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NON-ALGEBRAIC INTEGRABILITY OF ONE REVERSIBLE DYNAMICAL SYSTEM OF THE CREMONA TYPE

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1 Introduction

Reversible dynamical systems (RDS) are qualitatively very similar to Hamiltonian systems [1]–[9]. In particular, the existence of the Kolmogorov–Arnold–Moser tori in reversible non-Hamiltonian flows [1]–[6], and non-symplectic mappings [4]–[7], has been proved. Reversible dynamical systems are of great importance in physics, which is a consequence of the time-reversible invariance of many physical laws (see surveys [9]).

One of examples of RDS is the system of nonlinear functional (difference) equations [12] of the static model of the dispersion approach [11]. This system is defined by a reversible mapping that is the composition of two involutions: the standard Cremona transformation [10] and the crossing-symmetry mapping. Dynamical systems (DS) of this type, distinguished only by the crossing-symmetry matrix A and defined by the quadratic Cremona mappings, have been investigated in papers [13] –[22]. Interest in integrating these DS is related, besides the physical one, with a problem of integrating general Cremona mappings which, according to the M. Noether theorem, can be constructed from different quadratic Cremona mappings [10]. Moreover, such DS do not belong to the Quispel 18-parametric family of integrable mappings [9], [23], and were not considered in the survey on integrable maps [24]. The problem of integrating these two-dimensional DS with the second-order matrix A was solved long ago in [14], [15], but only recently 3-dimensional DS with the third-order matrices $A(1,1)$ and $A^{Chew-Low}$ have been integrated in [17] and [18]. A general approach to integrating such n-dimensional DS was developed in [19]. This paper is devoted to the problem of integrating dynamical system with crossing-symmetry matrix $A(l,1)$, describing the scattering of two particles with spins equal l and 1 within in the framework of the static model.

2 The general formulation of the problem. Structure of the general solution

Definition 1 Let X be an arbitrary set. A one-to-one mapping $T : X \to X$ is said to be reversible if there exists another mapping $G : X \to X$ for which $T^{-1} = G \circ T \circ G$ and G is an involution: $G^2 = id$ [6], [8], [9].

These conditions imply that $T \circ G$ is also an involution and $T = (T \circ G) \circ G$ is the composition of two involutions. Conversely, the composition of any two involutions is reversible with respect to each of them.

Consider the following two involutions of the complex plane $\mathbf{C}$: the linear one

$$A_1 : \mathbf{C} \to \mathbf{C}, \quad A_1 : w \mapsto -w,$$

and the affine one

$$I_1 : \mathbf{C} \to \mathbf{C}, \quad I_1 : w \mapsto 1 - w.$$

These involutions do not commute. Their compositions $A_1 \circ I_1$ and $I_1 \circ A_1$ define a couple of two reciprocal reversible dynamical systems on $\mathbf{C}$.

On the other hand, consider the following two involutions of the space $\mathbf{C}^3$: the linear one, defined by the crossing symmetry involutive matrix $A \stackrel{\text{def}}{=} A(l, 1)$

$$z \mapsto Az, \qquad A = \begin{pmatrix} 1 - \frac{(2l-1)(l+1)}{(2l+1)l} & -\frac{1}{l} & \frac{2l+3}{2l+1} \\ -\frac{2l-1}{(2l+1)l} & 1 - \frac{1}{l(l+1)} & \frac{2l+3}{(2l+1)(l+1)} \\ \frac{2l-1}{2l+1} & \frac{1}{l+1} & 1 - \frac{(2l+3)l}{(2l+1)(l+1)} \end{pmatrix} \tag{1}$$

with $\det A = -1$, and the nonlinear one

$$I : \mathbf{C}^3 \to \mathbf{C}^3, \qquad I(z_1, z_2, z_3) = \left(\frac{1}{z_1}, \frac{1}{z_2}, \frac{1}{z_3}\right)$$

(the latter being the standard Cremona transformation). These involutions do not commute either. Their compositions $A \circ I$ and $I \circ A$ define a couple of two reciprocal reversible dynamical systems on $\mathbf{C}^3$. Let us remark that the parameter $l \in \mathbf{N}$ though one can consider $l \in \mathbf{R}$.

We look for a meromorphic mapping $\mathbf{C} \to \mathbf{C}^3$ which realizes the equivalence of these two couples of reversible dynamical systems. In other words, we seek a meromorphic mapping $S : \mathbf{C} \to \mathbf{C}^3$ satisfying the following system of functional equations (FE):

$$S(-w) = AS(w), \tag{2}$$

$$S(w+1) = I \circ AS(w), \tag{3}$$

where $w \in \mathbf{C}$. This system describes a scattering of two particles with spins equal l and 1 within the framework of the static model.

