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R.Scherer*, S.Serdyukova

NECESSARY AND SUFFICIENT CONDITIONS FOR MINIMALITY OF LINEAR DYNAMIC SYSTEMS

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*Institut für Praktische Mathematik, Universität Karlsruhe, Germany

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1 Introduction

Consider the linear time–invariant system $\{F, G, H\}$ specified by

(1.1) $$\dot{x} = Fx + Gu, \quad y = Hx,$$

where x is an n–dimensional state vector, u is p–dimensional input vector, and F, G and H are real constant $n \times n, n \times p$ and $p \times n$ matrices respectively. The transfer function matrix is the $p \times p$ rational matrix

(1.2) $$Z(s) = H(sI - F)^{-1}G.$$

The realization $\{F, G, H\}$ is called *minimal*, if the transfer function matrix $Z(s)$ has no pole–zero cancellation. Recall that $Q_{con} = [G, FG, \dots, F^{n-1}G]$ and

$$Q_{obs} = \begin{bmatrix} H \\ HF \\ \vdots \\ HF^{n-1} \end{bmatrix}$$

characterize controllability and observability by fullness of their ranks, and that the linear system $\{F, G, H\}$ is minimal if and only if it is observable and controllable (Anderson and Vongpanitlerd [2, p. 98]).

Two linear systems $\{F, G, H\}$ and $\{\hat{F}, \hat{G}, \hat{H}\}$ are called *equivalent*, if there exists a nonsingular $n \times n$ matrix T satisfying

(1.4). $$\hat{F} = T^{-1}FT, \quad \hat{G} = T^{-1}G, \quad \hat{H} = HT.$$

In this article we prove the necessary and sufficient conditions for controllability, observability and minimality by applying the Jordan canonical form of matrices. We start from discussing an interesting useful example (Scherer and Wendler [1]), which clears up the problem at once.

2 Numerical Example

Consider the linear system (1.1) resulting from an RLC network, where

(2.1) $$F = \begin{bmatrix} -2\alpha_0 & \alpha_0/\alpha_1 & 0 & 0 & 0 \\ -\alpha_0\alpha_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2\alpha_0 & \alpha_0/\alpha_2 & 0 \\ 0 & 0 & -\alpha_0\alpha_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, G = \begin{bmatrix} \alpha_0 \\ \alpha_0\alpha_1 \\ \alpha_0 \\ \alpha_0\alpha_2 \\ \alpha_3 \end{bmatrix}, H^T = \begin{bmatrix} -\alpha_1 \\ 1 \\ -\alpha_2 \\ 1 \\ 1 \end{bmatrix}$$

(here and below ^{T} is the sign of transposition) and $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are positive real numbers. The transfer function

(2.2) $$Z(s) = \frac{\alpha_3(s + \alpha_0)^4}{s(s + \alpha_0)^4} = \frac{\alpha_3}{s}$$

shows a pole–zero cancellation.

The matrix F has two different eigenvalues

$$\lambda_1 = -\alpha_0, \quad \lambda_2 = 0$$

of multiplicities 4 and 1 respectively. We reduce F to the Jordan form:

$$J = T^{-1}FT.$$

The matrix F has block–diagonal form and we find the Jordan form of the blocks. The first block B_1 takes the form of the Jordan box

$$\begin{bmatrix} \lambda_1 & \lambda_1 \\ 0 & \lambda_1 \end{bmatrix}$$

with respect to the basis

$$e_1 = \begin{bmatrix} 1 \\ \alpha_1 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ -\alpha_1 \end{bmatrix}.$$

Here e_1 is eigenvector: $B_1e_1 = \lambda_1e_1$, and e_2 is adjoint vector: $B_1e_2 = \lambda_1e_2 + \lambda_1e_1$. The vector e_2 is generating vector of invariant two–dimensional eigenspace:

$$e_1 = (B_1 - \lambda_1 I)(e_2/\lambda_1).$$

Symmetrically the block B_2 takes the same form as B_1 with respect to the basis

$$e_1 = \begin{bmatrix} 1 \\ \alpha_2 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ -\alpha_2 \end{bmatrix}.$$

Hence

$$E_1 = \begin{bmatrix} 1 \\ \alpha_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \; E_2 = \begin{bmatrix} 0 \\ -\alpha_1 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \; E_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \alpha_2 \\ 0 \end{bmatrix}, \; E_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\alpha_2 \\ 0 \end{bmatrix}; \; E_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

are bases of the invariant eigenspaces for F. The second eigenvalue $\lambda_2 = 0$ has clearly eigenvector E_5, which at the same time is generating vector of one-dimensional eigenspace. So the matrix, reducing F to the Jordan form, is

$$T = [E_1, E_2, E_3, E_4, E_5], \; det\; T = \alpha_1\alpha_2 \neq 0.$$

Calculating gives

$$T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1/\alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/\alpha_2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The Jordan form of F is

$$J = \begin{bmatrix} -\alpha_0 & -\alpha_0 & 0 & 0 & 0 \\ 0 & -\alpha_0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha_0 & -\alpha_0 & 0 \\ 0 & 0 & 0 & -\alpha_0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

We must verify the fullness of rank of

$$Q_{con} = [\hat{G}, J\hat{G}, \ldots, J^4\hat{G}], \quad \hat{G} = T^{-1}G,$$

$$Q_{obs} = [\hat{H}, \hat{H}J, \ldots, \hat{H}J^4]^T, \quad \hat{H} = HT.$$

In this case

$$\hat{G} = T^{-1}G = [\alpha_0,\ 0,\ \alpha_0,\ 0,\ \alpha_3]^T = [b_1,\ b_2,\ b_3,\ b_4,\ b_5]^T.$$

