_a, s_b)(R^{\tau_{cd}})\}_{<a,b,c,d>}\right)(\tau^k(s)).
\end{aligned}$$

Note that in the tangent bundle case the substitution $A_{abcd} = (\mathcal{D}^{\tau_a} \circ \mathcal{D}^{\tau_b} \circ \mathcal{D}^{\tau_c})(\dot{\tau}_d)$ leads to the trivial identity $0 \equiv 0$.

6 Conclusion

In this paper we have examined some natural properties of the curvature (resp. the torsion) of linear transports along paths in vector bundles (resp. in the tangent bundle to a manifold). These properties are similar to the ones in the theory of linear connections. The cause for this similarity is that in the case of the parallel transport assigned to a linear connection our results reproduce the corresponding ones in the classical tensor analysis. The reduction to the known classical results can easily be proved by applying the used in [3] method for introduction of curvature and torsion of a linear connection by means of its parallel transport.

In connection with this, below is presented the generalization of the theorem that a linear connection is flat iff the assigned to it parallel transport is independent of the path (curve) along which it acts and depends only on the initial and final points of the transportation.

Theorem 6.1 *An L-transport in (E, π, B) is flat on $U \subseteq B$ if and only if in U it is independent of the path (lying in U) along which it acts and depends only on its initial and final points, i.e. the set $\{L^{\gamma}_{s\to t}\}$ forms a flat L-transport in $U \subseteq B$ iff $L^{\gamma}_{s\to t}$ for $\gamma : J \to U$ depends only on the points $\gamma(s)$ and $\gamma(t)$, but not on the path γ itself.*

Remark. In this theorem we implicitly suppose U to be linearly connected, i.e. its every two points can be connected by a path lying entirely in U. Otherwise the theorem may not be true.

Proof. Let the L-transport L be flat, i.e. $\mathcal{R}^{\eta}(s, t) \equiv 0$ for $\eta : J \times J' \to U \subseteq B$. By [3, theorem 3.1] there is a field of bases $\{e_i\}$ on U in which the matrix of L is unit, i.e. $H(t,s;\gamma) = \mathbb{1}$, $\gamma : J \to U$. In these bases for $u \in \pi^{-1}(\gamma(s))$, we have $L^{\gamma}_{s\to t}u = H^i_j(t,s;\gamma)u^j \left(e_i \big|_{\gamma(t)}\right) = u^i \left(e_i \big|_{\gamma(t)}\right)$, which evidently depends on the points $\gamma(s)$ and $\gamma(t)$ but not on the path γ itself.

Conversely, let for $\gamma : J \to U$ the transport $L^{\gamma}_{s\to t}$ depends only on the points $\gamma(s)$ and $\gamma(t)$ and not on the path γ connecting them. For fixed $x_0 \in U$ and basis $\{e^0_i\}$ in $\pi^{-1}(x)$ we define on U the field of bases $\{e_i\}$ by $e_i\big|_x := L^{\beta}_{a\to b}e^0_i$, where β is any path in U joining x_0 and $x \in U$, and such that $\beta(a) := x_0$ and $\beta(b) := x$. By assumption $\{e_i\big|_x\}$ depends only on x but not on β. Using that $L^{\gamma}_{s\to t}$ depends only on $\gamma(s)$ and $\gamma(t)$, we have

$$\begin{aligned} L^{\gamma}_{s\to t}\left(e_i\big|_{\gamma(s)}\right) &= L^{\alpha}_{a\to b}\left(e_i\big|_{\alpha(a)}\right) = \\ &= L^{\alpha}_{a\to b}(L^{\alpha}_{c\to a}e^0_i) = L^{\alpha}_{c\to b}e^0_i = e_i\big|_{\alpha(b)} = e_i\big|_{\gamma(t)}, \end{aligned}$$

where α is any path in U such that $\alpha(a) = \gamma(s)$, $\alpha(b) = \gamma(t)$, and $\alpha(c) = x_0$. As $L^{\gamma}_{s\to t}\left(e_i\big|_{\gamma(s)}\right) = H^j_i(t,s;\gamma)e_j\big|_{\gamma(t)}$, we see that in $\{e_i\}$ the matrix of L is $H(t,s;\gamma) = \mathbb{1}$, which, again by [3, theorem 3.1], implies the flatness of L in U. ∎

In conclusion we have to note that all of the results of the present paper remain true in the complex case. For this purpose one has simply to replace in it the word 'real' with 'complex' and the symbols $\mathbb{R}$ and dim with $\mathbb{C}$ and $\dim_{\mathbb{C}}$ respectively.

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