Taking the liberty of speech, we will sometimes call the system of FE (2)–(3) a dynamical system (DS).

One can say that the involutions A and I of $\mathbf{C}^3$ are representations of the involutions A_1 and I_1 of $\mathbf{C}$, respectively. Meromorphic mappings $S : \mathbf{C} \to \mathbf{C}^3$ satisfying FE (2)–(3) realize these representations.

Each meromorphic mapping $S : \mathbf{C} \to \mathbf{C}^3$ defines some curve in $\mathbf{C}^3$. The system of FE (2)–(3) completely determines this curve up to the group of automorphisms. One can easily verify that every transformation of the form

$$S(w) \to S\left(w + \beta(w)\right) \exp\left(\alpha(w - 1/2)\right), \tag{4}$$

where

$$\begin{aligned} \beta(w+1) &= \beta(w), & \beta(-w) &= -\beta(w), \\ \alpha(w+1) &= -\alpha(w), & \alpha(-w) &= -\alpha(w), \end{aligned}$$

is an automorphism of the solution space for DS (2)–(3) (β defining an inner automorphism of the curve). Obviously, there should also exist an automorphism which depends on another arbitrary function in w with a period equal to 1.

As $A^2 = id$, the matrix A is of a simple structure (i.e., diagonalizable):

$$A = B\Lambda^a B^{-1}, \qquad \Lambda^a = \operatorname{diag}(\lambda_1^a, \lambda_2^a, \lambda_3^a), \qquad \lambda_1^a = -1, \quad \lambda_{2,3}^a = 1, \tag{5}$$

$$B_{ij} = \mu_i^{(j)}, \qquad A\mu^{(j)} = \lambda_j^a \mu^{(j)}, \quad i, j = 1, 2, 3, \tag{6}$$

where B is the fundamental matrix for A in the basis of eigenvectors $\mu^{(j)}$ of the matrix A [28], which are the following:

$$\mu^{(1)} = (-2(l+1), -2, 2l), \quad \mu^{(2)} = \frac{2l+1}{2l+2}\left((2l+3), -\frac{(2l+3)(2l-1)}{2l+1}, 1\right), \mu^{(3)} = (1, 1, 1). \tag{7}$$

According to the general approach [19], introduce the functions $z : \mathbf{C} \to \mathbf{C}^3$, $j : \mathbf{C}^3 \to \mathbf{C}^3$, $\pi : \mathbf{C}^3 \to \mathbf{C}$,

$$z(w) = B^{-1} S(w), \tag{8}$$

$$j(z) = ABz, \tag{9}$$

$$\pi(z) = \prod_{i=1}^{3} j_i(z).$$

Define the mapping $\Phi : \mathbf{C}^3 \to \mathbf{C}^3$:

$$\Phi : z \mapsto \tilde{z} = \Phi(z) = \phi(z)/\pi(z), \tag{10}$$

where $\phi : \mathbf{C}^3 \to \mathbf{C}^3$ is the mapping

$$\phi(z) = \pi(z) B^{-1} \left(Ij(z)\right). \tag{11}$$

The DS (2)–(3) can be rewritten as:

$$z(-w) = \Lambda^a z(w), \quad (12)$$

$$z(w+1) = \phi(z(w))/\pi(z(w)). \quad (13)$$

Remark 1 Note that the mapping (10) is a birational Cremona transformation in $\mathbf{C}^3$. Transformations of the form (10) with different denominators $\pi(z)$ (but the same numerator $\phi(z)$) induce the same projective Cremona transformation in $\mathbf{CP}^2$ [10]:

$$\tilde{z}_1 : \tilde{z}_2 : \tilde{z}_3 = \phi_1 : \phi_2 : \phi_3.$$

According to (5) – (9) and taking into account that $A^2 = id$, we have

$$\Lambda^a = B^{-1}AB, \qquad S(w) = j\left(\Lambda^a z(w)\right) \stackrel{\text{def}}{=} j'\left(z(w)\right). \quad (14)$$

2.1 Partial automorphic forms

Let $m = (m_1, m_2, m_3) \in \mathbf{Z}_+^3$ be a multiindex, $m_i \in \mathbf{Z}_+$, $|m| = m_1 + m_2 + m_3$, where $\mathbf{Z}_+$ is a set of nonnegative integral numbers.

Definition 2 A polynomial $P : \mathbf{C}^3 \to \mathbf{C}$ is said to be invariant if $P(\Phi(z)) = 0$ whenever $P(z) = 0$.