Remark, that $G = \sum_{j=1}^{5} b_j E_j$ (because of $T^{-1}T = I$). So the vector G has zero components with respect to the first and the second generating vectors in the basis of eigenvectors and ajoint vectors . Now consider

$$\hat{H} = HT = [0, -\alpha_1, 0, -\alpha_2, 1] = [d_1, d_2, d_3, d_4, d_5].$$

So $\hat{H}$ has zero components $d_1, d_3 : (H, E_1) = 0, (H, E_3) = 0$, i.e., H is orthogonal to E_1 and E_3 (the first and the second eigenvectors).

Using simple relation

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix},$$

we get

$$Q_{con} = \begin{bmatrix} b_1 & -\alpha_0(b_1+b_2) & \alpha_0{}^2(b_1+2b_2) & -\alpha_0{}^3(b_1+3b_2) & \alpha_0{}^4(b_1+4b_2) \\ b_2 & -\alpha_0 b_2 & \alpha_0{}^2 b_2 & -\alpha_0{}^3 b_2 & \alpha_0{}^4 b_2 \\ b_3 & -\alpha_0(b_3+b_4) & \alpha_0{}^2(b_3+2b_4) & -\alpha_0{}^3(b_3+3b_4) & \alpha_0{}^4(b_3+4b_4) \\ b_4 & -\alpha_0 b_4 & \alpha_0{}^2 b_4 & -\alpha_0{}^3 b_4 & \alpha_0{}^4 b_4 \\ b_5 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note, that the rows, corresponding to the generating vectors (with numbers 2,4,5), are proportional to b_2, b_4, b_5. So it's necessary for the controllability, that the vector G has nonzero components with respect to all generating vectors (in basis of eigenvectors and adjoint vectors). In the considered example this is not true: $b_2 = b_4 = 0$! But if even b_2, b_4 were nonzeros, the second and the fourth rows, corresponding to the different generating vectors with the same eigenvalues $-\alpha_0$, are proportional. So the second necessary condition for the controllability is following: each λ eigenvalue of F must have only one eigenvector. If λ has multiplicity $q > 1$, then it must have

on the diagonal of J the unique Jordan box (of order q). In the considered example $\lambda_1 = -\alpha_0$ of the multiplicity four has two Jordan boxes of second order.

Now turn to observability. In the considered case

$$Q_{obs} = \begin{bmatrix} d_1 & d_2 & d_3 & d_4 & d_5 \\ -\alpha_0 d_1 & -\alpha_0(d_1+d_2) & -\alpha_0 d_3 & -\alpha_0(d_3+d_4) & 0 \\ \alpha_0{}^2 d_1 & \alpha_0{}^2(2d_1+d_2) & \alpha_0{}^2 d_3 & \alpha_0{}^2(2d_3+d_4) & 0 \\ -\alpha_0{}^3 d_1 & -\alpha_0{}^3(3d_1+d_2) & -\alpha_0{}^3 d_3 & -\alpha_0{}^3(3d_3+d_4) & 0 \\ \alpha_0{}^4 d_1 & \alpha_0{}^4(4d_1+d_2) & \alpha_0{}^4 d_3 & \alpha_0{}^4(4d_3+d_4) & 0 \end{bmatrix}.$$

Now the columns corresponding to the eigenvectors (the first, the third and the fifth columns) are proportional to d_1, d_3, d_5. So for observability H must be nonorthogonal to all eigenvectors of F. But in the considered example it is wrong: $d_1 = d_3 = 0$! But if even they were nonzeros, the first and the third columns, corresponding to the different eigenvectors with the same eigenvalue $-\alpha_0$, are proportional. So the second necessary condition of observability exactly the same as in the case of controllability: each eigenvalue of F must have only one eigenvector.

We used here the Jordan form with $-\alpha_0 \neq 0$ on the second diagonals of the Jordan boxes. Let $E_j(c_j)$ be the generating vector of the Jordan box of order q_j with $c_j \neq 0$ on the second diagonal and the corresponding generating vector of the canonical Jordan form is $E_j(1)$. The following relation is true: $E_j(1) = E_j(c_j)/(c_j^{q_j-1})$. So the conditions of nonorthogonality to all generating vectors hold with respect to any Jordan form (any nonzero c_j), in parcular with respect to the canonical Jordan form (Gantmacher [3, VI, §6.3, p.153]), when all $c_j = 1$.

3 Main Results

Before presenting theorems let us formulate **three conditions**, which gives a possibility to say more short. We consider the linear system $\{F, G, H\}$ as in (1.1).