Theorem 1 [19], For any invariant irreducible homogeneous polynomial $P(z)$ of degree k there exists a multiindex m with $|m| = k$ such that $P(z)$ satisfies the following system of FE:

$$P(\phi(z)) = \varepsilon P(z) j^m(z), \qquad j^m = j_1^{m_1} j_2^{m_2} j_3^{m_3}, \qquad \varepsilon = \pm 1, \quad (15)$$

$$P(\Lambda^a z) = \nu P(z), \qquad \nu = \pm 1. \quad (16)$$

We will denote any solution of these equations by $P_m(z)$ provided that it is an irreducible homogeneous polynomial of degree $|m|$.

As the matrix A (1) has the only eigenvalue equal to -1, we can assume that $\nu = +1$ because for $\nu = -1$ any solution P of FE (15)–(16) will be reducible:

$$P(z) = z_1^{2n+1} \hat{P}(z), \quad \deg \hat{P} = |m| - 2n - 1.$$

Following [19], we shall call polynomials $P_m(z)$ the partial automorphic forms (PAF) for the DS (2)–(3) of weight $m = (m_1, m_2, m_3)$.

Definition 3 PAF $P_m(z)$ is said to be an automorphic form (AF) of weight q if

$$P_m(\Phi(z)) \equiv P_m(z) J^q(z)$$

where J denotes the Jacobian of the mapping Φ (10).

One can easily verify that the Jacobian J of the mapping Φ (10) is equal to $\pi^{-2}(z)$. Indeed,

$$J = -\det\left(B^{-1} J B \Lambda^a\right) = -\det\left(A\bar{J}\right) = \pi^{-2}(z),$$

where $\bar{J} = \mathrm{diag}\left(1/j_1^2(z), 1/j_2^2(z), 1/j_3^2(z)\right)$.

A mapping Φ has a polynomial AF of weight q if and only if it possesses a PAF of weight $m = \{q, q, q\}$. It has a rational AF of weight q if there exists a rational function of the PAF that satisfies equations (15)–(16) with $m_1 = m_2 = m_3 = q$.

According to [17], the DS (2)–(3) with the crossing-symmetry matrix $A(1,1), (l = 1)$ has an AF of weight one (recall that it has three PAF) and is equivalent to an APM but it is non-algebraically integrable, i.e., it possesses a first non-algebraic integral. From this it follows that the DS (2)–(3) with matrix (1) isn't also algebraic integrable. The one has the PAF ($z_1/z_3 = x$, $z_2/z_3 = y$):

$$P_{(1,1,0)}(z) = z_3 z_2 - z_1^2 = z_3^2(y - x^2),$$

— the solution of (15) with $\varepsilon = -1$.

2.2 Fundamental points and principal lines of a quadratic Cremona mapping

Let $\tilde{z} = z(w+1)$, then according to (14), (13), (9) we have

$$j'(\tilde{z}) = I j(z).$$

Consider some special points and lines on the projective plane $\mathbf{CP}^2$ of homogeneous coordinates z known from the theory of quadratic Cremona transformations. Introduce six principal lines (P-lines) J_k, J'_k $(k = 1,2,3)$ of the direct mapping Φ specified by (10) and the inverse one Φ^{-1}:

$$J_k = \{z :\ j_k(z) = 0\}, \qquad J'_k = \{z :\ j'_k(z) = 0\} \qquad (k = 1,2,3),$$

and six points of pairwise intersections of these lines $\{O_1, O_2, O_3\}$ and $\{O'_1, O'_2, O'_3\}$:

$$O_i = J_k \bigcap J_n, \qquad O'_i = J'_k \bigcap J'_n.$$

They are called fundamental points (F-points) [10] of the direct mapping (13) and the inverse one. Here $\{ikn\}$ is a permutation of $\{123\}$.

Under the direct mapping (13) and its inverse

$$z(w) = \frac{\Lambda^a \phi(\Lambda^a \tilde{z})}{\pi(\Lambda^a \tilde{z})},$$

the images of the F-points O_i, O'_i and those of the P-lines J_i, J'_i are

$$\Phi : O_i \mapsto J'_i, \quad \Phi : J_i \mapsto O'_i, \qquad \Phi^{-1} : O'_i \mapsto J_i, \quad \Phi^{-1} : J'_i \mapsto O_i.$$

Thus, the F-points and P-lines $\{O_i, J_i\}$ and $\{O'_i, J'_i\}$ are the only elements of the mapping (13) where the one-to-one correspondence is violated.

Notice that the mappings Φ and Φ^{-1} realize blowing up and contraction (see the Codaira theorem [27], Ch. 1, §4) and also the concept of σ-process of blowing up of singularities in the theory of ordinary differential equations [26], §2.