Let F has the Jordan canonical form J with respect to the basis $E_1, \ldots, E_n$ and let there are s Jordan boxes J_j, $j = 1, \ldots, s$. The Jordan box J_j is in q_j rowes of J with the numbers $\mu_j, \mu_j + 1, \ldots, \xi_j$. In particular J_1 begins in the first row, i.e., $\mu_1 = 1$ and J_s ends in the last row, i.e., $\xi_s = n$. The basic vectors $E_{\mu_j}, \ldots, E_{\xi_j}$ are connected by the relations:

$$FE_{\mu_j} = \lambda_j E_{\mu_j},$$

$$FE_{\mu_j+1} = \lambda_j E_{\mu_j+1} + E_{\mu_j},$$

$$FE_{\mu_j+2} = \lambda_j E_{\mu_j+2} + E_{\mu_j+1},$$

$$\ldots$$

$$FE_{\xi_j} = \lambda_j E_{\xi_j} + E_{\xi_j-1}.$$

Thus $E_{\mu_j}, j = 1, \ldots, s$, are eigenvectors. The relations above imply

$$E_{\xi_j-1} = (F - \lambda_j I)E_{\xi_j},$$

$$E_{\xi_j-2} = (F - \lambda_j I)^2 E_{\xi_j},$$

$$\dots,$$

$$E_{\mu_j} = (F - \lambda_j I)^{q_j-1} E_{\xi_j}.$$

Here I are the unit matrices of orders $q_j = \xi_j - \mu_j + 1$. So vectors E_{ξ_j}, $j = 1, ..., s$, generate "Jordan chaines of vectors" (Gantmacher [3, VII, §7, p.188]) which are bases of cyclic invariant subspases corresponding to J_j. We call them generating vectors. The union of "Jordan chaines" is the Jordan basis.

Condition 1. Each eigenvalue of F has only one eigenvector (or, equivalently, each eigenvalue has only one generating vector: in the case of simple eigenvalue the eigenvector is the generating vector of the one–dimensional eigenspace at the same time).

Condition 2. For each generating vector $E_{\xi_j}, j = 1, \dots, s$, there is found at least one column of the matrix G having nonzero component with respect to E_{ξ_j} in the Jordan basis of eigenvectors and adjoint vectors.

Condition 3. For every eigenvector $E_{\mu_j}, j = 1, \dots, s$, there is found at least one (of p) row of the matrix H, nonorthogonal to E_{μ_j}.

Theorem of controllability. The system $\{F, G, H\}$ is controllable if and only if conditions 1 and 2 hold.

Theorem of observability. The system $\{F, G, H\}$ is observable if and only if conditions 1 and 3 hold.

Theorem of minimality. The system $\{F, G, H\}$ is minimal if and only if conditions 1, 2 and 3 hold.

The proof is presented for the controllability only. The theorem of observability and the theorem of minimality are proved by analogy.

The proof of the theorem of controllability.
First we consider the case of $p = 1$. The general case reduces to the considered one. This will be showed at the end. As above let us denote by $b_j, j = 1, \dots, n$, the components of $\hat{G} = T^{-1}G$, where $T = [E_1, \dots, E_n]$, $T^{-1}FT = J$. We simply calculate

$$det\ [\hat{G}, \hat{G}J, \dots, \hat{G}J^{n-1}].$$

The Jordan canonical form J has s Jordan boxes J_j, $j = 1, \dots, s$ of orders $q_j = \xi_j - \mu_j + 1$. But in the case of simple eigenvalues $\xi_j = \mu_j$ and the Jordan boxes

degenerate to scalar λ_j. Respectively the vector

$$\hat{G} = [b_1, b_2, \dots, b_n]^T.$$

is divided into s vectors $\hat{G} = [B_1, ..., B_s]$, where

$$B_j = [b_{\mu_j}, b_{\mu_j+1}, \dots, b_{\xi_j}]^T.$$

As result we have that

$$Q_{con} = \begin{bmatrix} B_1 & B_1 J_1 & \dots & B_1 J_1^{n-1} \\ B_2 & B_2 J_2 & \dots & B_2 J_2^{n-1} \\ . & \dots & \dots & \dots \\ B_s & B_s J_s & \dots & B_s J_s^{n-1} \end{bmatrix}. \tag{3.1}$$

We show, that if **conditions 1 and 2** hold, then $det\ Q_{con} \neq 0$. Let us clear up the structure of matrix rows of Q_{con}. Let $D = [d_1, d_2, \dots, d_q]^T$, $d_q \neq 0$, and J be the Jordan box of order q. Using well known formula (Gantmacher [3, VI, §7.1, p.155])

$$J^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \dots & C_k^{q-1}\lambda^{k-q+1} \\ 0 & \lambda^k & \dots & C_k^{q-2}\lambda^{k-q+2} \\ . & . & \dots & \dots \\ 0 & 0 & \dots 0 & \lambda^k \end{bmatrix},$$

we get

$$[D, DJ, \dots, DJ^{n-1}] = \tag{3.2}$$

$$= \begin{bmatrix} d_1 & d_1\lambda + d_2 & . & d_1\lambda^{q-1} + d_2 C_{q-1}^1 \lambda^{q-2} + \dots + d_q C_{q-1}^{q-1} \\ d_2 & d_2\lambda + d_3 & . & d_2\lambda^{q-1} + d_3 C_{q-1}^1 \lambda^{q-2} + \dots + d_q C_{q-1}^{q-2}\lambda \\ \vdots & \vdots & \vdots & \vdots \\ d_{q-1} & d_{q-1}\lambda + d_q & . & d_{q-1}\lambda^{q-1} + d_q C_{q-1}^1 \lambda^{q-2} \\ d_q & d_q\lambda & . & d_q\lambda^{q-1} \end{bmatrix.}$$

$$\begin{matrix} d_1\lambda^q + d_2 C_q^1 \lambda^{q-1} + \dots + d_q C_q^{q-1}\lambda & . & d_1\lambda^{n-1} + d_2 C_{n-1}^1 \lambda^{n-2} + \dots + d_q C_{n-1}^{q-1}\lambda^{n-q} \\ d_2\lambda^q + d_3 C_q^1 \lambda^{q-1} + \dots + d_q C_q^{q-2}\lambda^2 & . & d_2\lambda^{n-1} + d_3 C_{n-1}^1 \lambda^{n-2} + \dots + d_q C_{n-1}^{q-2}\lambda^{n-q+1} \\ \vdots & \vdots & \vdots \\ d_{q-1}\lambda^q + d_q C_q^1 \lambda^{q-1} & . & d_{q-1}\lambda^{n-1} + d_q C_{n-1}^1 \lambda^{n-2} \\ d_q\lambda^q & . & d_q\lambda^{n-1} \end{matrix} \Bigg] .$$