The F- and P-elements have the form

$$J_i = \left\{ z : \; j_i(z) = \mu_i^{(k)} \lambda_k^{(a)} z_k = 0 \right\}, \qquad J'_i = \left\{ z : \; j'_i(z) = \mu_i^{(k)} z_k = 0 \right\},$$

$$O_1 = \left(\frac{2}{2l-1}, \frac{4}{(2l-1)^2}, 1 \right), \quad O_2 = \left(\frac{2}{4l^2+4l-1}, -\frac{4}{4l^2+4l-1}, 1 \right),$$
$$O_3 = \left(-\frac{2}{2l+3}, \frac{4}{(2l+3)^2}, 1 \right), \qquad O'_i = \Lambda^a O_i. \tag{17}$$

Remark 2 Similarly to [21], [17], the multiindex $m = (m_1, m_2, m_3)$ has the meaning of multiplicities m_i of the algebraic invariant curve $P_m(z) = 0$ at the F-points O_i (see (15), (16)), and, consequently, at the points O'_i, according to the symmetry of (16).

2.3 Structure of the general solution

Let us proceed from homogeneous projective coordinates $z(w) \in \mathbf{C}^3$ to the coordinates

$$((x_1(w), x_2(w), z_3(w)), \quad x_i = \frac{z_i}{z_3} \quad (i = 1, 2), \quad x(w) = (x_1(w), x_2(w)) \in \mathbf{C}^2. \tag{18}$$

Then instead of (12), (13) we obtain

$$x_i(-w) = \Lambda^a_{ii} x_i(w) \quad (i = 1, 2), \qquad z_3(-w) = z_3(w), \tag{19}$$

$$x_i(w+1) = \frac{\phi_i(x z_3, z_3)}{\phi_3(x z_3, z_3)} \stackrel{\text{def}}{=} f_i(x) \quad (i = 1, 2), \qquad z_3(w+1) = \Phi_3(x z_3, z_3), \tag{20}$$

where ϕ_i and Φ_3 are defined by (5)-(11) and are (below and in the sequel, we set $x_1 \equiv x$, $x_2 \equiv y$)

$$\phi_1(x, 1) = x + 3x^2 - \frac{4l^2+4l-1}{4} xy - \frac{(2l+3)(2l-1)}{4} y^2, \tag{21}$$

$$\phi_2(x, 1) = -y + 2x^2 + 2xy - \frac{(2l-1)(2l+3)}{4} y^2, \tag{22}$$

$$\phi_3(x, 1) = 1 + 4x + \frac{4l^2+4l+5}{4} - \frac{4l^2+4l-9}{2} x^2 - \frac{4l^2+4l-7}{2} xy - \frac{(2l+3)(2l-1)}{4} y^2. \tag{23}$$

The following theorem on the structure of the general solution of the FE (19)–(20) is valid.

Theorem 2 The general solution of FE (19)–(20) has the form:

$$x(w) = \frac{2\left(F(w+\frac{1}{2}) + F(w-\frac{1}{2})\right)}{2(l-1) - (2l-3)(F(w+\frac{1}{2}) - F(w-\frac{1}{2})) + 4F(w+\frac{1}{2})F(w-\frac{1}{2})}, \tag{24}$$
$$y(w) = \frac{1}{2l-1} \frac{4\left(2(l-1) + F(w+\frac{1}{2}) - F(w-\frac{1}{2})\right)}{2(l-1) - (2l-3)(F(w+\frac{1}{2}) - F(w-\frac{1}{2})) + 4F(w+\frac{1}{2})F(w-\frac{1}{2})},$$

where the meromorphic function $F(w)$ is a solution of the functional equations ($F'' = F(w+2)$, $F' = F(w+1)$, $F = F(w)$)

$$F'' = \frac{2(l-1)^2(2l+3) + (l-1)(2l^2+l-5)F - 2(4l-1)F'^2 + 2FF' - (2l-1)FF'^2}{(l-1)(2l^2+l-5) - 2F' - 4(l-1)F - (2l-1)F'^2}, \tag{25}$$

$$F(-w) = -F(w). \tag{26}$$

Proof. Introduce the quadratic Cremona mapping casting the functions $x(w), y(w)$ to $u_1(w), u_2(w)$:

$$u_1 = \frac{y - \frac{4}{(2l-1)^2}}{y - \frac{2}{2l-1} x}, \qquad u_2 = \frac{y - \frac{4}{(2l-1)^2}}{y + \frac{2}{2l-1} x}. \tag{27}$$

The inverse mapping is single-valued and has the form:

$$x = \frac{2}{2l-1} \frac{u_1 - u_2}{u_1 + u_2 - 2u_1u_2}, \qquad y = \frac{4}{(2l-1)^2} \frac{u_1 + u_2}{u_1 + u_2 - 2u_1u_2}. \tag{28}$$