Turn to Q_{con} (see (3.1)). The row of Q_{con} with number j is obtained from (3.2) by replacing D by B_j, J by J_j, λ by λ_j, q by $q_j = \xi_j - \mu_j + 1$, $[d_1, \dots, d_q]$ by $[b_{\mu_j}, b_{\mu_j+1}, \dots, b_{\xi_j}]$. By assumption $d_q \neq 0$. We subtract from the rows with numbers $1, \dots, q-1$ on the right side of (3.2) the row with number q, multipled by

d_i/d_q, $i = 1, \ldots, q-1$, respectively. As result all terms of the first $q-1$ rows lost their first summands:

$$\begin{bmatrix} 0 & d_2 & . & d_2C^1_{q-1}\lambda^{q-2} + \ldots + d_qC^{q-1}_{q-1} & . & d_2C^1_{n-1}\lambda^{n-2} + \ldots + d_qC^{q-1}_{n-1}\lambda^{n-q} \\ 0 & d_3 & . & d_3C^1_{q-1}\lambda^{q-2} + \ldots + d_qC^{q-2}_{q-1}\lambda & . & d_3C^1_{n-1}\lambda^{n-2} + \ldots + d_qC^{q-2}_{n-1}\lambda^{n-q+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & d_q & . & d_qC^1_{q-1}\lambda^{q-2} & . & d_qC^1_{n-1}\lambda^{n-2} \\ d_q & d_q\lambda & . & d_q\lambda^{q-1} & . & d_q\lambda^{n-1} \end{bmatrix}.$$

Now we remove the first summands in all terms of the first $q-2$ rows by subtracting of them the $(q-1)$-*th* row, multiplied by d_{i+1}/d_q, $i = 1, \ldots, q-2$. After $q-1$ such manipulations $[D, DJ, \ldots, DJ^{n-1}]$ is transformed to the column of matrices (column–matrix below) $[b_{\xi_1}R^T(\lambda_1, q_1), \ldots, b_{\xi_s}R^T(\lambda_s, q_s)]^T$ with

$$d_qR^T(\lambda, q) = d_q \begin{bmatrix} 0 & 0 & . & 1 & C^{q-1}_q\lambda & . & C^{q-1}_{n-1}\lambda^{n-q} \\ 0 & 0 & . & C^{q-2}_{q-1}\lambda & C^{q-2}_q\lambda^2 & . & C^{q-2}_{n-1}\lambda^{n-q+1} \\ . & . & . & . & . & . & . \\ 0 & 1 & . & C^1_{q-1}\lambda^{q-2} & C^1_q\lambda^{q-1} & . & C^1_{n-1}\lambda^{n-2} \\ 1 & \lambda & . & \lambda^{q-1} & \lambda^q & . & \lambda^{n-1} \end{bmatrix}.$$

We calculate $det\ (Q_{con})$. Above we produced a number of manipulations with rows of Q_{con}. The value of the determinant did not change. Now we alter (into inverse) order of rows in all R^T matrix elements of the column–matrix. Every altering order of two neighbouring rows leads to changing the sign of the determinant. Altering (into inverse) order of q neighbouring rows of the matrix gives the multiplier $(-1)^{q(q-1)/2}$ into the determinant value. Then we transpose the obtained column–matrix into row–matrix with matrix elements $b_{\xi_j}R(\lambda_j, q_j)$:

$$det\ (Q_{con}) = det[b_{\xi_1}R^T(\lambda_1, q_1), \ldots, b_{\xi_s}R^T(\lambda_s, q_s)]^T =$$

(3.3) $$= (-1)^Q b^{q_1}_{\xi_1} b^{q_2}_{\xi_2} \ldots b^{q_s}_{\xi_s} det[R(\lambda_1, q_1), \ldots, R(\lambda_s, q_s)],\ Q = \frac{1}{2}\sum_{j=1}^{s} q_j(q_j - 1).$$

Here

(3.4) $$R(\lambda, q) = \begin{bmatrix} 1 & 0 & \ldots & 0 \\ \lambda & 1 & \ldots & 0 \\ \lambda^2 & C^1_2\lambda & \ldots & 0 \\ . & . & \ldots & . \\ \lambda^{q-1} & C^1_{q-1}\lambda^{q-2} & \ldots & 1 \\ \lambda^q & C^1_q\lambda^{q-1} & \ldots & C^{q-1}_q\lambda \\ . & . & \ldots & . \\ \lambda^{n-1} & C^1_{n-1}\lambda^{n-2} & \ldots & C^{q-1}_{n-1}\lambda^{n-q} \end{bmatrix}.$$

Put

(3.5) $$S(\lambda, q) = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ \lambda(1) & \lambda(2) & \ldots & \lambda(q) \\ \lambda(1)^2 & \lambda(2)^2 & \ldots & \lambda(q)^2 \\ . & . & \ldots & . \\ \lambda(1)^{n-1} & \lambda(2)^{n-1} & \ldots & \lambda(q)^{n-1} \end{bmatrix},$$

with $\lambda(l) = \lambda + l\epsilon$, $l = 1, \ldots, q$.