Using (27), (28) and (21)-(23) one can obtain a simpler expression for the system (20):

$$u_1(w+1) = \frac{\left(4l(2l+1)u_1 + 4u_2 - (2l-1)^2 u_1 u_2\right)\left(\frac{4(l-1)}{2l-1} - u_1\right)}{4l(2l-3)u_1 + 8(l-1)u_2 - (2l-1)^2 u_1^2}, \quad u_2(w+1) = \frac{4(l-1)}{2l-1} - u_1, \tag{29}$$

where in the right-hand side we have set $u_1 = u_1(w)$, $u_2 = u_2(w)$. According to (19) and (27) we have:

$$u_1(w) = u_2(-w). \tag{30}$$

Substituting (30) into the second eq. of (29) and assuming

$$u_2(w) = \frac{2(l-1)}{2l-1} - \frac{2}{2l-1}F(w-1/2), \tag{31}$$

we get (26). Eqs. (28), (30) and (31) imply (24). Substituting (30), (31) into the first eq. of (29) we obtain equation (25) for $F(w)$.

This equation has a particular solution $F(w) = w$ corresponding to particular rational solution for $S(w)$ in [16] .

3 Integration of the equation for the function $F(w)$

Let $F(w-1) = u(w)$, $F(w) = v(w)$ and consider the mapping T corresponding to equation (25) for $F(w)$ $\left(\tilde{U} = (\tilde{u}, \tilde{v}) = (u(w+1), v(w+1)),\ U = (u, v)\right)$

$$\begin{aligned} T : \tilde{U} &= T(U), \quad \tilde{u} = v, \\ \tilde{v} &= \frac{2(l-1)^2(2l+3) + (l-1)(2l^2+l-5)u - 2(4l-1)v^2 + 2uv - (2l-1)uv^2}{(l-1)(2l^2+l-5) - 2v - 4(l-1)u - (2l-1)v^2}. \end{aligned} \tag{32}$$

The mapping T is reversible since it can be represented in the form $T = R \circ W$ where R and W are involutions:

$$\begin{aligned} R : U' &= R(U), \quad u' = -u, \\ v' &= \frac{2(l-1)^2(2l+3) - (l-1)(2l^2+l-5)v - 2(4l-1)u^2 + 2uv + (2l-1)vu^2}{(l-1)(2l^2+l-5) + 2v + 4(l-1)v - (2l-1)u^2}; \end{aligned}$$

$$W : U' = W(U), \quad u' = -v, \quad v' = -u. \tag{33}$$

Besides the parabolic fixed point $d_1 : u = v = \infty$, the mapping (32) has two symmetrical hyperbolic fixed points

$$d_2 : \ u = v = (l-1)\sqrt{\frac{2l+3}{2l-1}}, \qquad d_3 : \ u = v = -(l-1)\sqrt{\frac{2l+3}{2l-1}}, \tag{34}$$

$$R(d_2) = d_3, \qquad R(d_3) = d_2, \qquad W(d_2) = d_3, \qquad W(d_3) = d_2. \tag{35}$$

The eigenvalues λ_1, λ_2 of the linear part of the mapping (32) at the fixed point d_2 (34) and λ_3, λ_4 at d_3 are the roots of fundamental polynomial $P_4(\lambda)$ of algebraic number λ:

$$\lambda_{1,2} = \frac{2\sqrt{(2l+3)(2l-1)} \pm \sqrt{3(2l+3)(2l-1)+1}}{\sqrt{(2l+3)(2l-1)}+1} \quad (+ \text{ for } \lambda_1); \quad \lambda_3 = \lambda_1^{-1}, \quad \lambda_4 = \lambda_2^{-1}. \tag{36}$$

Notice that the numbers λ_i are called conjugate algebraic numbers (see [29]). The eigenvalues λ_1, λ_2 of the mapping (32) at d_2 for $l \in \mathbf{N}$ belong to Siegel's domain ($0 < \lambda_2 < 1 < \lambda_1$). The same holds for the eigenvalues λ_3, λ_4.

Theorem 3 The set of eigenvalues λ_1, λ_2 at the fixed point d_2 and the set λ_3, λ_4 at the fixed point d_3 (36) are not resonant, i.e.

$$\lambda_j - \lambda_1^{m_1}\lambda_2^{m_2} \neq 0 \quad \text{for } j = 1,2 \text{ and } |m| = m_1 + m_2 \geq 2, m_i \in \mathbf{Z}_+,$$

$$\lambda_j - \lambda_3^{m_1}\lambda_4^{m_2} \neq 0 \quad \text{for } j = 3,4 \text{ and } |m| = m_1 + m_2 \geq 2, m_i \in \mathbf{Z}_+.$$

Proof. We have to prove that algebraic numbers $\chi_1 = 1 - \lambda_1^{m_1-1}\lambda_2^{m_2}$ and $\chi_2 = 1 - \lambda_1^{m_1}\lambda_2^{m_2-1}$ are not equal to zero for any integers m_1 and m_2, $|m| \geq 2$. Consider the number χ_1. As $\lambda_1, \lambda_2 \in \mathbf{Q}(\sqrt{N_1}, \sqrt{N_2})$, $N_1 = (2l+3)(2l-1)$, $N_2 = 3(2l+3)(2l-1)+1$, a quadratic extension of the rational number field [29] (see §17), then χ_1 belongs to $\mathbf{Q}(\sqrt{N_1}, \sqrt{N_2})$ and has relative degree ≤ 4. The equality $\chi_1 = 0$ implies $\chi_1 \in \mathbf{Q}(\sqrt{N_1})$. Since $\lambda_{1,2} = R_1 \pm R_2\sqrt{N_2}$, where $R_1, R_2 \in \mathbf{Q}(\sqrt{N_1})$, then, obviously, $\chi_1 \in \mathbf{Q}(\sqrt{N_1})$ if and only if $m_2 = m_1 - 1$. Let $m_2 = m_1 - 1$. Then $\chi_1 = 1 - ((N_1 + 1 - 2\sqrt{N_1})/(N_1 - 1))^{m_2}$. It is clear that for $|m| = 2m_2 + 1 \geq 2$ $(m_2 \geq 1)$ one has $\chi_1 \notin \mathbf{Q}$ and, consequently, $\chi_1 \neq 0$. Analogously, one can prove that $\chi_2 \neq 0$. It is obvious that also the set λ_3, λ_4 is non-resonant at the fixed point d_3. ◁

The problem of linearization of the mapping (32) will be solved on the basis of the following Siegel theorem.

Definition 4 A set $(\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$ is of the multiplicative type (C, ν), if the following inequality:

$$|\lambda_j - \lambda^m| \geq C|m|^{-\nu}, \quad (|m| = m_1 + \ldots + m_n, \ \lambda^m = \lambda_1^{m_1} \ldots \lambda_n^{m_n}) \tag{37}$$

is satisfied for all $j \in (1, 2, \ldots, n)$, $m_i \in \mathbf{Z}_+$, $|m| \geq 2$ $(C > 0, \nu > 0)$.

Theorem 4 (C.L. Siegel for $n = 1$, E. Zehnder's generalization for $n > 1$ [30])
If the set $\{\lambda\}$ of eigenvalues of the linear part of a mapping, holomorphic near a fixed point, is of the multiplicative type (C, ν) for some $C > 0, \nu > 0$, then the mapping is biholomorphically equivalent to its linear part in some neighborhood of the fixed point (see [26], §28).

The following theorem on sets $\{\lambda\}$ of the multiplicative type (C, ν) is valid [18]).

Theorem 5 (see [18].) A set $(\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$ is of the multiplicative type (C, ν), if this set is multiplicatively non-resonant and numbers $\lambda_j \in \mathbf{A}$, where $\mathbf{A}$ is an algebraic number field.

This theorem is an corollary of the Feldman theorem [25] (see chapter 10, §4.10) about the evaluation of linear forms of the logarithms of algebraic numbers.

Now we can make a general proposition about the solutions of equations (25)–(26) for the function $F(w)$, defining the general solution (24) of the initial FE (20).

Theorem 6 The function $F_\delta(w)$, as a function of a one-parametric family of solutions of the system (25)–(26), is a holomorphic, in certain finite neighborhoods of the origin in $\mathbf{C}^2$, function of variables $z_1,\ z_2$ or $z_3,\ z_4$:

$$z_i(w) = \delta(w) \exp\left(\left(w + \beta(w) - \frac{1}{2}\right) \ln \lambda_i\right), \quad 1 \le i \le 4, \tag{38}$$

where λ_i are determined by (36) while arbitrary functions $\delta(w)$ and $\beta(w)$ (cf. (4)) have the following properties:

$$\begin{aligned} \delta(-w) &= \delta(w), & \delta(w+1) &= \delta(w), \\ \beta(-w) &= -\beta(w), & \beta(w+1) &= \beta(w). \end{aligned} \tag{39}$$

The family $F_\delta(w)$ is defined by the Taylor expansion in z_1, z_2

$$F(w) = \sum_k f_k z^k, \qquad k \in \mathbf{Z}^2_+, \qquad f_k = f_{k_1,k_2}, \qquad z^k = z_1^{k_1} z_2^{k_2} \tag{40}$$

with the coefficients f_k determined by the recurrent relation

$$\begin{aligned} f_k \;=\; & \left\{ \sum_{|m|=1}^{|k|-1} f_{k-m} \Big[\Big(-((2l-1)f_0 + 1)\lambda^{2k-m} + ((2l-1)f_0 - 1)\lambda^m - 2(l-1)\lambda^{2k-2m} \right. \\ & \left. + (4l-1)\lambda^k \Big) f_m - (2l-1)/2(\lambda^{2k-m} - \lambda^m) \sum_{|l|=1}^{|m|-1} f_{m-l} f_l \Big] \right\} \\ & \Big[\Big((2l-1)f_0 + l - 1 \Big)(\lambda^k - \lambda_1)(\lambda^k - \lambda_2) \Big]^{-1}, \end{aligned} \tag{41}$$