We calculate $det[R(\lambda_1, q_1), \ldots, R(\lambda_s, q_s)]$, starting from

$$det[S(\lambda_1, q_1), \ldots, S(\lambda_s, q_s)] = W(\kappa_1, \kappa_2, \ldots, \kappa_n).$$

Here

$$(\kappa_{\mu_j}, \kappa_{\mu_j+1}, \ldots, \kappa_{\xi_j}) = (\lambda_j(1), \lambda_j(2), \ldots, \lambda_j(q_j)) = \Lambda_j,$$

$$\lambda_j(l) = \lambda_j + l\epsilon_j, \quad j = 1, \ldots, s, \quad l = 1, \ldots, q_j.$$

Naturally, $\lambda_j(i) \to \lambda_j$, when $\epsilon_j \to 0$. We can choose ϵ_j satisfying inequalities

$$\epsilon_1 p_1 < \epsilon_2,\ \ \epsilon_2 q_2 < \epsilon_3, \ldots,\ \ \epsilon_{s-1}q_{s-1} < \epsilon_s.$$

Then for sufficiently small ϵ_s all κ_j are distinct. And when ϵ_j going to zero one by one, starting from ϵ_1, all κ_j stay distinct. In general ϵ_j can go to zero in any order. All κ_j stay distinct, if only we choose ϵ_j satisfying the proper chain of inequalities.

By well known Vandermonde determinant formula

$$W(\Lambda_1, \Lambda_2, \ldots, \Lambda_s) = W(\kappa_1, \kappa_2, \ldots, \kappa_n) = \prod_{1 \le i < j \le n} (\kappa_j - \kappa_i).$$

Let us separate factors, containing $\lambda_j(m)$ from any Λ_j, $m = 1, \ldots, q_j$:

$$\omega_{j_0} = \prod_{1 \le l < m \le q_j} (\lambda_j(m) - \lambda_j(l)) = W(\lambda_j(1), \lambda_j(2), \ldots, \lambda_j(q_j)) = W(\Lambda_j),$$

$$\omega_{j_1} = \prod_{\substack{1 \le i < j, \\ l=1,\ldots,q_i,\ m=1,\ldots,q_j}} (\lambda_j(m) - \lambda_i(l)); \quad \omega_{j_2} = \prod_{\substack{j < i \le s, \\ l=1,\ldots,q_i,\ m=1,\ldots,q_j}} (\lambda_i(l) - \lambda_j(m)).$$

Remark, that ω_{j_1} is the product of differences of $\lambda_j(m) \in \Lambda_j$ and $\lambda_i(l) \in \Lambda_i$ with $i < j$, and ω_{j_2} is the product of differences of $\lambda_j(m) \in \Lambda_j$ and $\lambda_i(l) \in \Lambda_i$ with $i > j$. So we have

(3.6) $$W(\Lambda_1, \ldots, \Lambda_s) = W(\Lambda_j)\omega_{j_1}\omega_{j_2}W(\Lambda_1, \ldots, \Lambda_{j-1}, \Lambda_{j+1}, \ldots, \Lambda_s).$$

We need an information about symmetric polynomials of order k with l variables

$$\theta_k(x_1, \ldots, x_l) = \sum_{k_1+k_2+\ldots+k_l=k} x_1^{k_1}x_2^{k_2}\ldots x_l^{k_l}.$$

Lemma. Let $N_k(l)$ be the number of terms in $\theta_k(x_1,\dots,x_l)$, then

$$N_k(l) = C^{l-1}_{k+l-1} = \binom{k+l-1}{l-1}. \tag{3.7}$$

The proof is shown by induction with respect to k. Formula (3.7) is true for $k=1$: $x_1+x_2+\ldots+x_l$ has l terms and $C^{l-1}_l = l$. Let (3.7) is true for the polynomials of order less or equal to $(k-1)$, show that then it's true for the polynomials of order k. It's clear, that $\theta_k(x_1,\dots,x_l) = \theta_k(x_1,\dots,x_{l-1}) + x_l\theta_{k-1}(x_1,\dots,x_l)$. This implies

$$N_k(l) = N_k(l-1) + N_{k-1}(l). \tag{3.8}$$

For all m,l the following relations hold

$$C^{l-1}_{m+l-1} - C^{l-2}_{m+l-2} = \frac{(m+l-1)(m+l-2)\dots(m+1)}{(l-1)(l-2)\dots 1} - \frac{(m+l-2)\dots(m+1)}{(l-2)\dots 1} =$$

$$= \frac{(m+l-1-(l-1))(m+l-2)\dots(m+1)}{(l-1)(l-2)\dots 1} = C^{l-1}_{m+l-2}.$$

We have, in particular, that

$$C^{l-1}_{k+l-1} - C^{l-2}_{k+l-2} = C^{l-1}_{k+l-2},$$
$$C^{l-2}_{k+l-2} - C^{l-3}_{k+l-3} = C^{l-2}_{k+l-3},$$
$$\dots\dots\dots,$$
$$C^2_{k+2} - C^1_{k+1} = C^2_{k+1},$$
$$C^1_{k+1} - C^0_k = C^1_k.$$