where $\lambda^k = \lambda_1^{k_1}\lambda_2^{k_2}$, $f_0 = (l-1)\sqrt{\frac{2l+3}{2l-1}}$, $f_{0,1}$ and $f_{1,0}$ are arbitrary, and the sum is taken over all permissible values. The coefficients $\tilde{f}_k$ of the Taylor expansion of F in z_3, z_4 satisfy the same recurrent relation with $\lambda_1,\ \lambda_2$ replaced by $\lambda_3,\ \lambda_4$ and they are

$$\tilde{f}_0 = -(l-1)\sqrt{\frac{2l+3}{2l-1}}, \qquad \tilde{f}_{0,1} = -\lambda_3 f_{0,1}, \qquad \tilde{f}_{1,0} = -\lambda_4 f_{1,0}, \qquad \tilde{f}_k = -f_k/\lambda^k.$$

Each local solution of the family $F_\delta(w)$ (the germ), defined by (38)-(41), can be extended up to the global one by using $F_\delta(w), F_\delta(w-1)$ and iterations of equation (25) and lies on the invariant manifold Γ_δ

$$\Gamma_\delta : \qquad z_1^{\gamma_1}(w+1) z_2^{\gamma_2}(w+1) = z_1^{\gamma_1}(w) z_2^{\gamma_2}(w) = \delta^2(w), \tag{42}$$

where

$$\gamma_1 = \frac{-2\ln\lambda_2}{\ln\lambda_1 - \ln\lambda_2}, \qquad \gamma_2 = \frac{2\ln\lambda_1}{\ln\lambda_1 - \ln\lambda_2}.$$

Proof. According to Theorems 3 and 5, the set (λ_1, λ_2) from (36) of the eigenvalues of the mapping T (32) at the hyperbolic fixed point d_2 given by (34) is of the multiplicative type (C, ν). Therefore, Theorem 4 guarantees the existence of a biholomorphic mapping $U = G(z)$, $U = (u, v)$, $z = (z_1, z_2)$, $G(0) = d_2$, which transforms a neighborhood of the origin in $\mathbf{C}^2$ to a neighborhood of the point d_2 and reduces the mapping T to the normal form at d_2

$$G^{-1}TG : \ (z_1, z_2) \mapsto (\lambda_1 z_1, \lambda_2 z_2).$$

On the other hand, the biholomorphic mapping WG, where W is involution (33), transforms a neighborhood of the origin in $\mathbf{C}^2$ to a neighborhood of the point d_3 (34) and reduces the mapping T to the normal form at d_3:

$$G^{-1}WTWG : (z_3, z_4) \mapsto (\lambda_3 z_3, \lambda_4 z_4).$$

Indeed, since involution W reverses mapping T, the mappings $G^{-1}TG$ and $G^{-1}WTWG$ are inverse to each other.

Set $(z_3, z_4) = \zeta$ and

$$G(z) = (g_1(z), g_2(z)), \qquad WG(\zeta) = (-g_2(\zeta), -g_1(\zeta)).$$

If some functions $z_i(w)$, $1 \le i \le 4$, satisfy FE

$$z_i(w+1) = \lambda_i z_i(w), \tag{43}$$

the functions $F(w) = g_2(z(w))$ or $F(w) = -g_1(\zeta(w))$ satisfy FE (25). Moreover, one will have $g_1(z(w)) = F(w-1)$ or $-g_2(\zeta(w)) = F(w-1)$, respectively.

Now we should determine the relations between $z(w)$ and $\zeta(w)$ which make $F(w)$ satisfy FE (26), i.e., make $F(w)$ odd. Let $F(w) = g_2(z(w))$ be close to $1/\sqrt{2}$. Then, we have $F(-w) = -g_2(\zeta(1-w))$. The equality $F(-w) = -F(w)$ is therefore ensured by

$$\zeta(1-w) = z(w). \tag{44}$$

If

$$z_i(w) = \delta_i(w)\exp((w+\alpha_i)\ln\lambda_i), \qquad 1 \le i \le 4, \tag{45}$$

then (43) is equivalent to $\delta_i(w+1) = \delta_i(w)$ and (44) is equivalent to $\zeta(w) = \Lambda z(-w)$, where $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$, i.e.,

$$\delta_{i+2}(-w) = \lambda_i \delta_i(w)\exp((\alpha_i + \alpha_{i+2})\ln\lambda_i), \qquad i = 1, 2.$$

For $\alpha_i = -1/2$ $(1 \le i \le 4)$ we have

$$\delta_3(-w) = \delta_1(w), \qquad \delta_4(-w) = \delta_2(w), \tag{46}$$

Eqs. (43), (45) result in the existence of the invariant manifold Γ_δ (cf. (42))

$$\Gamma_\delta: \quad \tilde z_1^{\gamma_1}\tilde z_2^{\gamma_2} = z_1^{\gamma_1} z_2^{\gamma_2} = \delta^2(w), \qquad \tilde z_3^{\gamma_3}\tilde z_4^{\gamma_4} = z_3^{\gamma_3} z_4^{\gamma_4} = \delta^2(w), \tag{47}$$

where $\tilde z \equiv z(w+1)$, exponents γ_1, γ_2 are determined by (42) and are invariant relative to the change $\lambda_1, \lambda_2 \to \lambda_3, \lambda_4$. Substituting (45), (46) into (47) we have

$$\delta_1^{\gamma_1}(w)\delta_2^{\gamma_2}(w) = \delta^2(w), \qquad \delta_1^{\gamma_1}(-w)\delta_2^{\gamma_2}(-w) = \delta^2(w). \tag{48}$$

From (48) we have

$$\left(\frac{\delta_1(w)}{\delta_1(-w)}\right)^{\gamma_1}\left(\frac{\delta_2(w)}{\delta_2(-w)}\right)^{\gamma_2} = 1. \tag{49}$$

Since $\delta_i(w+1) = \delta_i(w)$, then supposing

$$\frac{\delta_i(w)}{\delta_i(-w)} = \exp(\beta_i(w)\ln\lambda_i), \qquad i = 1, 2, \tag{50}$$

we obtain from (49)

$$\exp[(-\beta_1(w) + \beta_2(w))\ln\lambda_1 \ln\lambda_2] = 1. \tag{51}$$

From (49)-(51) it follows that

$$\beta_1(w) = \beta_2(w) = 2\beta(w), \tag{52}$$

where $\beta(w)$ is odd function with period equal to 1 (cf. (39)). From (50), (52) we have

$$\delta_i(w) = \delta(w)\exp(\beta(w)\ln\lambda_i), \qquad \delta(-w) = \delta(w)$$

and we obtain (38) and (39).

Finally, let

$$F(w) = g_2(z(w)) = \sum_k f_k\,(z(w))^k$$

for $F(w)$ close to $1/\sqrt{2}$ and

$$F(w) = -g_1(\zeta(w)) = \sum_k \tilde f_k\,(\zeta(w))^k$$

for $F(w)$ close to $-1/\sqrt{2}$. Substituting these expansions into (25), we derive the desired recurrent relations on f_k and $\tilde f_k$. The arbitrariness in the choice of $f_{1,0}$ and $f_{0,1}$ corresponds to the ambiguity in the choice of the normalizing mapping G. ◁

Let us consider eq. (25) at $l = -3/2$. It has the form

$$F'' = \frac{5F + 14F'^2 + 2FF' + 4FF'^2}{5 - 2F' + 10F + 4F'2} \tag{53}$$

and admit a reduction of the order of (53)

$$\Phi(F'', F') = \Phi(F', F) + 1, \qquad \Phi(F', F) = \frac{2F'F + (5F' - 7F - 5)/2}{F' + F}. \tag{54}$$

It follows from (54) that $\Phi(F', F) = w + \beta(w)$ and we have the equation of lower order for the function $F(w)$ (we omit below $\beta(w)$)

$$F(w+1) = \frac{(w + 7/2)F(w) + 5/2}{2F(w) - w + 5/2}. \tag{55}$$

General solution of (55) has the form

$$F(w) = w\frac{f(w) - 1}{f(w) + 1}, \qquad f(-w) = f(w), \tag{56}$$

where the function $f(w)$ has the form

$$f(w) = \frac{(w^2 - 1/4)(w^2 - 9/4)}{w}\left(p\left(\frac{w + 1/2}{2}\right) - p\left(\frac{w - 1/2}{2}\right) + \gamma\left(\frac{w + 1/2}{2}\right)\right),$$

$$p(w) = \frac{1}{2}(\Psi(w) + \Psi(-w)) + \frac{1}{2}(\Psi(w + 1/2) + \Psi(-w + 1/2)),$$

where

$$\gamma(-w) = \gamma(w), \qquad \gamma(w + 1/2) = -\gamma(w).$$

The function $\Psi(z) \overset{\mathrm{def}}{=} \frac{d}{dz}\ln\Gamma(z)$ is a solution of functional equation

$$\Psi(z+1) - \Psi(z) = \frac{1}{z}.$$

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