After summing these relations we get

$$C^{l-1}_{k+l-1} = 1 + C^1_k + C^2_{k+1} + C^3_{k+2} + \ldots + C^{l-1}_{k+l-2}. \tag{3.9}$$

From the other hand (3.8) gives

$$N_k(l) = N_k(l-1) + N_{k-1}(l) = N_k(l-2) + N_{k-1}(l-1) + N_{k-1}(l) = \dots$$

$$= N_k(1) + N_{k-1}(2) + N_{k-1}(3) + \ldots + N_{k-1}(l).$$

The induction assumption and the trivial relation $N_k(1) = 1$ imply

$$N_k(l) = 1 + C^1_k + C^2_{k+1} + \ldots + C^{l-1}_{k+l-2}.$$

This and (3.9) give $N_k(l) = C^{l-1}_{k+l-1}$. The proof of **Lemma** is finished.

We produce a number of manipulations with the columns of $S(\lambda_j,q_j)$. For the sake of simplicity we do this with $S(\lambda,q)$ (see (3.5)). We subtract the first column from the rest $q-1$ columns and take out $(\lambda(2)-\lambda(1)),\dots,(\lambda(q)-\lambda(1))$ respectively:

$$W(\Lambda_1,\dots,\Lambda_s) =$$

$$= \prod_{l=2}^{q_j}(\lambda_j(l)-\lambda_j(1))det[S(\lambda_1,q_1),.,S(\lambda_{j-1},q_{j-1}),S^1(\lambda_j,q_j),S(\lambda_{j+1},q_{j+1}),.,S(\lambda_s,q_s)],$$

$$S^1(\lambda,q) = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \lambda(1) & 1 & \dots & 1 \\ \lambda^2(1) & \theta_1(\lambda(1),\lambda(2)) & \dots & \theta_1(\lambda(1),\lambda(q)) \\ \lambda^3(1) & \theta_2(\lambda(1),\lambda(2)) & \dots & \theta_2(\lambda(1),\lambda(q)) \\ . & . & \dots & . \\ \lambda^{n-1}(1) & \theta_{n-2}(\lambda(1),\lambda(2)) & \dots & \theta_{n-2}(\lambda(1),\lambda(q)) \end{bmatrix}.$$

On the second step we subtract the second column of $S^1(\lambda,q)$ from the last $q-2$ columns and take out $(\lambda(3)-\lambda(2)),\dots,(\lambda(q)-\lambda(2))$ respectively.

Remark. Note, that

$$\theta_2(x_1,x_2) = x_1^2 + x_1x_2 + x_2^2, \quad \theta_2(x_1,x_m) = x_1^2 + x_1x_m + x_m^2.$$

This gives

$$\frac{\theta_2(x_1,x_m) - \theta_2(x_1,x_2)}{x_m - x_2} = x_1 + x_2 + x_m = \theta_1(x_1,x_2,x_m).$$

In general case

$$\theta_k(x_1,..,x_l,x_{l+1}) = \theta_k(x_1,..,x_l) + x_{l+1}\theta_{k-1}(x_1,..,x_l) + \ldots + x_{l+1}^{k-1}\theta_1(x_1,..,x_l) + x_{l+1}^k,$$

$$\theta_k(x_1,..,x_l,x_m) = \theta_k(x_1,..,x_l) + x_m\theta_{k-1}(x_1,..,x_l) + \ldots + x_m^{k-1}\theta_1(x_1,..,x_l) + x_m^k.$$

This gives

$$\frac{\theta_k(x_1,..,x_l,x_m) - \theta_k(x_1,..,x_l,x_{l+1})}{x_m - x_{l+1}} = \theta_{k-1}(x_1,..,x_l) + (x_{l+1}+x_m)\theta_{k-2}(x_1,..,x_l) +$$

$$\ldots + \theta_{k-2}(x_{l+1},x_m)\theta_1(x_1,..,x_l) + \theta_{k-1}(x_{l+1},x_m) = \theta_{k-1}(x_1,..,x_l,x_{l+1},x_m).$$

After the second step of the manipulations we have

$$W(\Lambda_1,\dots,\Lambda_s) = \prod_{l=2}^{q_j}(\lambda_j(l)-\lambda_j(1))\prod_{l=3}^{q_j}(\lambda_j(l)-\lambda_j(2))\times$$

$$\times det[S(\lambda_1,q_1),.,S(\lambda_{j-1},q_{j-1}),S^2(\lambda_j,q_j),S(\lambda_{j+1},q_{j+1}),.,S(\lambda_s,q_s)],$$

where $S^2(\lambda,q) =$

$$= \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \lambda(1) & 1 & 0 & \dots & 0 \\ \lambda^2(1) & \theta_1(\lambda(1),\lambda(2)) & 1 & \dots & 1 \\ \lambda^3(1) & \theta_2(\lambda(1),\lambda(2)) & \theta_1(\lambda(1),\lambda(2),\lambda(3)) & \dots & \theta_1(\lambda(1),\lambda(2),\lambda(q)) \\ . & . & . & \dots & . \\ \lambda^{n-1}(1) & \theta_{n-2}(\lambda(1),\lambda(2)) & \theta_{n-3}(\lambda(1),\lambda(2),\lambda(3)) & \dots & \theta_{n-3}(\lambda(1),\lambda(2),\lambda(q)) \end{bmatrix}.$$

After $q_j - 1$ steps we have

$$W(\Lambda_1,\dots,\Lambda_s) = \prod_{l=2}^{q_j}(\lambda_j(l)-\lambda_j(1))\prod_{l=3}^{q_j}(\lambda_j(l)-\lambda_j(2))\dots\prod_{l=q_j}^{q_j}(\lambda_j(l)-\lambda_j(q_{j-1}))\times$$

(3.10) $\times det[S(\lambda_1,q_1),.,S(\lambda_{j-1},q_{j-1}),S^{q_j-1}(\lambda_j,q_j),S(\lambda_{j+1},q_{j+1}),.,S(\lambda_s,q_s)],$

where $S^{q-1}(\lambda,q) =$

$$=\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \lambda(1) & 1 & 0 & \dots & 0 \\ \lambda^2(1) & \theta_1(\lambda(1),\lambda(2)) & 1 & \dots & 1 \\ . & . & . & \dots & . \\ \lambda^{q-1}(1) & \theta_{q-2}(\lambda(1),\lambda(2)) & \theta_{q-3}(\lambda(1),\lambda(2),\lambda(3)) & \dots & 1 \\ \lambda^q(1) & \theta_{q-1}(\lambda(1),\lambda(2)) & \theta_{q-2}(\lambda(1),\lambda(2),\lambda(3)) & \dots & \theta_1(\lambda(1),\lambda(2),\dots,\lambda(q)) \\ . & . & . & \dots & . \\ \lambda^{n-1}(1) & \theta_{n-2}(\lambda(1),\lambda(2)) & \theta_{n-3}(\lambda(1),\lambda(2),\lambda(3)) & \dots & \theta_{n-q}(\lambda(1),\lambda(2),\dots,\lambda(q)) \end{bmatrix}$$

Remark that in (3.10) we have

$$\prod_{l=2}^{q_j}(\lambda_j(l)-\lambda_j(1))\prod_{l=3}^{q_j}(\lambda_j(l)-\lambda_j(2))\dots(\lambda_j(q_j)\prod_{l=q_j}^{q_j}(\lambda_j(l)-\lambda_j(q_j-1)) = W(\Lambda_j).$$

By using (3.6) and (3.10) we find

$$det[S(\lambda_1,q_1),.,S(\lambda_{j-1},q_{j-1}),S^{q_j-1}(\lambda_j,q_j),S(\lambda_{j+1},q_{j+1}),.,S(\lambda_s,q_s)] =$$

$$= \omega_{j_1}\omega_{j_2}W(\Lambda_1,\dots,\Lambda_{j-1},\Lambda_{j+1},\dots,\Lambda_s).$$

When $\epsilon_j \to 0$ all $\lambda_j(i)$, $i=1,\dots,q_j$, go to λ_j, and all $\theta_k(\lambda_j(1),\lambda_j(2),\dots,\lambda_j(l))$ go to $N_k(l)\lambda_j^k = C_{k+l-1}^{l-1}\lambda_j^k$. (see (3.7)). The row of $S^{q_j-1}(\lambda_j,q_j)$ with number $i+1$ takes the form

$$[\lambda_j^i, C_i^1\lambda_j^{i-1}, C_i^2\lambda_j^{i-2},\dots,C_i^{i-1}\lambda_j,1,0,\dots,0].$$

This is exactly the row of $R(\lambda_j,q_j)$ with number $i+1$ (see (3.4)).

$$\omega_{j_1} = \prod_{\substack{1\le i<j,\\ l=1,\dots,q_i,\ m=1,\dots,q_j}}(\lambda_j(m)-\lambda_i(l)) \quad\to\quad \prod_{\substack{1\le i<j,\\ l=1,\dots,q_i,}}(\lambda_j-\lambda_i(l))^{q_j}.$$

At last, when $\epsilon_j \to 0$,

$$\omega_{j_2} = \prod_{\substack{j<i\le s,\\ l=1,\dots,q_i,\ m=1,\dots,q_j}}(\lambda_i(l)-\lambda_j(m)) \quad\to\quad \prod_{\substack{j<i\le s,\\ l=1,\dots,q_i,}}(\lambda_i(l)-\lambda_j)^{q_j}.$$

As result we proved, that

$$det\ [S(\lambda_1,q_1),.,S(\lambda_{j-1},q_{j-1}),R(\lambda_j,q_j),S(\lambda_{j+1},q_{j+1}),.,S(\lambda_s,q_s)] =$$

$$= \prod_{\substack{1\le i<j,\\ l=1,\dots,q_i,}}(\lambda_j-\lambda_i(l))^{q_j}\prod_{\substack{j<i\le s,\\ l=1,\dots,q_i,}}(\lambda_i(l)-\lambda_j)^{q_j}W(\Lambda_1,\dots,\Lambda_{j-1},\Lambda_{j+1},\dots,\Lambda_s).$$

We produce the same manipulations with all $S(\lambda_l,q_l)$, $l=1,2,\dots,s$, and by using (3.3) we get the following:

(3.11) $$det(Q_{con}) = (-1)^Q b_{\xi_1}^{q_1}\dots b_{\xi_s}^{q_s}\prod_{1\le i<j\le s}(\lambda_j-\lambda_i)^{q_i+q_j}.$$

It is clear now, that $det(Q_{con})\ne 0$, if and only if $b_{\xi_1},\dots,b_{\xi_s}$ are nonzeros and λ_j, corresponding to the different Jordan boxes (including degenerate "scalar boxes") are distinct.

The controllability theorem is proved in the case, when the matrix G has the unique column.

Now we consider the case, when the matrix G has p columns and **condition 2** holds: for each $j=1,2,\dots,s$ there is at least one g_i (column of $G=[g_1,g_2,\dots,g_p]$), which has nonzero $g_i(\xi_j)$ component with respect to the generating vector E_{ξ_j} (in the Jordan basis of eigenvectors and adjoint vectors). First we show, that there are found constants $c_1,c_2,\dots,c_p$ such that vector $v=\sum_{j=1}^p c_jg_j$ has nonzero components with respect to all generating vectors: $v(\xi_j)\ne 0$, $j=1,\dots,s$. Let g_{i_1} be first of $g_1,g_2,\dots,g_p$, having nonzero component with respect to the first generating vector: $g_{i_1}(\xi_1)\ne 0$, but $g_j(\xi_1)=0$, $j<i_1$. If $g_{i_1}(\xi_j)\ne 0$, $j=1,2,\dots,s$, then $v=g_{i_1}$. Otherwise let η_1 is the number of first (among $\xi_1,\dots,\xi_s$) zero component of g_{i_1} with respect to to the generating vectors: $g_{i_1}(\eta_1)=0$ but $g_{i_1}(\xi_j)\ne 0$, $\xi_j<\eta_1$. By **condition 2** there is found $i_2\ne i_1$, $1\le i_2\le p$ such that $g_{i_2}(\eta_1)\ne 0$. Consider vector

$$v_1 = g_{i_1}+\frac{1}{2}\left(\min_{\xi_j<\eta_1}|g_{i_1}(\xi_j)|\frac{g_{i_2}}{\max_{1\le j\le s}|g_{i_2}(\xi_j)|}\right).$$

The component $v_1(\eta_1)\ne 0$ surely. Moreover, the vector v_1 has nonzero components in all places ξ_j, where $g_{i_1}(\xi_j)\ne 0$ or $g_{i_2}(\xi_j)\ne 0$. So the number of nonzero components with respect to the generating vectors increases. The next step is following. If $v_1(\xi_j)\ne 0$, for $j=1,\dots,s$, then $v=v_1$. Otherwise let η_2 is the number of first (among $\xi_1,\dots,\xi_s$) zero component of v_1 with respect to the generating vectors: $v_1(\eta_2)=0$ but $v_1(\xi_j)\ne 0$, $\xi j<\eta_2$. By **condition 2** there is found $i_3\ne i_1$, $i_3\ne i_2$, $1\le i_3\le p$ such that $g_{i_3}(\eta_2)\ne 0$. Consider vector

$$v_2 = v_1+\frac{1}{2}\left(\min_{\xi_j<\eta_2}|v_1(\xi_j)|\frac{g_{i_3}}{\max_{1\le j\le s}|g_{i_3}(\xi_j)|}\right).$$

The component $v_2(\eta_2) \neq 0$. Moreover, the vector v_2 has nonzero components in all places ξ_j, where $v_1(\xi_j) \neq 0$ or $g_{i_3}(\xi_j) \neq 0$. So the number of nonzero components with respect to the generating vectors increases again. And not more than in s steps we get $v = v_r$, $r < s$, with $v_r(\xi_j) \neq 0$, $j = 1, 2, \ldots, s$. And found v is linear combination of the columns of the matrix G: $v = v_r = \sum_{j=1}^{p} c_j g_j$. We proved above, that if **condition 1** holds, then

$$det\,[\sum_{j=1}^{p} c_j g_j, F\sum_{j=1}^{p} c_j g_j, \ldots, F^{n-1}\sum_{j=1}^{p} c_j g_j] \neq 0.$$

This gives, that each $n-$dimentional vector can be presented as linear combination of the vectors

$$\sum_{j=1}^{p} c_j g_j, \quad F\sum_{j=1}^{p} c_j g_j, \quad \ldots, \quad F^{n-1}\sum_{j=1}^{p} c_j g_j,$$

what gives at the same time that each $n-$dimentional vector can be presented as linear combination of the vectors

$$g_1, \ldots, g_p, Fg_1, \ldots, Fg_p, \ldots, F^{n-1}g_1, \ldots, F^{n-1}g_p.$$

In other words, $rank[G, FG, \ldots, F^{n-1}G] = n$. **The controllability theorem** is proved.

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Шерер Р., Сердюкова С.И. E5-96-445
Необходимые и достаточные условия минимальности линейных динамических систем

Доказаны необходимые и достаточные условия минимальности линейных динамических систем $\{F,G,H\}$. Они формулируются в виде условий на G,H,J (нормальную жорданову форму матрицы F) и жорданов базис собственных и присоединенных векторов, в котором F имеет вид J. Проблема проверки минимальности сводится к классической проблеме нахождения собственных векторов и генерирующих векторов циклических собственных инвариантных подпространств матрицы F.

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Scherer R., Serdyukova S. E5-96-445
Necessary and Sufficient Conditions for Minimality of Linear Dynamic Systems

The necessary and sufficient conditions for minimality of linear dynamic systems $\{F,G,H\}$ are presented. They are formulated in terms of G,H,J (the Jordan canonical form of F) and the basis of eigenvectors and adjoint vectors, in which F takes form J. The problem of minimality verification reduces to the classic problem of finding eigenvalues, eigenvectors and generating vectors of cyclic invariant eigenspaces of the matrix F.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR and at the Institut für Praktische Mathematik, Universität Karlsruhe, Germany.

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